Modern ECC signatures

2011 Bernstein–Duif–Lange–Schwabe–Yang:
Ed25519 signature scheme = EdDSA using conservative Curve25519 elliptic curve.
https://ed25519.cr.yp.to

32-byte public keys,
64-byte signatures,
$\approx 2^{125.8}$ security level.

Deployed in SSH, Signal, many more applications:
https://ianix.com/pub/ed25519-deployment.html

Many papers have explored Curve25519/Ed25519 speed.

e.g. 2015 Chou software:
on Intel Sandy Bridge (2011),
57164 cycles for keygen,
63526 cycles for signature,
205741 cycles for verification,
159128 cycles for ECDH.

Compare to, e.g., 2000 Brown–Hankerson–López–Menezes:
on Intel Pentium II (1997),
1920000 cycles for ECDH
using NIST P-256 curve.

ECC signatures

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9 signature scheme =
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No! Beware change in CPU.

Maybe $A_C > B_C$; $A_D > B_D$; C does more work per cycle than D, thanks to CPU manufacturer.

Sometimes people measure cost in seconds instead of cycles. Then they benefit from more work per cycle and from more cycles per second.

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Does $A_C < B_D$ prove that A is better than B?

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Better comparisons (still raising many questions):

ECDH on Intel Pentium II/III (still not exactly the same):
1920000 cycles for NIST P-256,
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ECDH on Sandy Bridge:
374000 cycles for NIST P-256 (from 2013 Gueron–Krasnov),
159128 cycles for Curve25519.

Verification on Sandy Bridge:
529000 cycles for ECDSA-P-256,
205741 cycles for Ed25519.

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For each of these operations
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Simplest implementations
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Questions in algorithm desig
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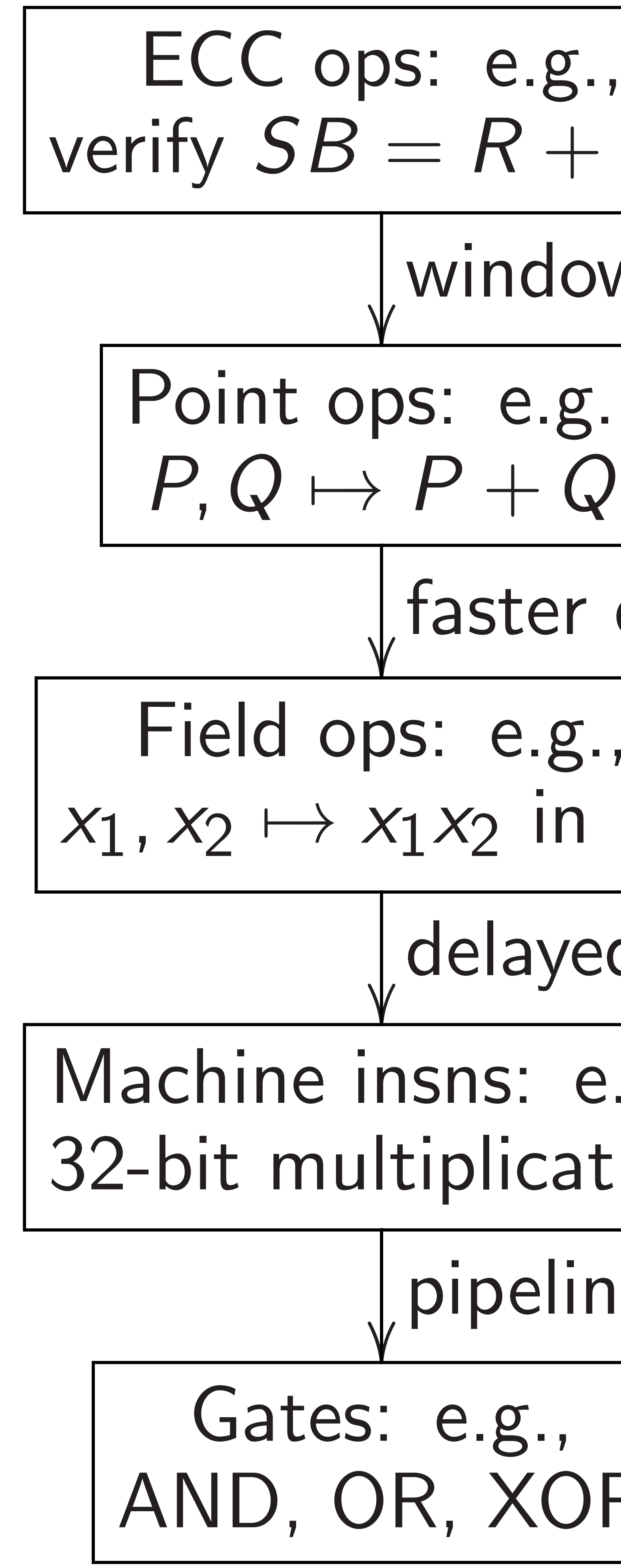

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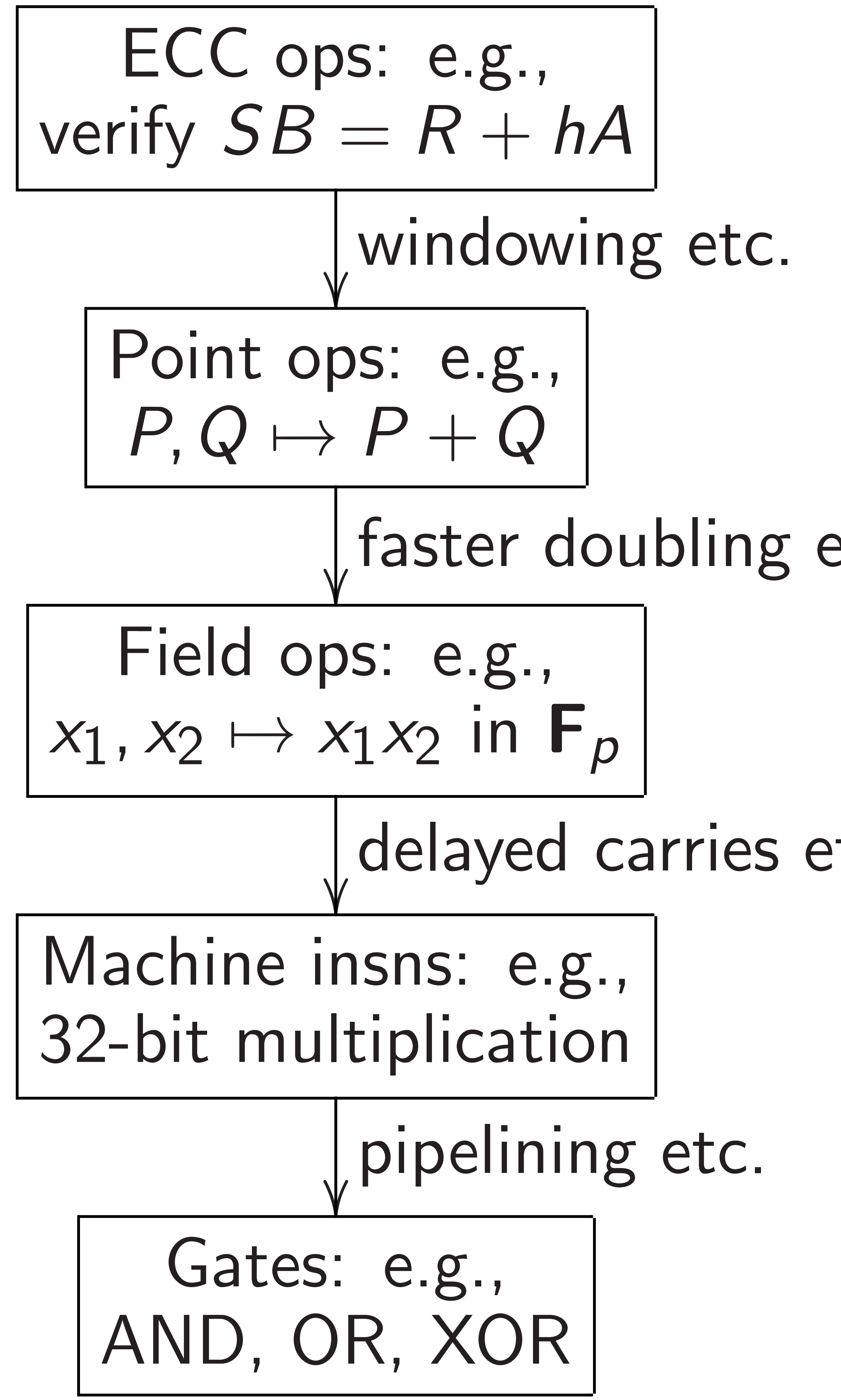

