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The Directed Orienteering Problem

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Abstract This paper studies vehicle routing problems on asymmetric metrics. Our starting point is the *directed k-TSP* problem: given an asymmetric metric (V,d), a root $r \in V$ and a target $k \le |V|$, compute the minimum length tour that contains r and at least k other vertices. We present a polynomial time $O(\frac{\log^2 n}{\log\log n} \cdot \log k)$-approximation algorithm for this problem. We use this algorithm for directed k-TSP to obtain an $O(\frac{\log^2 n}{\log\log n})$-approximation algorithm for the *directed orienteering* problem. This answers positively, the question of poly-logarithmic approximability of directed orienteering, an open problem from Blum *et al.* (SIAM J. Comput. 37(2):653–670, 2007). The previously best known results were quasi-polynomial time algorithms with approximation guarantees of $O(\log^2 k)$ for directed k-TSP, and $O(\log n)$ for directed orienteering (Chekuri and Pal in IEEE Symposium on Foundations in Computer Science, pp. 245–253, 2005). Using the algorithm for directed orienteering within the framework of Blum *et al.* (SIAM J. Comput. 37(2):653–670, 2007) and Bansal *et al.* (ACM Symposium on Theory of Computing, pp. 166–174, 2004), we also obtain poly-logarithmic approximation algorithms for the directed versions of discounted-reward TSP and vehicle routing problem with time-windows.

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1 Introduction

Vehicle routing problems (VRPs) form a large set of variants of the basic Traveling Salesman Problem, that are also encountered in practice. Some of the problems in this class are the capacitated VRP [13], the distance constrained VRP [19], the Dial-a-Ride problem [24], and the orienteering problem [12]. Many different objectives are encountered in VRPs: for example, minimizing cost of a tour (capacitated VRP and the Dial-a-Ride problem), minimizing number of vehicles (distance constrained VRP), and maximizing profit (the orienteering problem).

The Operations Research literature contains several papers dealing with exact or heuristic approaches for VRPs [8, 16, 18, 23]. The techniques used in these papers include dynamic programming, local search, simulated annealing, genetic algorithms, branch and bound, and cutting plane algorithms. There has also been some interesting work in approximation algorithms for VRPs [2, 5, 13]. The problem most relevant to this paper is orienteering, which involves finding a bounded length path starting at a fixed vertex that covers the maximum number of vertices; Blum *et al.* [4] obtained the first constant factor approximation algorithm for this problem (on symmetric metrics), which was improved to a factor of 3 in Bansal *et al.* [2]. Bansal *et al.* [2] then used orienteering as a subroutine to also obtain poly-logarithmic approximation algorithms for some generalizations of orienteering, namely deadline TSP and vehicle routing problem with time windows.

Most of the work on VRPs focuses on symmetric metric spaces. In asymmetric metrics, the best known approximation guarantee for even the basic traveling salesman problem until recently was $O(\log n)$ [10]; in a breakthrough result, Asadpour *et al.* [1] improved this bound to $O(\log n/\log\log n)$. Chekuri and Pal [7] obtained a general approximation algorithm for a class of VRPs on asymmetric metrics, that runs in *quasi-polynomial time*. In particular, their result implies an $O(\log n)$-approximation algorithm for the orienteering problem on directed graphs (in quasi-polynomial time). We are not aware of any previously known non-trivial polynomial-time approximation algorithms for this problem. In this paper, we study polynomial time approximation algorithms for directed orienteering and related problems on asymmetric metrics.

1.1 Problem Definition

All the problems that we consider are defined over an asymmetric metric space (V,d) on $|V|=n$ vertices. In the *directed k-TSP* problem, we are given a root $r\in V$ and a target $k\le n$, and the goal is to compute a minimum length tour that contains r and at least k other vertices. Directed k-TSP is a generalization of the asymmetric traveling salesman problem (ATSP). A related problem is the *minimum ratio ATSP* problem, which involves finding a tour containing the root r that minimizes the ratio of the length of the tour to the number of vertices in it. If the requirement that the tour contain the root is dropped, the ratio problem becomes the minimum mean weight cycle problem, which is solvable in polynomial time [17]. However, the rooted version which we are interested in is NP-complete.

In the *orienteering* problem, we are given a metric space, a specified origin s and a length bound D, and the goal is to find a path of length at most D, that starts at s and visits the maximum number of vertices. We actually consider a more general version of this problem, which is the directed version of point-to-point orienteering [2]. In the *directed orienteering* problem, we are given specified origin s and destination t vertices, and a length bound D, and the goal is to compute a path from s to t of length at most D, that visits the maximum number of vertices. The orienteering problem can also be extended to the setting where there is some profit at each vertex, and the goal is to maximize total profit.

