10-751 Assignment 3

Due: Tuesday October 7

Problem 1.

1. Suppose that X, Y, and Z are independent and uniformly distributed in $[0, 1]$. Derive the density function of $X + Y + Z$.

2. If $X_1, X_2, \ldots, X_r$ are independent and uniformly distributed on $[0, 1]$, show that the density $f_r(x)$ of $\sum_{i=1}^r X_i$ is a polynomial in x of degree $r - 1$.

Problem 2. There are $\binom{n}{m}$ items, each assigned a cost (with no equal costs). You seek to find the best item item—the item with the least cost. Suppose that a random walk among the items is constructed so that from the j-th best item (that is, the j-th item ranked according to cost), you step to any one of the $j - 1$ less costly items with uniform probability.

1. Let r_j be the expected number of steps to reach the best item from the j-th best item. Show that $r_j = \sum_{k=1}^{j-1} k^{-1}$.

2. Given an asymptotic expression for the expected time to reach the best item starting from the worst item, for large n and m.

(Note, this problem is related to the expected running time of the simplex algorithm.)

Problem 3. Let X and Y be independent Bernoulli random variables with parameter $\frac{1}{2}$. Show that $X + Y$ and $|X - Y|$ are dependent but uncorrelated.

Problem 4. Suppose that A, B and C are such that A and B are independent, $\Pr(ABC) = 0.04$, $\Pr(C \mid AB) = 0.25$, and $\Pr(B) = 4 \Pr(A)$. What is $\Pr(A \cup B)$?

Problem 5. (DeGroot 3.7.12) Let Y be the rate (calls per hour) at which calls arrive at a switchboard. Let X be the number of calls during a two-hour period. Suppose that the marginal p.d.f. of Y is

$$f_2(y) = \begin{cases} e^{-y} & \text{if } y > 0 \\ 0 & \text{otherwise} \end{cases}$$

and that the conditional p.f. of X given $Y = y$ is

$$g_1(x \mid y) = \begin{cases} \frac{(2y)^x}{x!} e^{-2y} & \text{if } x = 0, 1, \ldots, \\ 0 & \text{otherwise} \end{cases}$$

1. Find the marginal p.f. of X.
2. Find the conditional p.d.f. $g_2(y \mid 0)$ of Y given $X = 0$.
3. Find the conditional p.d.f. $g_2(y \mid 1)$ of Y given $X = 1$.
4. For what values of y is $g_2(y \mid 1) > g_2(y \mid 0)$? Does this agree with the intuition that the more calls you see, the higher you should think the rate is?

Problem 6. Let $\beta > 1$, let $p_1, p_2, \ldots$ denote the prime numbers, and let $N(1), N(2), \ldots$ be independent random variables having probability mass function

$$\Pr(N(i) = k) = \left(1 - p_i^{-\beta}\right) p_i^{-\beta k}$$

Show that $M = \prod_{i=1}^{\infty} p_i^{N(i)}$ is a random integer having probability mass fuction $\Pr(M = m) = Cm^{-\beta}$ for $m \geq 1$, where C is the constant given by

$$\begin{aligned} C &= \prod_{i=1}^{\infty} \left(1 - \frac{1}{p_i^{\beta}}\right) \\ &= \left(\sum_{m=1}^{\infty} \frac{1}{m^{\beta}}\right)^{-1} \end{aligned}$$