CMPUT 675: Approximation Algorithms **Winter 2005**

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Lecturer: Mohammad R. Salavatipour *Scribe: Jiyang Chen*

4.1 Designing algorithm using the primal-dual method

Consider the following example:

$$\begin{array}{ll} \text{minimize}: & 10x_1 + 6x_2 + 4x_3 \\ \text{subject to}: & 2x_1 + x_2 - x_3 \geq 2 \\ & x_1 + x_2 + x_3 \geq 3 \\ & x_i \geq 0 \end{array}$$

Suppose z^* is the value of the optimal solution and we are interested in finding upper and lower bounds for z^* without computing it. For instance, how can we say if $z^* \leq 40$? For this question, it is enough to find one feasible solution (not necessarily optimal) for which the objective function has value at most 40.

Now consider the following question: is $z^* \geq 10$? To answer this we need to find lower bounds for all feasible solutions. We can obtain lower bounds by looking at the constraints. For instance, from constraint 1, and because the coefficients of variables x_1, x_2, x_3 are all smaller in the constraint with respect to those in the objective function, it follows that 2 is a lower bound for z^*. By combining constraints 1 & 2 we obtain that $3x_1 + 2x_2 + 0x_3 \geq 5$ and therefore 5 is a new lower bound for z^*. In general, any linear combination of these constraints could lead to a lower bound, as long as the final coefficients of the variables are not larger than those in the objective function. For instance, using a y_1 factor of constraint 1 and y_2 factor of constraint 2, we get:

$$\begin{array}{ll} & y_1(2x_1 + x_2 - x_3) \geq 2y_1 \\ & y_2(x_1 + x_2 + x_3) \geq 3y_2 \\ \text{subject to} & 2y_1 + y_2 \leq 10 \\ & y_1 + y_2 \leq 6 \\ & -y_1 + y_2 \leq 4 \end{array}$$

And $2y_1 + 3y_2$ is a lower bound for z^* as long as the three new constraints on y_1, y_2 are satisfied. Since we want to find the best lower bound, we have to maximize $2y_1 + 3y_2$. This raises another linear program. The first problem is called the primal LP and the second is called the dual. Every LP has a dual, the optimal solution of the dual is the minimum of the primal. In general, for a primal LP of the form:

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^{n} c_i x_i \\ \text{subject to} & \forall i, \sum_{j=1}^{n} a_{ij} x_j \geq b_i \\ & x_j \geq 0 \end{array}$$

The dual has the form:

$$\begin{array}{ll} \text{maximize} & \sum_{i=1}^{n} b_i y_i \\ \text{subject to} & \forall j, \sum_{i=1}^{m} a_{ij} y_i \leq c_j \\ & y_i \geq 0 \end{array}$$

Any feasible solution of primal is an upper bound for every feasible solution to the dual. In our example, $x^* = (0, \frac{5}{2}, \frac{1}{2})$, gives $z^* = 17$ for the primal, $y^* = (1, 5)$ is the optimal dual with 17.

Theorem 4.1 (LP-Duality Theorem) *Primal has finite optimal solution iff the dual has finite optimal solution. Also, if $\vec{x^*}$ and $\vec{y^*}$ are optimal solutions for the primal and dual, respectively, then*

$$\sum_{j=1}^{n} c_j x_j = \sum_{i=1}^{m} b_i y_i$$

Theorem 4.2 (Weak Duality Theorem) *If $\vec{x}$ and $\vec{y}$ are feasible solutions to the primal and dual, then*

$$\sum_{j=1}^{n} c_j x_j \geq \sum_{i=1}^{m} b_i y_i$$

Proof: Since $\vec{y}$ is dual feasible and x_j's are nonnegative,

$$\sum_{j=1}^{n} c_j x_j \geq \sum_{j=1}^{n} (\sum_{i=1}^{m} a_{ij} y_i) x_j$$

Similarly, since x is primal feasible and y_i's are nonnegative,

$$\sum_{i=1}^{m} (\sum_{j=1}^{n} a_{ij} x_j) y_i \geq \sum_{i=1}^{m} b_i y_i$$

The theorem follows by observing that

$$\sum_{j=1}^{n} (\sum_{i=1}^{m} a_{ij} y_i) x_j = \sum_{i=1}^{m} (\sum_{j=1}^{n} a_{ij} x_j) y_i$$

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Theorem 4.3 *Let x and y be primal and dual feasible solutions, respectively. Then x and y are both optimal iff all of the following conditions are satisfied:*

Primal complementary slackness conditions:

$$\forall\ 1 \leq j \leq n, \textit{either } x_j = 0 \textit{ or } \sum_{i=1}^{m} a_{ij} y_i = c_j$$