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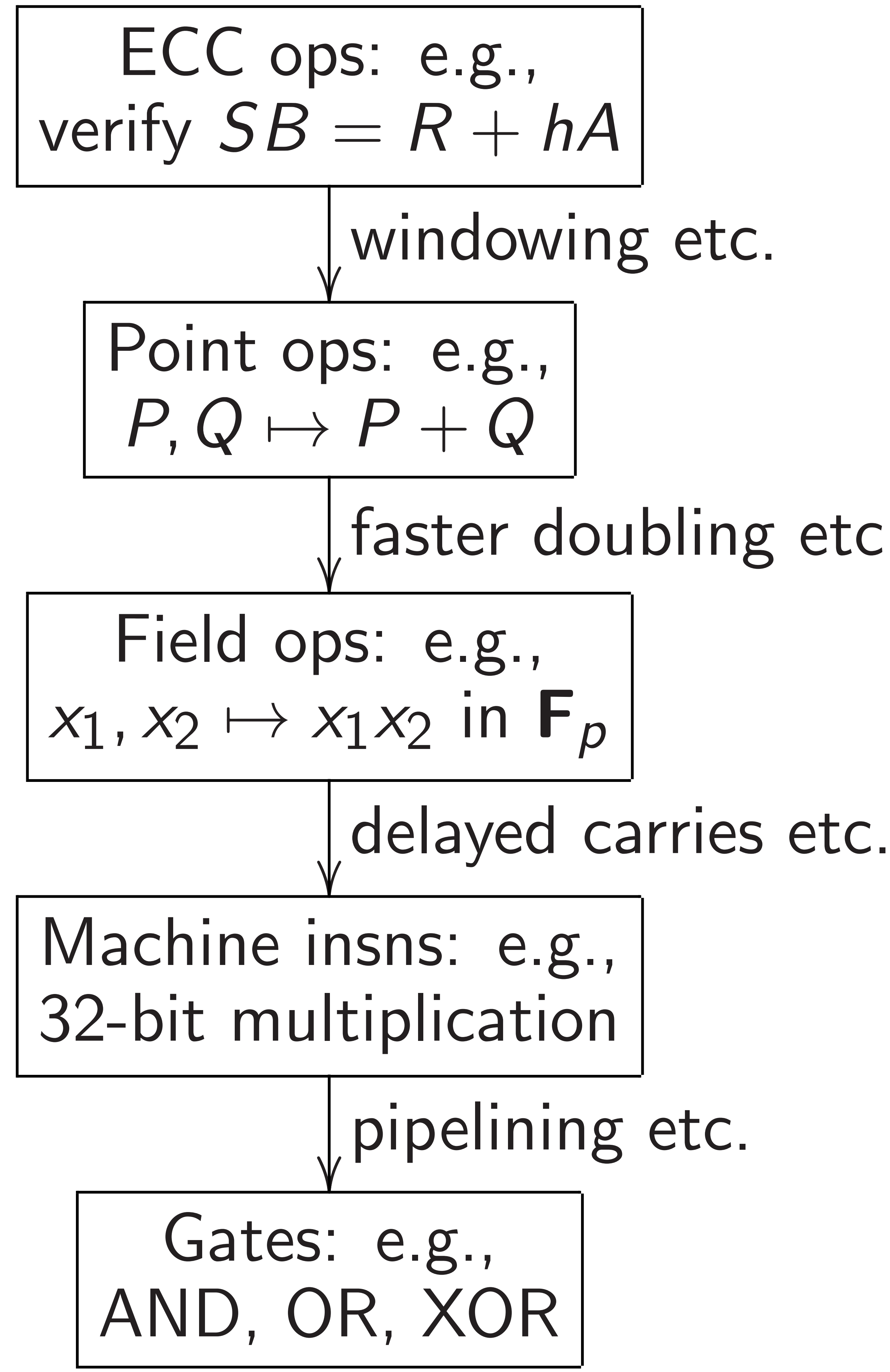

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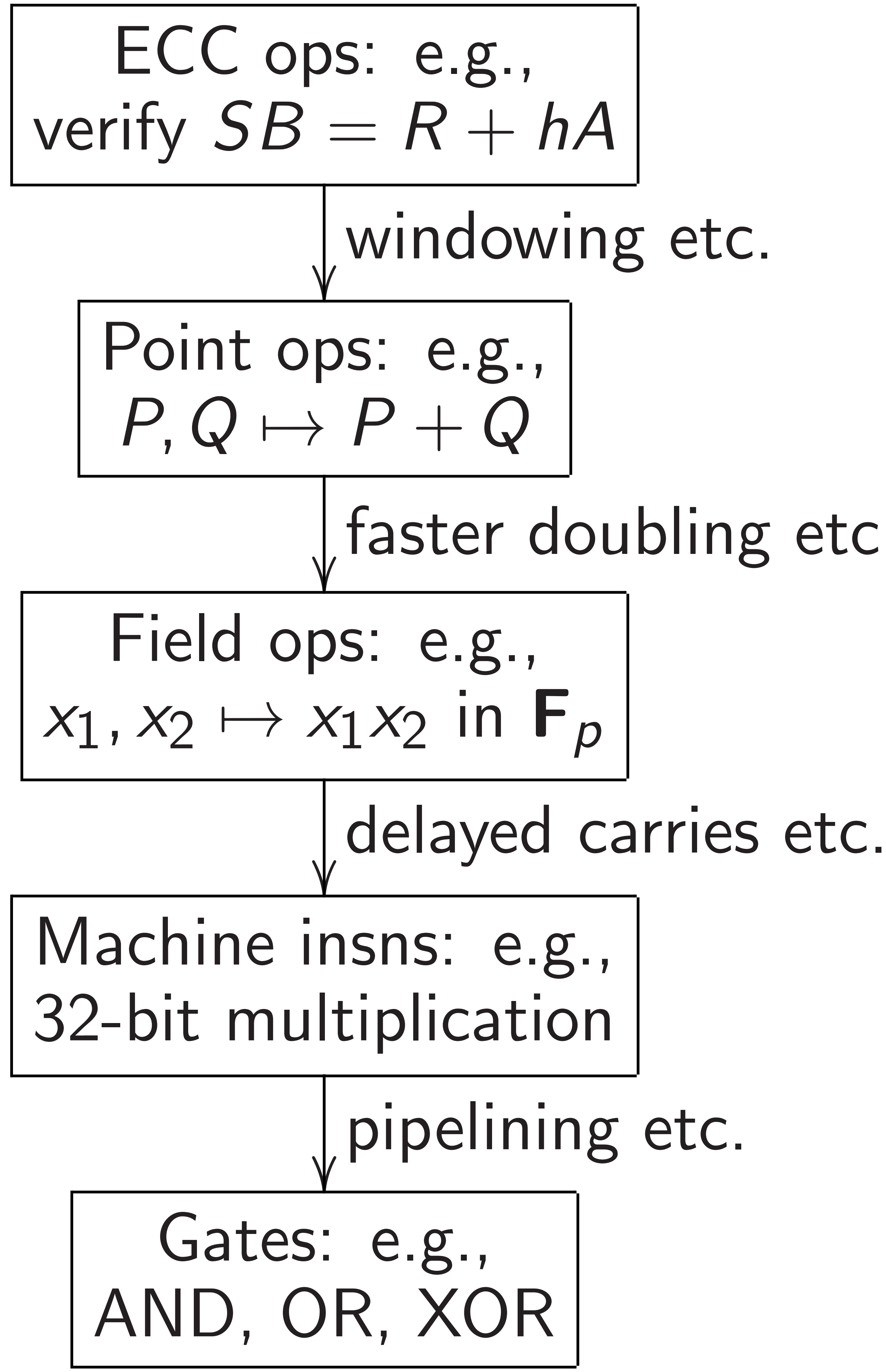

