

Warm-up: What to eat?

We are trying healthy by finding the optimal amount of food to purchase.

We can choose the amount of **stir-fry** (ounce) and **boba** (fluid ounces).

Healthy Squad Goals

- $2000 \leq \text{Calories} \leq 2500$
- $\text{Sugar} \leq 100 \text{ g}$
- $\text{Calcium} \geq 700 \text{ mg}$

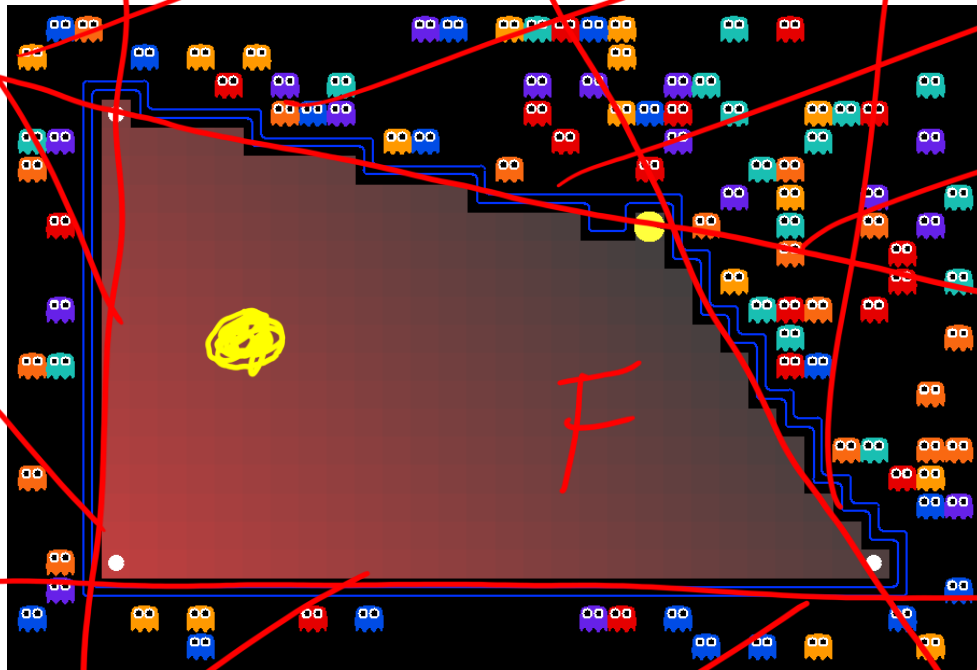
Food	Cost	Calories	Sugar	Calcium
Stir-fry (per oz)	1	100	3	20
Boba (per fl oz)	0.5	50	4	70

What is the cheapest way to stay “healthy” with this menu?

How much **stir-fry** (ounce) and **boba** (fluid ounces) should we buy?

AI: Representation and Problem Solving

Linear Programming



Instructor: Pat Virtue

Slide credits: CMU AI with drawings from <http://ai.berkeley.edu>

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Optimization

Problem
Description

Optimization
Representation

min.