Dual complementary slackness conditions:

$$\forall\ 1 \leq i \leq m, \textit{either } y_i = 0 \textit{ or } \sum_{j=1}^{n} a_{ij} x_j = b_i$$

To use the Primal-Dual method, we relax these conditions and obtain the relaxed complementary slackness conditions:

If $\vec{x}$ and $\vec{y}$ are feasible solutions to primal and dual, respectively and if they satisfy

Primal (relax) complementary slackness condition:

$$\alpha \geq 1, \forall\ 1 \leq j \leq n,\ \textit{either } x_j = 0 \textit{ or } \frac{c_j}{\alpha} \leq \sum_{i=1}^{m} a_{ij} y_i \leq c_j$$

Dual (relaxed) complementary slackness condition:

$$\beta \geq 1, \forall\ 1 \leq i \leq m,\ \textit{either } y_i = 0 \textit{ or } b_i \leq \sum_{j=1}^{n} a_{ij} x_j \leq b_i \beta$$

then

$$\sum_{j=1}^{n} c_j x_j \leq \alpha \cdot \beta \cdot \sum_{i=1}^{m} b_i y_i$$

where $\alpha \cdot \beta$ is the approximation factor.

Remark: If $\alpha = \beta = 1$, then we have the original complementary slackness conditions.

The general idea of primary-dual method is to start with a primal infeasible and a dual feasible solution (usually the trivial solution $\vec{x} = 0$ and $\vec{y} = 0$). Then we iteratively improve the feasibility of primal and optimality of dual. Primal is always extended integrally at the end Primal is the feasible solution. At each iteration, we use relaxed slackness conditions to help to find feasible solutions to the primal.

Big advantage of Primal-Dual over rounding: we don't have to solve LP (which is time consuming although polynomial time solvable).

4.2 PrimalDual applied to Set Cover

Recall the Set Cover problem and consider the LP-relaxation:

- Universe: $e_1, e_2 ... e_n$
- Sets: $S_1, S_2 ... S_k$
- Cost: $S \to Q^+$

Primal:

$$\begin{array}{ll} \text{minimize} & \sum_{s \in S} cost(s) \cdot x_s \\ \text{subject to} & \forall e \in U : \sum_{s: e \in s} x_s \geq 1 \\ & x_s \geq 0 \end{array}$$

Dual:

$$\begin{array}{ll} \text{Maximize} & \sum_{e \in U} y_e \\ \text{subject to} & \forall S \in \mathcal{S} : \sum_{S: e \in S} y_e \leq cost(S) \\ & y_e \geq 0 \end{array}$$

Note that the dual problem is a packing problem. We can say, we are going to pack stuff into elements so that the total amount packed is maximized without overpacking any set.

Remark: The dual of a covering problem is a packing problem and the dual of a packing problem is a covering problem. Packing problems are typically harder.

We are going to use the Primal-Dual method with $\alpha = 1$ and $\beta = f$. Primal Relaxed Slackness Condition:

$$\forall S \in \mathcal{S} : x_S \neq 0 \Rightarrow \sum_{e \in S} y_e = cost(S)$$

We call a set S for which the right-hand-side equality holds a tight set (intuitively we cannot pack more stuff into elements of that set).

Dual Relaxed Slackness Condition:

$$\forall e \in U : y_e \neq 0 \Rightarrow 1 \leq \sum_{S:e \in S} x_S \leq f$$

The following is the **PrimalDual Set Cover Algorithm**.

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x⃗ ← 0, y⃗ ← 0
while not all elements are covered do
  pick an uncovered element e, raise y_e until some set goes tight
  pick all tight sets and update x⃗
  declare all the elements in those sets as covered
endwhile
output the set cover x⃗
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Theorem 4.4 *This algorithm is an f approximation for set cover*

Proof: The solution is clearly a set cover by the condition of the loop, i.e. as long as there is an uncovered set we raise the corresponding value until some set that contains it goes tight and then it is picked. Thus all elements are covered at the end and we have a feasible primal solution. No set is going to be overpacked, so the dual is always feasible too. At the end, we have a feasible primal and a feasible dual solution. Moreover, the algorithm picks tight sets only. Thus the primal slackness condition is satisfied. Also, for every element e, it is covered (i.e. $1 \leq \sum_{S:e \in S} x_s$) and there are at most f elements covering it (i.e. $\sum_{S:e \in S} x_s \leq f$). Thus the primal and dual complementary slackness conditions are satisfied. Therefore, we get an $\alpha \cdot \beta = f$ approximation. ■