Happy New Year!

Structured Prediction for Computer Vision

MLSS, Sydney 2015

Stephen Gould

19 February 2015

Structured Models are Pervasive in Computer Vision

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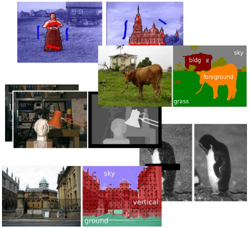

pixel labeling

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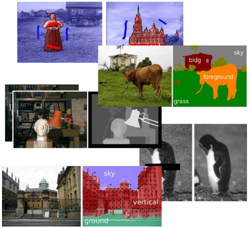

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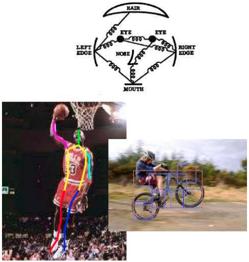

object detection, pose estimation

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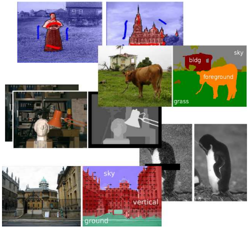

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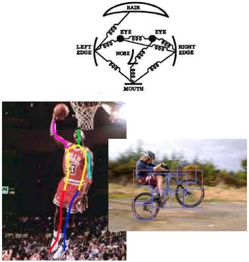

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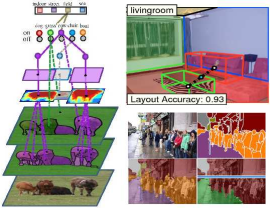

scene understanding

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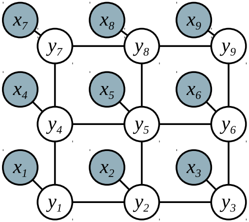

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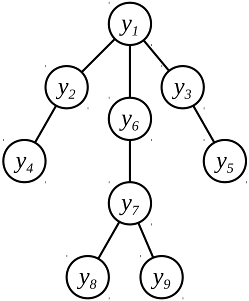

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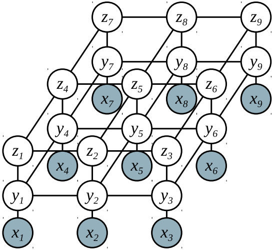

scene understanding

Demonstration: Pixel Labeling

[Agarwala et al., 2004]

- 640 × 480 image ≈ 300k pixels
- 4 possible labels per pixel
- $4^{300,000}$ label configurations
- inference in under 30 seconds (unoptimized code)

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Conditional Markov Random Fields

- Also known as:
  - Markov Networks, Undirected Graphical Models, MRFs, Structured Prediction models
  - I make no distinction between these (in this tutorial)
- $\mathbf{X} \in \mathcal{X}$ are the observed random variables (always)
- $\mathbf{Y} = (Y_1, \ldots, Y_n) \in \mathcal{Y}$ are the output random variables
- $\mathbf{Y}_c$ are a subset of variables for clique $c \subseteq \{1, \ldots, n\}$
- Define a factored probability distribution

$$P(\mathbf{Y} \mid \mathbf{X}) = \frac{1}{Z(\mathbf{X})} \prod_c \Psi_c(\mathbf{Y}_c; \mathbf{X})$$

  where $Z(\mathbf{X}) = \sum_{\mathbf{Y} \in \mathcal{Y}} \prod_c \Psi_c(\mathbf{Y}_c; \mathbf{X})$ is the partition function
- Main difficulty is the exponential number of configurations

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Machine Learning Tasks

There are two main tasks that we are interested in when talking about conditional Markov random fields (machine learning, more generally):

- **Learning:** Given data (and a problem specification), how do we choose the structure and set the parameters of our model?
- **Inference:** Given our model, how do we answer queries about instances of our problem?

MAP Inference

We will mainly be interested in maximum a posteriori (MAP) inference

$$\begin{aligned}
\mathbf{y}^\star &= \operatorname*{argmax}_{\mathbf{y}\in\mathcal{Y}} P(\mathbf{y} \mid \mathbf{x}) \\
&= \operatorname*{argmax}_{\mathbf{y}\in\mathcal{Y}} \frac{1}{Z(\mathbf{X})} \prod_c \Psi_c(\mathbf{Y}_c; \mathbf{X}) \\
&= \operatorname*{argmax}_{\mathbf{y}\in\mathcal{Y}} \log \left(\frac{1}{Z(\mathbf{X})} \prod_c \Psi_c(\mathbf{Y}_c; \mathbf{X}) \right) \\
&= \operatorname*{argmax}_{\mathbf{y}\in\mathcal{Y}} \sum_c \log \Psi_c(\mathbf{Y}_c; \mathbf{X}) - \log Z(\mathbf{X}) \\
&= \operatorname*{argmax}_{\mathbf{y}\in\mathcal{Y}} \sum_c \log \Psi_c(\mathbf{Y}_c; \mathbf{X})
\end{aligned}$$

Energy Functions

- Define an energy function

$$E(\mathbf{Y}; \mathbf{X}) = \sum_{c} \psi_c(\mathbf{Y}_c; \mathbf{X})$$

where $\psi_c(\cdot) = -\log \Psi_c(\cdot)$

- Then

$$P(\mathbf{Y} \mid \mathbf{X}) = \frac{1}{Z(\mathbf{X})} \exp\left\{-E(\mathbf{Y}; \mathbf{X})\right\}$$

- And

$$\underset{\mathbf{y} \in \mathcal{Y}}{\operatorname{argmax}} P(\mathbf{y} \mid \mathbf{x}) = \underset{\mathbf{y} \in \mathcal{Y}}{\operatorname{argmin}} E(\mathbf{y}; \mathbf{x})$$

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energy minimization ‘equals’ MAP inference

Clique Potentials

- A clique potential $\psi_c(\mathbf{y}_c; \mathbf{x})$ defines a mapping from an assignment of the random variables to a real number

$$\psi_c : \mathcal{Y}_c \times \mathcal{X} \to \mathbb{R}$$

- The clique potential encodes a preference for assignments to the random variables (lower value is more preferred)
- Often parameterized as

$$\psi_c(\mathbf{y}_c; \mathbf{x}) = \mathbf{w}_c^T \phi_c(\mathbf{y}_c; \mathbf{x})$$

- In this tutorial is suffices to think of the clique potentials as big lookup tables
- We will also ignore the explicit conditioning on $\mathbf{X}$

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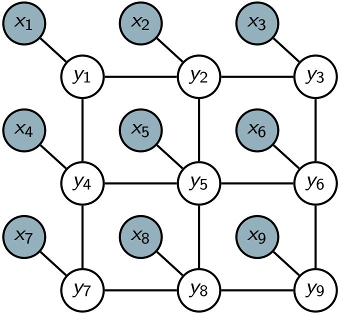
Clique Potential Arity

$$
\begin{aligned}
E(\mathbf{y}; \mathbf{x}) &= \sum_c \psi_c(\mathbf{y}_c; \mathbf{x}) \\
&= \underbrace{\sum_{i \in \mathcal{V}} \psi_i^U(y_i; \mathbf{x})}_{\text{unary}} + \underbrace{\sum_{ij \in \mathcal{E}} \psi_{ij}^P(y_i, y_j; \mathbf{x})}_{\text{pairwise}} + \underbrace{\sum_{c \in \mathcal{C}} \psi_c^H(\mathbf{y}_c; \mathbf{x})}_{\text{higher-order}}.
\end{aligned}
$$



Example Energy Functions

Semantic Segm.

Labels: $\mathcal{L} = \{\text{sky}, \text{tree}, \text{grass}, \ldots\}$
Unary: classifier, $\psi_i^U(y_i = \ell; \mathbf{x}) = \log \mathrm{P}\left(\phi_i(\mathbf{x}) \mid \ell\right)$
Pairwise: contrast-dependent smoothness prior,

$$\psi_{ij}^P(y_i, y_j; \mathbf{x}) = \begin{cases} \lambda_0 + \lambda_1 \exp\left(-\frac{\|x_i - x_j\|^2}{2\beta}\right), & \text{if } y_i \neq y_j \\ 0, & \text{otherwise} \end{cases}$$

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Object Detection

Labels: $\mathcal{L} = [0, W] \times [0, H] \times \mathbb{R}_+$
Unary: part detector/filter response, $\psi_i^U = \phi_i(\mathbf{x}) * w_i(\ell)$
Pairwise: deformation cost,

$$\psi_{ij}^P(y_i, y_j; \mathbf{x}) = \begin{cases} \lambda \|y_i - y_j\|_2^2, & \text{same scale} \\ \infty, & \text{otherwise} \end{cases}$$

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Photo Montage

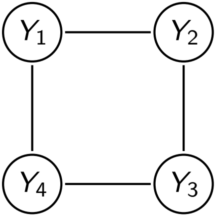
Labels: $\mathcal{L} = \{1, 2, \ldots, K\}$
Unary: none!
Pairwise: seam penalty

$$\psi_{ij}^P(y_i, y_j; \mathbf{x}) = \|\mathbf{x}_{y_i}(i) - \mathbf{x}_{y_j}(i)\| + \|\mathbf{x}_{y_i}(j) - \mathbf{x}_{y_j}(j)\|$$

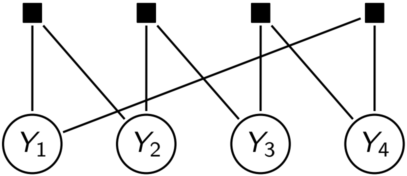
(or edge-normalized variant)

Graphical Representation

$$E(\mathbf{y}) = \psi(y_1, y_2) + \psi(y_2, y_3) + \psi(y_3, y_4) + \psi(y_4, y_1)$$



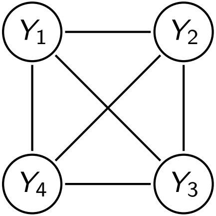
graphical model

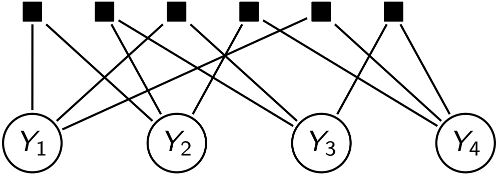


factor graph

Graphical Representation

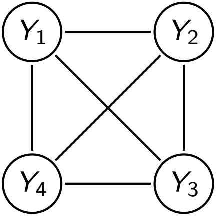
$$E(\mathbf{y}) = \sum_{i,j} \psi(y_i, y_j)$$

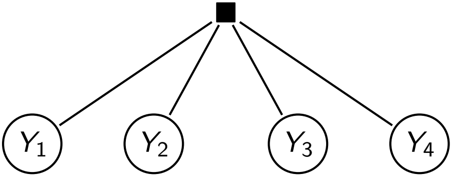




Graphical Representation

$$E(\mathbf{y}) = \psi(y_1, y_2, y_3, y_4)$$





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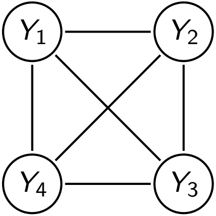

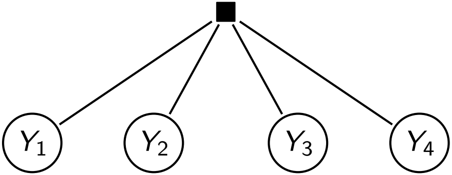

don't worry too much about the graphical representation, look at the form of the energy function

MAP Inference / Energy Minimization

- Computing the energy minimizing assignment is NP-hard

$$\operatorname*{argmin}_{\mathbf{y}\in\mathcal{Y}} E(\mathbf{y};\mathbf{x}) = \operatorname*{argmax}_{\mathbf{y}\in\mathcal{Y}} P(\mathbf{y} \mid \mathbf{x})$$

- Some structures admit tractable exact inference algorithms
  - low treewidth graphs → message passing
  - submodular potentials → graph-cuts
- Moreover, efficent approximate inference algorithms exist
  - message passing on general graphs
  - move making inference (submodular moves)
  - linear programming relaxations

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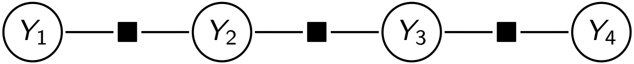
exact inference

An Example: Chain Graph

$$E(\mathbf{y}) = \psi_A(y_1, y_2) + \psi_B(y_2, y_3) + \psi_C(y_3, y_4)$$



$$\begin{aligned}
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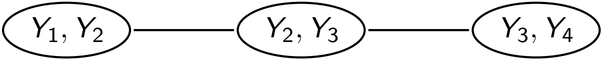

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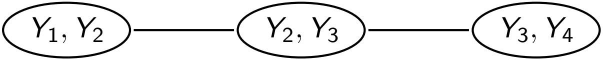

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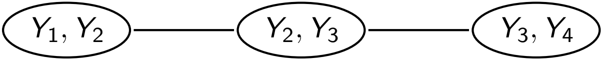

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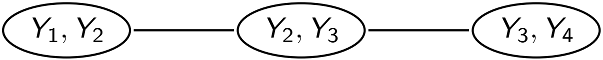

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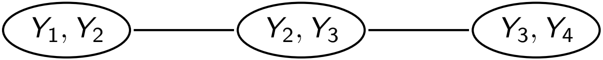

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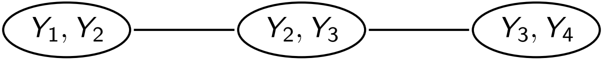

The energy minimizing assignment can be decoded as

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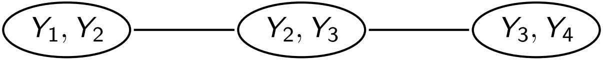

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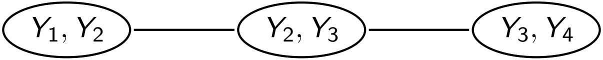

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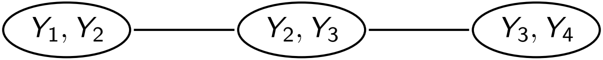

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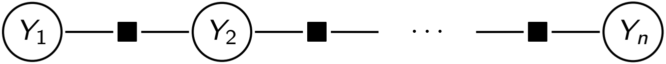

For a chain of length n with L labels per variable:

- Brute force enumeration would cost $|\mathcal{Y}| = L^n$
- Viterbi decoding (message passing) costs $O(nL^2)$
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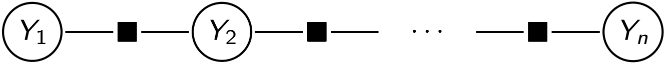

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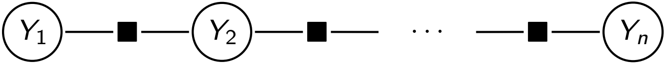

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Factor Operations

The preceeding inference algorithm was based on two important operations defined on factors (clique potentials).

