Warm-up: What to eat?

We are trying healthy by finding the optimal amount of food to purchase.

We can choose the amount of stir-fry (ounce) and boba (fluid ounces).

Healthy Squad Goals

- $2000 \leq \text{Calories} \leq 2500$
- $\text{Sugar} \leq 100$ g
- $\text{Calcium} \geq 700$ mg

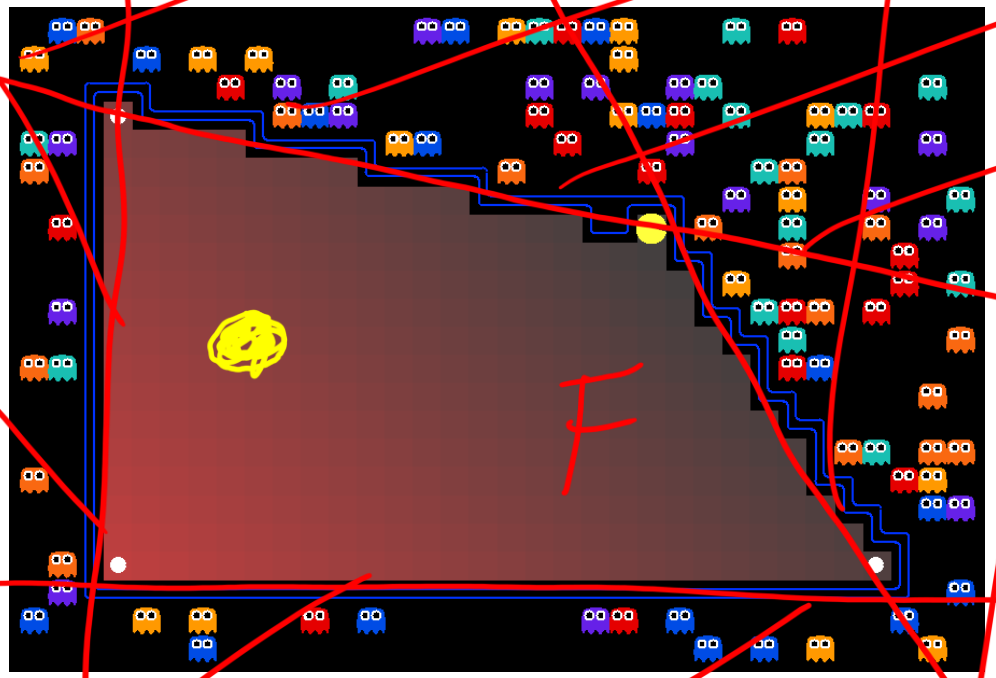
Food	Cost	Calories	Sugar	Calcium
Stir-fry (per oz)	1	100	3	20
Boba (per fl oz)	0.5	50	4	70

What is the cheapest way to stay "healthy" with this menu?

How much stir-fry (ounce) and boba (fluid ounces) should we buy?