Many problems we deal with in this paper have the following form, where $\mathcal{S}$ is a feasible set, $C : \mathcal{S} \to \mathbb{R}^+$ is a cost function, $N : \mathcal{S} \to \mathbb{N}$ is a coverage function, and k is the target:

$$\min\{C(x) : x \in \mathcal{S}, N(x) \geq k\}$$

For example, in the k-TSP problem, $\mathcal{S}$ is the set of all tours containing r, and for any $x \in \mathcal{S}$, $C(x)$ is the length of tour x and $N(x)$ is the number of vertices (other than r) covered in the tour. For any problem of the above form, a polynomial time algorithm $\mathcal{A}$ is said to be an (α, β) bi-criteria approximation if on each problem instance, $\mathcal{A}$ obtains a solution $y \in \mathcal{S}$ satisfying $C(y) \leq \alpha \cdot OPT$ and $N(y) \geq \frac{k}{\beta}$, where $OPT = \min\{C(x) : x \in \mathcal{S}, N(x) \geq k\}$ is the optimal value of this instance.

1.2 Results and Paper Outline

We present a polynomial time $O(\frac{\log^2 n}{\log\log n} \cdot \log k)$-approximation algorithm for the directed k-TSP problem. This is based on an $O(\frac{\log^2 n}{\log\log n})$-approximation algorithm for the minimum ratio ATSP problem. To the best of our knowledge, this problem has not been studied earlier. An important ingredient in this algorithm is a splitting-off theorem on directed Eulerian graphs due to Frank [9] and Jackson [15]. This algorithm is described in Sect. 2. In the preliminary version [21] we gave an $O(\log^2 n)$-approximation algorithm for minimum ratio ATSP. Using the recent $O(\log n/\log\log n)$-approximation algorithm for ATSP due to Asadpour *et al.* [1], our approximation factor for minimum ratio ATSP also improves to $O(\frac{\log^2 n}{\log\log n})$. In this paper we give a self-contained proof of the $O(\log^2 n)$ guarantee, and note that the algorithm of [1] can be used to obtain the $O(\frac{\log^2 n}{\log\log n})$ approximation.

We then use the approximation algorithm for minimum ratio ATSP, to obtain a bi-criteria approximation algorithm for the *directed k-path* problem (Sect. 3). We also observe that the reductions in Blum *et al.* [4] and Bansal *et al.* [2] (in undirected metrics) from the k-path problem to the orienteering problem, can be easily adapted to the directed case. Together with the approximation algorithm for the directed k-path problem, we obtain an $O(\frac{\log^2 n}{\log\log n})$-approximation guarantee for directed orienteering. This answers in the affirmative, the question of poly-logarithmic approximability of directed orienteering [4].

Finally, we note that the techniques used for discounted-reward TSP [4], and vehicle routing with time-windows [2], also work in the directed setting (see Sect. 3.1 for definitions of these problems). Since these algorithms use the orienteering (or the related minimum excess) problem in a black-box fashion, our results imply approximation algorithms with guarantees $O(\frac{\log^2 n}{\log\log n})$ for discounted-reward TSP, and $O(\frac{\log^4 n}{\log\log n})$ for VRP with time-windows.

Other Related Results In independent work, Chekuri *et al.* [6] also obtained many of the results reported in this paper, although via different techniques. They obtain an $O(\log^3 k)$-approximation algorithm for the directed k-TSP problem, and an $O(\log^2 OPT)$-approximation algorithm for directed orienteering (where $OPT \leq n$ is the optimal value of the orienteering instance). Recently Bateni and Chuzhoy [3] gave an improved $O(\frac{\log^2 n}{\log\log n})$-approximation algorithm for directed k-TSP, by showing that our approach for minimum ratio ATSP (Theorem 5) can be directly applied to directed k-TSP (instead of the set-cover based reduction we use in Theorem 6). More interestingly, they also gave an $O(\frac{\log^2 n}{\log\log n} \log k)$-approximation algorithm for the "directed k-path" problem (studied in Sect. 3), which is the first true approximation ratio for this problem.

2 Directed *k*-TSP

The *directed k-TSP* problem is a generalization of the asymmetric traveling salesman problem (ATSP), for which the best known approximation guarantee is $O(\frac{\log n}{\log\log n})$ [1]. In this section, we obtain an $O(\frac{\log^2 n}{\log\log n} \cdot \log k)$-approximation algorithm for directed k-TSP. We first obtain an $O(\frac{\log n}{\log\log n})$-approximation algorithm for minimum ratio ATSP (Theorem 5), and then show how this implies the result for directed k-TSP (Theorem 6). Our algorithm for minimum ratio ATSP is based on a bound on the integrality gap (Theorem 2) of a suitable LP relaxation for ATSP, which we study next.