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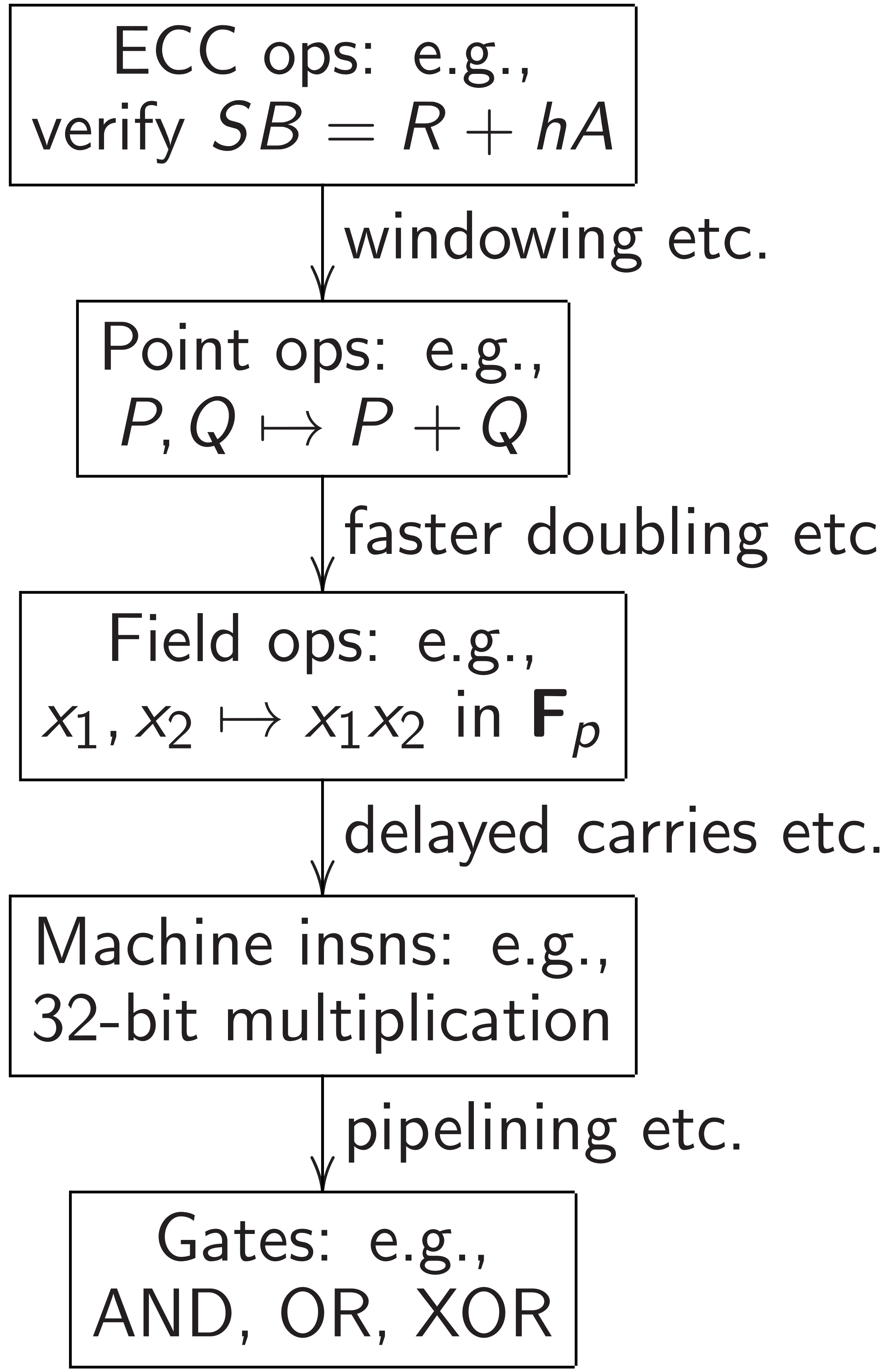

Single-scalar multi

Fundamental ECC
$n, P \mapsto nP$.

Input n is integer
$\{0, 1, \ldots, 2^{256} - 1$

Input P is point o

Will build $n, P \mapsto$
using additions $P,$
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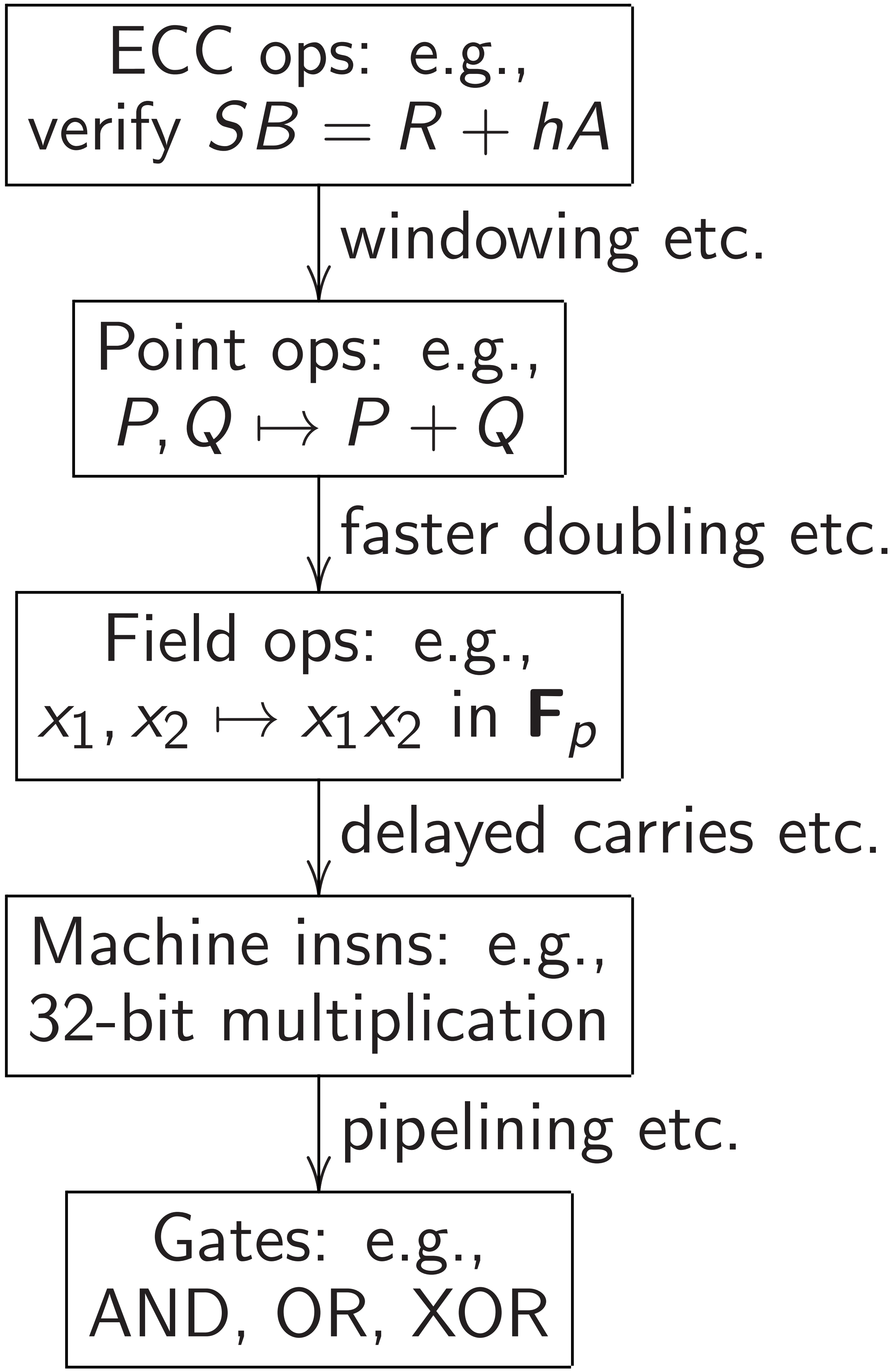

Single-scalar multiplication

Fundamental ECC operation
$n, P \mapsto nP$.

Input n is integer in, e.g.,
$\{0, 1, \ldots, 2^{256} - 1\}$.

Input P is point on elliptic c

Will build $n, P \mapsto nP$
using additions $P, Q \mapsto P +$
and subtractions $P, Q \mapsto P -$

Later will also look at
double-scalar multiplication
$m, P, n, Q \mapsto mP + nQ$.

Several levels to optimize:

ECC ops: e.g., verify $SB = R + hA$

↓ windowing etc.

Point ops: e.g., $P, Q \mapsto P + Q$

↓ faster doubling etc.