AI: Representation and Problem Solving

Linear Programming



Instructor: Pat Virtue

Slide credits: CMU AI with drawings from http://ai.berkeley.edu

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Optimization

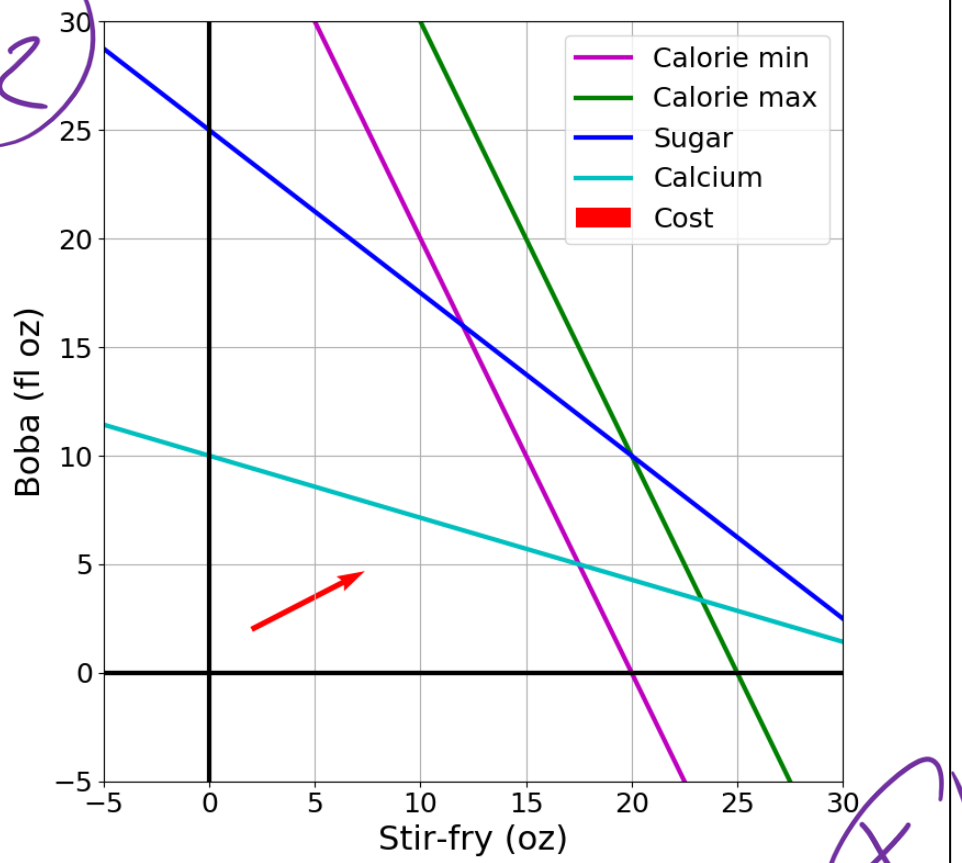
Problem Description

Optimization Representation

$$\min_{\mathbf{x}} \quad \mathbf{c}^\top \mathbf{x}$$

$$\text{s.t.} \quad A\mathbf{x} \preceq \mathbf{b}$$

Graphical Representation



Constraint Satisfaction Problems

Map coloring

$Any \quad \mathbf{x}$

s.t. $\mathbf{x}$ satisfies constraints

"such that"

Assume vectors are column vectors

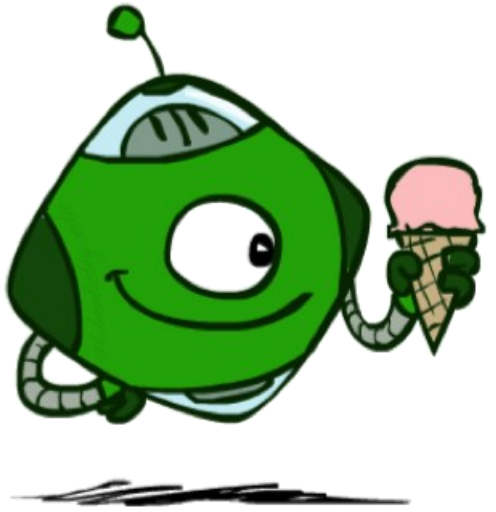
Notation Alert!

Optimization Formulation

Diet Problem

$Any \quad \mathbf{x}$

s.t. $\mathbf{x}$ satisfies constraints



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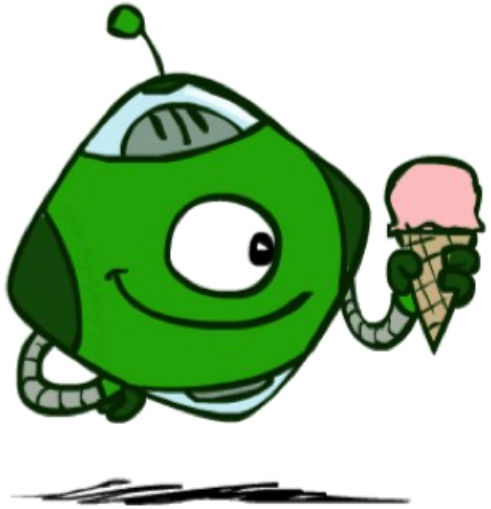
Optimization Formulation

Diet Problem

$$\min_{\mathbf{x}} \quad cost(\mathbf{x}) \qquad \text{Objective function}$$

$$\text{s.t.} \quad \mathbf{x} \text{ satisfies constraints}$$



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Boba (per fl oz)	0.5	50	4	70

x_1

x_2

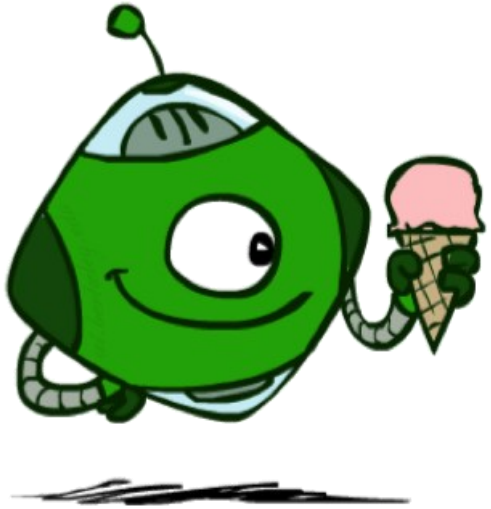
Notation Alert!