2.1 A Linear Relaxation for ATSP

In this section, we consider the following LP relaxation for ATSP, where (V, d) is the input metric. Let $\delta^+(S)$ denote the edges leaving the set S. Similarly, $\delta^-(S)$ is the set of edges entering S. We use $z(\delta^+(S))$ to denote the sum of the z-values of the edges in $\delta^+(S)$.

$$\begin{aligned}
&\min \sum_e d_e \cdot z_e \\
&\text{s.t.} \\
&\quad z(\delta^+(v)) = z(\delta^-(v)) \qquad \forall v \in V \\
&\quad z(\delta^+(S)) \geq 1 \qquad \forall \emptyset \neq S \neq V \\
&\quad z_e \geq 0 \qquad \forall \text{ arc } e
\end{aligned} \tag{ALP}$$

This relaxation was also studied in Vempala and Yannakakis [25], where the authors proved a structural property about basic solutions to (ALP). We are not aware of any previous result bounding the integrality gap of (ALP). However, the following stronger LP relaxation (ALP′), with additional degree equals 1 constraints, was shown to have an integrality gap of at most $\lceil \log n \rceil$ by Williamson [26]. It was also shown [26] that (ALP′) is equivalent to the Held-Karp bound [14].

$$
\begin{aligned}
&\min \sum_e d_e \cdot z_e \\
&\text{s.t.} \\
&\quad z(\delta^+(v)) = 1 \quad \forall v \in V \\
&\quad z(\delta^-(v)) = 1 \quad \forall v \in V \\
&\quad z(\delta^+(S)) \geq 1 \quad \forall \emptyset \neq S \neq V \\
&\quad z_e \geq 0 \quad \forall \text{ arc } e
\end{aligned}
\tag{ALP$'$}
$$

We first give a proof of a $\lceil \log n \rceil$ upper bound on the integrality gap of the weaker (ALP) relaxation (Theorem 2), and then show that for any asymmetric metric (V, d), the optimal values of (ALP) and (ALP′) coincide (Theorem 3). This gives an independent proof of the same upper bound for the stronger (ALP′) relaxation. Our proof makes use of the following directed splitting-off theorem due to Mader [20].

Theorem 1 (Mader [20]) *Let $D = (U + x, A)$ be a directed graph such that indegree equal to outdegree at x, and the directed connectivity between any pair of vertices in U is at least k. Then for every arc $(x, v) \in A$ there exists an arc $(u, x) \in A$ so that after replacing the two arcs (u, x) and (x, v) by an arc (u, v), the directed connectivity between every pair of vertices in U remains at least k.*

This operation of replacing two arcs (u, x) and (x, v) by the single arc (u, v) is called *splitting-off*.

Theorem 2 *The integrality gap of (ALP) is at most $\lceil \log n \rceil$.*

Proof This proof has the same outline as the proof for the stronger LP relaxation (ALP′) in Williamson [26]. We use the $\lceil \log n \rceil$ approximation algorithm for ATSP due to Frieze *et al.* [10], which works by computing minimum-length cycle covers[1] repeatedly (in at most $\lceil \log n \rceil$ iterations). Briefly, the algorithm is as follows: initially set R of *representatives* is V; in each iteration compute a minimum cycle cover on R, and retain one (arbitrary) representative from each cycle into the set R for the next iteration. In this algorithm, if $U \subseteq V$ is the set of representative vertices in some iteration, the cost incurred in this iteration equals the minimum cycle cover on U. Let ALP(U) denote the LP relaxation ALP restricted to a subset U of the original vertices (and arcs induced on U), and opt(ALP(U)) its optimal value. Then we have:

Claim 1 *For any subset $U \subseteq V$, the minimum cycle cover on U has cost at most* opt(ALP(U)).

[1] A cycle cover is a subgraph in which every vertex has in-degree and out-degree exactly one, and hence is a cover of the vertices by directed cycles.

Proof Consider the following linear relaxation for cycle cover.

$$\begin{aligned}
&\min \sum_e d_e \cdot x_e \\
&\text{s.t.} \\
&\quad x(\delta^+(v)) - x(\delta^-(v)) = 0 \quad && \forall v \in U \\
&\quad x(\delta^+(v)) \geq 1 && \forall v \in U \\
&\quad x_e \geq 0 && \forall \text{ arc } e
\end{aligned} \tag{CLP}$$

These constraints are equivalent to a circulation problem on network N which contains two vertices v_{in} and v_{out} for each vertex $v \in U$. The arcs in N are: $\{(u_{out}, v_{in}) : \forall u, v \in U,\ u \neq v\}$, and $\{(v_{in}, v_{out}) : \forall v \in U\}$. The cost of each (u_{out}, v_{in}) arc is $d(u, v)$, and each (v_{in}, v_{out}) arc costs 0. It is easy to see that the minimum cost circulation on N that places at least one unit of flow on each arc in $\{(v_{in}, v_{out}) : \forall v \in U\}$ is exactly the optimal solution to (CLP). But the linear program for minimum cost circulation is integral (network matrices are totally unimodular, cf. [22]), and so is (CLP).

