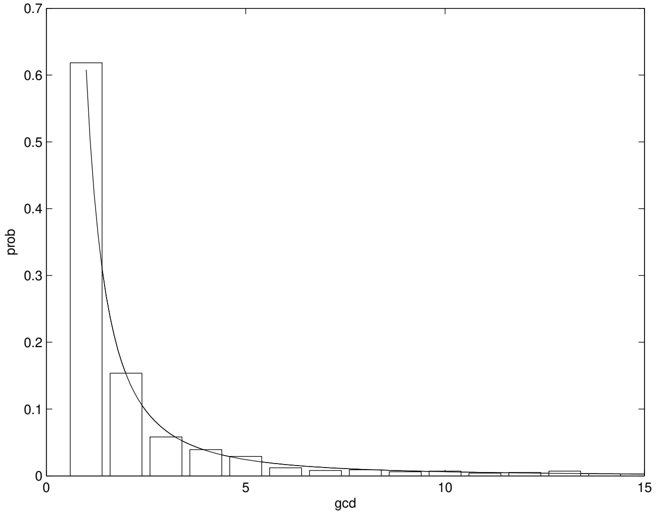
9/17-19: Random Variables

Comment on Plots



Random Variables

A *random variable* is a function $X : S \longrightarrow \mathbb{R}$

Random Variables

Let S be the set of all sequences of three rolls of a die Let $X(\omega)$ be the sum of the number of dots on the three rolls.

Discrete RVs

A random variable is *discrete* if it can take on only a finite or countably infinite set of values (as above).

The *probability mass function* of a discrete X is defined by

$$f_X(x) = \Pr(X = x)$$

Note: we often talk about a random variable via its probability mass function, without explicitly describing the underlying sample space and events.

Discrete RVs

For a discrete r.v., the probability of a subset $A \subset \mathbb{R}$ is

$$\Pr(X \in A) = \sum_{x_i \in A} f_X(x_i)$$

Discrete RVs

Uniform:

$$f_X(x) = \begin{cases} \frac{1}{N} & x = 1, \ldots, N \\ 0 & \text{otherwise} \end{cases}$$

Bernoulli: $\Pr(X = 0) = p \in [0,1]$ and $\Pr(X = 1) = 1 - p$.

$$f_X(x) = p^x (1-p)^{1-x} \text{ for } x \in \{0, 1\}$$

Bionomial: n draws of a Bernoulli. Parameters are n and $p \in [0,1]$.

$$f_X(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & x = 0, 1, \ldots, n \\ 0 & \text{otherwise} \end{cases}$$

Discrete RVs (cont)

Poisson

$$f_X(x) = e^{-\lambda} \frac{\lambda^x}{x!} \quad \text{for } x = 0, 1, 2, \ldots$$

Geometric

$$f_X(x) = p(1-p)^{x-1} \quad \text{for } x = 1, 2, \ldots$$

Number of flips until the first head appears

Multinomial

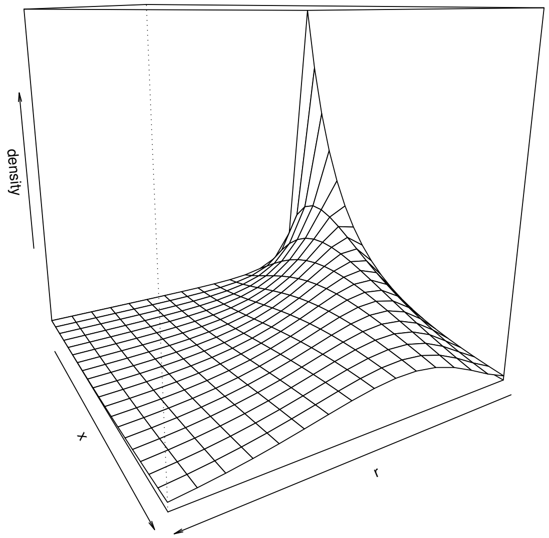
$$f_X(x_1, x_2, \ldots, x_k) = \binom{n}{x_1\, x_2 \cdots x_k} p_1^{x_1} p_2^{x_2} \cdots p_k^{x_k}$$

Discrete RVs (cont)