Optimization Formulation

Diet Problem

$$\min_{\mathbf{x}} \quad cost(\mathbf{x})$$

$$\text{s.t.} \quad calories(\mathbf{x})\ contained$$
$$sugar(\mathbf{x}) \leq limit$$
$$\text{calcium}(\mathbf{x}) \geq limit$$



Healthy Squad Goals

- $2000 \leq$ Calories ≤ 2500
- Sugar ≤ 100 g
- Calcium ≥ 700 mg

Food	Cost	Calories	Sugar	Calcium
Stir-fry (per oz)	1	100	3	20
Boba (per fl oz)	0.5	50	4	70

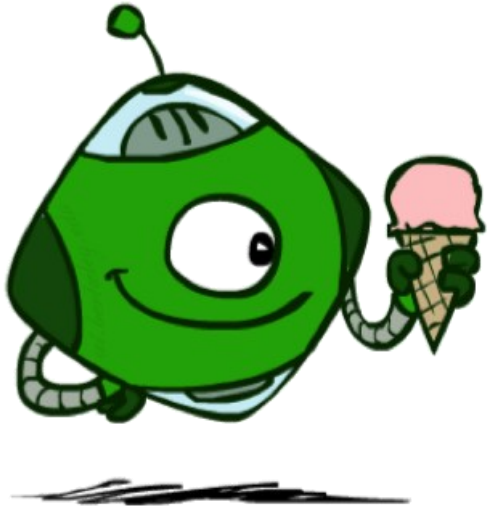
Optimization Formulation

Diet Problem

$$\min_{x_1, x_2} \quad 1\,x_1 + 0.5\,x_2$$

$$\text{s.t.} \quad \begin{aligned} 100\,x_1 + 50\,x_2 &\geq 2000 \\ 100\,x_1 + 50\,x_2 &\leq 2500 \\ 3\,x_1 + 4\,x_2 &\leq 100 \\ 20\,x_1 + 70\,x_2 &\geq 700 \end{aligned}$$

$A\vec{x} \quad \vec{b}$



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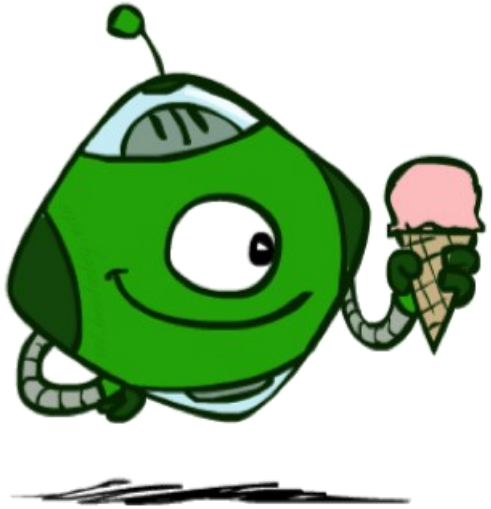
Notation Alert!

Optimization Formulation

Diet Problem

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$



$$\min_{x_1, x_2} \quad c_1 x_1 + c_2 x_2$$

$$\text{s.t.} \quad a_{1,1} x_1 + a_{1,2} x_2 \geq b_1$$
$$a_{2,1} x_1 + a_{2,2} x_2 \leq b_2$$
$$a_{3,1} x_1 + a_{3,2} x_2 \leq b_3$$
$$a_{4,1} x_1 + a_{4,2} x_2 \geq b_4$$

Cost

$$\mathbf{c} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ a_{41} & a_{42} \end{bmatrix}$$

Stir-fry Boba

$$A = \begin{bmatrix} 100 & 50 \\ 100 & 50 \\ 3 & 4 \\ 20 & 70 \end{bmatrix}$$

Limit

$$\mathbf{b} = \begin{bmatrix} 2000 \\ 2500 \\ 100 \\ 700 \end{bmatrix}$$

Calorie min
Calorie max
Sugar
Calcium

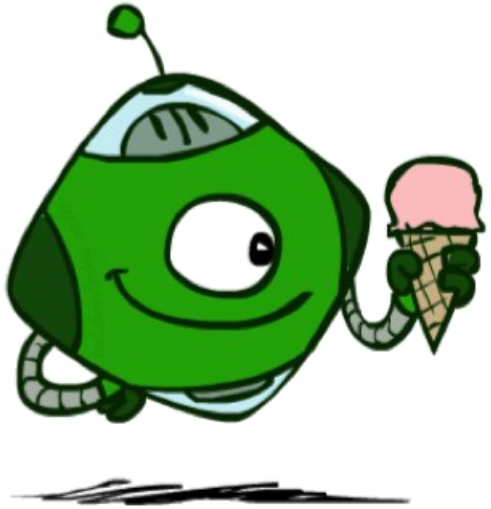
✓ Notation Alert!