Any integral solution to (CLP) defines an Eulerian subgraph H with each vertex in U having degree at least 1. Each connected component C of H is Eulerian and can be shortcut to get a cycle on the vertices of C. Since triangle inequality holds, the cost of each such cycle is at most that of the original component. So this gives a cycle cover of U of cost at most opt(CLP(U)), the optimal value of (CLP). But the linear program ALP(U) is more constrained than CLP(U); so the minimum cycle cover on U costs at most opt(ALP(U)). □

We now establish the *monotonicity property* of ALP, namely:

$$\text{opt}(\text{ALP}(U)) \leq \text{opt}(\text{ALP}(V)) \quad \forall U \subseteq V$$

Consider any subset $U \subseteq V$, vertex $v \in U$, and $U' = U - v$; we will show that opt(ALP(U')) $\leq$ opt(ALP(U)). Let z be any fractional solution to ALP(U) so that $L \cdot z$ is integral for some large enough $L \in \mathbb{N}$. Define a multigraph H on vertex set U with $L \cdot z_{w_1, w_2}$ arcs going from w_1 to w_2 (for all $w_1, w_2 \in U$). From the feasibility of z in ALP(U), H is Eulerian and has arc-connectivity at least L. Now applying Theorem 1 repeatedly on vertex $v \in U$ (until its degree is zero), we obtain a multigraph H' on $U' = U - v$ such that the arc-connectivity of H' is still at least L. Further, due to the triangle inequality, the total cost of H' is at most that of H. Finally, scaling down H' by L we obtain a fractional solution to ALP(U') of cost at most $d \cdot z$. Thus, opt(ALP(U')) $\leq$ opt(ALP(U)), and using this inductively we have monotonicity for ALP.

This suffices to prove the theorem, as the cost incurred in each iteration of the Frieze *et al.* [10] algorithm can be bounded by opt(ALP(V)), and there are at most $\lceil \log n \rceil$ iterations. □

We note that in order to prove the monotonicity property for the linear program (ALP′), Williamson [26] used the equivalence of (ALP′) and the Held-Karp bound [14], and showed that the Held-Karp lower bound is monotone. Using splitting-off, we obtained a more direct proof of monotonicity. In fact, we can prove a stronger statement than Theorem 2, which relates the optimal values of (ALP) and (ALP′). It was shown in [26] that the optimal value of (ALP′) equals the Held-Karp lower bound [14]; so the next theorem shows that for any ATSP instance, the values of the Held-Karp bound, (ALP′) and (ALP) are all equal. A similar result for the symmetric case was proved in Goemans and Bertsimas [11], which was also based on splitting-off (for undirected graphs).

Theorem 3 *The optimal values of* (ALP) *and* (ALP′) *are equal.*

Proof Clearly the optimal value of (ALP′) is at most that of (ALP). We will show that any fractional solution z to (ALP) can be modified to a fractional solution z' to (ALP′), such that $\sum_e d_e \cdot z'_e \le \sum_e d_e \cdot z_e$, which would prove the theorem. As in the proof of Theorem 2, let $L \in \mathbb{N}$ be large enough so that $L \cdot z$ is integral, and let H denote a multi di-graph with $L \cdot z_{u,v}$ arcs from u to v, for all $u, v \in V$. From the feasibility of z in (ALP), we know that H is Eulerian and has arc-connectivity at least L.

If some $v \in V$ has degree strictly greater than L, we reduce its degree by one as follows. Let v' be any vertex in $V \setminus v$, and $\mathcal{P}_{v,v'}$ denote a minimal set of arcs that constitutes exactly L arc-disjoint paths from v to v'. Due to minimality, the number of arcs in $\mathcal{P}_{v,v'}$ incident to v is exactly L and they are all arcs leaving v. Since the degree of v is at least $L+1$, there is an arc $(v, w) \in H \setminus \mathcal{P}_{v,v'}$. Applying Theorem 1 to arc (v, w), we obtain arc $(u, v) \in H \setminus \mathcal{P}_{v,v'}$ such that the arc-connectivity of vertices $V \setminus v$ in $H' = (H \setminus \{(u, v), (v, w)\}) \cup (u, w)$ remains at least L. Further, by the choice of (v, w), $\mathcal{P}_{v,v'} \subseteq H'$; so the arc-connectivity from v to v' in H' is at least L. Since H' is Eulerian, it now follows that the arc-connectivity of vertices V in H' is also at least L. Thus we obtain a multigraph H' from H which maintains connectivity and decreases the degree of vertex v by 1. Repeating this procedure for all vertices in V having degree greater than L, we obtain (an Eulerian) multigraph G having arc-connectivity L such that the degree of each vertex equals L.