Field ops: e.g., $x_1, x_2 \mapsto x_1 x_2$ in $\mathbf{F}_p$

↓ delayed carries etc.

Machine insns: e.g., 32-bit multiplication

↓ pipelining etc.

Gates: e.g., AND, OR, XOR

Single-scalar multiplication

Fundamental ECC operation: $n, P \mapsto nP$.

Input n is integer in, e.g., $\{0, 1, \ldots, 2^{256} - 1\}$.

Input P is point on elliptic curve.

Will build $n, P \mapsto nP$ using additions $P, Q \mapsto P + Q$ and subtractions $P, Q \mapsto P - Q$.

Later will also look at double-scalar multiplication $m, P, n, Q \mapsto mP + nQ$.

levels to optimize:

ops: e.g.,
$\ldots B = R + hA$

↓ windowing etc.

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$\mapsto P + Q$

↓ faster doubling etc.

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$\to x_1 x_2$ in $\mathbf{F}_p$

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def scal
  if n =
  if n =
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Later will also look at
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$m, P, n, Q \mapsto mP + nQ$.

Left-to-right binar

```
def scalarmult(n
  if n == 0: ret
  if n == 1: ret
  R = scalarmult
  R = R + R
  if n % 2: R =
  return R
```

Two Python notes

- `n//2` in Python
- Recursion depth
  See `sys.setrec`

etc.

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Single-scalar multiplication

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Later will also look at
double-scalar multiplication
$m, P, n, Q \mapsto mP + nQ$.

Left-to-right binary method

```
def scalarmult(n,P):
  if n == 0: return 0
  if n == 1: return P
  R = scalarmult(n//2,P)
  R = R + R
  if n % 2: R = R + P
  return R
```

Two Python notes:

- n//2 in Python means ⌊n/
- Recursion depth is limited
  See sys.setrecursionli

Single-scalar multiplication

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- `n//2` in Python means $\lfloor \texttt{n}/2 \rfloor$.
- Recursion depth is limited.
  See `sys.setrecursionlimit`.

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- $2\left(\frac{n}{2}P\right.$

  e.g. 20

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- `n//2` in Python means $\lfloor \mathtt{n}/2 \rfloor$.
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This recursion con

- $2\left(\frac{n}{2}P\right)$ if $n \in 2$

  e.g. $20P = 2 \cdot 1$

- $2\left(\frac{n-1}{2}P\right) +$

  e.g. $21P = 2 \cdot 1$

Base cases in recu

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Otherwise use nP

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Left-to-right binary method

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  return R
```

Two Python notes:

- `n//2` in Python means $\lfloor n/2 \rfloor$.
- Recursion depth is limited.
  See `sys.setrecursionlimit`.

This recursion computes nP

- $2\left(\frac{n}{2}P\right)$ if $n \in 2\mathbf{Z}$.
  e.g. $20P = 2 \cdot 10P$.
- $2\left(\frac{n-1}{2}P\right) + P$ if $n \in 1$
  e.g. $21P = 2 \cdot 10P + P$.

Base cases in recursion:

$0P = 0$. For Edwards: $0 = ($

$1P = P$. Could omit this ca

Assuming $n \geq 0$ for simplicit

Otherwise use $nP = -(-n)$

Left-to-right binary method

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def scalarmult(n,P):
  if n == 0: return 0
  if n == 1: return P
  R = scalarmult(n//2,P)
  R = R + R
  if n % 2: R = R + P
  return R
```

Two Python notes:

- `n//2` in Python means $\lfloor \mathtt{n}/2 \rfloor$.
- Recursion depth is limited. See `sys.setrecursionlimit`.

This recursion computes nP as

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- $2\left(\frac{n-1}{2}P\right) + P$ if $n \in 1 + 2\mathbf{Z}$.
  e.g. $21P = 2 \cdot 10P + P$.

Base cases in recursion:
$0P = 0$. For Edwards: $0 = (0, 1)$.
$1P = P$. Could omit this case.

Assuming $n \geq 0$ for simplicity.
Otherwise use $nP = -(-n)P$.

ight binary method

```
larmult(n,P):
== 0: return 0
== 1: return P
calarmult(n//2,P)
 + R
% 2: R = R + P
n R
```

thon notes:

n Python means $\lfloor n/2 \rfloor$.

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This recursion computes nP as

- $2\left(\frac{n}{2}P\right)$ if $n \in 2\mathbf{Z}$.

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Assuming $n \geq 0$ for simplicity.

Otherwise use $nP = -(-n)P$.

If $0 \leq n$

this algo

$\leq 2b - 2$

$\leq b - 1$

$\leq b - 1$

Example

$31P = 2$

$31 = (11$

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Average

$35P = 2$

$35 = (10$

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,P):
urn 0
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means $\lfloor n/2 \rfloor$.

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This recursion computes nP as

- $2\left(\frac{n}{2}P\right)$ if $n \in 2\mathbf{Z}$.
  e.g. $20P = 2 \cdot 10P$.

- $2\left(\frac{n-1}{2}P\right) + P$ if $n \in 1 + 2\mathbf{Z}$.
  e.g. $21P = 2 \cdot 10P + P$.

Base cases in recursion:
$0P = 0$. For Edwards: $0 = (0, 1)$.
$1P = P$. Could omit this case.

Assuming $n \geq 0$ for simplicity.
Otherwise use $nP = -(-n)P$.

If $0 \leq n < 2^b$ the
this algorithm use
$\leq 2b - 2$ addition
$\leq b - 1$ doublings
$\leq b - 1$ additions

Example of worst
$31P = 2(2(2(2P+$
$31 = (11111)_2$; b
4 doublings; 4 mo

Average case is be
$35P = 2(2(2(2(2P$
$35 = (100011)_2$; b
5 doublings; 2 add

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$\leq b - 1$ additions of P.

Example of worst case:
$31P = 2(2(2(2P+P)+P)+$
$31 = (11111)_2$; $b = 5$;
4 doublings; 4 more additior

Average case is better: e.g.
$35P = 2(2(2(2(2P))) + P)$
$35 = (100011)_2$; $b = 6$;
5 doublings; 2 additions.

This recursion computes nP as

- $2\left(\frac{n}{2}P\right)$ if $n \in 2\mathbf{Z}$.
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$P\Big)$ if $n \in 2\mathbf{Z}$.

$0P = 2 \cdot 10P$.

$\frac{-1}{2} P\Big) + P$ if $n \in 1 + 2\mathbf{Z}$.

$1P = 2 \cdot 10P + P$.

ses in recursion:

For Edwards: $0 = (0, 1)$.

. Could omit this case.

g $n \geq 0$ for simplicity.

se use $nP = -(-n)P$.

If $0 \leq n < 2^b$ then
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$35 = (100011)_2$; $b = 6$;
5 doublings; 2 additions.

Non-adj

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  if n =
  if n =
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```

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$= -(-n)P$.

If $0 \le n < 2^b$ then
this algorithm uses
$\le 2b - 2$ additions: specifically
$\le b - 1$ doublings and
$\le b - 1$ additions of P.

Example of worst case:
$31P = 2(2(2(2P{+}P){+}P){+}P){+}P$.
$31 = (11111)_2$; $b = 5$;
4 doublings; 4 more additions.

Average case is better: e.g.
$35P = 2(2(2(2(2P))) + P) + P$.
$35 = (100011)_2$; $b = 6$;
5 doublings; 2 additions.

Non-adjacent form

```
def scalarmult(n
  if n == 0: ret
  if n == 1: ret
  if n % 4 == 1:
    R = scalarmu
    R = R + R
    return (R +
  if n % 4 == 3:
    R = scalarmu
    R = R + R
    return (R +
  R = scalarmult
  return R + R
```

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If $0 \le n < 2^b$ then this algorithm uses $\le 2b - 2$ additions: specifically $\le b - 1$ doublings and $\le b - 1$ additions of P.

Example of worst case:
$31P = 2(2(2(2P{+}P){+}P){+}P){+}P$.
$31 = (11111)_2$; $b = 5$;
4 doublings; 4 more additions.

Average case is better: e.g.
$35P = 2(2(2(2(2P))) + P) + P$.
$35 = (100011)_2$; $b = 6$;
5 doublings; 2 additions.

