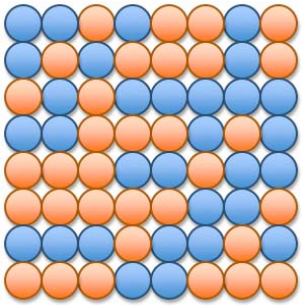


UNIT 5B
Binary Search

Course Announcements

- Sunday's review sessions at 5-7pm and 7-9 pm moved to GHC 4307
- Sample exam available at the SCHEDULE & EXAMS page

http://www.cs.cmu.edu/~15110-f12/schedule.html

This Lecture

- A new search technique for arrays called binary search
- Application of recursion to binary search
- Logarithmic worst-case complexity

Binary Search

- Input: Array A of n unique elements.
  - The elements are sorted in increasing order.
- Result: The index of a specific element called the key or nil if the key is not found.
- Algorithm uses two variables lower and upper to indicate the range in the array where the search is being performed.
  - lower is always one less than the **start** of the range
  - upper is always one more than the **end** of the range

Algorithm

1. Set lower = -1.
2. Set upper = the length of the array a
3. Return BinarySearch(list, key, lower, upper).

BinSearch(list, key, lower, upper):

1. Return nil if the range is empty.
2. Set mid = the midpoint between lower and upper
3. Return mid if a[mid] is the key you're looking for.
4. If the key is less than a[mid],
   return BinarySearch(list,key,lower,mid)
   Otherwise, return BinarySearch(list,key,mid,upper).

Example 1: Search for 73

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
12	25	32	37	41	48	58	60	66	73	74	79	83	91	95

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
12	25	32	37	41	48	58	60	66	73	74	79	83	91	95

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
12	25	32	37	41	48	58	60	66	73	74	79	83	91	95

Found: return 9

Example 2: Search for 42

```
 0  1  2  3  4  5  6  7  8  9 10 11 12 13 14
12 25 32 37 41 48 58 60 66 73 74 79 83 91 95

 0  1  2  3  4  5  6  7  8  9 10 11 12 13 14
12 25 32 37 41 48 58 60 66 73 74 79 83 91 95

 0  1  2  3  4  5  6  7  8  9 10 11 12 13 14
12 25 32 37 41 48 58 60 66 73 74 79 83 91 95

 0  1  2  3  4  5  6  7  8  9 10 11 12 13 14
12 25 32 37 41 48 58 60 66 73 74 79 83 91 95

 0  1  2  3  4  5  6  7  8  9 10 11 12 13 14
12 25 32 37 41 48 58 60 66 73 74 79 83 91 95
Not found:  return nil
```

Finding `mid`

- How do you find the midpoint of the range?
  mid = (lower + upper) / 2
  - Example: lower = -1, upper = 9
  - (range has 9 elements)
  - mid = 4
- What happens if the range has an even number of elements?

Range is empty

- How do we determine if the range is empty?

  lower + 1 == upper

Reccursive Binary Search in Ruby

```
def bsearch(list, key)
   return bs_helper(list, key, -1, list.length)
end
def bs_helper(list, key, lower, upper)
   return nil if lower + 1 == upper
   mid = (lower + upper)/2
   return mid if key == list[mid]
   if key < list[mid] then
      return bs_helper(list, key, lower, mid)
   else
       return bs_helper(list, key, mid, upper)
    end
end
```

Example 1: Search for 73

```
 0  1  2  3  4  5  6  7  8  9 10 11 12 13 14
12 25 32 37 41 48 58 60 66 73 74 79 83 91 95

                key lower upper
bs_helper(list, 73,  -1,   15)
                    mid = 7 and 73 > a[7]
bs_helper(list, 73,   7,   15)
                    mid = 11 and 73 < a[11]
bs_helper(list, 73,   7,   11)
                    mid = 9 and 73 == a[9]
                    ---> return 9
```

Example 2: Search for 42

```
 0  1  2  3  4  5  6  7  8  9 10 11 12 13 14
12 25 32 37 41 48 58 60 66 73 74 79 83 91 95

              key lower upper
bs_helper(list, 73,  -1,   15)
                             mid = 7 and 42 < a[7]
bs_helper(list, 73,  -1,    7)
                             mid = 3 and 42 > a[3]
bs_helper(list, 73,   3,    7)
                             mid = 5 and 42 < a[5]
bs_helper(list, 73,   3,    5)
                             mid = 4 and 42 > a[4]
bs_helper(list, 73,   4,    5)
                             lower+1 == upper
                             --->    Return nil.
```

