Short generators without quantum computers: the case of multiquadratics

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`https://multiquad.cr.yp.to`

Joint work with:
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Followup to my SIAM AG15 talk
"Computational algebraic number theory tackles lattice-based cryptography":
`https://cr.yp.to/talks.html#2015.08.06`

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Sounds like SVP is claimed to be a hard problem. How hard is it?

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Who thinks this is the end of the story?
Is $2^{(0.1+o(1))N}$ possible? $2^{\Theta(N/\log N)}$? $2^{N^{1/2+o(1)}}$?

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2002 Micciancio–Goldwasser (emphasis added): "To date, the best known polynomial time (possibly randomized) approximation algorithms for SVP and CVP achieve worst-case (over the choice of the input) approximation factors $\gamma(n)$ that are **essentially exponential** in the rank n."

2007 Regev:

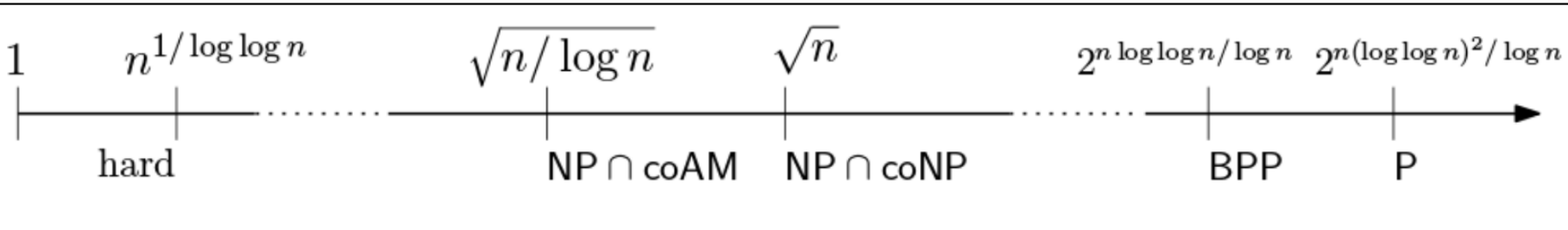

Figure 1: The complexity of lattice problems (some constants omitted)

2013 Micciancio: "Smooth trade-off between running time and approximation: $\gamma \approx 2^{O(n \log\log T/\log T)}$"

Quantum attacks against cyclotomic lattice problems

STOC 2014 Eisenträger–Hallgren–Kitaev–Song:
poly-time quantum algorithm for $K \mapsto \mathcal{O}_K^\times$.

K: number field.
$\mathcal{O}_K$: ring of algebraic integers in K.
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also using an idea from 2014 Campbell–Groves–Shepherd:
poly-time quantum algorithm for $K, g\mathcal{O}_K \mapsto \zeta_m^j g$ for some j,
assuming cyclotomic $K = \mathbf{Q}(\zeta_m)$, small h_m^+, very short g.

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This recovers secret keys in, e.g.,
STOC 2009 Gentry homomorphic-encryption system using cyclotomics,
Eurocrypt 2013 Garg–Gentry–Halevi multilinear-map system, etc.

Is the attack idea limited to very short generators?

More lattice problems of interest:

$I \mapsto$ shortest nonzero vector in I. ("Exact Ideal-SVP".)

$I \mapsto$ close to shortest nonzero vector in I. ("Approximate Ideal-SVP".)

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Counterargument: attack is poly time against **arbitrary principal ideals** for approx factor $2^{N^{1/2+o(1)}}$ in degree-N cyclotomics, assuming small h^+. See, e.g., 2016 Cramer–Ducas–Peikert–Regev.

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2015 Peikert:
"Although cyclotomics have a lot of structure, nobody has yet found a way to exploit it in attacking Ideal-SVP/BDD . . . For commonly used rings, principal ideals are an extremely small fraction of all ideals. . . . The weakness here is not so much due to the structure of cyclotomics, but rather to the extra structure of principal ideals that have short generators."

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Counterargument, 2016 Cramer–Ducas–Wesolowski:
fast Ideal-SVP attack for approx factor $2^{N^{1/2+o(1)}}$ in degree-N cyclotomics, under plausible assumptions about class-group generators etc.
Starts from Biasse–Song, uses more features of cyclotomic fields.