Optimization Formulation

Diet Problem

$$\min_{\mathbf{x}} \quad \mathbf{c}^\top \mathbf{x}$$

$$\text{s.t.} \quad \begin{aligned} a_{1,1}\, x_1 + a_{1,2}\, x_2 &\geq b_1 \\ a_{2,1}\, x_1 + a_{2,2}\, x_2 &\leq b_2 \\ a_{3,1}\, x_1 + a_{3,2}\, x_2 &\leq b_3 \\ a_{4,1}\, x_1 + a_{4,2}\, x_2 &\geq b_4 \end{aligned}$$



Cost

$$\mathbf{c} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$$

Notation Alert!

$$A = \begin{bmatrix} 100 & 50 \\ 100 & 50 \\ 3 & 4 \\ 20 & 70 \end{bmatrix}$$

Stir-fry Boba

Limit

$$\mathbf{b} = \begin{bmatrix} 2000 \\ 2500 \\ 100 \\ 700 \end{bmatrix}$$

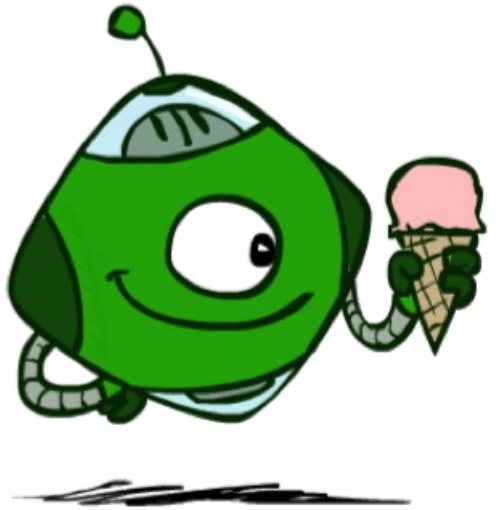
Calorie min
Calorie max
Sugar
Calcium

Optimization Formulation

Diet Problem

$$\min_{\mathbf{x}} \quad \mathbf{c}^\top \mathbf{x}$$

$$\text{s.t.} \quad -a_{1,1}\, x_1 - a_{1,2}\, x_2 \le -b_1$$
$$a_{2,1}\, x_1 + a_{2,2}\, x_2 \le b_2$$
$$a_{3,1}\, x_1 + a_{3,2}\, x_2 \le b_3$$
$$-a_{4,1}\, x_1 - a_{4,2}\, x_2 \le -b_4$$



Cost

$$\mathbf{c} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$$

Stir-fry Boba

$$A = \begin{bmatrix} 100 & 50 \\ 100 & 50 \\ 3 & 4 \\ 20 & 70 \end{bmatrix}$$

Limit

$$\mathbf{b} = \begin{bmatrix} 2000 \\ 2500 \\ 100 \\ 700 \end{bmatrix}$$

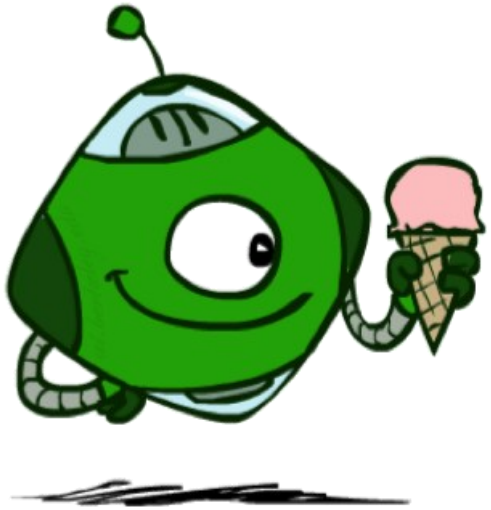
Calorie min
Calorie max
Sugar
Calcium

Optimization Formulation

Diet Problem

$$\min_{\mathbf{x}} \quad \mathbf{c}^\top \mathbf{x}$$

$$\text{s.t.} \quad \begin{aligned} a_{1,1}\, x_1 + a_{1,2}\, x_2 &\le b_1 \\ a_{2,1}\, x_1 + a_{2,2}\, x_2 &\le b_2 \\ a_{3,1}\, x_1 + a_{3,2}\, x_2 &\le b_3 \\ a_{4,1}\, x_1 + a_{4,2}\, x_2 &\le b_4 \end{aligned}$$



