10-751 Assignment 5

Due: Tuesday, November 12

Problem 1. Suppose that we independent samples $X_1, X_2, \ldots, X_n$, where $X_i \sim \mathcal{N}(\mu, \sigma^2)$.

1. Show that $\overline{X}_n = \frac{1}{n}\sum_i X_i$ is an unbiased estimator of μ.
2. Show that the estimator
$$\hat{\sigma}_n^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X}_n)^2$$
is a biased estimator of σ^2 and that the estimator
$$\hat{\sigma}_{n-1}^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X}_n)^2$$
is an unbiased estimator of σ^2.
3. Show that $\hat{\sigma}_n^2$ has lower mean squared error than $\hat{\sigma}_{n-1}^2$, thus illustrating that biased estimators may be more "efficient" than unbiased estimators.

Problem 2. Suppose that $X_1, \ldots, X_n$ are independent with $X_i \sim \mathcal{N}(\mu, \sigma^2)$. Let $\hat{\mu}$ and $\hat{\sigma}$ denote the maximum likelihood estimates of μ and σ. With $n = 17$, find a value of k such that $\Pr(\hat{\mu} > \mu + k\hat{\sigma}) = 0.95$. (You will need to use a table of probabilities for the t distribution, or calculate these numerically.)

Problem 3. Let $X_1, \ldots, X_n$ be independent with $X_i \sim \text{Uniform}(0, \theta)$, and let $\hat{\theta}_n = \max_i X_i$.

1. Compute the bias of $\hat{\theta}_n$.
2. Compute the standard error of $\hat{\theta}_n$.
3. Computer the mean squared error of $\hat{\theta}_n$.

Problem 4. Suppose that $X_1, \ldots, X_{n+1}$ form a sample from a normal distribution, and let $\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $T_n = \left(\frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X}_n)^2\right)^{1/2}$. Determine the value of a constant c such that $c(X_{n+1} - \overline{X}_n)/T_n$ will have a t distribution. (Remember that if Z is standard normal and Y is a χ^2 distribution with n degrees of freedom, $Z/\sqrt{\frac{1}{n}Y}$ has a t distribution with n degrees of freedom.)

Problem 5. Using Matlab, generate $n = 20$ samples from a Poisson(λ_0) distribution, where $\lambda_0 = 1$. Perform the Wald test with $\alpha = 0.05$. Repeat many times and count how many times the null hypothesis $\lambda = \lambda_0$ is rejected. How close is the type I error rate to 0.05? Give a description of your results and include your Matlab code with your assignment.