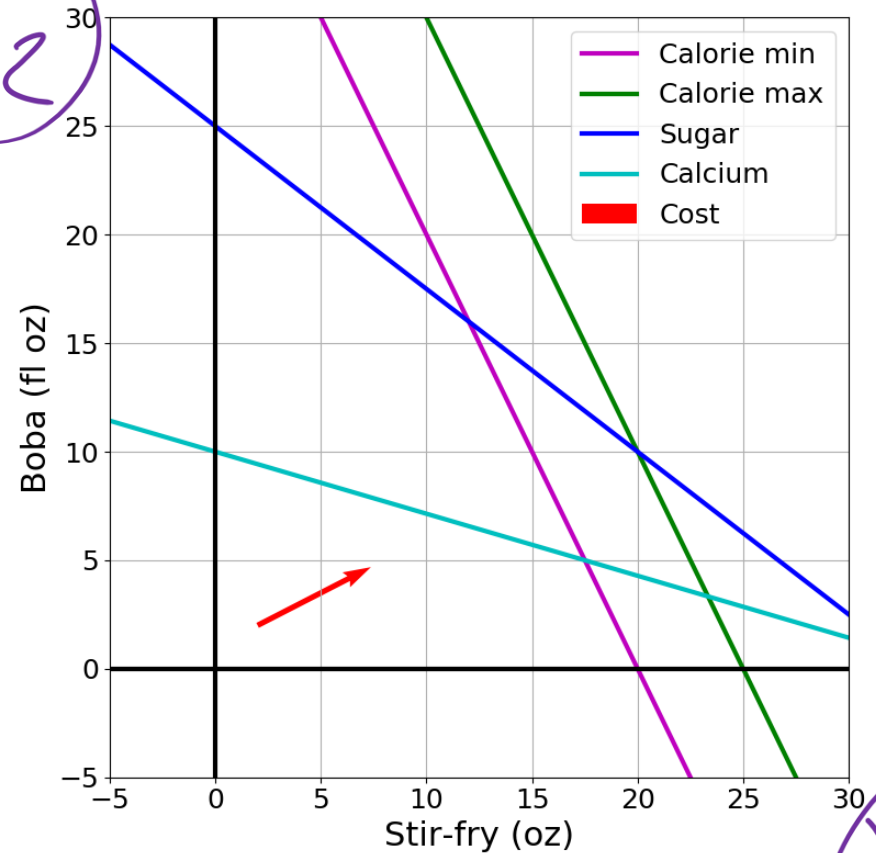
$\mathbf{x}$

s.t.

$\mathbf{c}^T \mathbf{x}$

$A\mathbf{x} \leq \mathbf{b}$

Graphical Representation



Constraint Satisfaction Problems

Map coloring

Any

$\mathbf{x}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

s.t.

$\mathbf{x}$ satisfies constraints

"such that"

Assume vectors are column vectors

Notation Alert!

$$\begin{matrix} \uparrow & \uparrow & \uparrow \\ [1 \times 3] & [3 \times 1] & [u_1, u_2, u_3] \end{matrix} \begin{matrix} \xrightarrow{T} \xrightarrow{V} \\ \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \end{matrix}$$

$$\begin{matrix} \xrightarrow{u} \xrightarrow{v} \\ \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \end{matrix}$$

$$\begin{matrix} \text{frown} \\ [3 \times 1] [3 \times 1] \end{matrix}$$

$$\begin{matrix} uv & \times & u^T v \\ u \cdot v & \times & \\ u \circ v & ? & \\ u \times v & \times & \text{(cross)} \end{matrix}$$

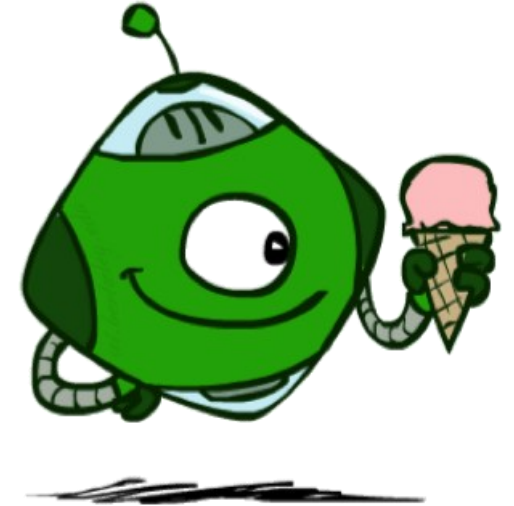


Optimization Formulation

Diet Problem

Any $\mathbf{x}$

s.t. $\mathbf{x}$ satisfies constraints



Healthy Squad Goals

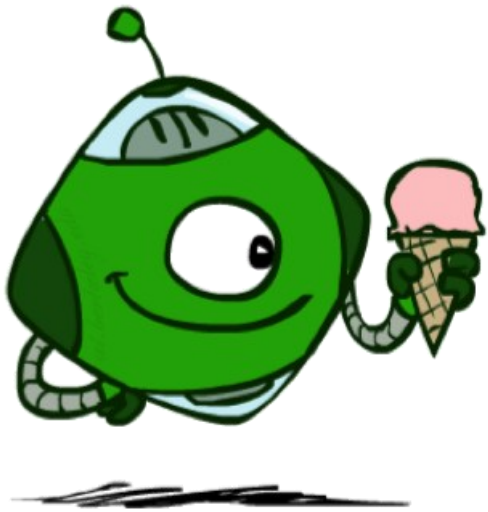
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Optimization Formulation

Diet Problem

$$\begin{array}{ll} \min_{\mathbf{x}} & \text{cost}(\mathbf{x}) \quad \text{Objective function} \\ \text{s.t.} & \mathbf{x} \text{ satisfies constraints} \end{array}$$



Healthy Squad Goals

- $2000 \leq \text{Calories} \leq 2500$
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- $\text{Calcium} \geq 700 \text{ mg}$

Notation Alert!