- **Factor addition** creates an outut whose scope is the union of the scope of its inputs. Each element of the output is the sum of the corresponding (projected) elements of the inputs.

$$\mathbf{Y}_c = \mathbf{Y}_a \cup \mathbf{Y}_b \quad : \quad \psi_c(\mathbf{y}_c) = \psi_a([\mathbf{y}_c]_a) + \psi_b([\mathbf{y}_c]_b)$$

- **Factor minimization** creates an output where one or more input variables are removed. Each element of the output is the result of minimizing over values of the removed variables.

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Factor Operations Worked Example

y_1	y_2	ψ_a
0	0	1
0	1	4
1	0	7
1	1	2

plus

y_2	y_3	ψ_b
0	0	5
0	1	-3
1	0	1
1	1	8

=

y_1	y_2	y_3	$\psi_c = \psi_a + \psi_b$
0	0	0	$1 + 5 = 6$
0	0	1	1 - 3 = -2
0	1	0	$4 + 1 = 5$
0	1	1	$4 + 8 = 12$
1	0	0	$7 + 5 = 12$
1	0	1	7 - 3 = 4
1	1	0	$2 + 1 = 3$
1	1	1	$2 + 8 = 10$

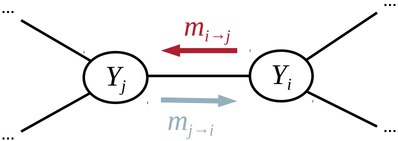
Clique Trees

A *clique tree* (or *tree decomposition*) for an energy function $E(\mathbf{y})$ is a pair $(\mathcal{C}, \mathcal{T})$, where $\mathcal{C} = \{C_1, \ldots, C_M\}$ is a family of subsets of $\{1, \ldots, n\}$ and $\mathcal{T}$ is a tree with nodes C_m satisfying:

- **Family Preserving:** if $\mathbf{Y}_c$ is a clique in $E(\mathbf{y})$ then there must exist a subset $C_m \in \mathcal{C}$ with $\mathbf{Y}_c \in C_m$;
- **Running Intersection Property:** if C_m and $C_{m'}$ both contain Y_i then there is a unique path through $\mathcal{T}$ between C_m and $C_{m'}$ such that Y_i is in every node along the path.

These properties are sufficient to ensure the message passing correctness of message passing.

Min-Sum Message Passing on Clique Trees



- messages sent in reverse then forward topological ordering
- message from clique i to clique j calculated as

$$m_{i\to j}(\mathbf{Y}_j \cap \mathbf{Y}_i) = \min_{\mathbf{Y}_i \setminus \mathbf{Y}_j} \left(\psi_i(\mathbf{Y}_i) + \sum_{k \in \mathcal{N}(i)\setminus\{j\}} m_{k\to i}(\mathbf{Y}_i \cap \mathbf{Y}_k) \right)$$

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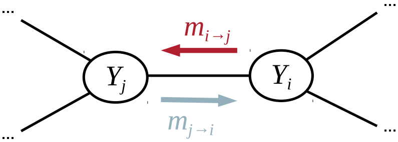

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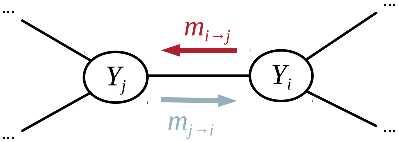

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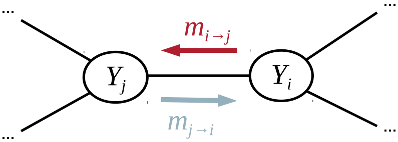

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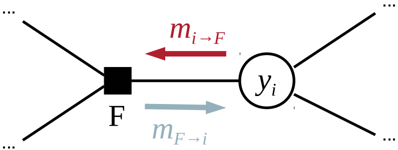
Min-Sum Message Passing on Factor Graphs (Trees)



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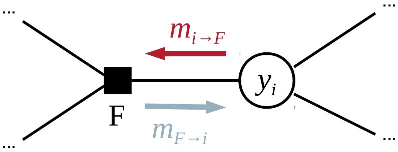

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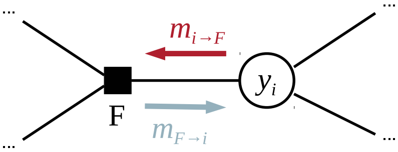

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- Message passing can be generalized to graphs with loops
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- Otherwise run message passing anyway
  - **loopy belief propagtaion**
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graph-cut based methods

Binary MRF Example

Consider the following energy function for two binary random variables, y_1 and y_2.

0	5
1	2

0	1
1	3

	0	1
0	0	3
1	4	0

$$E(y_1, y_2) = \psi_1(y_1) + \psi_2(y_2) + \psi_{12}(y_1, y_2)$$
$$= \underbrace{5\bar{y}_1 + 2y_1}_{\psi_1} + \underbrace{\bar{y}_2 + 3y_2}_{\psi_2} + \underbrace{3\bar{y}_1 y_2 + 4y_1\bar{y}_2}_{\psi_{12}}$$

where $\bar{y}_1 = 1 - y_1$ and $\bar{y}_2 = 1 - y_2$.

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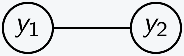
Graphical Model



Probability Table

y_1	y_2	E	P
0	0	6	0.244
0	1	11	0.002
1	0	7	0.090
1	1	5	0.664

Pseudo-boolean Functions [Boros and Hammer, 2001]

Pseudo-boolean Function

A mapping $f : \{0,1\}^n \rightarrow \mathbb{R}$ is called a *pseudo-Boolean function*.

- Pseudo-boolean functions can be uniquely represented as *multi-linear polynomials*, e.g., $f(y_1, y_2) = 6 + y_1 + 5y_2 - 7y_1y_2$.
- Pseudo-boolean functions can also be represented in *posiform*, e.g., $f(y_1, y_2) = 2y_1 + 5\bar{y}_1 + 3y_2 + \bar{y}_2 + 3\bar{y}_1 y_2 + 4y_1\bar{y}_2$. This representation is not unique.
- **A binary pairwise Markov random field (MRF) is just a quadratic pseudo-Boolean function.**

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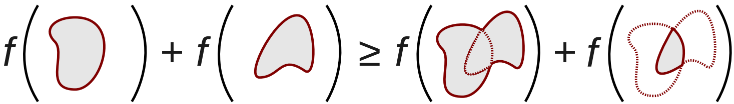
Submodular Functions

Submodularity

Let $\mathcal{V}$ be a set. A set function $f : 2^{\mathcal{V}} \to \mathbb{R}$ is called *submodular* if $f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y)$ for all subsets $X, Y \subseteq \mathcal{V}$.



Submodular Binary Pairwise MRFs

Submodularity

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Submodularity checks for pairwise binary MRFs:

- polynomial form (of pseudo-boolean function) has negative coefficients on all bi-linear terms;
- posiform has pairwise terms of the form $u\bar{v}$;
- all pairwise potentials satisfy

$$\psi^P_{ij}(0, 1) + \psi^P_{ij}(1, 0) \geq \psi^P_{ij}(1, 1) + \psi^P_{ij}(0, 0)$$

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Submodularity of Binary Pairwise Terms

To see the equivalence of the last two conditions consider the following pairwise potential

	0	1
0	α	β
1	γ	δ

$$\alpha + \begin{array}{|c|c|}\hline 0 & 0 \\ \hline \gamma-\alpha & \gamma-\alpha \\ \hline\end{array} + \begin{array}{|c|c|}\hline 0 & \delta-\gamma \\ \hline 0 & \delta-\gamma \\ \hline\end{array} + \begin{array}{|c|c|}\hline 0 & \beta+\gamma-\alpha-\delta \\ \hline 0 & 0 \\ \hline\end{array}$$

$$E(y_1, y_2) = \alpha + (\gamma - \alpha)y_1 + (\delta - \gamma)y_2 + (\beta + \gamma - \alpha - \delta)\bar{y}_1 y_2$$

[Kolmogorov and Zabih, 2004]

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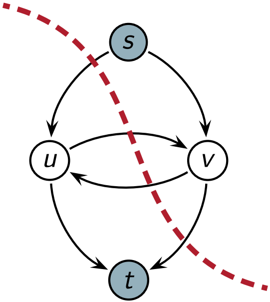
Minimum-cut Problem

Graph Cut

Let $\mathcal{G} = \langle \mathcal{V}, \mathcal{E} \rangle$ be a capacitated digraph with two distinguished vertices s and t. An st-cut is a partitioning of $\mathcal{V}$ into two disjoint sets $\mathcal{S}$ and $\mathcal{T}$ such that $s \in \mathcal{S}$ and $t \in \mathcal{T}$. The cost of the cut is the sum of edge capacities for all edges going from $\mathcal{S}$ to $\mathcal{T}$.



Quadratic Pseudo-boolean Optimization

Main idea:

- construct a graph such that every *st*-cut corresponds to a joint assignment to the variables $\mathbf{y}$
- the cost of the cut should be equal to the energy of the assignment, $E(\mathbf{y}; \mathbf{x})$.*
- the minimum-cut then corresponds to the the minimum energy assignment, $\mathbf{y}^\star = \operatorname{argmin}_{\mathbf{y}} E(\mathbf{y}; \mathbf{x})$.

*Requires non-negative edge weights.