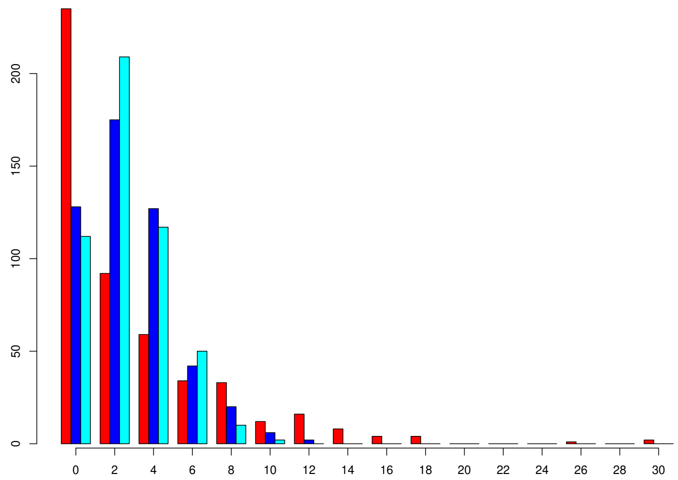
Negative Binomial: Flip coins and record number of tails (failures) before the r-th head (success)

$$f_X(x) = \binom{r+x-1}{x} p^r (1-p)^x \quad x = 0, 1, 2, \ldots$$

Discrete RVs (cont)



Negative Binomial Family



Cumulative Distribution Functions

The *cumulative distribution function* (cdf) of random variable is defined by

$$F_X(x) = \Pr(X \le x)$$

X is *continuous*, with density f_X, if the cdf is given by

$$F_X(x) = \int_{-\infty}^{x} f_X(u)\, du$$

What's continuous about a continuous random variable? Its distribution function.

Continuous RVs

For a continuous random variable X with density f_X,

$$\Pr(a \le X \le b) = \int_a^b f_X(x)\,dx$$

In particular, for each $a \in \mathbb{R}$, $\Pr(X = a) = 0$

The density can be changed at finite number of x without changing the distribution of the random variable. (Example: coin on a table.)

Must have

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

Continuous RVs

More generally, (for Borel sets B)

$$\Pr(X \in B) = \int_B f_X(x)\,dx$$

Continuous RVs

Uniform: on interval $[a, b]$

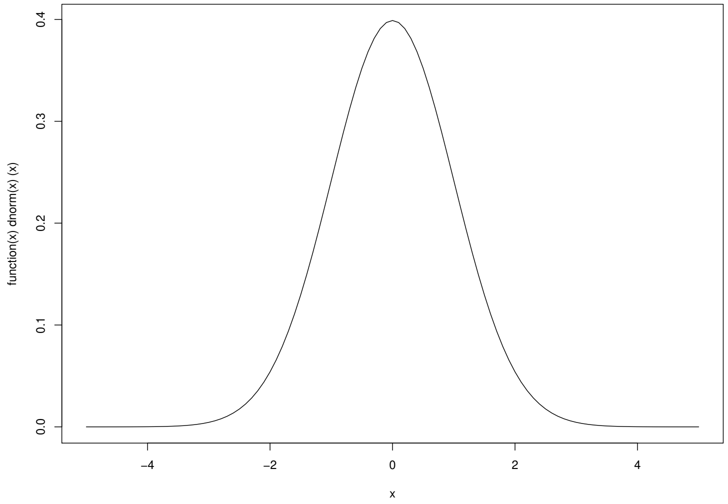
$$f(x) = \begin{cases} \frac{1}{b-a} & \text{for } x \in [a, b] \\ 0 & \text{otherwise} \end{cases}$$

Gaussian:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right)$$

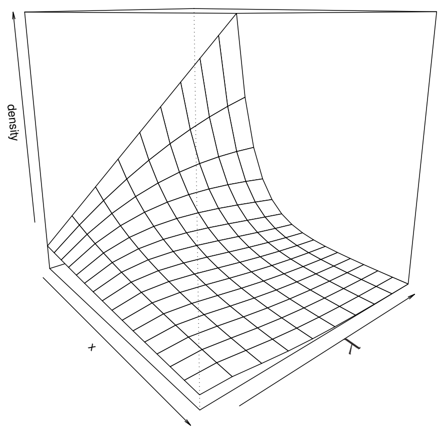
(limit of binomials)

Continuous RVs



Continuous RVs

Exponential: $F_X(x) = 1 - e^{-\lambda x}$



Continuous or Discrete

Are all random variables either continuous or discrete?