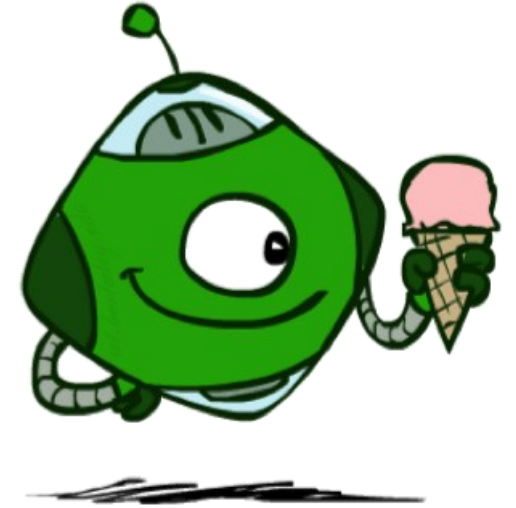
x_1
 x_2

Food	Cost	Calories	Sugar	Calcium
Stir-fry (per oz)	1	100	3	20
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Optimization Formulation

Diet Problem

$$\begin{array}{ll}\min_{\mathbf{x}} & \text{cost}(\mathbf{x}) \\ \text{s.t.} & \text{calories}(\mathbf{x}) \text{ contained} \\ & \text{sugar}(\mathbf{x}) \leq \text{limit} \\ & \text{calcium}(\mathbf{x}) \geq \text{limit}\end{array}$$



Healthy Squad Goals

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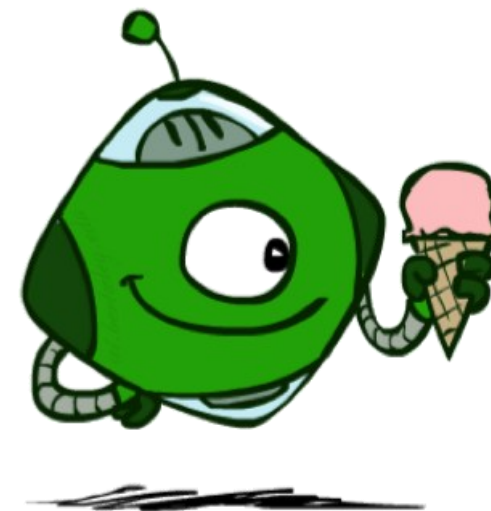
Food	Cost	Calories	Sugar	Calcium
→ Stir-fry (per oz)	1	100	3	20
→ Boba (per fl oz)	0.5	50	4	70

Optimization Formulation

Diet Problem

$$\begin{array}{ll}\min_{x_1, x_2} & 1x_1 + 0.5x_2 \\ \text{s.t.} & 100x_1 + 50x_2 \geq 2000 \\ & 100x_1 + 50x_2 \leq 2500 \\ & 3x_1 + 4x_2 \leq 100 \\ & 20x_1 + 70x_2 \geq 700\end{array}$$

$A \vec{x} \leq \vec{b}$



Healthy Squad Goals

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Notation Alert!

Optimization Formulation

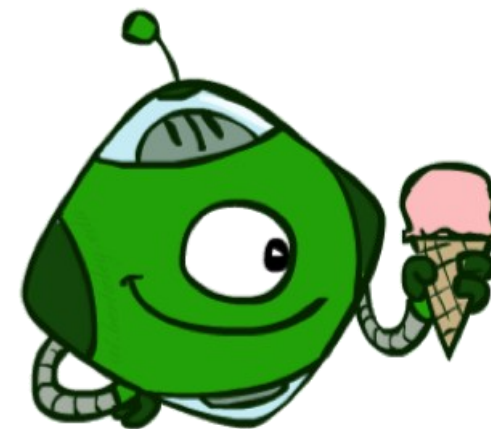
Diet Problem

$$\begin{aligned} \min_{x_1, x_2} \quad & c_1 x_1 + c_2 x_2 \\ \text{s.t.} \quad & a_{1,1} x_1 + a_{1,2} x_2 \geq b_1 \\ & a_{2,1} x_1 + a_{2,2} x_2 \leq b_2 \\ & a_{3,1} x_1 + a_{3,2} x_2 \leq b_3 \\ & a_{4,1} x_1 + a_{4,2} x_2 \geq b_4 \end{aligned}$$

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} \\ a_{4,1} & a_{4,2} \end{bmatrix}$$

✓ Notation Alert!