Note that in the degree reducing procedure above, the only operation we used was splitting-off. Since d satisfies triangle inequality, the total cost of arcs in G (under length d) is at most that of H. Finally, scaling down G by L, we obtain the claimed fractional solution z' to (ALP′). □

Using this correspondence, we obtain the following improvement:

Corollary 1 ([1]) *The integrality gap of* (ALP) *is* $O(\log n / \log\log n)$.

Proof Asadpour *et al.* [1] gave an $O(\log n / \log\log n)$-approximation algorithm for ATSP relative to the LP relaxation (ALP′). Using Theorem 3 we obtain the corollary. □

2.2 Minimum Ratio ATSP

We now describe the approximation algorithm for minimum ratio ATSP, which uses Theorem 2. We call any tour containing the root r an r-tour. In addition, we require the following strengthening of Mader's splitting-off Theorem, in the case of Eulerian digraphs.

Theorem 4 (Frank [9] (Theorem 4.3) and Jackson [15]) *Let $D=(U+x,A)$ be a directed Eulerian graph. For each arc $f=(x,v)\in A$ there exists an arc $e=(u,x)\in A$ so that after replacing arcs eandf by arc (u,v), the directed connectivity between every pair of vertices in U is preserved.*

Theorem 5 *There is an $O(\log^2 n)$-approximation algorithm for the minimum ratio ATSP problem.*

Proof The approximation algorithm for minimum ratio ATSP is based on the following LP relaxation for this problem.

$$
\begin{aligned}
&\min \sum_e d_e \cdot x_e \\
&\text{s.t.} \\
&\quad x(\delta^+(v)) = x(\delta^-(v)) && \forall v \in V \\
&\quad x(\delta^+(S)) \geq y_v && \forall S \subseteq V-\{r\}\ \forall v \in S \\
&\quad \sum_{v\neq r} y_v \geq 1 \\
&\quad x_e \geq 0 && \forall \text{ arc } e \\
&\quad 0 \leq y_v \leq 1 && \forall v \in V-\{r\}
\end{aligned}
\tag{RLP}
$$

To see that this is indeed a relaxation, consider the optimal integral r-tour C^* that covers l vertices (excluding r). We construct a solution to (RLP) by setting $y_v=\frac{1}{l}$ for all vertices $v\in C^*$, and $x_e=\frac{1}{l}$ for all arcs $e\in C^*$. It is easy to see that this solution is feasible and has cost $\frac{d(C^*)}{l}$ which is the optimal ratio. The linear program (RLP) can be solved in polynomial time using the Ellipsoid algorithm. The algorithm is as follows:

1. Let (x,y) denote an optimal solution to (RLP).
2. Discard all vertices $v\in V\setminus r$ with $y_v\leq\frac{1}{2n}$; all remaining vertices have y-values in the interval $[\frac{1}{2n},1]$.
3. Define $g=\lceil\log_2 n+1\rceil$ groups of vertices where group G_i (for $i=1,\dots,g$) consists of all vertices v having $y_v\in(\frac{1}{2^i},\frac{1}{2^{i-1}}]$.
4. Run the Frieze *et al.* [10] algorithm on each of $G_i\cup\{r\}$ and output the r-tour with the smallest ratio.

Note that the total y-value of vertices remaining after step 2 is at least $1/2$. Consider any group G_i; let $L_i\in\mathbb{N}$ be large enough so that $L_i\cdot 2^i\cdot x$ is integral. We note that L_is are not required to by polynomially bounded- we use them only in the analysis and not in the algorithm. Define a multigraph H_i with $L_i\cdot 2^i\cdot x_{u,v}$ arcs from u to v for all $u,v\in V$. Below, for a directed graph D and vertices $u,v\in D$ the directed arc-connectivity from u to v is denoted $\lambda(u,v;D)$. From the feasibility of x in RLP, it is clear that H_i is Eulerian. Further, for all $v\in G_i$, $\lambda(r,v;H_i)=$