Continuous or Discrete

A coin is flipped onto a table. We record whether it was heads or tails, and the position of the coin on the table. So, the sample space is $S = \{\{H, T\} \times \mathbb{R}^2\}$.

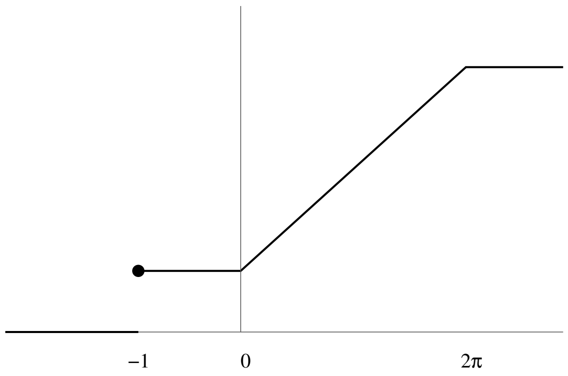
Define a random variable

$$\begin{aligned} X(T, x, y) &= -1 \\ X(H, x, y) &= \theta(x, y) \end{aligned}$$

where $\theta(x, y) \in [0, 2\pi)$ is the angle formed by the position of the coin with respect to the positive x-axis of the table.

Continuous or Discrete

The distribution has a point mass at -1.



Properties of Distributions

$F_X(x)$ is non-decreasing, approaches 1 as $x \to \infty$ and approaches 0 as $x \to -\infty$.

A distribution is right-continuous: $F_X(x) = \lim_{y>x, y\to x} F_X(x) = F_X(x^+)$ since

$$\begin{aligned} \Pr(X \le x) &= \Pr(\cap_n \{X \le y_n\}) \\ &= \lim_{n\to\infty} \Pr(X \le y_n) \\ &= F_X(x^+) \end{aligned}$$

Jumps only where there is a point mass

Properties of Distributions

$$\begin{aligned} \Pr(X > x) &= 1 - F(x) \\ \Pr(X < x) &= F(x^-) \\ \Pr(X = x) &= F(x) - F(x^-) \\ \Pr(a < X \le b) &= F(b) - F(a) \end{aligned}$$

Bivariate Distributions

Take the discrete case. Two random variables X and Y — for example, hash values output by two different "random" hash functions on input strings.

Define *joint probability mass function* as

$$f(x, y) = \Pr(X = x \text{ and } Y = y)$$

Could think of vector-valued random variable $Z = (X, Y)$ (though this may sacrifice some intuition)

Defined over *the same sample* space.

Example

Consider (X, Y) where X and Y are two independent coin flips (Bernoulli r.v.'s) and (X, Z) where $Z = (1 - X)$.

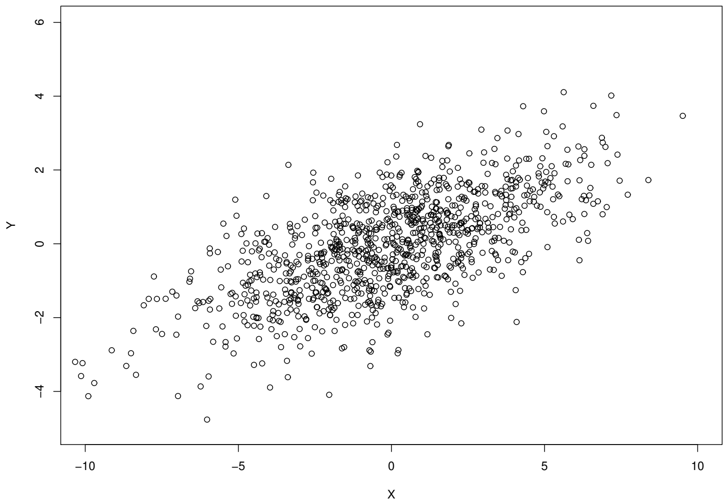
Then X, Y, and Z all have the same distribution, but the pairs (X, Y) and (X, Z) have very different distributions.