<u>Non-adjacent form (NAF)</u>

```
def scalarmult(n,P):
  if n == 0: return 0
  if n == 1: return P
  if n % 4 == 1:
    R = scalarmult((n-1)/
    R = R + R
    return (R + R) + P
  if n % 4 == 3:
    R = scalarmult((n+1)/
    R = R + R
    return (R + R) - P
  R = scalarmult(n/2,P)
  return R + R
```

If $0 \leq n < 2^b$ then
this algorithm uses
$\leq 2b - 2$ additions: specifically
$\leq b - 1$ doublings and
$\leq b - 1$ additions of P.

Example of worst case:
$31P = 2(2(2(2P+P)+P)+P)+P$.
$31 = (11111)_2$; $b = 5$;
4 doublings; 4 more additions.

Average case is better: e.g.
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Non-adjacent form (NAF)

```
def scalarmult(n,P):
  if n == 0: return 0
  if n == 1: return P
  if n % 4 == 1:
    R = scalarmult((n-1)/4,P)
    R = R + R
    return (R + R) + P
  if n % 4 == 3:
    R = scalarmult((n+1)/4,P)
    R = R + R
    return (R + R) - P
  R = scalarmult(n/2,P)
  return R + R
```

$< 2^b$ then
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2 additions: specifically
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e of worst case:
$2(2(2(2P+P)+P)+P)+P$.
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$2(2(2(2(2P))) + P) + P$.
$00011)_2$; $b = 6$;
ngs; 2 additions.

Non-adjacent form (NAF)

```
def scalarmult(n,P):
  if n == 0: return 0
  if n == 1: return P
  if n % 4 == 1:
    R = scalarmult((n-1)/4,P)
    R = R + R
    return (R + R) + P
  if n % 4 == 3:
    R = scalarmult((n+1)/4,P)
    R = R + R
    return (R + R) - P
  R = scalarmult(n/2,P)
  return R + R
```

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$31 = (10$

$35P = 2$
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$P))) + P) + P$.
$b = 6$;
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Non-adjacent form (NAF)

```
def scalarmult(n,P):
  if n == 0: return 0
  if n == 1: return P
  if n % 4 == 1:
    R = scalarmult((n-1)/4,P)
    R = R + R
    return (R + R) + P
  if n % 4 == 3:
    R = scalarmult((n+1)/4,P)
    R = R + R
    return (R + R) - P
  R = scalarmult(n/2,P)
  return R + R
```


Subtraction on the
is as cheap as add
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$31P = 2(2(2(2(2P$
$31 = (10000\bar{1})_2$; $\bar{1}$

$35P = 2(2(2(2(2P$
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“Non-adjacent”: ±
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plus $\approx b/2$ additio
On average $\approx b/3$

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Non-adjacent form (NAF)

```
def scalarmult(n,P):
  if n == 0: return 0
  if n == 1: return P
  if n % 4 == 1:
    R = scalarmult((n-1)/4,P)
    R = R + R
    return (R + R) + P
  if n % 4 == 3:
    R = scalarmult((n+1)/4,P)
    R = R + R
    return (R + R) - P
  R = scalarmult(n/2,P)
  return R + R
```


Subtraction on the curve
is as cheap as addition.
NAF takes advantage of this

$31P = 2(2(2(2(2P)))) - P$.
$31 = (10000\bar{1})_2$; $\bar{1}$ denotes

$35P = 2(2(2(2(2P)) + P))$
$35 = (10010\bar{1})_2$.

"Non-adjacent": $\pm P$ ops ar
separated by $\geq$2 doublings.

Worst case: $\approx b$ doublings
plus $\approx b/2$ additions of $\pm P$.
On average $\approx b/3$ additions.

Non-adjacent form (NAF)

```
def scalarmult(n,P):
  if n == 0: return 0
  if n == 1: return P
  if n % 4 == 1:
    R = scalarmult((n-1)/4,P)
    R = R + R
    return (R + R) + P
  if n % 4 == 3:
    R = scalarmult((n+1)/4,P)
    R = R + R
    return (R + R) - P
  R = scalarmult(n/2,P)
  return R + R
```


Subtraction on the curve is as cheap as addition. NAF takes advantage of this.

$31P = 2(2(2(2(2P)))) - P$.
$31 = (10000\bar{1})_2$; $\bar{1}$ denotes -1.

$35P = 2(2(2(2(2P)) + P)) - P$.
$35 = (10010\bar{1})_2$.

"Non-adjacent": $\pm P$ ops are separated by ≥ 2 doublings.

Worst case: $\approx b$ doublings plus $\approx b/2$ additions of $\pm P$.
On average $\approx b/3$ additions.

acent form (NAF)

```
larmult(n,P):
== 0: return 0
== 1: return P
% 4 == 1:
 scalarmult((n-1)/4,P)
 R + R
urn (R + R) + P
% 4 == 3:
 scalarmult((n+1)/4,P)
 R + R
urn (R + R) - P
calarmult(n/2,P)
n R + R
```


Subtraction on the curve is as cheap as addition. NAF takes advantage of this.

$31P = 2(2(2(2(2P)))) - P$.
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Worst case: $\approx b$ doublings plus $\approx b/2$ additions of $\pm P$.
On average $\approx b/3$ additions.

Width-2

```
def wind
  if n =
  if n =
  if n =
  if n %
    R =
    R =
    R =
    retu
  if n %
    R =
    R =
    R =
    retu
```

n (NAF)

```
,P):
urn 0
urn P

lt((n-1)/4,P)

R) + P

lt((n+1)/4,P)

R) - P
(n/2,P)
```


Subtraction on the curve
is as cheap as addition.
NAF takes advantage of this.

$31P = 2(2(2(2(2P)))) - P$.
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"Non-adjacent": $\pm P$ ops are separated by ≥ 2 doublings.

Worst case: $\approx b$ doublings plus $\approx b/2$ additions of $\pm P$.
On average $\approx b/3$ additions.

Width-2 signed sli

```
def window2(n,P,
  if n == 0: ret
  if n == 1: ret
  if n == 3: ret
  if n % 8 == 1:
    R = window2(
    R = R + R
    R = R + R
    return (R +
  if n % 8 == 3:
    R = window2(
    R = R + R
    R = R + R
    return (R +
```

/4,P)

/4,P)

Subtraction on the curve is as cheap as addition. NAF takes advantage of this.

$31P = 2(2(2(2(2P)))) - P$.
$31 = (10000\bar{1})_2$; $\bar{1}$ denotes -1.

$35P = 2(2(2(2(2P)) + P)) - P$.
$35 = (10010\bar{1})_2$.

"Non-adjacent": $\pm P$ ops are separated by ≥ 2 doublings.

Worst case: $\approx b$ doublings plus $\approx b/2$ additions of $\pm P$. On average $\approx b/3$ additions.

Width-2 signed sliding windo

```
def window2(n,P,P3):
  if n == 0: return 0
  if n == 1: return P
  if n == 3: return P3
  if n % 8 == 1:
    R = window2((n-1)/8,P
    R = R + R
    R = R + R
    return (R + R) + P
  if n % 8 == 3:
    R = window2((n-3)/8,P
    R = R + R
    R = R + R
    return (R + R) + P3
```

Subtraction on the curve is as cheap as addition. NAF takes advantage of this.

$31P = 2(2(2(2(2P)))) - P$.
$31 = (10000\bar{1})_2$; $\bar{1}$ denotes -1.

$35P = 2(2(2(2(2P)) + P)) - P$.
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"Non-adjacent": $\pm P$ ops are separated by ≥ 2 doublings.

Worst case: $\approx b$ doublings plus $\approx b/2$ additions of $\pm P$. On average $\approx b/3$ additions.

Width-2 signed sliding windows

```
def window2(n,P,P3):
  if n == 0: return 0
  if n == 1: return P
  if n == 3: return P3
  if n % 8 == 1:
    R = window2((n-1)/8,P,P3)
    R = R + R
    R = R + R
    return (R + R) + P
  if n % 8 == 3:
    R = window2((n-3)/8,P,P3)
    R = R + R
    R = R + R
    return (R + R) + P3
```

tion on the curve
eap as addition.
kes advantage of this.

$2(2(2(2(2P)))) - P$.
$00001\bar{1})_2$; $\bar{1}$ denotes -1.