Instrumenting Binary Search

```
def bsearch(list, key)
    return bs_helper(list, key, -1, list.length, 1)
end

def bs_helper(list, key, lower, upper, count)
    print "lower\t", "upper\t", "iteration\t\n"
    print lower, "\t", upper, "\t", count, "\n“
    return nil if lower + 1 == upper
    mid = (lower + upper)/2
    return mid if key == list[mid]
     if key < list[mid] then
         return bs_helper(list, key, lower, mid, count + 1)
     else
         return bs_helper(list, key, mid, upper, count + 1)
     end
end

a = TestArray.new(100).sort
```


Iterative Binary Search in Ruby

```
def bsearch(list,key)
    lower = -1
    upper = list.length
    while true do
        mid = (lower+upper) / 2
        return nil if upper == lower + 1
        return mid if key == list[mid]
        if key < list[mid] then
            upper = mid
        else
            lower = mid
        end
    end
end
```

Analyzing Efficiency

- For binary search, consider the worst-case scenario (target is not in array)
- How many times can we split the search area in half before the array becomes empty?
- For the previous examples:
  15 --> 7 --> 3 --> 1 --> 0 ... 4 times

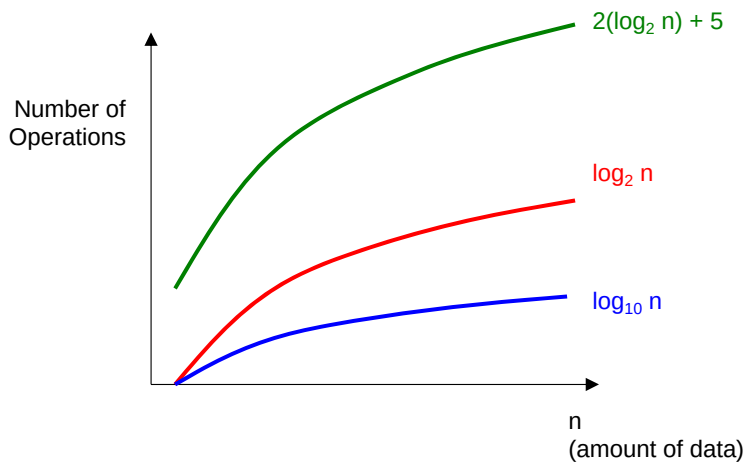
In general...

- In general, we can split search region in half $\lfloor \log_2 n \rfloor + 1$ times before it becomes empty.
- Recall the log function:
  $\log_a b = c$ is equivalent to **$a^c = b$**
  Examples:
  $\log_2 128 = 7$
  $\log_2 n = 5$ implies $n = 32$
- In our example: when there were 15 elements, we needed 4 comparisons: $\lfloor \log_2 15 \rfloor + 1 = 3 + 1 = 4$

Big O

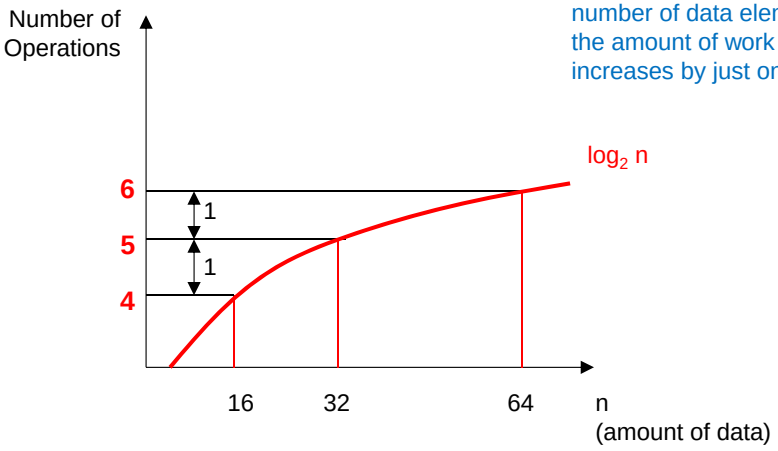
- In the worst case, binary search requires O(log n) time on a sorted array with n elements.
  - Note that in Big O notation, we do not usually specify the base of the logarithm. (It's usually 2.)
Number of operations	Order of Complexity
$\log_2 n$	O(log n)
$\log_{10} n$	O(log n)
$2(\log_2 n) + 5$	O(log n)

O(log n) ("logarithmic")



O(log n)

For a $\log_2$ algorithm,
If you double the
number of data elements
the amount of work you do
increases by just one unit



Binary Search (Worst Case)

Number of elements	Number of Comparisons
15	4
31	5
63	6
127	7
255	8
511	9
1023	10
1 million	20

Binary Search Pays Off

- Finding an element in an array with a million elements requires only 20 comparisons!
- BUT....
  - The array must be sorted.
  - What if we sort the array first using insertion sort?
    - Insertion sort $O(n^2)$ (worst case)
    - Binary search $O(\log n)$ (worst case)
    - Total time: $O(n^2) + O(\log n) = O(n^2)$

    Luckily there are faster ways to sort in the worst case...