15-451 Algorithms, Fall 2003

Homework # 1 **due: September 9, 2003**

Please hand in each problem on a separate sheet and put your **name** and **recitation** (time or letter) at the top of your handin. You will be handing each problem into a separate box, and we will then give homeworks back in recitation.

Remember: written homeworks are to be done individually. Group work is only for the oral-presentation assignments.

Problems:

(40 pts) 1. **Recurrences.** Solve the following recurrences, giving your answer in Θ notation. For each of them, assume the base case $T(x) = 1$ for $x \leq 2$. Show your work.

(a) $T(n) = 3T(n/5) + n$.

(b) $T(n) = 3T(n-5)$.

(c) $T(n) = T(n-3) + n^3$.

(d) $T(n) = \sqrt{n}\, T(\sqrt{n}) + n$. (E.g., we might get this from a divide-and-conquer procedure that uses linear time to break the problem into $\sqrt{n}$ pieces of size $\sqrt{n}$ each. Hint: write out the recursion tree.)

(30 pts) 2. **Recurrences and proofs by induction.** Consider the following recurrence:

$$T(n) \quad = \quad 2T(n/2) + n \lg n.$$

(The base case isn't so important, but you can think of $T(2) = 2$ if you like.) We would like you to solve this recurrence using the "guess and prove by induction" method.

(a) Try a proof by induction using the guess "$T(n) \leq cn \lg n$." This guess is *incorrect* and so your proof should *fail*. (If your proof succeeds, then there is a problem!!) Explain where this proof fails.

(b) Use the way the above proof failed to suggest a better guess. Explain how you arrived at this guess and prove by induction that your new guess is correct.

(c) Give a Θ bound for the recurrence. In particular, give a proof by induction to show that $T(n) \geq c' f(n)$ where $c' > 0$ is some constant, and $f(n)$ is your guess from (b).

(30 pts) 3. **Probability and expectation**.

(a) A carnival game consists of three dice in a cage. A player can bet a dollar on any of the numbers 1 through 6. The cage is shaken, and the payoff is as follows. If the player's number does not appear on any of the dice, he loses his dollar. Otherwise, if his number appears on exactly k of the three dice, for $k = 1, 2, 3$, he keeps his dollar and wins $k - 1$ more dollars. What is his expected gain from playing the carnival game once?

(b) Suppose you wanted to make the above game fair. What should you replace the "$k - 1$" with so that the expected gain from playing this game is 0? Your answer should be a (simple) function of k.