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$



Cost

$$\mathbf{c} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Limit

$$A = \begin{matrix} \begin{matrix} \text{Stir-fry} & \text{Boba} \end{matrix} \\ \begin{bmatrix} 100 & 50 \\ 100 & 50 \\ 3 & 4 \\ 20 & 70 \end{bmatrix} \end{matrix}$$

$$\mathbf{b} = \begin{bmatrix} 2000 \\ 2500 \\ 100 \\ 700 \end{bmatrix} \begin{matrix} \text{Calorie min} \\ \text{Calorie max} \\ \text{Sugar} \\ \text{Calcium} \end{matrix}$$

Optimization Formulation

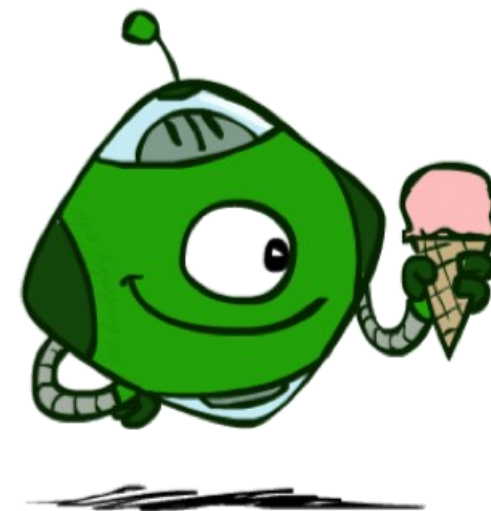
Diet Problem

$$\min_{\mathbf{x}} \quad \mathbf{c}^T \mathbf{x}$$

$$\begin{aligned} \text{s.t.} \quad & a_{1,1} x_1 + a_{1,2} x_2 \geq b_1 \\ & a_{2,1} x_1 + a_{2,2} x_2 \leq b_2 \\ & a_{3,1} x_1 + a_{3,2} x_2 \leq b_3 \\ & a_{4,1} x_1 + a_{4,2} x_2 \geq b_4 \end{aligned}$$

$$A = \begin{array}{cc} \text{Stir-fry} & \text{Boba} \\ \begin{bmatrix} 100 & 50 \\ 100 & 50 \\ 3 & 4 \\ 20 & 70 \end{bmatrix} \end{array}$$

$$\mathbf{b} = \begin{array}{c} \text{Limit} \\ \begin{bmatrix} 2000 \\ 2500 \\ 100 \\ 700 \end{bmatrix} \end{array} \begin{array}{l} \text{Calorie min} \\ \text{Calorie max} \\ \text{Sugar} \\ \text{Calcium} \end{array}$$



Cost

$$\mathbf{c} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$$

Notation Alert!

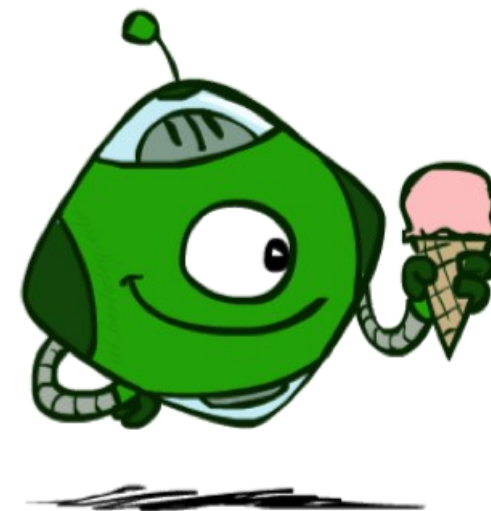
Optimization Formulation

Diet Problem

$$\begin{array}{ll}\min_{\mathbf{x}} & \mathbf{c}^T \mathbf{x} \\ \text{s.t.} & -a_{1,1} x_1 - a_{1,2} x_2 \leq -b_1 \\ & a_{2,1} x_1 + a_{2,2} x_2 \leq b_2 \\ & a_{3,1} x_1 + a_{3,2} x_2 \leq b_3 \\ & -a_{4,1} x_1 - a_{4,2} x_2 \leq -b_4\end{array}$$

$$A = \begin{array}{cc} \text{Stir-fry} & \text{Boba} \\ \begin{bmatrix} 100 & 50 \\ 100 & 50 \\ 3 & 4 \\ 20 & 70 \end{bmatrix} \end{array}$$

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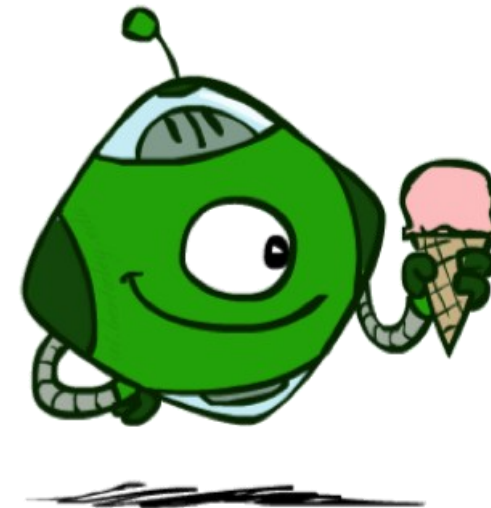
$$\mathbf{c} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}^T \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Optimization Formulation