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$$E(y_1, y_2) = \psi_1(y_1) + \psi_2(y_2) + \psi_{ij}(y_1, y_2)$$
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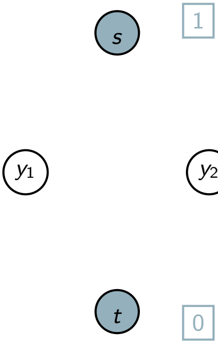

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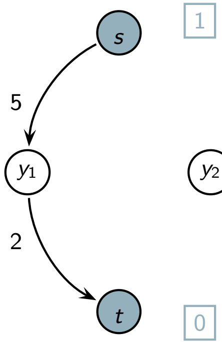

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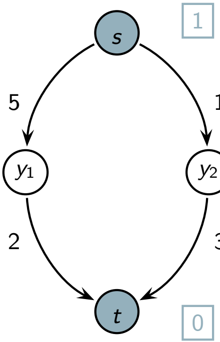

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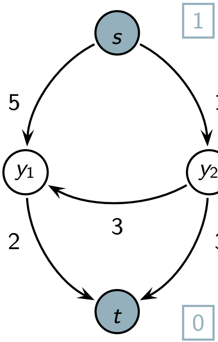

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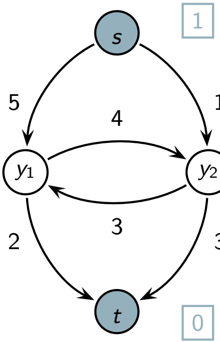
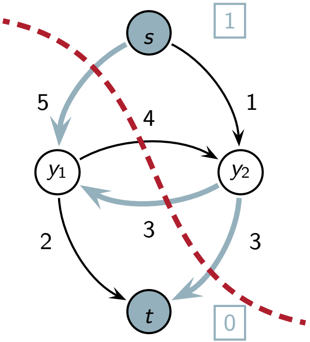


An Example *st*-Cut

$$\begin{aligned} E(0,1) &= \psi_1(0) + \psi_2(1) + \psi_{ij}(0,1) \\ &= 2y_1 + 5\bar{y}_1 + 3y_2 + \bar{y}_2 + 3\bar{y}_1 y_2 + 4y_1\bar{y}_2 \end{aligned}$$



Another *st*-Cut

$$E(1,1) = \psi_1(1) + \psi_2(1) + \psi_{ij}(1,1)$$
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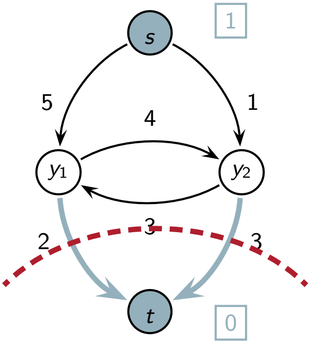
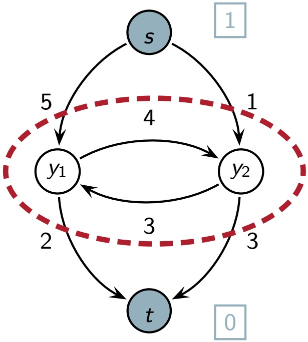


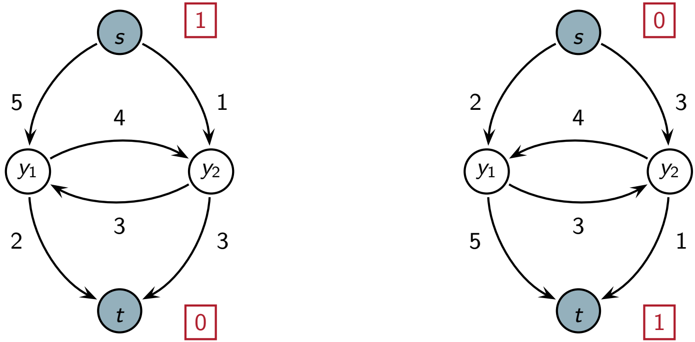
Invalid *st*-Cut

This is not a valid cut, since it does not correspond to a partitioning of the nodes into two sets—one containing s and one containing t.



Alternative *st*-Graph Construction

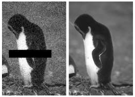
Sometimes you will see the roles of s and t switched.

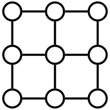


These graphs represent the same energy function.

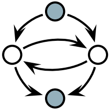
Big Picture: Where are we?

We can now formulate inference in a submodular binary pairwise MRF as a minimum-cut problem.





$\{0,1\}^n \to \mathbb{R}$

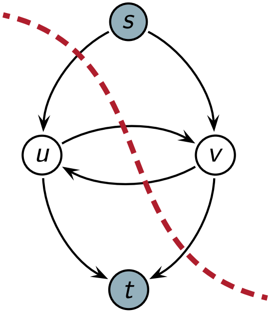


How do we solve the minimum-cut problem?

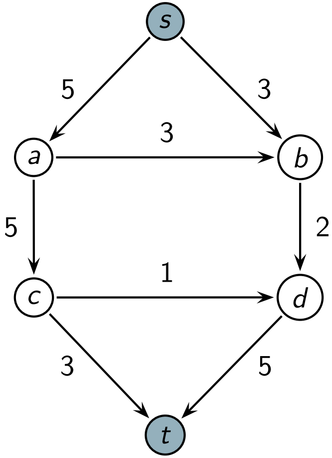
Max-flow/Min-cut Theorem

Max-flow/Min-cut Theorem [Fulkerson, 1956]

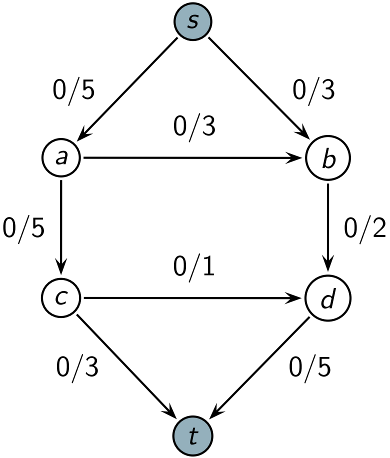
The maximum flow f from vertex s to vertex t is equal to the minimum cost st-cut.



Maximum Flow Example



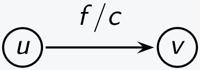
Maximum Flow Example (Augmenting Path)



flow

0

notation



edge with capacity c, and current flow f.

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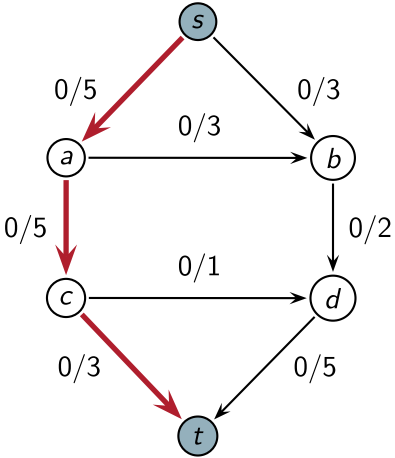

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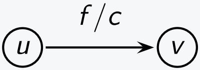

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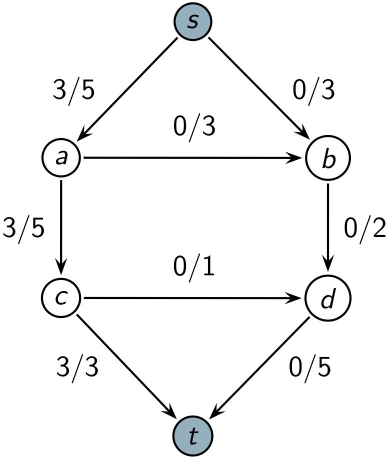

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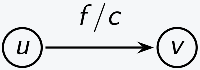
3

notation



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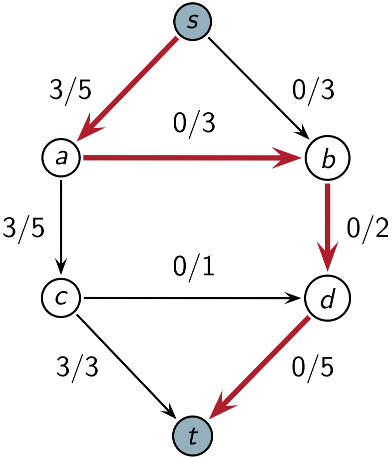
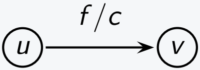


flow

3

notation



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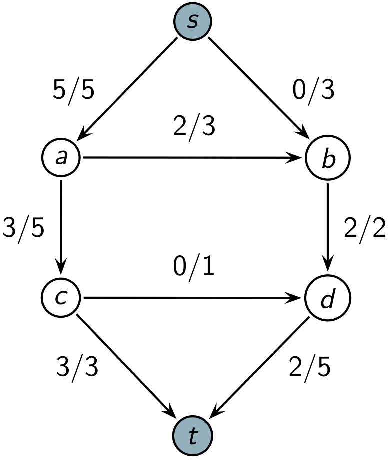
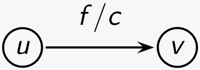


flow

5

notation



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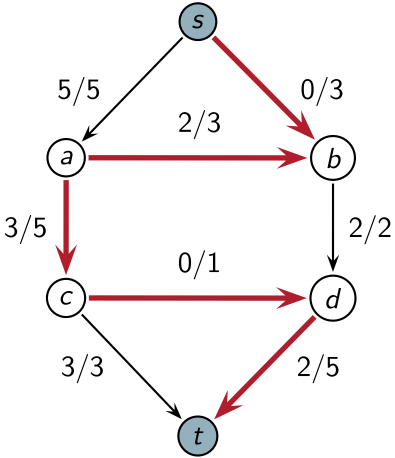

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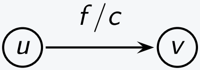

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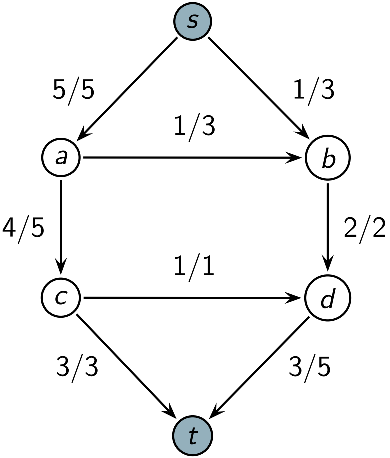
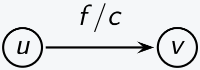


flow

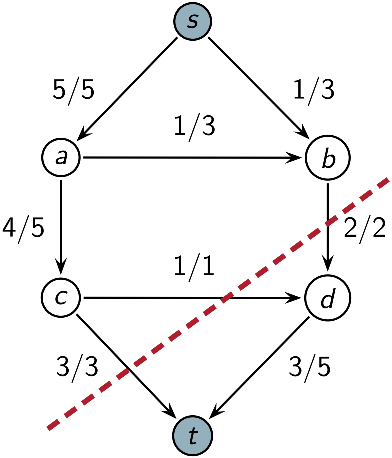
6

notation



edge with capacity c, and current flow f.

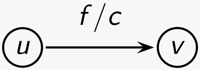
Maximum Flow Example (Augmenting Path)



flow

6

notation

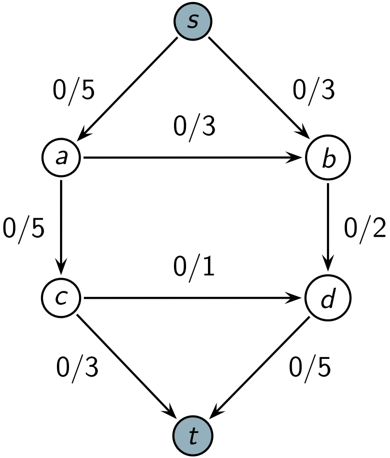


edge with capacity c, and current flow f.

Augmenting Path Algorithm Summary

- while an **augmenting path** exists (directed path with positive capacity between the source and sink)
    - send **flow** along the augmenting path updating **edge capacities** to produce a **residual graph**
- put all nodes reachable from the source in $\mathcal{S}$
- put all nodes that can reach the sink in $\mathcal{T}$

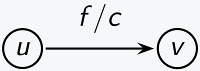
Maximum Flow Example (Push-Relabel)



state

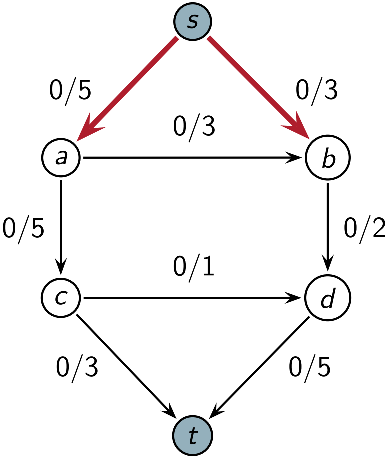
	$h(\cdot)$	$e(\cdot)$
s	6	∞
a	0	0
b	0	0
c	0	0
d	0	0
t	0	0

notation



edge with capacity c, current flow f.

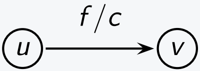
Maximum Flow Example (Push-Relabel)



state

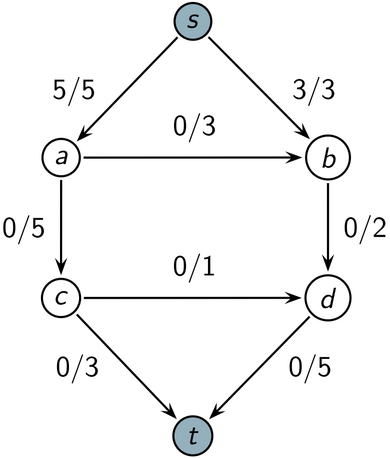
	$h(\cdot)$	$e(\cdot)$
s	6	∞
a	0	0
b	0	0
c	0	0
d	0	0
t	0	0

notation



edge with capacity c, current flow f.

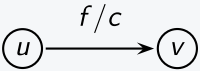
Maximum Flow Example (Push-Relabel)



state

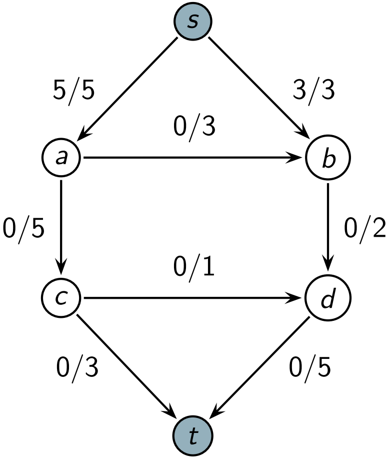
	$h(\cdot)$	$e(\cdot)$
s	6	∞
a	0	5
b	0	3
c	0	0
d	0	0
t	0	0

notation



edge with capacity c, current flow f.

