10-751 Assignment 4

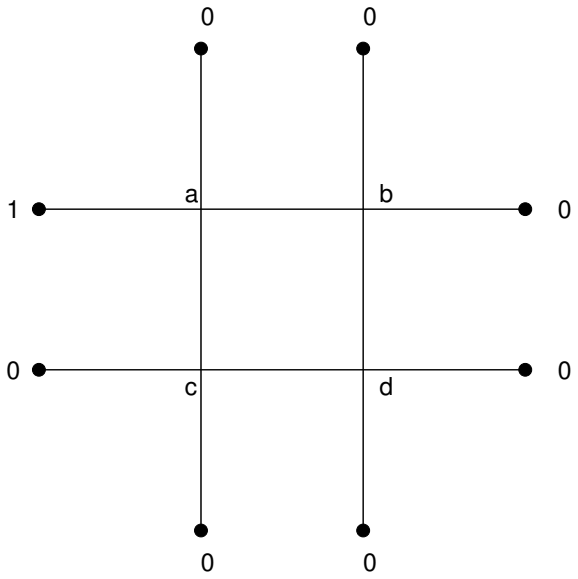
Due: Thursday October 17

The following problems are concerned with random walks and the Dirichlet problem, as presented in class. This is discussed at length in Doyle and Snell's *Random Walks and Electric Networks*, from which the following problems are adapted. This book is available online from `http://front.math.ucdavis.edu/math.PR/0001057`.

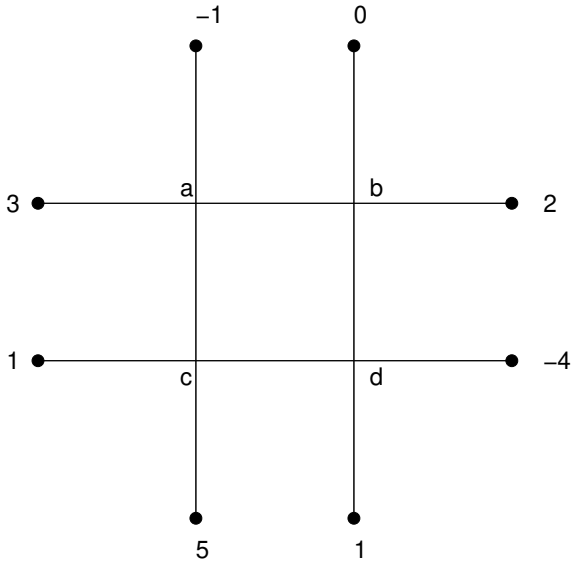
Problem 1. Show that if f and g are harmonic functions so is $h = af + bg$ for constants a and b. (This is called the *superposition principle*.)

Problem 2. Let $B_1, B_2, \ldots, B_n$ be the boundary points for a region S. Let $e_j(a, b)$ be a function that is harmonic in S and has boundary value 1 at B_j and 0 at the other boundary points. Show that if arbitrary boundary values $v_1, v_2, \ldots, v_n$ are assigned, we can find the harmonic function v with these values from the solutions $e_1, e_2, \ldots e_n$.

Problem 3. Consider a random walk on the graph below. Find the probability of reaching the point with a 1 before any of the points with 0's for each starting point a, b, c, and d.



Problem 4. Solve the Dirichlet problem for the graph shown below.

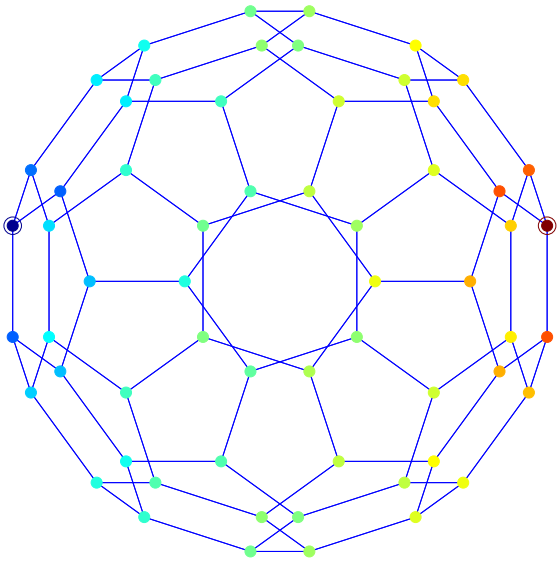


Problem 5. In this problem, you will implement the random walk approach to the Dirichlet problem in Matlab.

Given a graph G and a set of distinguished boundary vertices with prescribed values f_B, when the random walk is at vertex x it steps to one of the neighbors of x (with uniform probability), until it hits one of the boundary vertices. The *payoff* is the value of the boundary vertex where the walk stops. The solution to the Dirichlet is the function $f_D : G \longrightarrow \mathbb{R}$ such that $f_D(x)$ is the expected payoff when the random walk starts at vertex x.

You are to write a matlab function `dirichlet` that takes three arguments: the $n \times n$ weighted adjacency matrix `A` of the graph, an $n \times 2$ matrix `XY` of coordinates, and a list of boundary vertices and their values, in the form of a $k \times 2$ matrix `bndry`, where k is the number of boundary vertices. The matrix of coordinates `XY` is such that $(\texttt{XY}(j,1), \texttt{XY}(j,2))$ is the (x,y)-coordinate of vertex j when the graph is drawn in the plane. Your function should compute the solution f_D using the Markov chain method, and then plot the result, using the `gcplot` Matlab function which we have provided for you.

As an example of a graph, try setting `[A XY] = bucky` in Matlab and then evaluating `gplot(A, XY)`. Using this graph as input, an example call would be `dirichlet(A, XY, [[9,1];[53,0]])`, specifying that vertex 9 has value 1 and vertex 53 has value 0. The output of your program should then look something like the figure below, where the color of vertex x indicates the value of $f_D(x)$.



A call to `gcplot` looks like `gcplot(A, XY, bndry, f)` where `A` is the adjacency matrix, `XY` is the matrix of coordinates, `bndry` is a list of the boundary vertices (so that the graphing function can mark them with special symbols), and `f` is your computed solution to the Dirichlet problem.

The Matlab code `gcplot.m` and sample graphs are available off of the course Web page. Please email your Matlab code for the `dirichlet` function to Guy Lebanon, `lebanon@cs.cmu.edu`.