This shreds the standard approx-Ideal-SVP tradeoff picture.

Non-cyclotomic lattice-based cryptography

Cyclotomics are scary. Let's explore alternatives:

- Eliminate the ideal structure.
  e.g., use LWE instead of Ring-LWE.
  But this limits the security achievable for key size K.

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- This talk: Switch from cyclotomics to other Galois number fields. Another popular example in algebraic-number-theory textbooks: multiquadratics; e.g., $\mathbf{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{7}, \sqrt{11}, \sqrt{13}, \sqrt{17}, \sqrt{19}, \sqrt{23})$.

A reasonable multiquadratic cryptosystem

Case study of a lattice-based cryptosystem
that was already defined in detail for arbitrary number fields:
2010 Smart–Vercauteren, optimized version of 2009 Gentry.

Parameter: $R = \mathbf{Z}[\alpha]$ for an algebraic integer α.
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To handle multiquadratics better,
we generalized beyond $\mathbf{Z}[\alpha]$; fixed a keygen speed problem;
used twisted Hadamard transforms as replacement for FFTs;
adapted 2011 Gentry–Halevi cyclotomic speedups to multiquadratics.

Like Smart–Vercauteren, we took $N \in \lambda^{2+o(1)}$ for target security 2^λ.
Checked security against standard lattice attacks:
nothing better than exponential time.

Our main multiquadratic results

See `https://multiquad.cr.yp.to` for paper and software:

- **Huge parameter range for this cryptosystem is now broken.**
- Applicability: Attack is non-quantum. Works on your PC.
- Verifiability: We implemented the attack. It works as predicted.
- Fits the pattern motivating the "NTRU prime" recommendations: subfields and automorphisms tend to damage security.
- Speed: Quasipoly for $\mathbb{Q}(\sqrt{d_1}, \ldots, \sqrt{d_n})$ if $d_1, \ldots, d_n$ are quasipoly. (Quasipoly as function of 2^n: i.e., logs are bounded by $n^{O(1)}$.)
- Analysis assuming standard heuristics:
  - Attack always finds unit group $\mathcal{O}_L^\times$.
  - Attack always finds some generator of input ideal.
  - Attack always finds short generator if $d_1, \ldots, d_n > 2^{1.03n}$.
- Experiments find short generator even for much smaller d's. Some failures for tiny d's—presumably should do better BDD.

Conventional techniques to find generators

Want to find g given gR.

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Much more common: can factor $(\alpha/g)R$ into prime ideals.
Multiply and divide these equations sensibly.

Example from SIAM AG15 talk:
If $\alpha_1 R = gR \cdot P^2 \cdot Q^2$ and $\alpha_2 R = gR \cdot P \cdot Q^3$ and $\alpha_3 R = gR \cdot P \cdot Q^2$
then $P = \alpha_1 \alpha_3^{-1} R$ and $Q = \alpha_2 \alpha_3^{-1} R$ and $gR = \alpha_1^{-1} \alpha_2^{-2} \alpha_3^4 R$.

This is the core of the conventional methods of
computing units (find *two* generators of same ideal; divide);
computing class group (obstruction to all ideals having generators);
computing discrete logarithms by NFS; factoring by NFS; etc.

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(Can't find all units of "hard" cyclotomics without solving this!)