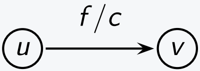
Maximum Flow Example (Push-Relabel)



state

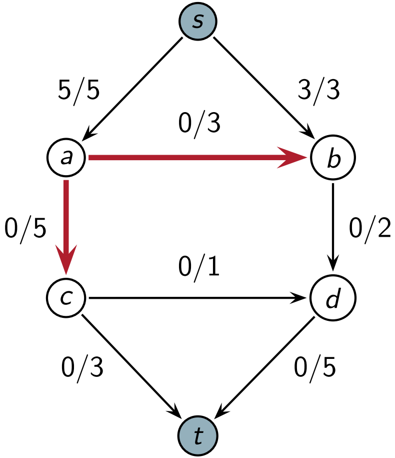
	$h(\cdot)$	$e(\cdot)$
s	6	∞
a	1	5
b	0	3
c	0	0
d	0	0
t	0	0

notation



edge with capacity c, current flow f.

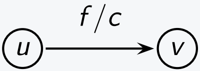
Maximum Flow Example (Push-Relabel)



state

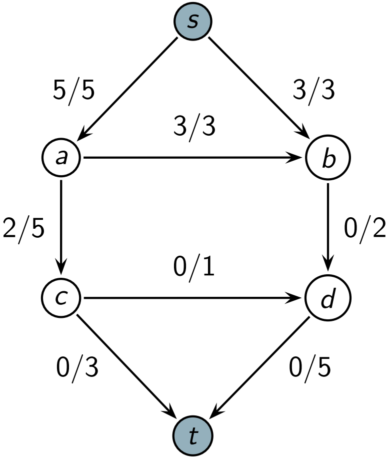
	$h(\cdot)$	$e(\cdot)$
s	6	∞
a	1	5
b	0	3
c	0	0
d	0	0
t	0	0

notation



edge with capacity c, current flow f.

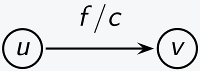
Maximum Flow Example (Push-Relabel)



state

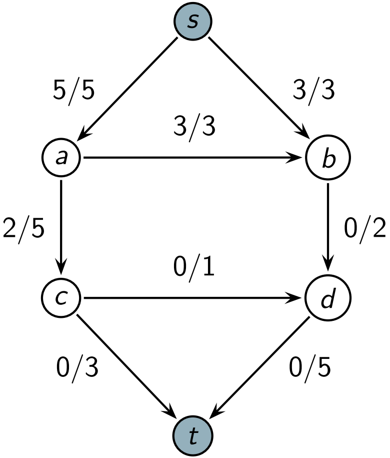
	$h(\cdot)$	$e(\cdot)$
s	6	∞
a	1	0
b	0	6
c	0	2
d	0	0
t	0	0

notation



edge with capacity c, current flow f.

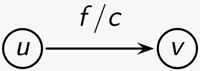
Maximum Flow Example (Push-Relabel)



state

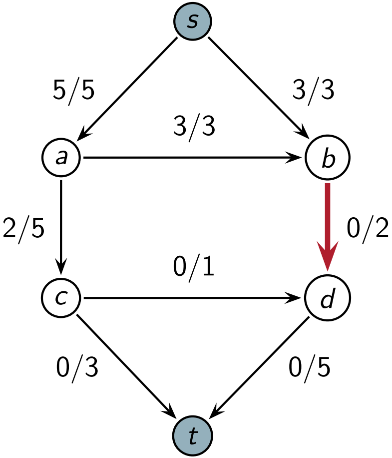
	$h(\cdot)$	$e(\cdot)$
s	6	∞
a	1	0
b	1	6
c	0	2
d	0	0
t	0	0

notation



edge with capacity c, current flow f.

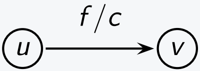
Maximum Flow Example (Push-Relabel)



state

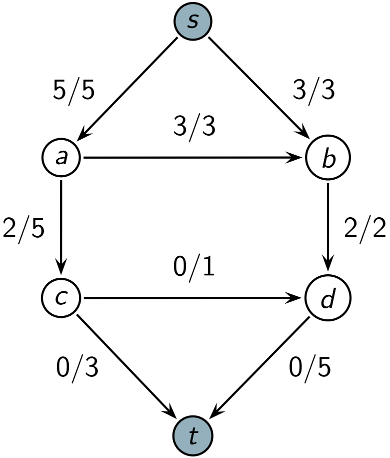
	$h(\cdot)$	$e(\cdot)$
s	6	∞
a	1	0
b	1	6
c	0	2
d	0	0
t	0	0

notation



edge with capacity c, current flow f.

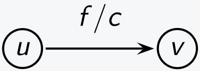
Maximum Flow Example (Push-Relabel)



state

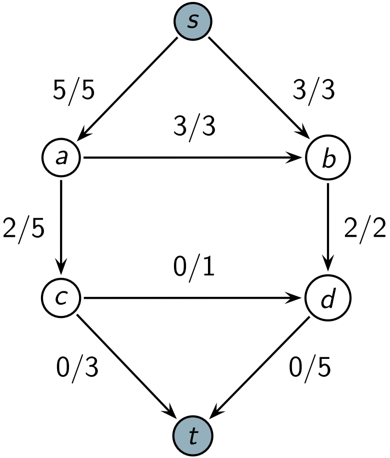
	$h(\cdot)$	$e(\cdot)$
s	6	∞
a	1	0
b	1	4
c	0	2
d	0	2
t	0	0

notation



edge with capacity c, current flow f.

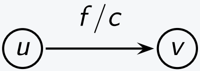
Maximum Flow Example (Push-Relabel)



state

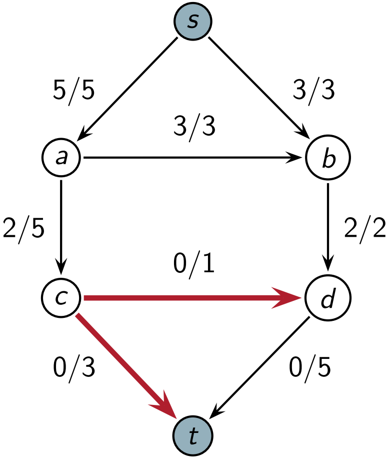
	$h(\cdot)$	$e(\cdot)$
s	6	∞
a	1	0
b	1	4
c	1	2
d	0	2
t	0	0

notation



edge with capacity c, current flow f.

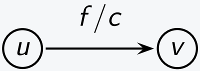
Maximum Flow Example (Push-Relabel)



state

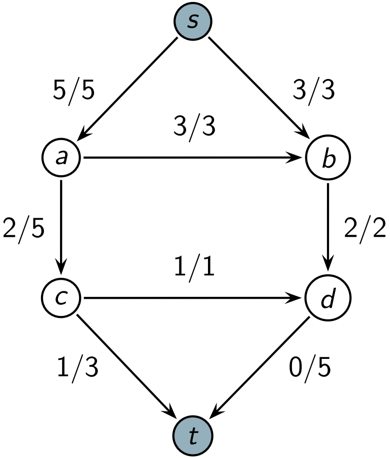
	$h(\cdot)$	$e(\cdot)$
s	6	∞
a	1	0
b	1	4
c	1	2
d	0	2
t	0	0

notation



edge with capacity c, current flow f.

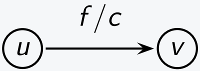
Maximum Flow Example (Push-Relabel)



state

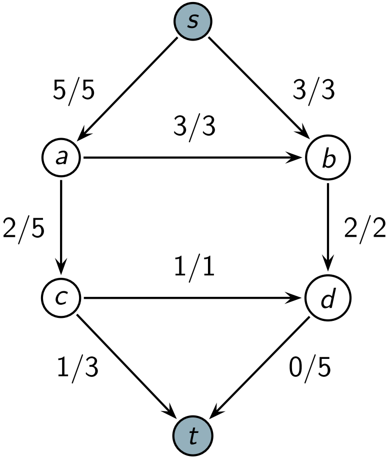
	$h(\cdot)$	$e(\cdot)$
s	6	∞
a	1	0
b	1	4
c	1	0
d	0	3
t	0	1

notation



edge with capacity c, current flow f.

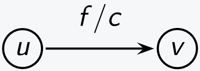
Maximum Flow Example (Push-Relabel)



state

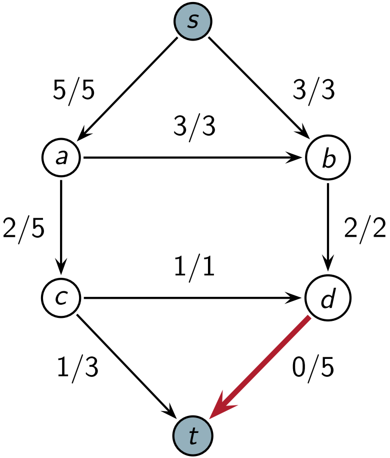
	$h(\cdot)$	$e(\cdot)$
s	6	∞
a	1	0
b	1	4
c	1	0
d	1	3
t	0	1

notation



edge with capacity c, current flow f.

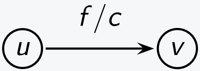
Maximum Flow Example (Push-Relabel)



state

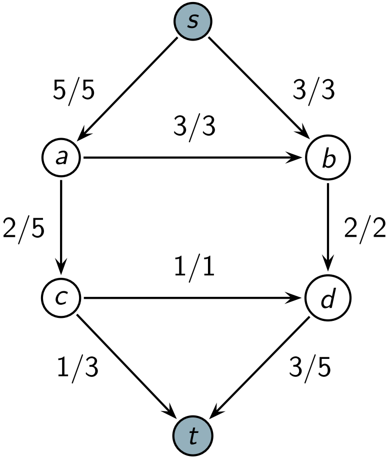
	$h(\cdot)$	$e(\cdot)$
s	6	∞
a	1	0
b	1	4
c	1	0
d	1	3
t	0	1

notation



edge with capacity c, current flow f.

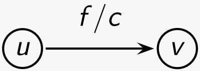
Maximum Flow Example (Push-Relabel)



state

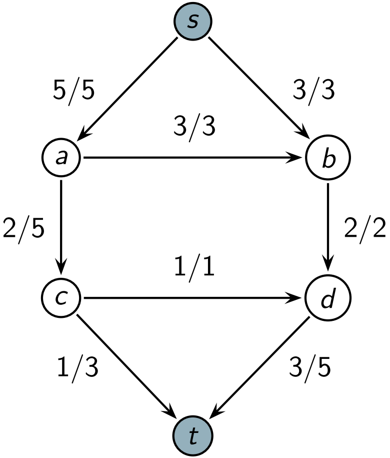
	$h(\cdot)$	$e(\cdot)$
s	6	∞
a	1	0
b	1	4
c	1	0
d	1	0
t	0	4

notation



edge with capacity c, current flow f.

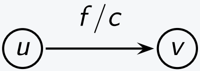
Maximum Flow Example (Push-Relabel)



state

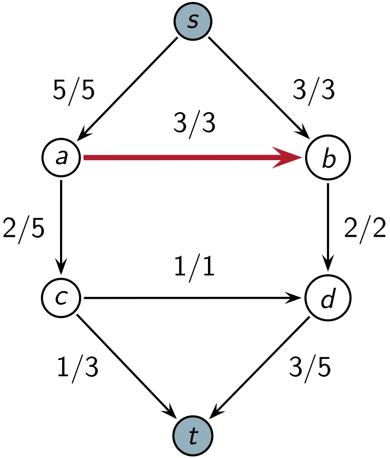
	$h(\cdot)$	$e(\cdot)$
s	6	∞
a	1	0
b	2	4
c	1	0
d	1	0
t	0	4

notation



edge with capacity c, current flow f.

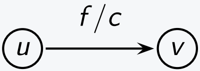
Maximum Flow Example (Push-Relabel)



state

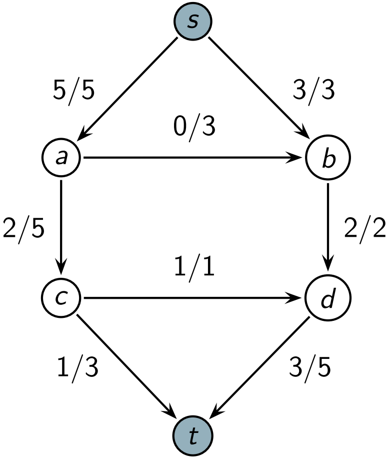
	$h(\cdot)$	$e(\cdot)$
s	6	∞
a	1	0
b	2	4
c	1	0
d	1	0
t	0	4

notation



edge with capacity c, current flow f.

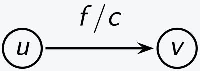
Maximum Flow Example (Push-Relabel)



state

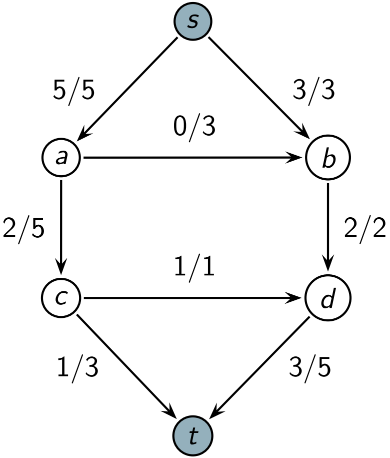
	$h(\cdot)$	$e(\cdot)$
s	6	∞
a	1	3
b	2	1
c	1	0
d	1	0
t	0	4

notation



edge with capacity c, current flow f.