General Case

The natural definition of a bivariate distribution is: given random variables X and Y,

$$F_{X,Y}(x, y) = \Pr(X \le x \text{ and } Y \le y)$$

Bivariate Normal



Independence

To discrete random variables X and Y are *independent* in case the events

$$\{X = x\} \quad \text{and} \quad \{Y = y\}$$

are independent for all x and $y \in \mathbb{R}$

Independent Random Variables

Two random variables X and Y are *independent* in case

$$\{X \le x\} \quad \text{and} \quad \{Y \le y\}$$

are independent events for all x and y.

- For discrete r.v.s this is equivalent to the definition we saw earlier
- Above definition is general
- $\Pr(X \in A \text{ and } Y \in B) = \Pr(X \in A)\,\Pr(Y \in B)$

Sums of Random Variables

Again, first the discrete case:

$$f_{X+Y}(z) = \sum_x f_{X,Y}(x, z-x)$$

If X and Y are independent, then

$$\begin{aligned}
\Pr(X+Y=z) &= f_{X+Y}(z) \\
&= \sum_x f_{X,Y}(x, z-x) \\
&= \sum_x f_X(x)\, f_Y(z-x) \\
&= \sum_x \Pr(X=x)\Pr(Y=z-x)
\end{aligned}$$

Example

Let X and Y be two independent geometric random variables with parameter p.

$$\begin{aligned}
\Pr(X+Y=z) &= \sum_{x=1}^{z-1} p\,(1-p)^{x-1}\, p\,(1-p)^{z-x-1} \\
&= p^2 \sum_{x=1}^{z-1} (1-p)^{z-2} \\
&= (z-1)\, p^2\, (1-p)^{z-2}
\end{aligned}$$

The negative binomial is a sum (convolution) of geometrics.

Continuous Case

Nothing new in the continuous case. Work with the joint density function:

$$f_{X+Y}(z) = \int_{-\infty}^{\infty} f_{X,Y}(x, z-x)\, dx$$

In case X and Y are independent, get a convolution:

$$\begin{aligned}
f_{X+Y}(z) &= \int_{-\infty}^{\infty} f_{X,Y}(x, z-x)\, dx \\
&= \int_{-\infty}^{\infty} f_X(x)\, f_Y(z-x)\, dx \\
&= (f_X \star f_Y)(z)
\end{aligned}$$

Functions of Random Variables

Let $Y = g(X)$. What is the distribution function of Y? Think probabilistically:

$$\begin{aligned}
\Pr(Y \in A) &= \Pr(g(X) \in A) \\
&= \Pr\left(X \in g^{-1}(A) = \{x : g(x) \in A\}\right) \\
&= \int_{g^{-1}(A)} f_X(x)\, dx
\end{aligned}$$

In particular, $F_Y(y) = \int_{g^{-1}(-\infty, y]} f_X(x)\, dx$

Example

Let $X \sim \mathcal{N}(0,1)$ and $Y = X^2$. Then

$$
\begin{aligned}
F_Y(y) &= \Pr(X^2 \in (-\infty, y]) \\
&= \Pr(X \in [-\sqrt{y}, \sqrt{y}]) \\
&= \Phi(\sqrt{y}) - \Phi(-\sqrt{y}) \\
&= \Phi(\sqrt{y}) - (1 - \Phi(\sqrt{y})) \\
&= 2\Phi(\sqrt{y}) - 1
\end{aligned}
$$

Example

So, the density of Y is

$$
\begin{aligned}
f_Y(y) &= \frac{d}{dy}\Phi(\sqrt{y}) \\
&= \frac{1}{\sqrt{y}}\Phi'(\sqrt{y}) \\
&= \frac{1}{\sqrt{2\pi y}} e^{-\frac{y}{2}}
\end{aligned}
$$

Quantiles

If F is invertible, $F^{-1}(p)$ is called the *p quantile* of X.

More generally, the p quantile is defined as *the smallest x for which $F(x) \geq p$*

Example: test scores report quantiles. If you got an 800 on the quantitative GRE, you were reported to be in the 98-th percentile (0.98 quartile)

Random Factored Numbers

Some comments on Adam Kalai's algorithm – let's go to the board...

To Do

- Read Chapter 3 in DeGroot & Schervish
- Assignment 2 due next Thursday