1966 Wada: Use subfield relation $u^2 = N_\sigma(u)N_\tau(u)/\sigma(N_{\sigma\tau}(u))$.
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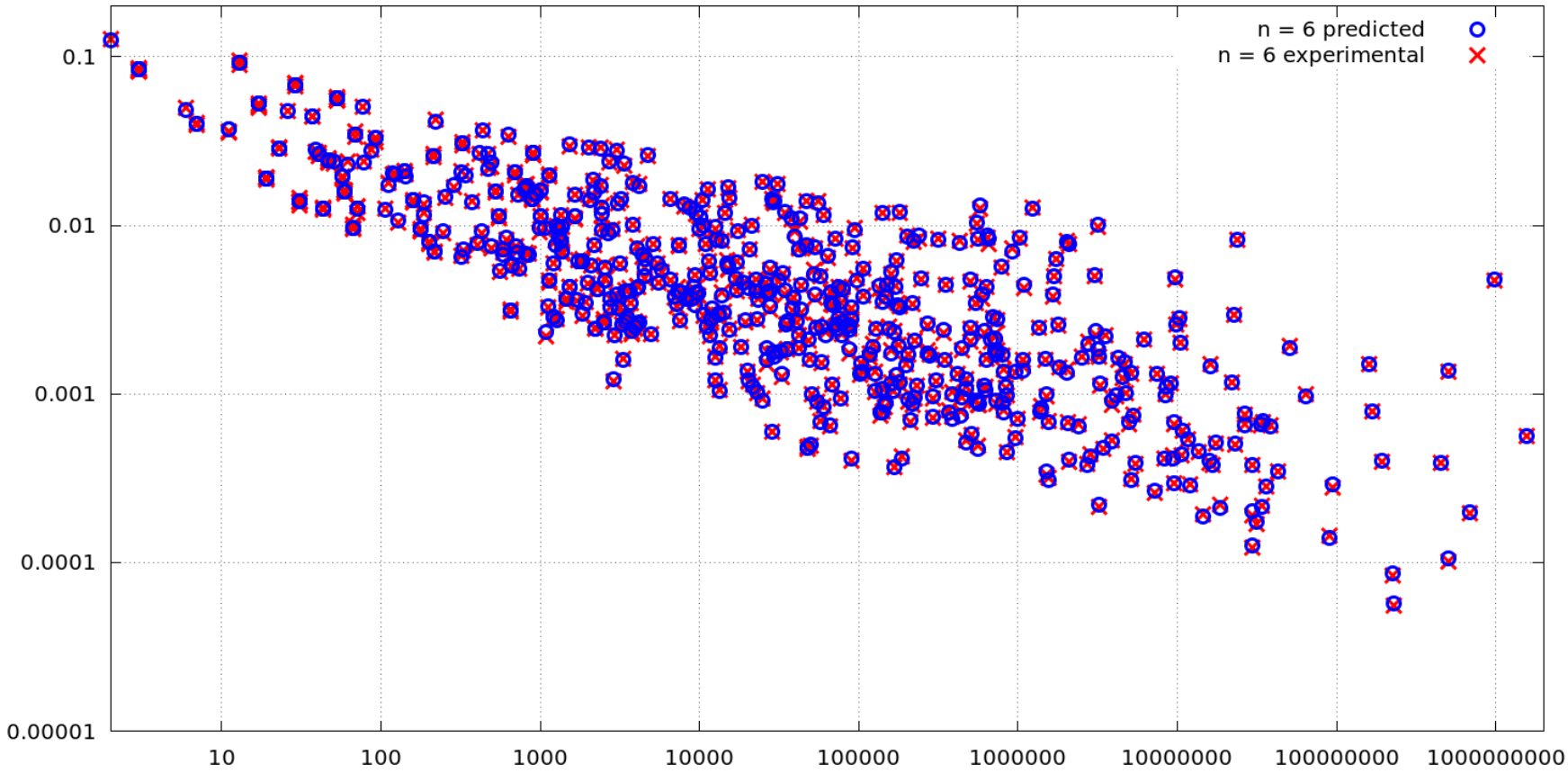
4. Similarly find generators starting with generators of norms.