$\lambda(v, r; H_i) \geq L_i \cdot 2^i \cdot y_v \geq L_i$. Now we split-off vertices in $V \setminus (G_i \cup \{r\})$ one by one, using Theorem 4, which preserves the arc-connectivity of $G_i \cup \{r\}$. This results in an Eulerian multigraph H_i' on vertices $G_i \cup r$ satisfying $\lambda(r, v; H_i'), \lambda(v, r; H_i') \geq L_i$ for all $v \in G_i$. Further, due to triangle inequality the total weight of arcs in H_i' is at most that in H_i. Now, scaling down H_i' by L_i, we obtain a fractional solution z^i to ALP$(G_i \cup \{r\})$ of cost $d \cdot z^i \leq 2^i(d \cdot x)$. Now Theorem 2 implies that there exists an r-tour on G_i of cost at most $\beta = \lceil \log n \rceil$ times $d \cdot z^i$. In fact, the Frieze *et al.* [10] algorithm applied on $G_i + r$ produces such a tour. We now claim that one of the r-tours found in step 4 (over all $i = 1, \ldots, g$) has a small ratio:

$$\min_{i=1}^{g} \frac{\beta(d \cdot z^i)}{|G_i|} \leq \min_{i=1}^{g} \frac{2^i \beta(d \cdot x)}{|G_i|} \leq \frac{\beta \sum_{i=1}^{g} d \cdot x}{\sum_{i=1}^{g} |G_i|/2^i} \leq 4g\beta \cdot (d \cdot x)$$

The last inequality follows from the fact that after step 2,

$$\frac{1}{2} \leq \sum_{v \neq r} y_v \leq \sum_{i=1}^{g} \frac{1}{2^{i-1}} |G_i| = 2 \sum_{i=1}^{g} \frac{|G_i|}{2^i},$$

since there is a total y-weight of at least $1/2$ even after step 2. Thus we have a $4g\beta = O(\log^2 n)$ approximation algorithm for minimum ratio ATSP. □

We note that for this proof of Theorem 5 to work, just a bound on the integrality gap of (ALP′) [26] is insufficient. The Eulerian multigraph H_i' that gives rise to the fractional ATSP solution z^i on $G_i \cup \{r\}$ may not have degree L_i at all vertices; so z^i may be infeasible for $ALP'(G_i \cup \{r\})$. This is the reason we need to consider the LP relaxation (ALP).

Corollary 2 *There is an $O(\frac{\log^2 n}{\log \log n})$-approximation algorithm for minimum ratio ATSP.*

Proof This is identical to the algorithm in Theorem 6, except that we use the Asadpour *et al.* [1] algorithm in Step 4, and use Corollary 1 instead of Theorem 2 in the analysis. □

2.3 Application to Directed k-TSP

We now describe how minimum ratio ATSP can be used to obtain an approximation algorithm for the directed k-TSP problem.

Theorem 6 *There is a polynomial time $O(\log^2 n \cdot \log k)$ approximation algorithm for the directed k-TSP problem.*

Proof We use the $\alpha = O(\log^2 n)$-approximation algorithm for the related minimum ratio ATSP problem. Let *OPT* denote the optimal value of the directed k-TSP instance. By performing binary search, we may assume that we know the value of *OPT* within a factor 2. We only consider vertices $v \in V$ satisfying $d(r, v), d(v, r) \leq OPT$;

this does not affect the optimal solution. Then we invoke the minimum ratio ATSP algorithm repeatedly (each time restricted to the currently uncovered vertices) until the total number of covered vertices $t \geq \frac{k}{2}$. Note that for every instance of the ratio problem that we solve, there is a feasible solution of ratio $\leq \frac{2 \cdot OPT}{k}$ (namely, the optimal k-TSP tour covering at least $k/2$ residual vertices). Thus we obtain an r-tour on $t \geq \frac{k}{2}$ vertices having ratio $\leq \frac{2\alpha \cdot OPT}{k}$; so the length of this r-tour is at most $\frac{2\alpha t \cdot OPT}{k}$. Note that t may be much larger than k. Therefore, we split this r-tour into $l = \lceil \frac{2t}{k} \rceil$ di-paths, each containing at least $\frac{t}{l} \geq \frac{k}{4}$ vertices (this can be done in a greedy fashion). By averaging, the minimum length di-path in this collection has length at most $\frac{2\alpha t OPT/k}{l} \leq \alpha \cdot OPT$. Joining the first and last vertices in this di-path to r, we obtain an r-tour containing at least $\frac{k}{4}$ vertices, of length at most $(\alpha + 2) \cdot OPT$. So we get an $(O(\alpha), 4)$ bi-criteria approximation for directed k-TSP. This algorithm can now be used as follows. Until k vertices are covered, repeat: if k' denotes the number of vertices covered so far, run the bi-criteria approximation algorithm with a target of $k - k'$, restricted to currently uncovered vertices. A standard set cover based analysis implies that this is an $O(\alpha \cdot \log k)$-approximation algorithm for directed k-TSP. $\square$

3 Directed Orienteering

In this section, we consider the orienteering problem in asymmetric metrics. As mentioned before, this is in fact the directed counterpart of the point-to-point orienteering problem [2]. In what follows, we adapt the framework of Blum *et al.* [4] (for undirected orienteering) to the directed case.