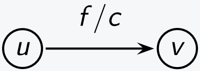
Maximum Flow Example (Push-Relabel)



state

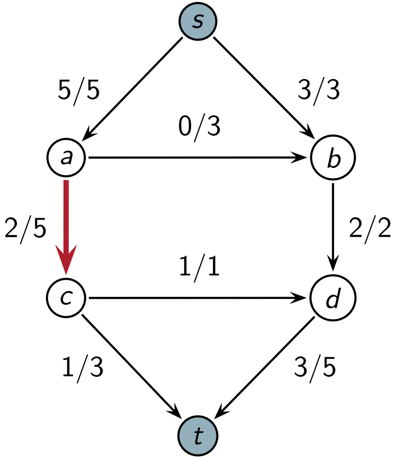
	$h(\cdot)$	$e(\cdot)$
s	6	∞
a	2	3
b	2	1
c	1	0
d	1	0
t	0	4

notation



edge with capacity c, current flow f.

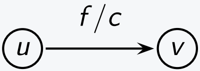
Maximum Flow Example (Push-Relabel)



state

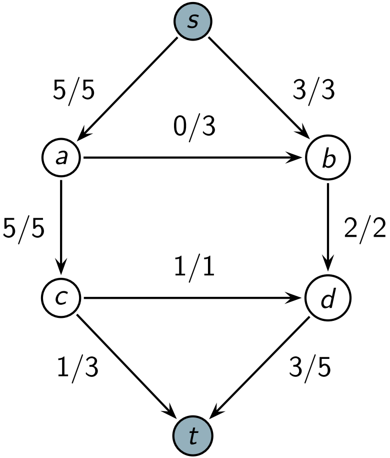
	$h(\cdot)$	$e(\cdot)$
s	6	∞
a	2	3
b	2	1
c	1	0
d	1	0
t	0	4

notation



edge with capacity c, current flow f.

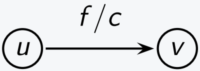
Maximum Flow Example (Push-Relabel)



state

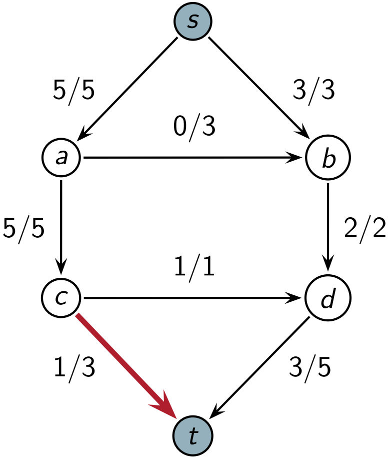
	$h(\cdot)$	$e(\cdot)$
s	6	∞
a	2	0
b	2	1
c	1	3
d	1	0
t	0	4

notation



edge with capacity c, current flow f.

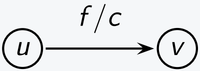
Maximum Flow Example (Push-Relabel)



state

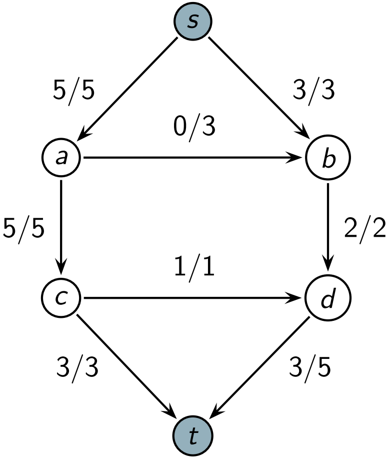
	$h(\cdot)$	$e(\cdot)$
s	6	∞
a	2	0
b	2	1
c	1	3
d	1	0
t	0	4

notation



edge with capacity c, current flow f.

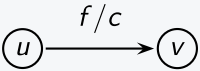
Maximum Flow Example (Push-Relabel)



state

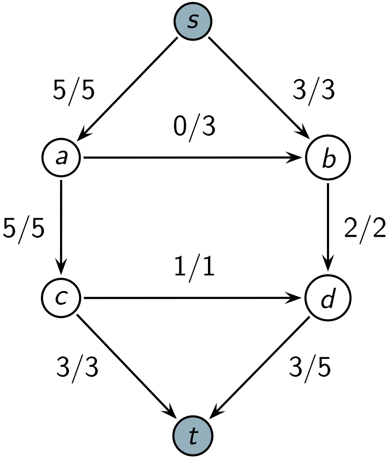
	$h(\cdot)$	$e(\cdot)$
s	6	∞
a	2	0
b	2	1
c	1	1
d	1	0
t	0	6

notation



edge with capacity c, current flow f.

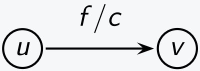
Maximum Flow Example (Push-Relabel)



state

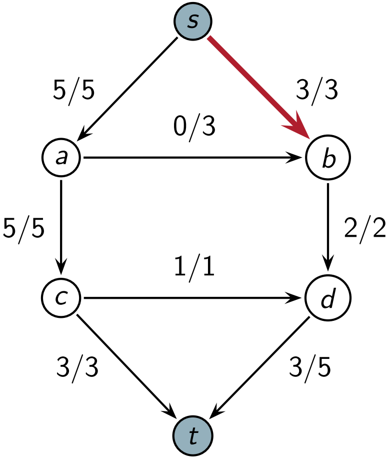
	$h(\cdot)$	$e(\cdot)$
s	6	∞
a	2	0
b	7	1
c	1	1
d	1	0
t	0	6

notation



edge with capacity c, current flow f.

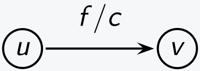
Maximum Flow Example (Push-Relabel)



state

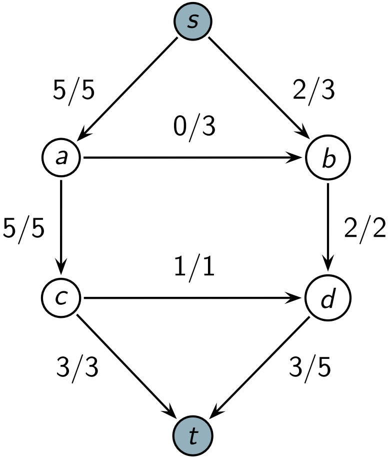
	$h(\cdot)$	$e(\cdot)$
s	6	∞
a	2	0
b	7	1
c	1	1
d	1	0
t	0	6

notation



edge with capacity c, current flow f.

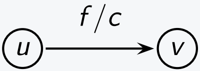
Maximum Flow Example (Push-Relabel)



state

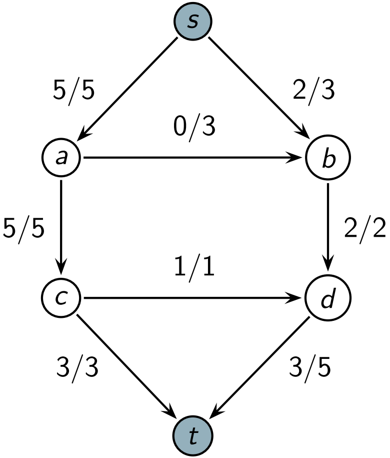
	$h(\cdot)$	$e(\cdot)$
s	6	∞
a	2	0
b	7	0
c	1	1
d	1	0
t	0	6

notation



edge with capacity c, current flow f.

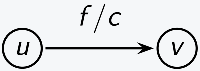
Maximum Flow Example (Push-Relabel)



state

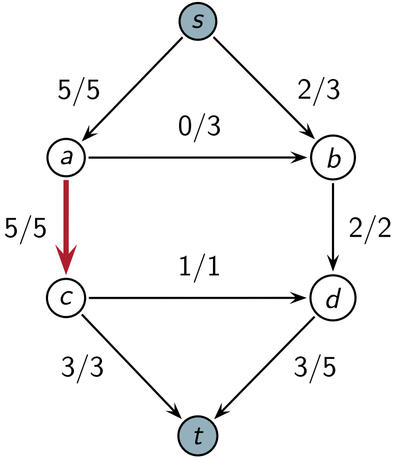
	$h(\cdot)$	$e(\cdot)$
s	6	∞
a	2	0
b	7	0
c	3	1
d	1	0
t	0	6

notation



edge with capacity c, current flow f.

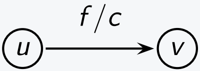
Maximum Flow Example (Push-Relabel)



state

	$h(\cdot)$	$e(\cdot)$
s	6	∞
a	2	0
b	7	0
c	3	1
d	1	0
t	0	6

notation



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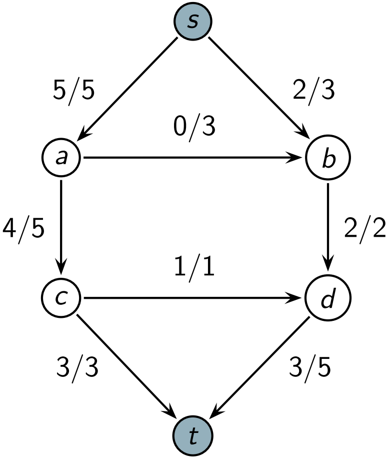

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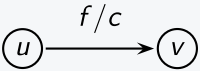

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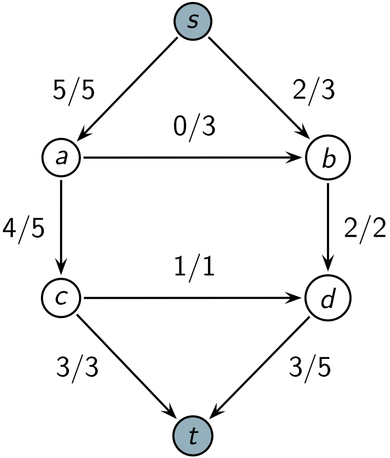

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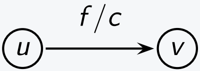

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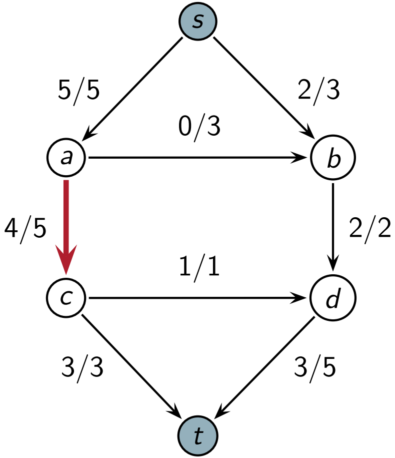

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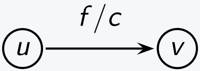

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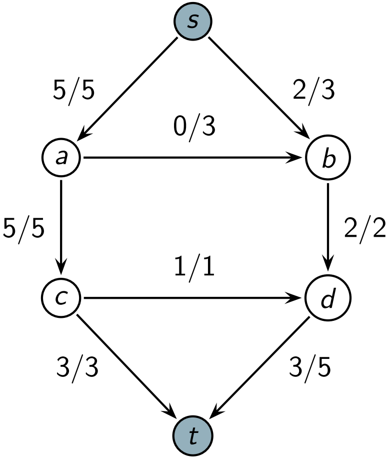

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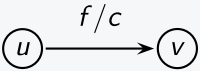

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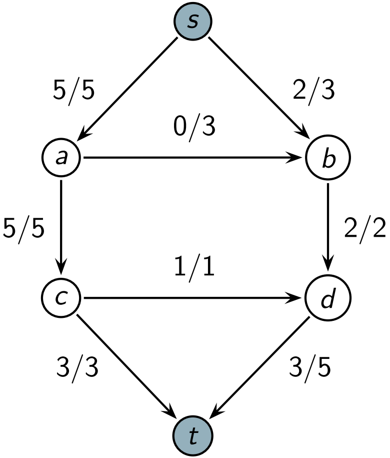

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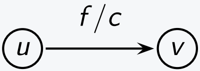

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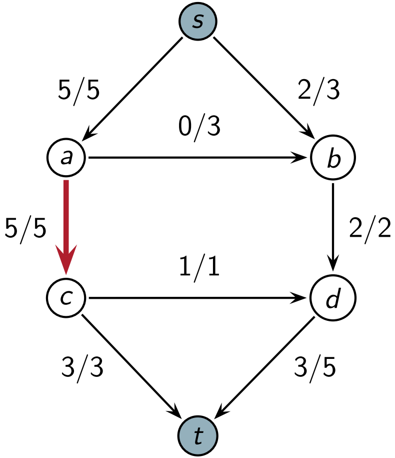

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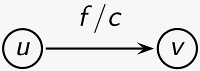