Cost

$$\mathbf{c} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$$

Stir-fry Boba

$$A = \begin{bmatrix} -100 & -50 \\ 100 & 50 \\ 3 & 4 \\ -20 & -70 \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Limit

$$\mathbf{b} = \begin{bmatrix} -2000 \\ 2500 \\ 100 \\ -700 \end{bmatrix}$$

Calorie min

Calorie max

Sugar

Calcium

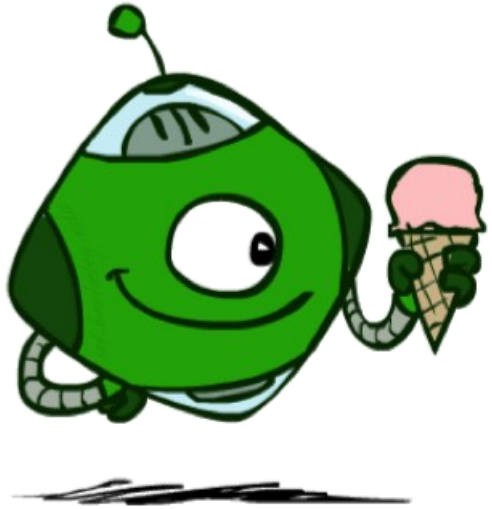
Fat +

Optimization Formulation

Diet Problem

$$\min_{\mathbf{x}} \quad \mathbf{c}^\top \mathbf{x}$$

$$\text{s.t.} \quad A\mathbf{x} \preceq \mathbf{b}$$



Cost

$$\mathbf{c} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$$

$\mathbf{x}$: Stir-fry, Boba (2×1)

Limit

$$A = \begin{bmatrix} -100 & -50 \\ 100 & 50 \\ 3 & 4 \\ -20 & -70 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} -2000 \\ 2500 \\ 100 \\ -700 \end{bmatrix}$$

Calorie min

Calorie max

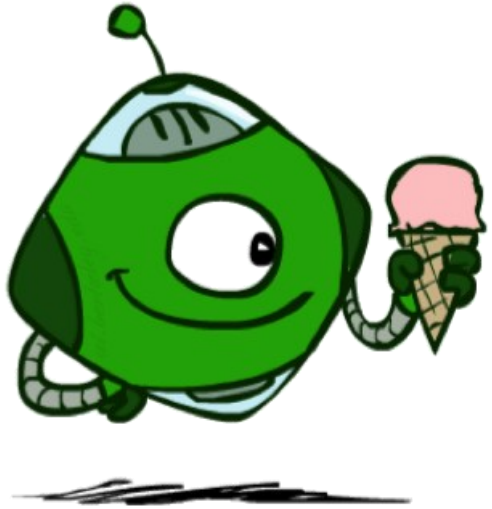
Sugar

Calcium

Notation Alert!

Poll 1

What has to increase to add more nutrition constraints?



$$\min_{\mathbf{x}} \quad \mathbf{c}^\top \mathbf{x}$$

$$\text{s.t.} \quad A\mathbf{x} \preceq \mathbf{b}$$

Select all that apply

A) length $\mathbf{x}$

B) length $\mathbf{c}$

C) height A

D) width A

E) length $\mathbf{b}$

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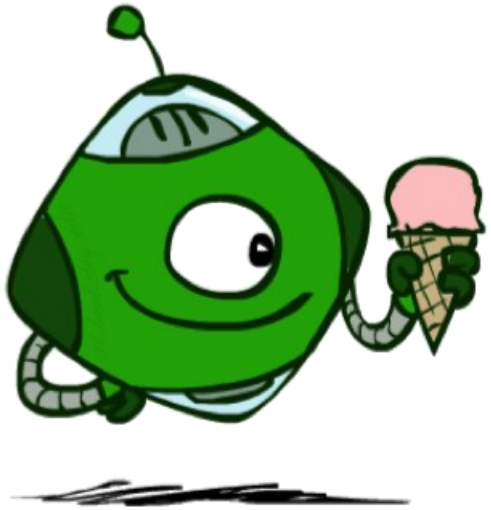

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$$\text{s.t.} \quad A\mathbf{x} \preceq \mathbf{b}$$