As in Blum *et al.* [4], we define the *excess* of an s-t di-path as the difference of the path length and the shortest path distance from s to t. The *directed min-excess* problem is defined as follows: given an asymmetric metric (V, d), origin (s) and destination (t) vertices, and a target k, find an s-t di-path of minimum excess that visits at least k other vertices.

The *directed k-path* problem is the following: given an asymmetric metric (V, d), origin (s) and destination (t) vertices, and a target k, find an s-t di-path of minimum length that visits at least k other vertices.

The algorithm of [4] for directed orienteering is based on the following sequence of reductions: directed k-path to minimum ratio ATSP (Theorem 7), directed minimum excess to directed k-path (Theorem 8), and directed orienteering to directed minimum excess (Theorem 9). The last two reductions are identical to the corresponding reductions for undirected orienteering in Blum *et al.* [4] and Bansal *et al.* [2].

We prove the following bi-criteria approximation guarantee for the directed k-path problem.

Theorem 7 *A ρ-approximation algorithm for minimum ratio ATSP implies a $(3, 4\rho)$ bi-criteria approximation algorithm for the directed k-path problem.*

Proof We assume (by performing a binary search) that we know the optimal value OPT of the directed k-path instance within a constant factor, and let G denote the directed graph corresponding to metric (V, d) (which has an arc of length $d(u, v)$ from u to v for every pair of vertices $u, v \in V$). We modify graph G to obtain graph H as follows: (a) discard all vertices v such that $d(s, v) > OPT$ or $d(v, t) > OPT$; and (b) add an extra arc from t to s of length OPT. In the rest of this proof, we refer to the shortest path metric induced by H as (V, l). Note that each tour in metric l corresponds to a tour in graph H (using shortest paths in H for each metric arc); below, any tour in metric l will refer to the corresponding tour in graph H. Since there is an s-t path of length OPT (in metric d) covering k vertices, appending the (t, s) arc, we have an s-tour σ^* of length at most $2 \cdot OPT$ (in metric l) covering $k+1$ vertices.

Now, we run the minimum ratio ATSP algorithm with root s in metric l repeatedly until either (1) $\frac{k}{2}$ vertices are covered and the extra (t, s) arc is never used in the current tour (in graph H); or (2) the extra (t, s) arc is used for the first time in the current tour (in H). Let σ be the s-tour obtained (in graph H) at the end of this iteration, and h the number of vertices covered. Note that each s-tour added in a single call to minimum ratio ATSP, may use the extra (t, s) arc at most once (by an averaging argument). So in case (1), the (t, s) arc is absent in σ, and in case (2), the (t, s) arc is used exactly once and it is the last arc in σ. Note also that during each call of minimum ratio ATSP, there is a feasible solution of ratio $\frac{2OPT}{k}$ (σ^* restricted to the remaining vertices); so the ratio of the s-tour σ, $\frac{l(\sigma)}{h} \le \rho \cdot \frac{2OPT}{k}$. From σ we now obtain a feasible s-t path τ in metric d as follows. In case (1), add a direct (s, t) arc: $\tau = \sigma \cdot (s, t)$; in case (2), remove the only copy of the extra (t, s) arc (occurring at the end of σ): $\tau = \sigma \setminus \{(t, s)\}$. In either case, s-t path τ contains h vertices and has length $d(\tau) \le \frac{2\rho h}{k} OPT + OPT$. Note that in case (1), $h \ge \frac{k}{2}$; and in case (2), since the extra (t, s) arc is used, $\frac{OPT}{h} \le \frac{l(\sigma)}{h} \le 2\rho \frac{OPT}{k}$, so $h \ge \frac{k}{2\rho}$. Hence in either case, τ contains $h \ge \frac{k}{2\rho}$ vertices and $d(\tau) \le \frac{4\rho h}{k} OPT$. We now greedily split τ into maximal paths, each of which has length at most OPT; the number of subpaths obtained is at most $\frac{d(\tau)}{OPT} \le \frac{4\rho h}{k}$. So one of these paths contains at least $h/(\frac{4\rho h}{k}) = \frac{k}{4\rho}$ vertices. Adding direct arcs from s to the first vertex on this path and from the last vertex on this path to t, we obtain an s-t path of length at most $3 \cdot OPT$ containing at least $\frac{k}{4\rho}$ vertices. □

The next two theorems together reduce the directed orienteering problem to the directed k-path problem, for which we just obtained an approximation algorithm.