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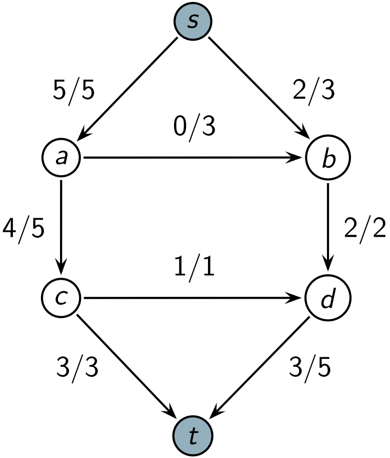

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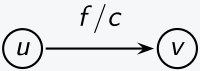

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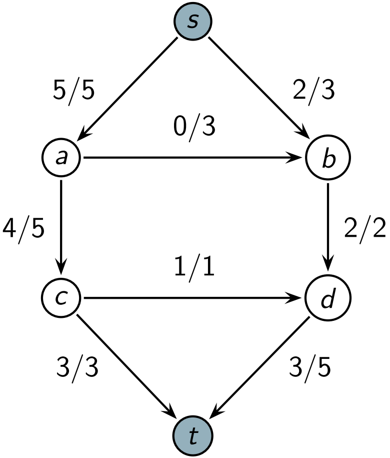

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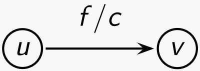

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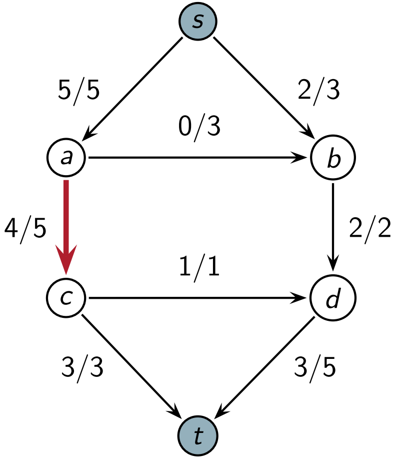

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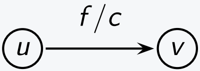

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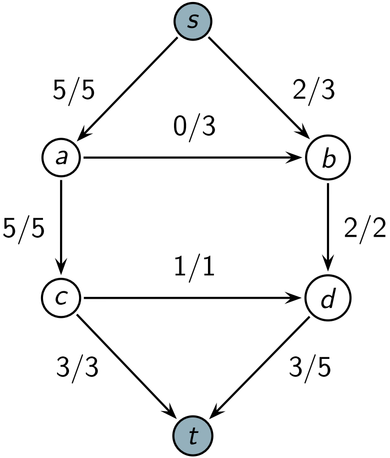

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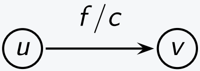

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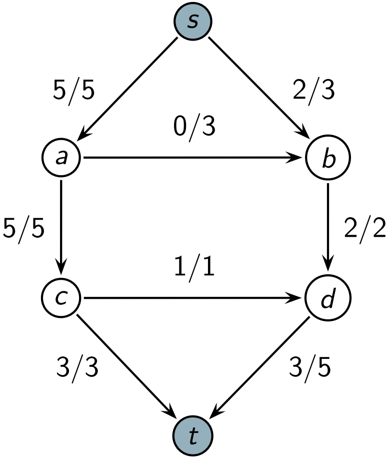

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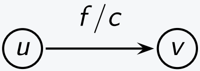

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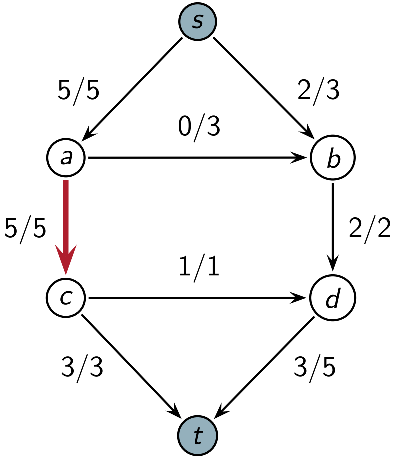

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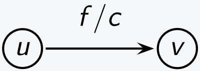

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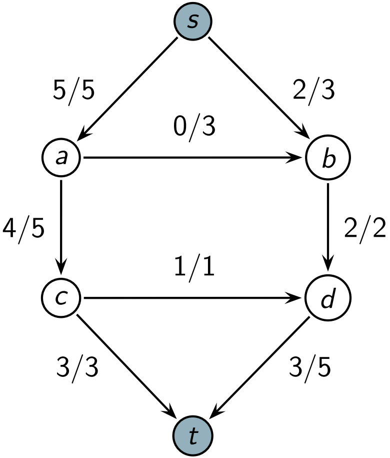

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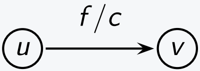

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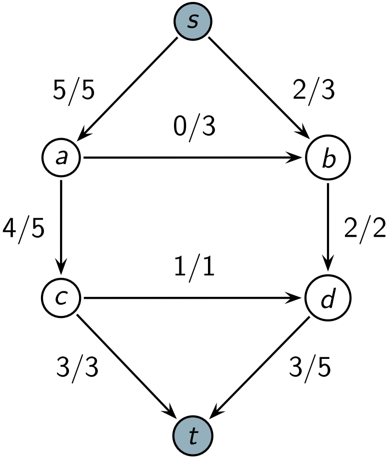

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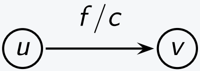

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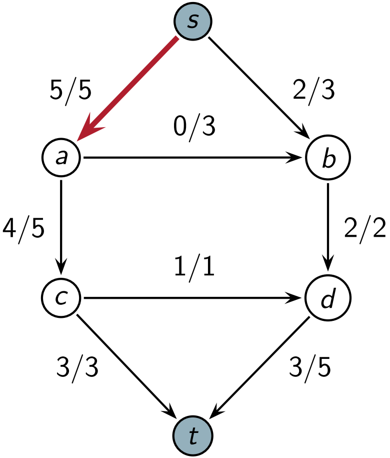

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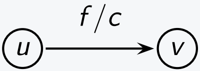

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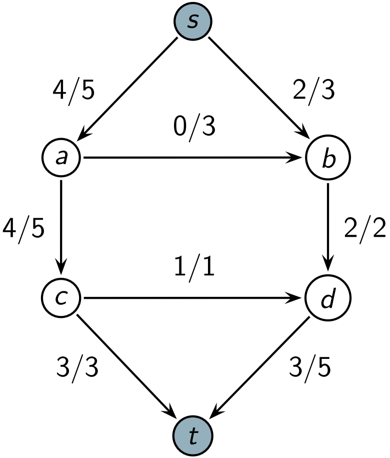

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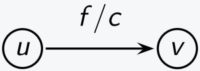

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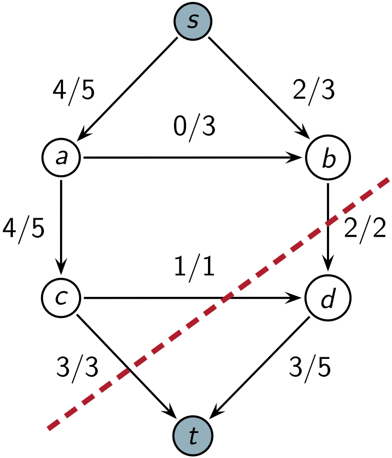

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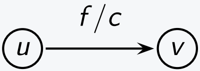

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Push-Relabel Algorithm Summary

- **Initialize:** set height of s to number of nodes in the graph; set excess for all nodes to zero.
- **Push:** for a node with excess capacity, push as much flow as possible onto neighbours with lower height
- **Relabel:** for a node with excess capacity and no neighbours with lower height, increase its height to one more than its lowest neighbour (with residual capacity).

Comparison of Maximum Flow Algorithms

Current state-of-the-art algorithm for exact minimization of general submodular pseudo-Boolean functions is $O(n^5 T + n^6)$, where T is the time taken to evaluate the function [Orlin, 2009].

†assumes integer capacities

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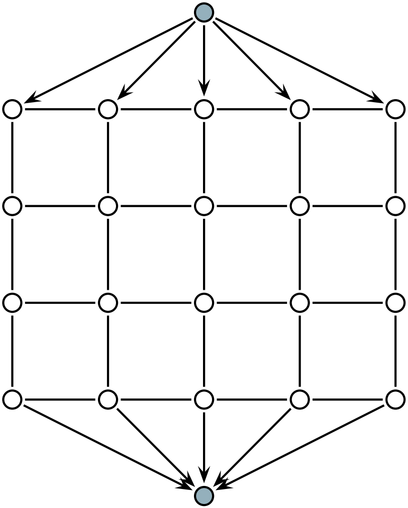
Algorithm	**Complexity**
Ford-Fulkerson	$O(E \max f)^\dagger$
Edmonds-Karp (BFS)	$O(VE^2)$
Push-relabel	$O(V^3)$
Boykov-Kolmogorov	$O(V^2 E \max f)$ ($\sim O(V)$ in practice)

† assumes integer capacities

Maximum Flow (Boykov-Kolmogorov, PAMI 2004)



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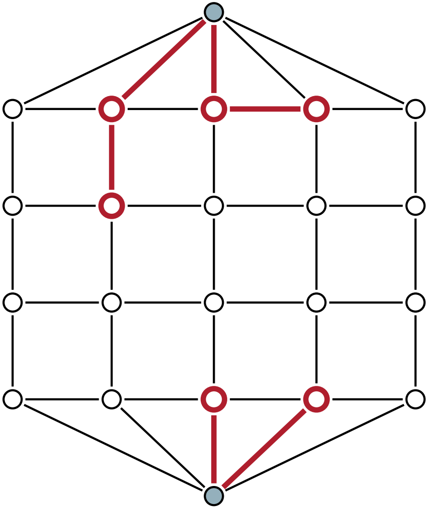

growth stage

search trees from s and t grow until they touch

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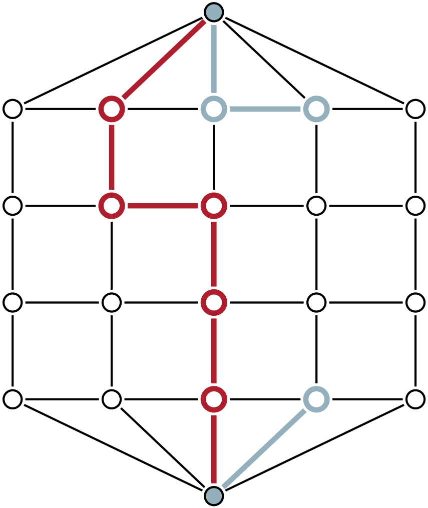

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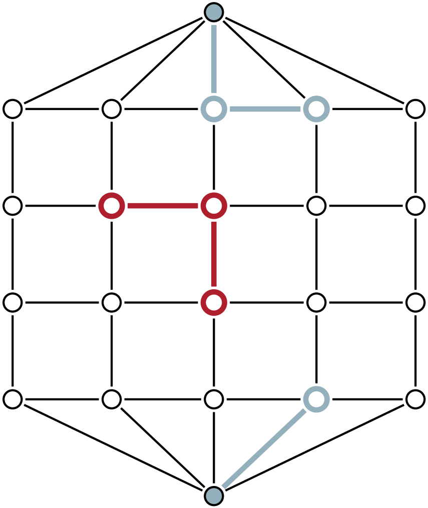

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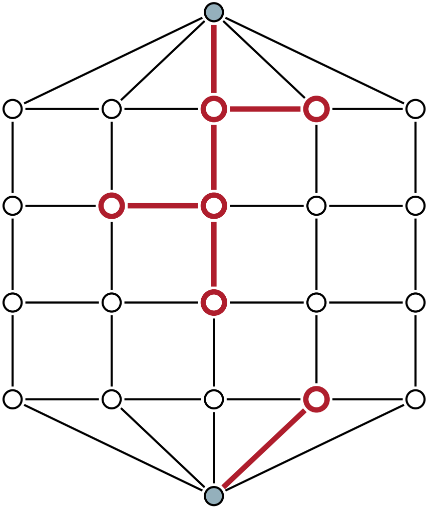


growth stage

search trees from s and t grow until they touch

augmentation stage

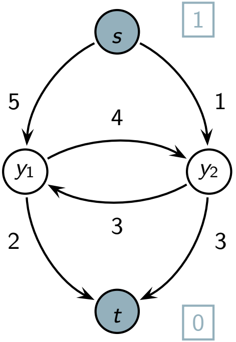
the path found is augmented; trees break into forests

adoption stage

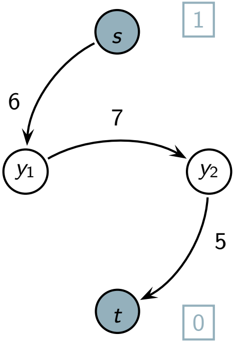
trees are restored

Reparameterization of Energy Functions

$$E(y_1, y_2) = 2y_1 + 5\bar{y}_1 + 3y_2 + \bar{y}_2 + 3\bar{y}_1 y_2 + 4y_1\bar{y}_2$$

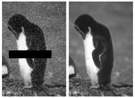


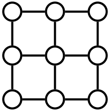
$$E(y_1, y_2) = 6\bar{y}_1 + 5y_2 + 7y_1\bar{y}_2$$



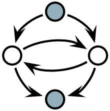
Big Picture: Where are we now?