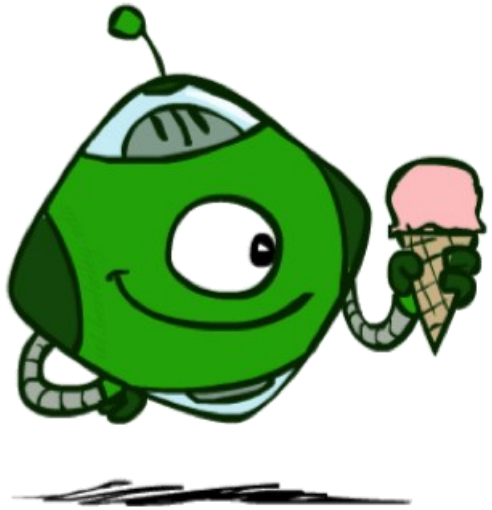
$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \mathbf{c} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$$

$$A = \begin{bmatrix} -100 & -50 \\ 100 & 50 \\ 3 & 4 \\ -20 & -70 \end{bmatrix}$$

$$\mathbf{b} = \begin{bmatrix} -2000 \\ 2500 \\ 100 \\ -700 \end{bmatrix}$$

~~Poll 2~~

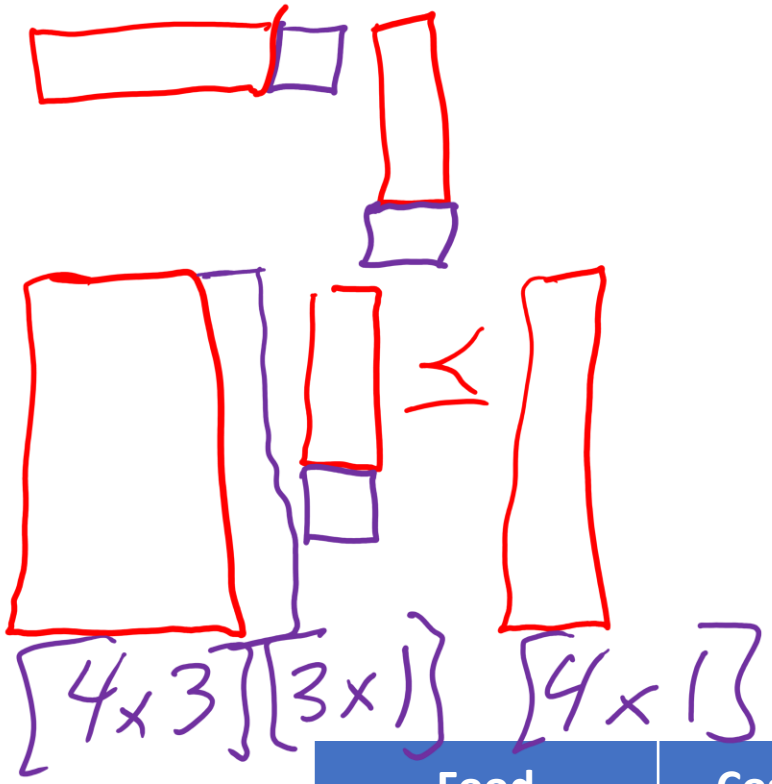
Bad Wifi



What has to increase to add more menu items?

$$\min_{\mathbf{x}} \quad \mathbf{c}^\top \mathbf{x}$$

$$\text{s.t.} \quad A\mathbf{x} \preceq \mathbf{b}$$



Select all that apply

- ✓ A) length $\mathbf{x}$
- ✓ B) length $\mathbf{c}$
- ✗ C) height A
- ✓ D) width A
- ✗ E) length $\mathbf{b}$

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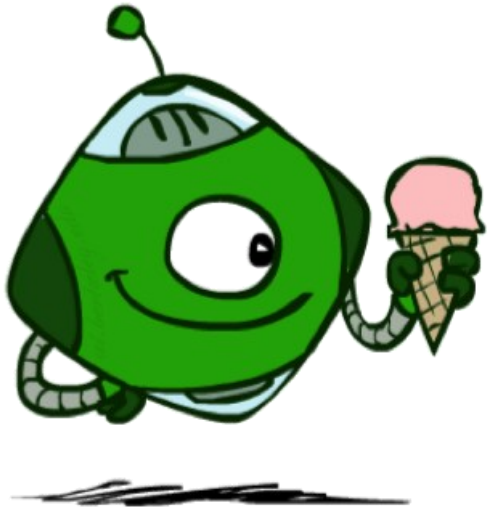
veg (oz)

Poll 2

What has to increase to add more nutrition constraints?