$2(2(2(2(2P)) + P)) - P$.
$00101\bar{1})_2$.

ljacent": $\pm P$ ops are
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ase: $\approx b$ doublings
$/2$ additions of $\pm P$.
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Width-2 signed sliding windows

```
def window2(n,P,P3):
  if n == 0: return 0
  if n == 1: return P
  if n == 3: return P3
  if n % 8 == 1:
    R = window2((n-1)/8,P,P3)
    R = R + R
    R = R + R
    return (R + R) + P
  if n % 8 == 3:
    R = window2((n-3)/8,P,P3)
    R = R + R
    R = R + R
    return (R + R) + P3
```

```
  if n %
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  if n %
    R =
    R =
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  R = wi
  return

def scal
  return
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ns of $\pm P$.
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Width-2 signed sliding windows

```
def window2(n,P,P3):
  if n == 0: return 0
  if n == 1: return P
  if n == 3: return P3
  if n % 8 == 1:
    R = window2((n-1)/8,P,P3)
    R = R + R
    R = R + R
    return (R + R) + P
  if n % 8 == 3:
    R = window2((n-3)/8,P,P3)
    R = R + R
    R = R + R
    return (R + R) + P3
```

```
  if n % 8 == 5:
    R = window2(
    R = R + R
    R = R + R
    return (R +
  if n % 8 == 7:
    R = window2(
    R = R + R
    R = R + R
    return (R +
  R = window2(n/
  return R + R

def scalarmult(n
  return window2
```

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Width-2 signed sliding windows

```
def window2(n,P,P3):
  if n == 0: return 0
  if n == 1: return P
  if n == 3: return P3
  if n % 8 == 1:
    R = window2((n-1)/8,P,P3)
    R = R + R
    R = R + R
    return (R + R) + P
  if n % 8 == 3:
    R = window2((n-3)/8,P,P3)
    R = R + R
    R = R + R
    return (R + R) + P3
```



```
  if n % 8 == 5:
    R = window2((n+3)/8,P
    R = R + R
    R = R + R
    return (R + R) - P3
  if n % 8 == 7:
    R = window2((n+1)/8,P
    R = R + R
    R = R + R
    return (R + R) - P
  R = window2(n/2,P,P3)
  return R + R

def scalarmult(n,P):
  return window2(n,P,P+P+
```

Width-2 signed sliding windows

```
def window2(n,P,P3):
  if n == 0: return 0
  if n == 1: return P
  if n == 3: return P3
  if n % 8 == 1:
    R = window2((n-1)/8,P,P3)
    R = R + R
    R = R + R
    return (R + R) + P
  if n % 8 == 3:
    R = window2((n-3)/8,P,P3)
    R = R + R
    R = R + R
    return (R + R) + P3
```



```
  if n % 8 == 5:
    R = window2((n+3)/8,P,P3)
    R = R + R
    R = R + R
    return (R + R) - P3
  if n % 8 == 7:
    R = window2((n+1)/8,P,P3)
    R = R + R
    R = R + R
    return (R + R) - P
  R = window2(n/2,P,P3)
  return R + R

def scalarmult(n,P):
  return window2(n,P,P+P+P)
```

signed sliding windows

```
dow2(n,P,P3):
== 0: return 0
== 1: return P
== 3: return P3
% 8 == 1:
 window2((n-1)/8,P,P3)
 R + R
 R + R
urn (R + R) + P
% 8 == 3:
 window2((n-3)/8,P,P3)
 R + R
 R + R
urn (R + R) + P3
```



```
  if n % 8 == 5:
    R = window2((n+3)/8,P,P3)
    R = R + R
    R = R + R
    return (R + R) - P3
  if n % 8 == 7:
    R = window2((n+1)/8,P,P3)
    R = R + R
    R = R + R
    return (R + R) - P
  R = window2(n/2,P,P3)
  return R + R

def scalarmult(n,P):
  return window2(n,P,P+P+P)
```

Worst ca

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ding windows

```
P3):
urn 0
urn P
urn P3

(n-1)/8,P,P3)

R) + P

(n-3)/8,P,P3)

R) + P3
```



```
  if n % 8 == 5:
    R = window2((n+3)/8,P,P3)
    R = R + R
    R = R + R
    return (R + R) - P3
  if n % 8 == 7:
    R = window2((n+1)/8,P,P3)
    R = R + R
    R = R + R
    return (R + R) - P
  R = window2(n/2,P,P3)
  return R + R

def scalarmult(n,P):
  return window2(n,P,P+P+P)
```

Worst case: $\approx b$ d

$\approx b/3$ additions of

On average $\approx b/4$

ows

,P3)

,P3)


```
  if n % 8 == 5:
    R = window2((n+3)/8,P,P3)
    R = R + R
    R = R + R
    return (R + R) - P3
  if n % 8 == 7:
    R = window2((n+1)/8,P,P3)
    R = R + R
    R = R + R
    return (R + R) - P
  R = window2(n/2,P,P3)
  return R + R

def scalarmult(n,P):
  return window2(n,P,P+P+P)
```

Worst case: $\approx b$ doublings p

$\approx b/3$ additions of $\pm P$ or $\pm$

On average $\approx b/4$ additions.

```
  if n % 8 == 5:
    R = window2((n+3)/8,P,P3)
    R = R + R
    R = R + R
    return (R + R) - P3
  if n % 8 == 7:
    R = window2((n+1)/8,P,P3)
    R = R + R
    R = R + R
    return (R + R) - P
  R = window2(n/2,P,P3)
  return R + R

def scalarmult(n,P):
  return window2(n,P,P+P+P)
```


Worst case: $\approx b$ doublings plus $\approx b/3$ additions of $\pm P$ or $\pm 3P$.

On average $\approx b/4$ additions.

```
  if n % 8 == 5:
    R = window2((n+3)/8,P,P3)
    R = R + R
    R = R + R
    return (R + R) - P3
  if n % 8 == 7:
    R = window2((n+1)/8,P,P3)
    R = R + R
    R = R + R
    return (R + R) - P
  R = window2(n/2,P,P3)
  return R + R

def scalarmult(n,P):
  return window2(n,P,P+P+P)
```


Worst case: $\approx b$ doublings plus $\approx b/3$ additions of $\pm P$ or $\pm 3P$.
On average $\approx b/4$ additions.

Width-3 signed sliding windows:
Precompute $P, 3P, 5P, 7P$.
On average $\approx b/5$ additions.

```
  if n % 8 == 5:
    R = window2((n+3)/8,P,P3)
    R = R + R
    R = R + R
    return (R + R) - P3
  if n % 8 == 7:
    R = window2((n+1)/8,P,P3)
    R = R + R
    R = R + R
    return (R + R) - P
  R = window2(n/2,P,P3)
  return R + R

def scalarmult(n,P):
  return window2(n,P,P+P+P)
```


Worst case: $\approx b$ doublings plus $\approx b/3$ additions of $\pm P$ or $\pm 3P$. On average $\approx b/4$ additions.

Width-3 signed sliding windows: Precompute $P, 3P, 5P, 7P$. On average $\approx b/5$ additions.

Width 4: Precompute $P, 3P, 5P, 7P, 9P, 11P, 13P, 15P$. On average $\approx b/6$ additions.

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Cost of precomputation eventually outweighs savings.
Optimal: $\approx b$ doublings plus roughly $b/\lg b$ additions.

```
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 R + R
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indow2(n/2,P,P3)
n R + R

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(n+1)/8,P,P3)

R) - P
2,P,P3)

,P):
(n,P,P+P+P)
```


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Double-scalar mul

Want to quickly c
$m, P, n, Q \mapsto mP$ -

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by computing $h =$
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Obvious approach
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Optimal: $\approx b$ doublings plus roughly $b/\lg b$ additions.

Double-scalar multiplication

Want to quickly compute
$m, P, n, Q \mapsto mP + nQ$.

e.g. verify signature (R, S)
by computing $h = H(R, M)$,
computing $SB - hA$,
checking whether $R = SB -$

Obvious approach:
Compute mP; compute nQ;

e.g. $b = 256$:
≈ 256 doublings for mP,
≈ 256 doublings for nQ,
≈ 50 additions for mP,
≈ 50 additions for nQ.

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Compute mP; compute nQ; add.

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$\approx$256 doublings for mP,
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$\approx$256 doublings for nQ,
$\approx$50 additions for mP,
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Joint do

Do much
$2X + 2Y$

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Double-scalar multiplication

Want to quickly compute
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e.g. $b = 256$:
$\approx$256 doublings for mP,
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Joint doublings

Do much better b
$2X + 2Y$ into $2(X$

```
def scalarmult2(
  if m == 0:
    return scala
  if n == 0:
    return scala
  R = scalarmult
  R = R + R
  if m % 2: R =
  if n % 2: R =
  return R
```

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Double-scalar multiplication

Want to quickly compute
$m, P, n, Q \mapsto mP + nQ$.

e.g. verify signature (R, S)
by computing $h = H(R, M)$,
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Joint doublings

Do much better by merging
$2X + 2Y$ into $2(X + Y)$.