We can perform inference in submodular binary pairwise Markov random fields exactly.





$\{0,1\}^n \to \mathbb{R}$



What about...

- non-submodular binary pairwise Markov random fields?
- multi-label Markov random fields?
- higher-order Markov random fields?

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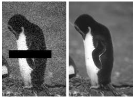

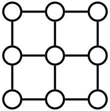

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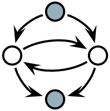

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Non-submodular Binary Pairwise MRFs

Non-submodular binary pairwise MRFs have potentials that do not satisfy $\psi_{ij}^P(0,1) + \psi_{ij}^P(1,0) \geq \psi_{ij}^P(1,1) + \psi_{ij}^P(0,0)$.

They are often handled in one of the following ways:

- approximate the energy function by one that is submodular (i.e., project onto the space of submodular functions);
- solve a relaxation of the problem using QPBO (Rother et al., 2007) or dual-decomposition (Komodakis et al., 2007).

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Approximating Non-submodular Binary Pairwise MRFs

Consider the non-submodular potential

A	B
C	D

with $A + D > B + C$.

We can project onto a submodular potential by modifying the coefficients as follows:

$$\Delta = A + D - C - B$$
$$A \leftarrow A - \frac{\Delta}{3}$$
$$C \leftarrow C + \frac{\Delta}{3}$$
$$B \leftarrow B + \frac{\Delta}{3}$$

QPBO (Roof Duality) [Rother et al., 2007]

Consider the energy function

$$E(\mathbf{y}) = \sum_{i \in \mathcal{V}} \psi_i^U(y_i) + \underbrace{\sum_{ij \in \mathcal{E}} \psi_{ij}^P(y_i, y_j)}_{\text{submodular}} + \underbrace{\sum_{ij \in \mathcal{E}} \tilde{\psi}_{ij}^P(y_i, y_j)}_{\text{non-submodular}}$$

We can introduce duplicate variables $\bar{y}_i$ into the energy function, and write

$$E'(\mathbf{y}, \bar{\mathbf{y}}) = \sum_{i \in \mathcal{V}} \frac{\psi_i^U(y_i) + \psi_i^U(1 - \bar{y}_i)}{2} + \sum_{ij \in \mathcal{E}} \frac{\psi_{ij}^P(y_i, y_j) + \psi_{ij}^P(1 - \bar{y}_i, 1 - \bar{y}_j)}{2} + \sum_{ij \in \mathcal{E}} \frac{\tilde{\psi}_{ij}^P(y_i, 1 - \bar{y}_j) + \tilde{\psi}_{ij}^P(1 - \bar{y}_i, y_j)}{2}$$

QPBO (Roof Duality) [Rother et al., 2007]

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QPBO (Roof Duality)

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Observations

- if $y_i = 1 - \bar{y}_i$ for all i, then $E(\mathbf{y}) = E'(\mathbf{y}, \bar{\mathbf{y}})$.
- $E'(\mathbf{y}, \bar{\mathbf{y}})$ is submodular.

Ignore the constraint on $\bar{y}_i$ and solve anyway. Result satisfies *partial optimality*: if $\bar{y}_i = 1 - y_i$ then y_i is the optimal label.

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Multi-label Markov Random Fields

The quadratic pseudo-Boolean optimization techniques described above cannot be applied directly to multi-label MRFs.

However...

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- ...we can project the multi-label problem onto a series of binary problems in a so-called *move-making* algorithm.

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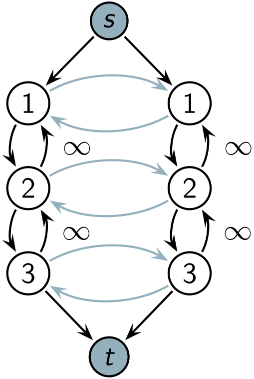
The "Battleship" Transform [Ishikawa, 2003]

If the multi-label MRFs has pairwise potentials that are convex functions over the label differences, i.e., $\psi^P_{ij}(y_i, y_j) = g(|y_i - y_j|)$ where $g(\cdot)$ is convex, then we can transform the energy function into an equivalent binary one.

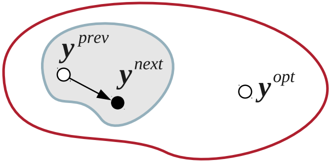
$$y = 1 \Leftrightarrow \mathbf{z} = (0, 0, 0)$$
$$y = 2 \Leftrightarrow \mathbf{z} = (1, 0, 0)$$
$$y = 3 \Leftrightarrow \mathbf{z} = (1, 1, 0)$$
$$y = 4 \Leftrightarrow \mathbf{z} = (1, 1, 1)$$



Move-making Inference

Idea:

- initialize $\mathbf{y}^{\text{prev}}$ to any valid assignment
- restrict the label-space of each variable y_i from $\mathcal{L}$ to $\mathcal{Y}_i \subseteq \mathcal{L}$ (with $y_i^{\text{prev}} \in \mathcal{Y}_i$)
- transform $E : \mathcal{L}^n \to \mathbb{R}$ to $\hat{E} : \mathcal{Y}_1 \times \cdots \times \mathcal{Y}_n \to \mathbb{R}$
- find the optimal assignment $\hat{\mathbf{y}}$ for $\hat{E}$ and repeat



each move results in an assignment with lower energy

Iterated Conditional Modes [Besag, 1986]

Reduce multi-variate inference to solving a series of univariate inference problems.

ICM move

For one of the variables y_i, set $\mathcal{Y}_i = \mathcal{L}$. Set $\mathcal{Y}_j = \{y_j^{\text{prev}}\}$ for all $j \neq i$ (i.e., hold all other variables fixed).

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Alpha Expansion and Alpha-Beta Swap [Boykov et al., 2001]

Reduce multi-label inference to solving a series of binary (submodular) inference problems.

α-expansion move

Choose some $\alpha \in \mathcal{L}$. Then for all variables, set $\mathcal{Y}_i = \{\alpha, y_i^{\text{prev}}\}$.

$\psi^P_{ij}(\cdot, \cdot)$ must be metric for the resulting move to be submodular

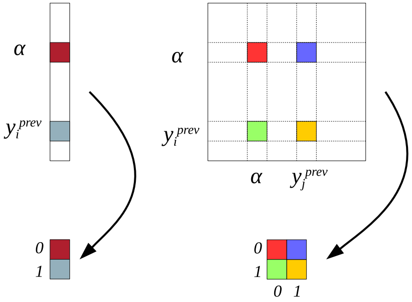
$\alpha\beta$-swap move

Choose two labels $\alpha, \beta \in \mathcal{L}$. Then for each variable y_i such that $y_i^{\text{prev}} \in \{\alpha, \beta\}$, set $\mathcal{Y}_i = \{\alpha, \beta\}$. Otherwise set $\mathcal{Y}_i = \{y_i^{\text{prev}}\}$.

$\psi^P_{ij}(\cdot, \cdot)$ must be semi-metric

Alpha Expansion Potential Construction

$$y_i^{\text{next}} = \begin{cases} y_i^{\text{prev}} & \text{if } t_i = 1 \\ \alpha & \text{if } t_i = 0 \end{cases}$$



$$E(\mathbf{t}) = \sum_i \psi_i(\alpha)\bar{t}_i + \psi_i(y_i^{\text{prev}})t_i + \sum_{ij} \psi_{ij}(\alpha,\alpha)\bar{t}_i\bar{t}_j$$
$$+ \psi_{ij}(\alpha, y_j^{\text{prev}})\bar{t}_i t_j + \psi_{ij}(y_i^{\text{prev}}, \alpha)t_i\bar{t}_j + \psi_{ij}(y_i^{\text{prev}}, y_j^{\text{prev}})t_i t_j$$

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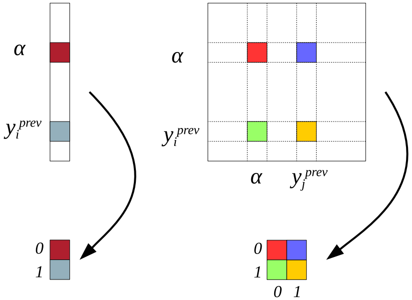

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A Note on Higher-Order Models

- **Order reduction.** [Ishikawa, 2009]

  Replace $-\prod_{i=1}^{n} y_i$ with $\underbrace{\bar{z} + \sum_{i=1}^{n} \bar{y}_i z - 1}_{*}$.

- **Special forms.** E.g., lower-linear envelopes [Gould, 2011]

$$\psi_c^H(\mathbf{y}_c) \triangleq \min_k \left\{ a_k \sum_{i \in c} y_i + b_k \right\} = \min_k \left\{ f_k(\mathbf{y}_c) \right\}$$

  Assume sorted on a_k. Then replace above with

$$f_1(\mathbf{y}_c) + \underbrace{\sum_k z_k \left(f_{k+1}(\mathbf{y}_c) - f_k(\mathbf{y}_c) \right)}_{\text{* submodular binary pairwise}}$$

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relaxations and dual decomposition

Mathematical Programming Formulation

- Let $\theta_{c,\mathbf{y}_c} \triangleq \psi_c(\mathbf{y}_c)$ and let $\mu_{c,\mathbf{y}_c} \triangleq \begin{cases} 1, & \text{if } \mathbf{Y}_c = \mathbf{y}_c \\ 0, & \text{otherwise} \end{cases}$

$$\operatorname*{argmin}_{\mathbf{y} \in \mathcal{Y}} \sum_c \psi_c(\mathbf{y}_c)$$

$$\Updownarrow$$

$$\begin{array}{lll} \text{minimize (over } \boldsymbol{\mu}) & \boldsymbol{\theta}^T \boldsymbol{\mu} & \\ \text{subject to} & \mu_{c,\mathbf{y}_c} \in \{0,1\}, & \forall c, \mathbf{y}_c \in \mathcal{Y}_c \\ & \sum_{\mathbf{y}_c} \mu_{c,\mathbf{y}_c} = 1, & \forall c \\ & \sum_{\mathbf{y}_c \setminus y_i} \mu_{c,\mathbf{y}_c} = \mu_{i,y_i}, & \forall i \in c, y_i \in \mathcal{Y}_i \end{array}$$

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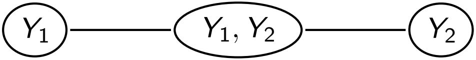
Binary Integer Program: Example

Consider energy function $E(y_1, y_2) = \psi_1(y_1) + \psi_{12}(y_1, y_2) + \psi_2(y_2)$ for binary variables y_1 and y_2.



$$\boldsymbol{\theta} = \begin{bmatrix} \psi_1(0) \\ \psi_1(1) \\ \psi_2(0) \\ \psi_2(1) \\ \psi_{12}(0,0) \\ \psi_{12}(1,0) \\ \psi_{12}(0,1) \\ \psi_{12}(1,1) \end{bmatrix} \qquad \boldsymbol{\mu} = \begin{bmatrix} \mu_{1,0} \\ \mu_{1,1} \\ \mu_{2,0} \\ \mu_{2,1} \\ \mu_{12,00} \\ \mu_{12,10} \\ \mu_{12,01} \\ \mu_{12,11} \end{bmatrix} \qquad \text{s.t.} \begin{cases} \mu_{1,0} + \mu_{1,1} = 1 \\ \mu_{2,0} + \mu_{2,1} = 1 \\ \mu_{12,00} + \mu_{12,10} \\ \quad + \mu_{12,01} + \mu_{12,11} = 1 \\ \mu_{12,00} + \mu_{12,01} = \mu_{1,0} \\ \mu_{12,10} + \mu_{12,11} = \mu_{1,1} \\ \mu_{12,00} + \mu_{12,10} = \mu_{2,0} \\ \mu_{12,01} + \mu_{12,11} = \mu_{2,1} \end{cases}$$

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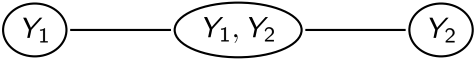

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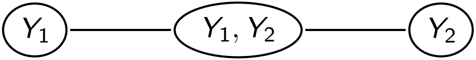

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Binary Integer Program: Example

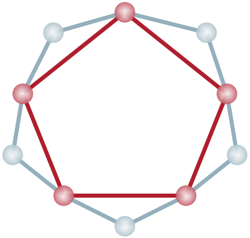
Let $y_1 = 1$ and $y_2 = 0$. Then

$$\boldsymbol{\mu} = \begin{bmatrix} \mu_{1,0} \\ \mu_{1,1} \\ \mu_{2,0} \\ \mu_{2,1} \\ \mu_{12,00} \\ \mu_{12,10} \\ \mu_{12,01} \\ \mu_{12,11} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \cdot \quad \boldsymbol{\theta} = \begin{bmatrix} \psi_1(0) \\ \psi_1(1) \\ \psi_2(0) \\ \psi_2(1) \\ \psi_{12}(0,0) \\ \psi_{12}(1,0) \\ \psi_{12}(0,1) \\ \psi_{12}(1,1) \end{bmatrix}$$

So $\boldsymbol{\theta}^T \boldsymbol{\mu} = \psi_1(1) + \psi_2(0) + \psi_{12}(1,0)$.