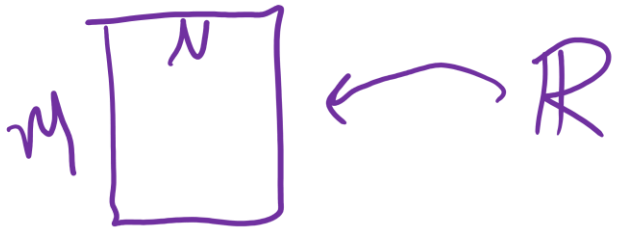


$$\min_{\mathbf{x}} \quad \mathbf{c}^\top \mathbf{x}$$

$$\text{s.t.} \quad A\mathbf{x} \preceq \mathbf{b}$$

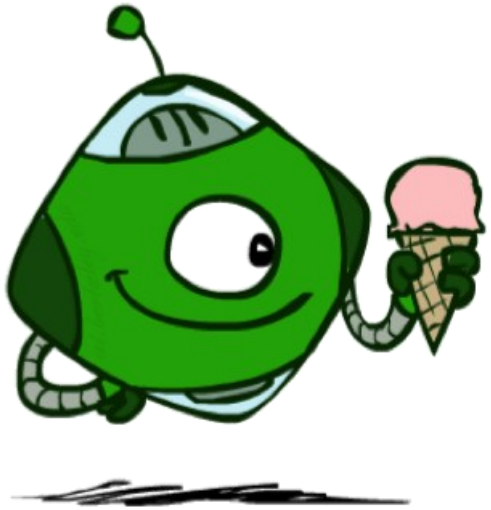
$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \mathbf{c} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} \quad A = \begin{bmatrix} -100 & -50 \\ 100 & 50 \\ 3 & 4 \\ -20 & -70 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} -2000 \\ 2500 \\ 100 \\ -700 \end{bmatrix}$$

Question



If $A \in \mathbb{R}^{M \times N}$, which of the following also equals N?

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$



$$\min_{\mathbf{x}} \quad \mathbf{c}^\top \mathbf{x}$$

$$\text{s.t.} \quad A\mathbf{x} \preceq \mathbf{b}$$

Select all that apply

A) length $\mathbf{x}$

B) length $\mathbf{c}$

C) length $\mathbf{b}$

Notation Alert!

Linear Programming

Linear objective with linear constraints

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{c}^\top \mathbf{x} \\ \text{s.t.} \quad & A\mathbf{x} \preceq \mathbf{b} \end{aligned}$$

As opposed to general optimization

$$\begin{aligned} \min_{\mathbf{x}} \quad & f_0(\mathbf{x}) \\ \text{s.t.} \quad & f_i(\mathbf{x}) \leq 0, \quad i = 1 \ldots M \\ & \mathbf{a}_i^\top \mathbf{x} = \mathbf{b}_i, \quad i = 1 \ldots P \end{aligned}$$

Linear Programming

Different formulations

Inequality form

$$\min_{\mathbf{x}} \quad \mathbf{c}^\top \mathbf{x}$$
$$\text{s.t.} \quad A\mathbf{x} \preceq \mathbf{b}$$

General form

$$\min_{\mathbf{x}} \quad \mathbf{c}^\top \mathbf{x} + \mathbf{d}$$
$$\text{s.t.} \quad G\mathbf{x} \preceq \mathbf{h}$$
$$A\mathbf{x} = \mathbf{b}$$

Standard form

$$\min_{\mathbf{x}} \quad \mathbf{c}^\top \mathbf{x}$$
$$\text{s.t.} \quad A\mathbf{x} = \mathbf{b}$$
$$\mathbf{x} \succeq 0$$

Important to pay attention to form!

Can switch between formulations!