Theorem 8 (Blum *et al.* [4]) *An (α, β) bi-criteria approximation algorithm for the directed k-path problem implies a $(2\alpha - 1, \beta)$ bi-criteria approximation algorithm for the directed minimum excess problem.*

Theorem 9 (Bansal *et al.* [2]) *An (α, β) bi-criteria approximation algorithm for the directed minimum excess problem implies an $\lceil \alpha \rceil \cdot \beta$ approximation algorithm for the directed orienteering problem.*

The proofs of Theorems 8 and 9 are identical to the corresponding proofs in the undirected setting, and are not repeated here. The only difference from the undirected case is that we consider bi-criteria guarantees for the directed k-path and minimum excess problems. We now obtain a result that relates the directed orienteering problem and minimum ratio ATSP.

Corollary 3 *A ρ-approximation algorithm for the minimum ratio ATSP problem implies an $O(\rho)$-approximation algorithm for the directed orienteering problem. Conversely, a ρ-approximation algorithm for directed orienteering implies an $O(\rho)$-approximation algorithm for minimum ratio ATSP.*

Proof The first direction follows directly from Theorems 7, 8 and 9. For the other direction, we are given a ρ-approximation algorithm for directed orienteering. Let D denote the length of some minimum ratio tour σ^*, t the last vertex visited by σ^* (before returning to the root r), and h the number of vertices it covers; so the optimal ratio is $\frac{D}{h}$. The algorithm for minimum ratio ATSP first guesses a value D' such that $D' \leq D \leq 2 \cdot D'$, and the last vertex t. Note that we can guess powers of 2 for the value of D', which gives $O(\log_2(n \cdot d_{\max}))$ possibilities for D' (where $d_{\max}$ is the length of the longest arc). Also, the number of possibilities for t is at most n; so the algorithm only makes a polynomial number of guesses. The algorithm then runs the directed orienteering algorithm with r and t as the start/end vertices and a length bound of $2D' - d(t,r) \geq D - d(t,r)$. Note that removing the last (t,r) arc from σ^* gives a feasible solution to this orienteering instance that covers h vertices. Hence the ρ-approximation algorithm is guaranteed to find an r-t di-path covering at least $\frac{h}{\rho}$ vertices, having length at most $2D' - d(t,r)$. Now, adding the (t,r) arc to this path gives an r-tour of ratio at most $2D'/(\frac{h}{\rho}) \leq 2\rho\frac{D}{h}$. □

Corollary 3 and Corollary 2 imply an $O(\log^2 n/\log\log n)$-approximation algorithm for the directed orienteering problem. Further, any improvement in the approximation guarantee of minimum ratio ATSP implies a corresponding improvement for directed orienteering.

3.1 Some Extensions

Discounted Reward TSP In this problem [4], we are given a metric space with rewards on vertices, and a discount factor $\gamma < 1$; the goal is to find a path that maximizes the total discounted reward (where the reward for a vertex visited at distance t is discounted by a factor γ^t). The approximation algorithm for the undirected version of this problem (Blum *et al.* [4]) uses the minimum excess problem as a subroutine within a dynamic program. It can be verified directly that this reduction also works in the directed case, and so the $(O(1), O(\log^2 n/\log\log n))$ bi-criteria approximation for directed minimum excess implies an $O(\log^2 n/\log\log n)$-approximation algorithm for directed discounted reward TSP.

Vehicle Routing Problem with Time Windows In this VRP, we are given a metric space with a specified depot vertex and all other vertices having a time window (that specifies a release time and a deadline), and the goal is to find a path starting at the depot that maximizes the number of vertices visited in their time window. Note that orienteering is a special case when all vertices have the same time window.

Bansal *et al.* [2] use the point-to-point orienteering problem as a subroutine, and show that an α-approximation algorithm for orienteering implies an $O(\alpha \cdot \log^2 n)$-approximation for vehicle routing with time-windows. In fact, all the steps used in these reductions can be adapted to the case of directed metrics as well. So there is an $O(\log^4 n/\log\log n)$-approximation algorithm for VRP with time-windows on asymmetric metrics.

A special case of the VRP with time-windows occurs when each vertex has the same release time, and only the deadline is vertex dependent; this problem is *deadline TSP*. The results of Bansal *et al.* [2] for this problem, along with the directed orienteering algorithm imply an $O(\log^3 n/\log\log n)$-approximation algorithm for directed deadline TSP.

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