```
def scalarmult2(m,P,n,Q):
  if m == 0:
    return scalarmult(n,Q
  if n == 0:
    return scalarmult(m,P
  R = scalarmult2(m//2,P,
  R = R + R
  if m % 2: R = R + P
  if n % 2: R = R + Q
  return R
```

Double-scalar multiplication

Want to quickly compute
$m, P, n, Q \mapsto mP + nQ$.

e.g. verify signature (R, S)
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e.g. $b = 256$:
≈256 doublings for mP,
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Joint doublings

Do much better by merging
$2X + 2Y$ into $2(X + Y)$.

```
def scalarmult2(m,P,n,Q):
  if m == 0:
    return scalarmult(n,Q)
  if n == 0:
    return scalarmult(m,P)
  R = scalarmult2(m//2,P,n//2,Q)
  R = R + R
  if m % 2: R = R + P
  if n % 2: R = R + Q
  return R
```

scalar multiplication

quickly compute
$Q \mapsto mP + nQ$.

fy signature (R, S)
puting $h = H(R, M)$,
ng $SB - hA$,
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Joint doublings

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def scalarmult2(m,P,n,Q):
  if m == 0:
    return scalarmult(n,Q)
  if n == 0:
    return scalarmult(m,P)
  R = scalarmult2(m//2,P,n//2,Q)
  R = R + R
  if m % 2: R = R + P
  if n % 2: R = R + Q
  return R
```

For exan
$35P = 2$
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Joint doublings

Do much better by merging $2X + 2Y$ into $2(X + Y)$.

```
def scalarmult2(m,P,n,Q):
  if m == 0:
    return scalarmult(n,Q)
  if n == 0:
    return scalarmult(m,P)
  R = scalarmult2(m//2,P,n//2,Q)
  R = R + R
  if m % 2: R = R + P
  if n % 2: R = R + Q
  return R
```

For example: merg

$35P = 2(2(2(2(2F$

$31Q = 2(2(2(2Q+$

into $35P + 31Q =$

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$+ hA$.

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Joint doublings

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def scalarmult2(m,P,n,Q):
  if m == 0:
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  if n == 0:
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  R = scalarmult2(m//2,P,n//2,Q)
  R = R + R
  if m % 2: R = R + P
  if n % 2: R = R + Q
  return R
```


For example: merge

$35P = 2(2(2(2(2P))) + P)$ +

$31Q = 2(2(2(2Q{+}Q){+}Q){+}$

into $35P + 31Q =$

$2(2(2(2(2P{+}Q){+}Q){+}Q){+}$

$\quad {+}P{+}Q$.

$\approx b$ doublings (merged!),

$\approx b/2$ additions of P,

$\approx b/2$ additions of Q.

Combine idea with windows:

≈ 256 doublings for $b = 256$

≈ 50 additions using P,

≈ 50 additions using Q.

Joint doublings

Do much better by merging $2X + 2Y$ into $2(X + Y)$.

```
def scalarmult2(m,P,n,Q):
  if m == 0:
    return scalarmult(n,Q)
  if n == 0:
    return scalarmult(m,P)
  R = scalarmult2(m//2,P,n//2,Q)
  R = R + R
  if m % 2: R = R + P
  if n % 2: R = R + Q
  return R
```


For example: merge

$35P = 2(2(2(2(2P))) + P) + P$,

$31Q = 2(2(2(2Q+Q)+Q)+Q)+Q$

into $35P + 31Q =$

$2(2(2(2(2P+Q)+Q)+Q)+P+Q)$
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$\approx b$ doublings (merged!),

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Combine idea with windows: e.g.,

≈ 256 doublings for $b = 256$,

≈ 50 additions using P,

≈ 50 additions using Q.

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Y into $2(X+Y)$.

```
larmult2(m,P,n,Q):
== 0:
urn scalarmult(n,Q)
== 0:
urn scalarmult(m,P)
calarmult2(m//2,P,n//2,Q)
 + R
% 2: R = R + P
% 2: R = R + Q
n R
```


For example: merge
$35P = 2(2(2(2(2P))) + P) + P$,
$31Q = 2(2(2(2Q{+}Q){+}Q){+}Q){+}Q$
into $35P + 31Q =$
$2(2(2(2(2P{+}Q){+}Q){+}Q){+}P{+}Q)$
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$\approx$256 doublings for $b = 256$,
$\approx$50 additions using P,
$\approx$50 additions using Q.

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Verifying
need to
$S_1 B =$
$S_2 B =$
$S_3 B =$
etc.

Obvious
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```
m,P,n,Q):

rmult(n,Q)

rmult(m,P)

2(m//2,P,n//2,Q)

R + P

R + Q
```


For example: merge

$35P = 2(2(2(2(2P))) + P) + P$,

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$\approx b$ doublings (merged!),

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$\approx b/2$ additions of Q.

Combine idea with windows: e.g.,

≈ 256 doublings for $b = 256$,

≈ 50 additions using P,

≈ 50 additions using Q.

Batch verification

Verifying many sig

need to be confide

$S_1 B = R_1 + h_1 A_1$

$S_2 B = R_2 + h_2 A_2$

$S_3 B = R_3 + h_3 A_3$

etc.

Obvious approach:

Check each equati

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n//2,Q)

For example: merge

$35P = 2(2(2(2(2P))) + P) + P$,

$31Q = 2(2(2(2Q+Q)+Q)+Q)+Q$

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$\approx b$ doublings (merged!),

$\approx b/2$ additions of P,

$\approx b/2$ additions of Q.

Combine idea with windows: e.g.,

≈ 256 doublings for $b = 256$,

≈ 50 additions using P,

≈ 50 additions using Q.

Batch verification

Verifying many signatures:

need to be confident that

$S_1 B = R_1 + h_1 A_1$,

$S_2 B = R_2 + h_2 A_2$,

$S_3 B = R_3 + h_3 A_3$,

etc.

Obvious approach:

Check each equation separat

For example: merge

$35P = 2(2(2(2(2P))) + P) + P$,

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$2(2(2(2(2P{+}Q){+}Q){+}Q){+}P{+}Q)$
$\quad {+}P{+}Q$.

$\approx b$ doublings (merged!),

$\approx b/2$ additions of P,

$\approx b/2$ additions of Q.

Combine idea with windows: e.g.,

≈ 256 doublings for $b = 256$,

≈ 50 additions using P,

≈ 50 additions using Q.

Batch verification

Verifying many signatures:

need to be confident that

$S_1 B = R_1 + h_1 A_1$,

$S_2 B = R_2 + h_2 A_2$,

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etc.

Obvious approach:

Check each equation separately.

For example: merge

$35P = 2(2(2(2(2P))) + P) + P$,

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$\approx b$ doublings (merged!),

$\approx b/2$ additions of P,

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Combine idea with windows: e.g.,

≈ 256 doublings for $b = 256$,

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Batch verification

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need to be confident that

$S_1 B = R_1 + h_1 A_1$,

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Much faster approach:
Check random linear combination
of the equations.

mple: merge

$2(2(2(2(2P))) + P) + P,$

$2(2(2(2Q{+}Q){+}Q){+}Q){+}Q$

$P + 31Q =$

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Batch verification

Verifying many signatures:
need to be confident that
$S_1B = R_1 + h_1A_1,$
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Batch verification

Verifying many signatures:
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etc.

Obvious approach:
Check each equation separately.

Much faster approach:
Check random linear combination
of the equations.

Pick independent
128-bit $z_1, z_2, z_3, .$

Check whether
$(z_1 S_1 + z_2 S_2 + z_3$
$z_1 R_1 + (z_1 h_1) A_1 -$
$z_2 R_2 + (z_2 h_2) A_2 -$
$z_3 R_3 + (z_3 h_3) A_3 -$

(If $\neq$: See 2012 B
Doumen–Lange–O

Easy to prove:
forgeries have prob
of fooling this che

$+ P$,

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$P+Q)$

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Batch verification

Verifying many signatures:
need to be confident that
$S_1 B = R_1 + h_1 A_1$,
$S_2 B = R_2 + h_2 A_2$,
$S_3 B = R_3 + h_3 A_3$,
etc.