Optimization

Problem Description

Optimization Representation

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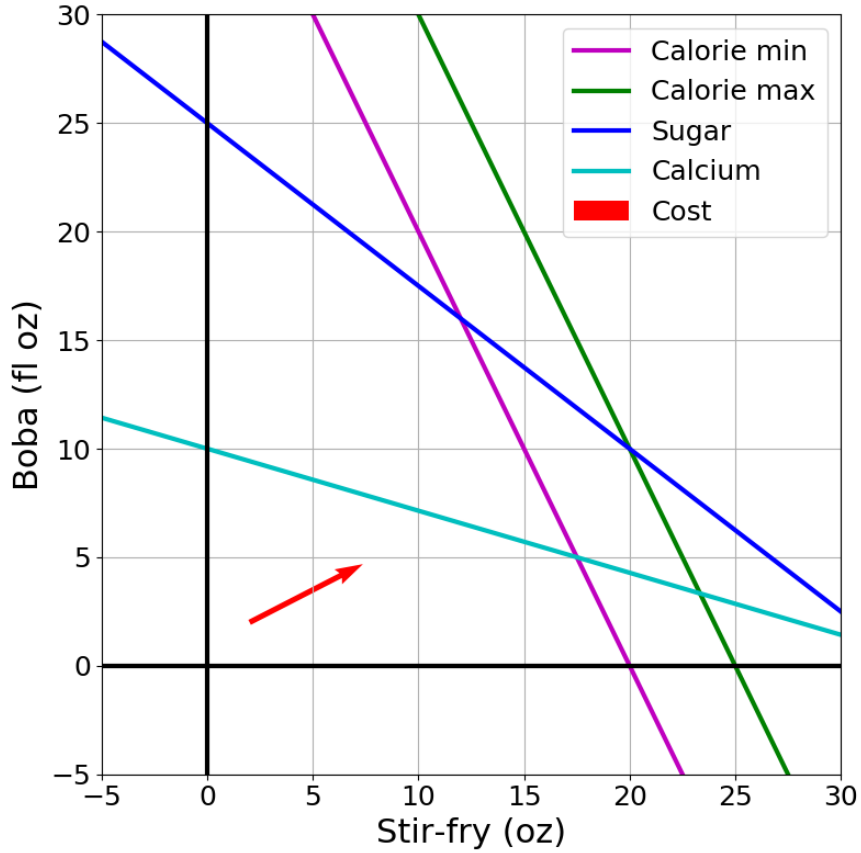
$$\text{s.t.} \quad A\mathbf{x} \preceq \mathbf{b}$$

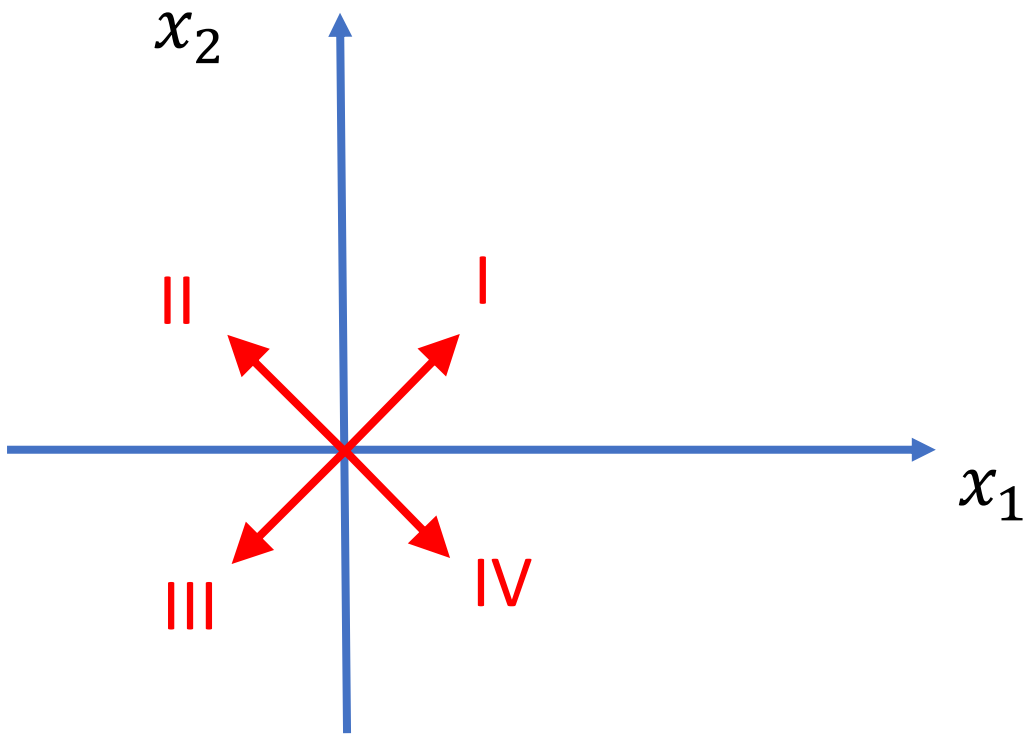
Graphical Representation



Poll 3

Which of these points have cost $\mathbf{c}^\top \mathbf{x} = 0$?

for cost vector: $\mathbf{c} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$



$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}^\top \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \rightarrow c_1 x_1 \ldots$