Diet Problem

$$\min_{\mathbf{x}} \quad \mathbf{c}^T \mathbf{x}$$

$$\text{s.t.} \quad \begin{aligned} a_{1,1} x_1 + a_{1,2} x_2 &\leq b_1 \\ a_{2,1} x_1 + a_{2,2} x_2 &\leq b_2 \\ a_{3,1} x_1 + a_{3,2} x_2 &\leq b_3 \\ a_{4,1} x_1 + a_{4,2} x_2 &\leq b_4 \end{aligned}$$



Cost

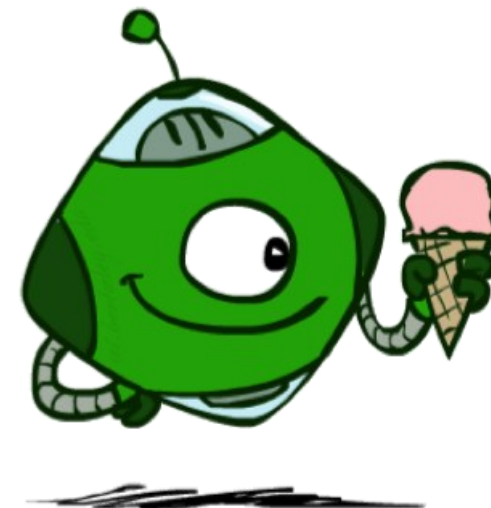
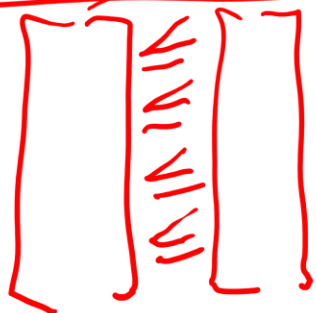
$$\mathbf{c} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$$

	Stir-fry	Boba		Limit	
$A =$	$\begin{bmatrix} -100 & -50 \\ 100 & 50 \\ 3 & 4 \\ -20 & -70 \end{bmatrix}$	$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$	$\mathbf{b} =$	$\begin{bmatrix} -2000 \\ 2500 \\ 100 \\ -700 \end{bmatrix}$	Calorie min ← Calorie max ← Sugar ← Calcium ← Fat +
	<u> </u>	<u> </u> X		<u> </u>	

Optimization Formulation

Diet Problem

$$\begin{array}{ll}\min_{\mathbf{x}} & \mathbf{c}^T \mathbf{x} \\ \text{s.t.} & A\mathbf{x} \leq \mathbf{b}\end{array}$$



Cost

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Calorie min
Calorie max
Sugar
Calcium

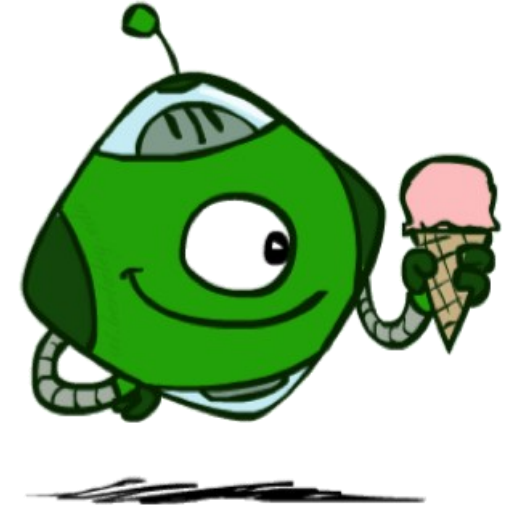
Handwritten red annotations: $[4 \times 2]$ and $[2 \times 1]$ are written above the matrix A . The words "Stir-fry" and "Boba" are written in blue and crossed out with red lines over the first two columns of A .

Notation Alert!

Poll 1

What has to increase to add more nutrition constraints?

$$\begin{array}{ll} \min_{\mathbf{x}} & \mathbf{c}^T \mathbf{x} \\ \text{s.t.} & A\mathbf{x} \leq \mathbf{b} \end{array}$$