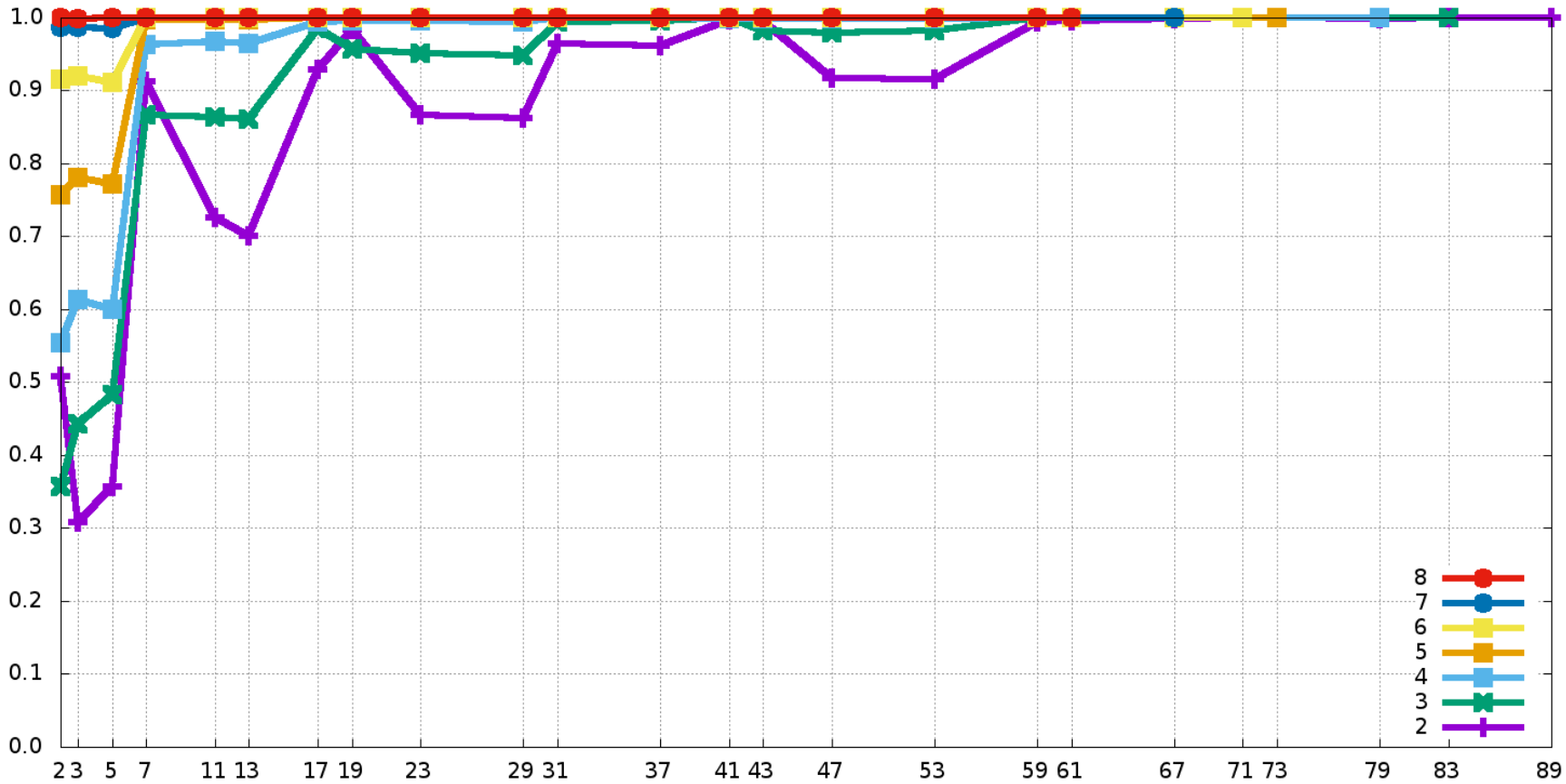
Coefficients for MQ lattice

Vertical axis: Average absolute coefficients of $\operatorname{Log} g$ on MQ basis.
Horizontal axis: $1.11/(2^{n/2}\log(u_D))$.



Success for MQ lattice

Vertical axis: Success probability of simple rounding (in the MQ lattice).
Horizontal axis: d_1, using n consecutive primes for $(d_1, \ldots, d_n)$.



Time (in seconds) to find full lattice and generator

2^n	Sage tower units	Sage absolute units	new units	new units2	new gen	new gen2
8	0.05	0.03	0.90	0.91	0.07	0.07
16	0.48	0.24	2.33	2.39	0.20	0.19
32	6.75	4.73	6.61	7.36	0.56	0.51
64	>700000	>700000	23.30	37.51	1.51	1.51
128			93.02	1560.49	4.95	7.29
256			463.91	31469.23	27.95	100.65

Table: Observed time to compute (once) the units of $\mathbb{Q}(\sqrt{d_1}, \ldots, \sqrt{d_n})$; and to find a generator for the public key in the cryptosystem.

Success at finding short generator of ideal

n	3	4	5	6	7	8
$p_{\mathrm{suc}}(L_1)$	0.122	0.137	0.132	0.036	0.001	0.000
$p_{\mathrm{suc}}(L_n)$	0.203	0.490	0.648	0.936	0.631	0.423
$p_{\mathrm{suc}}(L_{n^2})$	0.784	0.981	1.000	1.000	1.000	1.000

Table: Observed attack success probabilities for various multiquadratic fields.