Local Marginal Polytope

$$\mathcal{M} = \left\{ \boldsymbol{\mu} \geq \mathbf{0} \;\middle|\; \begin{array}{ll} \sum_{y_i} \mu_{i,y_i} = 1, & \forall i \\ \sum_{\mathbf{y}_c \backslash y_i} \mu_{c,\mathbf{y}_c} = \mu_{i,y_i}, & \forall i \in c, y_i \in \mathcal{Y}_i \end{array} \right\}$$



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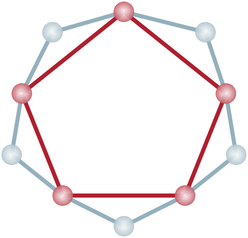

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Linear Programming (LP) Relaxation

- Binary integer program

$$\begin{array}{ll} \text{minimize (over } \boldsymbol{\mu}) & \boldsymbol{\theta}^T \boldsymbol{\mu} \\ \text{subject to} & \mu_{c,\mathbf{y}_c} \in \{0, 1\} \\ & \boldsymbol{\mu} \in \mathcal{M} \end{array}$$

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- Solution by standard LP solvers typically infeasible due to large number of variables and constraints
- More easily solved via coordinate ascent of the dual
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Dual Decomposition: Rewriting the Primal

$$\begin{array}{ll} \text{minimize (over } \boldsymbol{\mu}) & \sum_c \boldsymbol{\theta}_c^T \boldsymbol{\mu}_c \\ \text{subject to} & \boldsymbol{\mu} \in \mathcal{M} \end{array}$$

$\Updownarrow$ (pad $\boldsymbol{\theta}_c$)

$$\begin{array}{ll} \text{minimize (over } \boldsymbol{\mu}) & \sum_c \tilde{\boldsymbol{\theta}}_c^T \boldsymbol{\mu} \\ \text{subject to} & \boldsymbol{\mu} \in \mathcal{M} \end{array}$$

$\Updownarrow$ (introduce copies of $\boldsymbol{\mu}$)

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$$E(\mathbf{y}) = \sum_c \psi_c(\mathbf{y}_c)$$

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$$\min_{\mathbf{y}} E(\mathbf{y}) \geq \sum_c \min_{\mathbf{y}_c} \psi_c(\mathbf{y}_c) + \lambda_c(\mathbf{y}_c)$$

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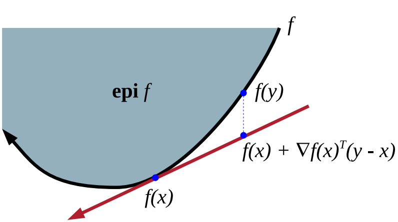
Subgradients

Subgradient

A subgradient of a function f at x is *any* vector g satisfying

$$f(y) \geq f(x) + g^T(y - x) \quad \text{for all } y$$



Subgradient Method

The basic subgradient method is a algorithm for minimizing a nondifferentiable convex function $f : \mathbb{R}^n \to \mathbb{R}$.

$$x^{(k+1)} = x^{(k)} - \alpha_k g^{(k)}$$

- $x^{(k)}$ is the k-th iterate
- $g^{(k)}$ is any subgradient of f at $x^{(k)}$
- $\alpha_k > 0$ is the k-th step size

It is possible that $-g^{(k)}$ is not a descent direction for f at $x^{(k)}$, so we keep track of the best point found so far

$$f_{\text{best}}^{(k)} = \min\left\{ f_{\text{best}}^{(k-1)}, f(x^{(k)}) \right\}$$

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Step Size Rules

Step sizes are chosen ahead of time (unlike line search is ordinary gradient methods). A few common step size schedules are:

- **constant step size:** $\alpha_k = \alpha$
- **constant step length:** $\alpha_k = \frac{\gamma}{\|g^{(k)}\|_2}$
- **square summable but not summable:**

$$\sum_{k=1}^{\infty} \alpha_k^2 < \infty, \qquad \sum_{k=1}^{\infty} \alpha_k = \infty$$

- **nonsummable diminishing:**

$$\lim_{k\to\infty} \alpha_k = 0, \qquad \sum_{k=1}^{\infty} \alpha_k = \infty$$

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Convergence Results

For constant step size and constant step length, the subgradient algorithm will converge to within some range of the optimal value,

$$\lim_{k \to \infty} f_{\text{best}}^{(k)} < f^\star + \epsilon$$

For the diminishing step size and step length rules the algorithm converges to the optimal value,

$$\lim_{k \to \infty} f_{\text{best}}^{(k)} = f^\star$$

but may take a very long time to converge.

Optimal Step Size for Known $f^\star$

Assume we know $f^\star$ (we just don't know $x^\star$). Then

$$\alpha_k = \frac{f(x^{(k)}) - f^\star}{\|g^{(k)}\|_2^2}$$

is an optimal step size in some sense. Called the Polyak step size.

A good approximation when $f^\star$ is not known (but non-negative) is

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Projected Subgradient Method

One extension of the subgradient method is the **projected subgradient method** which solves problems of the form

$$\begin{array}{rl} \text{minimize} & f(x) \\ \text{subject to} & x \in \mathcal{C} \end{array}$$

Here the updates are

$$x^{(k+1)} = P_{\mathcal{C}}\left(x^{(k)} - \alpha_k g^{(k)}\right)$$

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Supergradient of $\min_i\{a_i^T x + b_i\}$

Consider $f(\mathbf{x}) = \min_i\{\mathbf{a}_i^T\mathbf{x} + b_i\}$ and let $I(\mathbf{x}) = \text{argmin}_i\{\mathbf{a}_i^T x + b_i\}$. Then for any $i \in I(\mathbf{x})$, $\mathbf{g} = \mathbf{a}_i$ is a supergradient of f at $\mathbf{x}$.

$$\begin{aligned}
f(\mathbf{x}) + \mathbf{g}^T(\mathbf{z} - \mathbf{x}) &= f(\mathbf{x}) - \mathbf{a}_i^T(\mathbf{z} - \mathbf{x}) && i \in I(\mathbf{x}) \\
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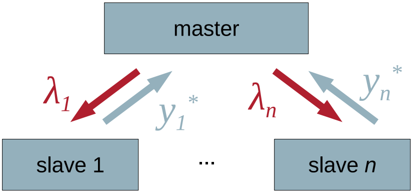
Dual Decomposition Inference [Komodakis et al., 2010]



- initialize $\lambda_c = 0$
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  - slaves solve $\min_{\mathbf{y}_c} \psi_c(\mathbf{y}_c) + \lambda_c(\mathbf{y}_c)$ (to get $\mu_c^\star$)
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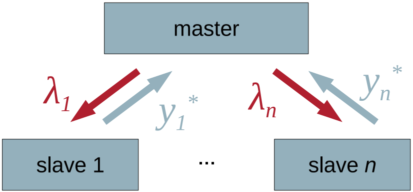

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parameter learning

Max-Margin Learning

- Assume we have an energy function which is linear in its parameters, $E_{\mathbf{w}}(\mathbf{y}; \mathbf{x}) = \mathbf{w}^T \phi(\mathbf{y}; \mathbf{x})$.
- Let $\mathcal{D} = \{(\mathbf{y}_t, \mathbf{x}_t)\}_{t=1}^T$ be our set of training examples.
- Our goal in learning is to find a parameter setting $\mathbf{x}^\star$ so that for each training example $E_{\mathbf{w}}(\mathbf{y}_t; \mathbf{x}_t)$ is lower than the energy of any other assignment $E_{\mathbf{w}}(\mathbf{y}; \mathbf{x}_t)$ by some margin.
- We formalise the notion of margin by defining a loss function $\Delta(\mathbf{y}_t, \mathbf{y})$, which is zero when $\mathbf{y} = \mathbf{y}_t$ and positive otherwise.
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Max-Margin Quadratic Program

Learning goal: Find $\mathbf{w}$ such that $E_{\mathbf{w}}(\mathbf{y}) - E_{\mathbf{w}}(\mathbf{y}^\dagger) \geq \Delta(\mathbf{y}^\dagger, \mathbf{y})$.

Relaxed and regularized learning goal:

$$\begin{array}{ll} \text{minimize} & \overbrace{\frac{1}{2}\|\mathbf{w}\|_2^2}^{\text{regularization}} + \overbrace{C\xi}^{\text{slack}} \\ \text{subject to} & \underbrace{\mathbf{w}^T\phi(\mathbf{y}) - \mathbf{w}^T\phi(\mathbf{y}^\dagger)}_{\text{energy difference}} \geq \underbrace{\Delta(\mathbf{y}, \mathbf{y}^\dagger) - \xi}_{\text{rescaled margin}}, \quad \overbrace{\forall \mathbf{y} \in \mathcal{Y}}^{\text{very large}} \\ & \xi \geq 0 \end{array}$$

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Relaxed and regularized learning goal:

$$\text{minimize} \quad \overbrace{\frac{1}{2}\|\mathbf{w}\|_2^2}^{\text{regularization}} + \overbrace{C\xi}^{\text{slack}}$$

$$\text{subject to} \quad \underbrace{\mathbf{w}^T\phi(\mathbf{y}) - \mathbf{w}^T\phi(\mathbf{y}^\dagger)}_{\text{energy difference}} \geq \underbrace{\Delta(\mathbf{y}, \mathbf{y}^\dagger) - \xi}_{\text{rescaled margin}}, \quad \overbrace{\forall \mathbf{y} \in \mathcal{Y}}^{\text{very large}}$$

$$\xi \geq 0$$

Re-writing Margin Constraints

Recognize that $\mathbf{w}^T\phi(\mathbf{y}) - \mathbf{w}^T\phi(\mathbf{y}^\dagger) \geq \Delta\left(\mathbf{y}, \mathbf{y}^\dagger\right) - \xi$ for all $\mathbf{y}$ so, in particular, it must hold for the worst case $\mathbf{y}$.

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}\|\mathbf{w}\|_2^2 + C\xi \\ \text{subject to} & \xi \geq \underbrace{\max_{\mathbf{y}\in\mathcal{Y}} \left\{\Delta(\mathbf{y}, \mathbf{y}^\dagger) - \mathbf{w}^T\phi(\mathbf{y})\right\}}_{\text{loss-augmented inference (for given } \mathbf{w})} + \mathbf{w}^T\phi(\mathbf{y}^\dagger) \\ & \xi \geq 0 \end{array}$$

As long as $\Delta(\mathbf{y}, \mathbf{y}_t)$ decomposes over cliques of E we can use inference to find the most violated constraint (for a fixed $\mathbf{w}$).

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Cutting-Plane Max-Margin Learning

- Start with active set $\mathcal{A} = \{\}$.
- Solve for **w** and ξ

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}\|\mathbf{w}\|_2^2 + C\xi \\ \text{subject to} & \mathbf{w}^T\phi(\mathbf{y}) - \mathbf{w}^T\phi(\mathbf{y}^\dagger) \geq \Delta(\mathbf{y}, \mathbf{y}^\dagger) - \xi, \quad \forall \mathbf{y} \in \mathcal{A} \\ & \xi \geq 0 \end{array}$$

- Find the most violated constraint,

$$\mathbf{y}^\star \in \operatorname*{argmin}_{\mathbf{y} \in \mathcal{Y}} \left\{ \mathbf{w}^T\phi(\mathbf{y}) - \Delta(\mathbf{y}, \mathbf{y}^\dagger) \right\}$$

- Add $\mathbf{y}^\star$ to active set $\mathcal{A}$ and repeat.

Subgradient Descent Max-Margin Learning

Recognize that $\xi^\star = \max_{\mathbf{y} \in \mathcal{Y}} \left\{ \Delta(\mathbf{y}, \mathbf{y}^\dagger) - \mathbf{w}^T \phi(\mathbf{y}) \right\}$. So rewrite the max-margin QP as the non-smooth optimization problem

$$\text{minimize} \quad \frac{1}{2}\|\mathbf{w}\|_2^2 + C \max_{\mathbf{y} \in \mathcal{Y}} \underbrace{\left\{ \Delta(\mathbf{y}, \mathbf{y}^\dagger) - \mathbf{w}^T \phi(\mathbf{y}) \right\}}_{\text{family of linear functions}}$$

which we can solve by the subgradient method.

Tutorial Summary

- Structured prediction models, or energy functions, are pervasive in computer vision (and other fields).
- Often we are interested in finding the energy minimizing assignment.
- Exact and approximate inference algorithms exploit structure:
  - message passing for low treewidth graphs
  - graph-cuts for submodular energies
  - dual decomposition for decomposeable energies
- Parameter learning within a max-margin setting.
- Still very active research in inference and learning.

Any Questions?

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