Select all that apply

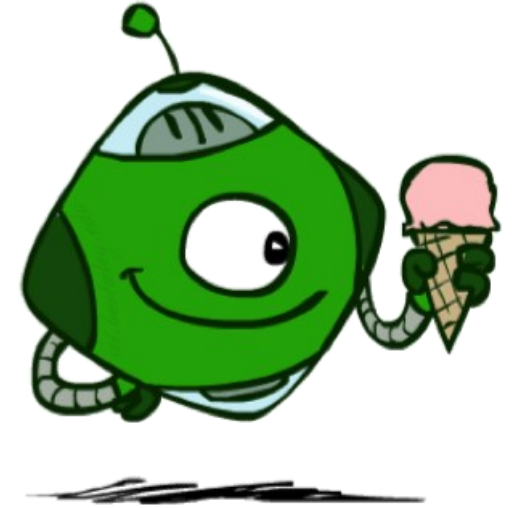
- ☒ A) length $\mathbf{x}$
- ☒ B) length $\mathbf{c}$
- ☒ C) height A
- ☐ D) width A
- ☒ E) length $\mathbf{b}$

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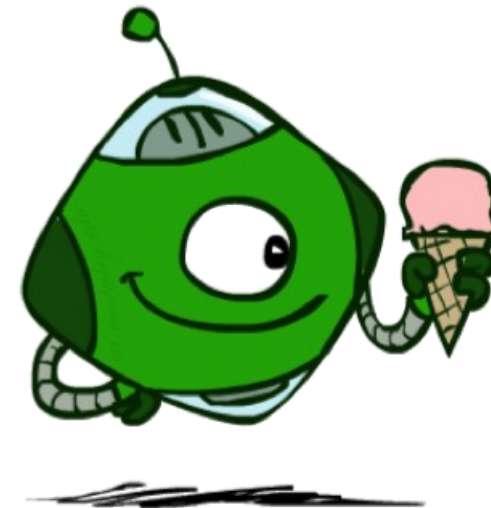
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~~Poll 2~~

Bad Wifi

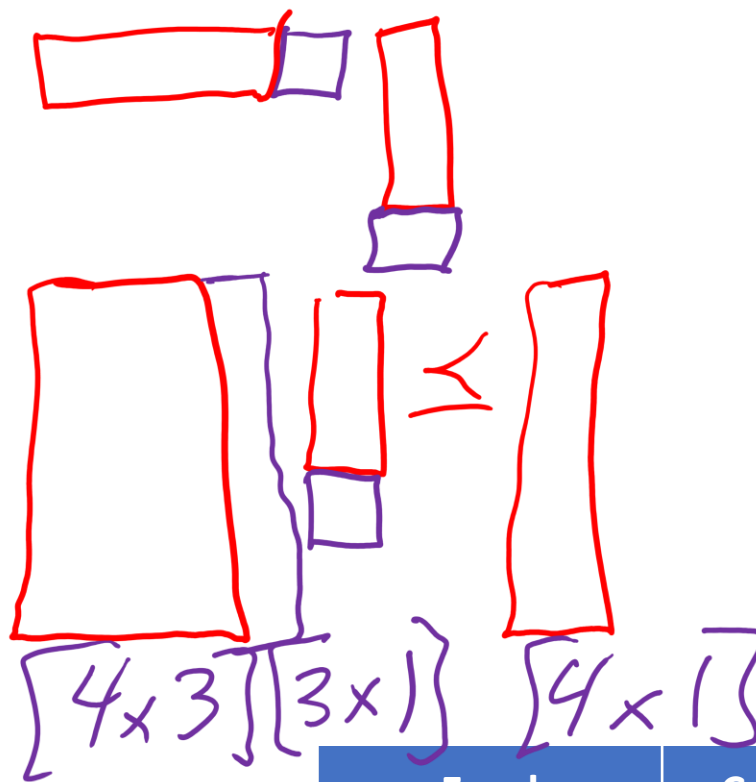


What has to increase to add more menu items?

$$\begin{array}{ll} \min_{\mathbf{x}} & \mathbf{c}^T \mathbf{x} \\ \text{s.t.} & A\mathbf{x} \leq \mathbf{b} \end{array}$$

Select all that apply

- ✓ A) length $\mathbf{x}$
- ✓ B) length $\mathbf{c}$
- ✗ C) height A
- ✓ D) width A
- ✗ E) length $\mathbf{b}$



Food	Cost	Calories	Sugar	Calcium
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veg (oz)



Poll 2

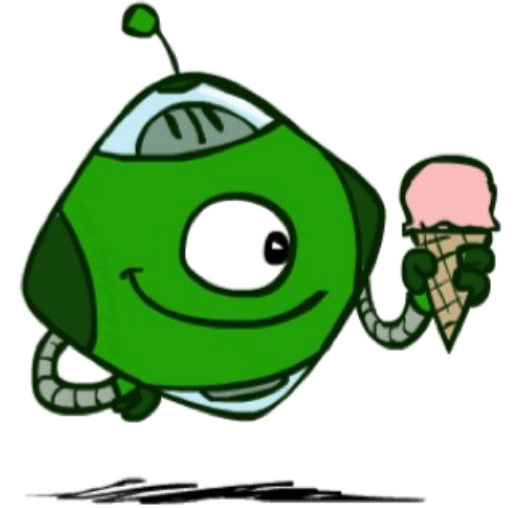
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$$\begin{array}{ll}\min_{\mathbf{x}} & \mathbf{c}^T \mathbf{x} \\ \text{s.t.} & A\mathbf{x} \leq \mathbf{b}\end{array}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \mathbf{c} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$$

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Question

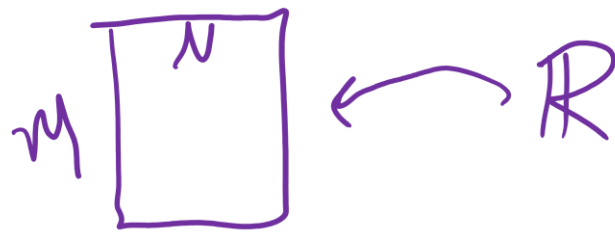
If $A \in \mathbb{R}^{M \times N}$, which of the following also equals N ?

$$\begin{array}{ll} \min_{\mathbf{x}} & \mathbf{c}^T \mathbf{x} \\ \text{s.t.} & A\mathbf{x} \leq \mathbf{b} \end{array}$$

Select all that apply

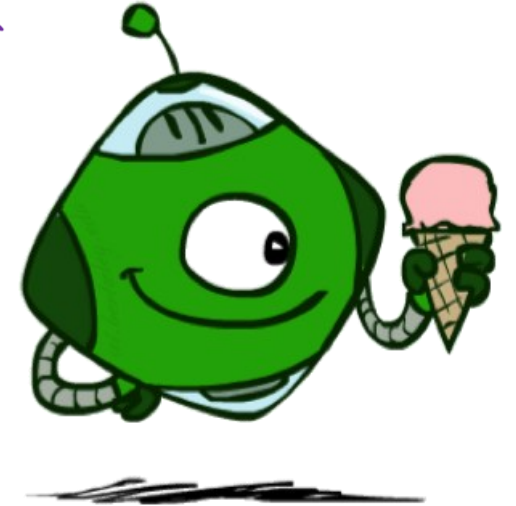
- A) length $\mathbf{x}$
- B) length $\mathbf{c}$
- C) length $\mathbf{b}$

Notation Alert!



$$\vec{x} \in \mathbb{R}^2$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$



Linear Programming

Linear objective with linear constraints

$$\begin{array}{ll}\min. & \mathbf{c}^\top \mathbf{x} \\ \text{s.t.} & A\mathbf{x} \leq \mathbf{b}\end{array}$$

As opposed to general optimization

$$\begin{array}{ll}\min. & f_0(\mathbf{x}) \\ \text{s.t.} & f_i(\mathbf{x}) \leq 0, \quad i = 1 \dots M \\ & \mathbf{a}_i^\top \mathbf{x} = \mathbf{b}_i, \quad i = 1 \dots P\end{array}$$

Linear Programming

Different formulations

Inequality form

$$\begin{array}{ll}\min. & \mathbf{c}^\top \mathbf{x} \\ \text{s.t.} & A\mathbf{x} \leq \mathbf{b}\end{array}$$

General form

$$\begin{array}{ll}\min. & \mathbf{c}^\top \mathbf{x} + \mathbf{d} \\ \text{s.t.} & G\mathbf{x} \leq \mathbf{h} \\ & A\mathbf{x} = \mathbf{b}\end{array}$$

Standard form

$$\begin{array}{ll}\min. & \mathbf{c}^\top \mathbf{x} \\ \text{s.t.} & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq 0\end{array}$$

Important to pay attention to form!

Can switch between formulations!

Optimization

Problem
Description

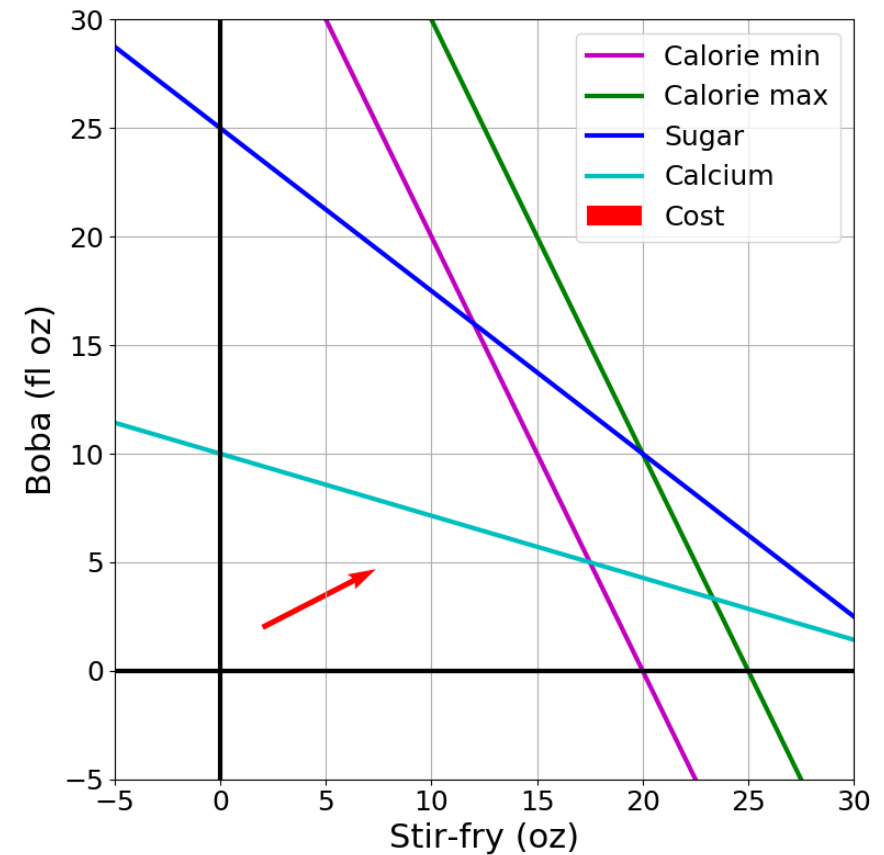


Optimization
Representation

$$\begin{array}{ll} \min_{\mathbf{x}} & \mathbf{c}^T \mathbf{x} \\ \text{s.t.} & A\mathbf{x} \leq \mathbf{b} \end{array}$$



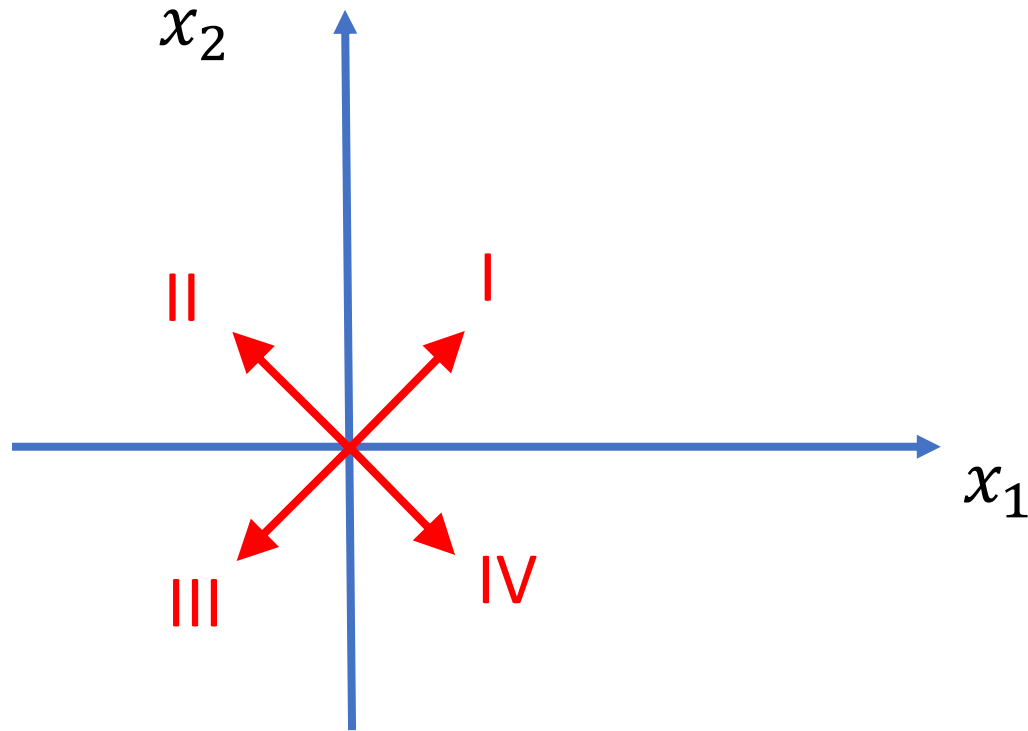
Graphical Representation



Poll 3

Which of these points have cost $\mathbf{c}^T \mathbf{x} = 0$?

for cost vector: $\mathbf{c} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$



$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}^T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \rightarrow c_1 x_1 + c_2 x_2$$