Obvious approach:
Check each equation separately.

Much faster approach:
Check random linear combination of the equations.

Pick independent uniform ra
128-bit $z_1, z_2, z_3, \ldots$.

Check whether
$(z_1 S_1 + z_2 S_2 + z_3 S_3 + \cdots)B$
$z_1 R_1 + (z_1 h_1) A_1 +$
$z_2 R_2 + (z_2 h_2) A_2 +$
$z_3 R_3 + (z_3 h_3) A_3 + \cdots$.

(If $\neq$: See 2012 Bernstein–
Doumen–Lange–Oosterwijk.

Easy to prove:
forgeries have probability ≤ 2
of fooling this check.

Batch verification

Verifying many signatures:
need to be confident that
$S_1 B = R_1 + h_1 A_1$,
$S_2 B = R_2 + h_2 A_2$,
$S_3 B = R_3 + h_3 A_3$,
etc.

Obvious approach:
Check each equation separately.

Much faster approach:
Check random linear combination of the equations.

Pick independent uniform random 128-bit $z_1, z_2, z_3, \ldots$.

Check whether
$(z_1 S_1 + z_2 S_2 + z_3 S_3 + \cdots)B =$
$z_1 R_1 + (z_1 h_1)A_1 +$
$z_2 R_2 + (z_2 h_2)A_2 +$
$z_3 R_3 + (z_3 h_3)A_3 + \cdots$.

(If $\neq$: See 2012 Bernstein–Doumen–Lange–Oosterwijk.)

Easy to prove:
forgeries have probability $\leq 2^{-128}$ of fooling this check.

erification

g many signatures:
be confident that
$R_1 + h_1 A_1$,
$R_2 + h_2 A_2$,
$R_3 + h_3 A_3$,

approach:
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Pick independent uniform random 128-bit $z_1, z_2, z_3, \ldots$.

Check whether

$$(z_1 S_1 + z_2 S_2 + z_3 S_3 + \cdots)B =$$
$$z_1 R_1 + (z_1 h_1) A_1 +$$
$$z_2 R_2 + (z_2 h_2) A_2 +$$
$$z_3 R_3 + (z_3 h_3) A_3 + \cdots.$$

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Multi-sc

Review o

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$\approx (1 + 1$
additions
$P \mapsto nP$

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Pick independent uniform random 128-bit $z_1, z_2, z_3, \ldots$.

Check whether

$(z_1 S_1 + z_2 S_2 + z_3 S_3 + \cdots)B =$
$z_1 R_1 + (z_1 h_1)A_1 +$
$z_2 R_2 + (z_2 h_2)A_2 +$
$z_3 R_3 + (z_3 h_3)A_3 + \cdots$.

(If $\neq$: See 2012 Bernstein–Doumen–Lange–Oosterwijk.)

Easy to prove:
forgeries have probability $\leq 2^{-128}$ of fooling this check.

Multi-scalar multi

Review of asympto

1939 Brauer (wind

$\approx (1 + 1/\lg b)b$
additions to comp
$P \mapsto nP$ if $n < 2^b$

1964 Straus (joint

$\approx (1 + k/\lg b)b$
additions to comp
$P_1, \ldots, P_k \mapsto n_1 P$
if $n_1, \ldots, n_k < 2^b$.

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Pick independent uniform random 128-bit $z_1, z_2, z_3, \ldots$.

Check whether
$(z_1 S_1 + z_2 S_2 + z_3 S_3 + \cdots)B =$
$z_1 R_1 + (z_1 h_1)A_1 +$
$z_2 R_2 + (z_2 h_2)A_2 +$
$z_3 R_3 + (z_3 h_3)A_3 + \cdots$.

(If $\neq$: See 2012 Bernstein–Doumen–Lange–Oosterwijk.)

Easy to prove:
forgeries have probability $\le 2^{-128}$ of fooling this check.

Multi-scalar multiplication

Review of asymptotic speeds

1939 Brauer (windows):

$\approx (1 + 1/\lg b)b$
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$P \mapsto nP$ if $n < 2^b$.

1964 Straus (joint doublings

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More generally, Pippenger's
algorithm computes
ℓ sums of multiples of k inp

$$\approx \left(\min\{k, \ell\} + \frac{k\ell}{\lg(k\ell b)}\right) b$$

if all coefficients are below 2
Within $1 + \epsilon$ of optimal.

1976 Yao:

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if all coefficients are below 2^b.
Within $1+\epsilon$ of optimal.

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Remarkably simple
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$(n_1 - qn_2)P_1 + n_2(qP_1 + P_2$

$n_3P_3 + \cdots$ where $q = \lfloor n_1/n$

Remarkably simple; competitive with Pippenger for random choices of n_i's; much better memory usage.

More generally, Pippenger's algorithm computes ℓ sums of multiples of k inputs.

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Example of Bos–C

000100000 = 32
000010000 = 16
100101100 = 300
010010010 = 146
001001101 = 77
000000010 = 2
000000001 = 1

Goal: Compute 32
$300P$, $146P$, $77P$,

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$n_1 P_1 + n_2 P_2 + n_3 P_3 + \cdots =$
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$100101100 = 300$
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Goal: Compute $32P$, $16P$, $300P$, $146P$, $77P$, $2P$, $1P$.

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Goal: Compute $32P$, $16P$,
$300P$, $146P$, $77P$, $2P$, $1P$.

39 Bos–Coster:

$n_2 \geq \cdots$ then
$n_2 P_2 + n_3 P_3 + \cdots =$
$n_2) P_1 + n_2(q P_1 + P_2) +$
$\cdots$ where $q = \lfloor n_1/n_2 \rfloor$.

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$000100000 = 32$
$000010000 = 16$
$100101100 = 300$
$010010010 = 146$
$001001101 = 77$
$000000010 = 2$
$000000001 = 1$

Goal: Compute $32P$, $16P$, $300P$, $146P$, $77P$, $2P$, $1P$.

Reduce

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$(qP_1 + P_2) +$

$q = \lfloor n_1/n_2 \rfloor$.

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Example of Bos–Coster:

$000100000 = 32$

$000010000 = 16$

$100101100 = 300$

$010010010 = 146$

$001001101 = 77$

$000000010 = 2$

$000000001 = 1$

Goal: Compute $32P$, $16P$, $300P$, $146P$, $77P$, $2P$, $1P$.

Reduce largest row

$000100000 = 32$

$000010000 = 16$

$010011010 = 154$

$010010010 = 146$

$001001101 = 77$

$000000010 = 2$

$000000001 = 1$

Goal: Compute 32

$154P$, $146P$, $77P$,

Plus one extra add

add $146P$ into 154

obtaining $300P$.

Example of Bos–Coster:

$000100000 = 32$

$000010000 = 16$

$100101100 = 300$

$010010010 = 146$

$001001101 = 77$

$000000010 = 2$

$000000001 = 1$

Goal: Compute $32P$, $16P$, $300P$, $146P$, $77P$, $2P$, $1P$.

Reduce largest row:

$000100000 = 32$

$000010000 = 16$

$010011010 = 154 \leftarrow$

$010010010 = 146$

$001001101 = 77$

$000000010 = 2$

$000000001 = 1$

Goal: Compute $32P$, $16P$, $154P$, $146P$, $77P$, $2P$, $1P$. Plus one extra addition: add $146P$ into $154P$, obtaining $300P$.

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$000100000 = 32$

$000010000 = 16$

$100101100 = 300$

$010010010 = 146$

$001001101 = 77$

$000000010 = 2$

$000000001 = 1$

Goal: Compute $32P$, $16P$, $300P$, $146P$, $77P$, $2P$, $1P$.

Reduce largest row:

$000100000 = 32$

$000010000 = 16$

$010011010 = 154 \leftarrow$

$010010010 = 146$

$001001101 = 77$

$000000010 = 2$

$000000001 = 1$

Goal: Compute $32P$, $16P$, $154P$, $146P$, $77P$, $2P$, $1P$.
Plus one extra addition: add $146P$ into $154P$, obtaining $300P$.

Example of Bos–Coster:

$000100000 = 32$

$000010000 = 16$

$100101100 = 300$

$010010010 = 146$

$001001101 = 77$

$000000010 = 2$

$000000001 = 1$

Goal: Compute $32P$, $16P$, $300P$, $146P$, $77P$, $2P$, $1P$.

Reduce largest row:

$000100000 = 32$

$000010000 = 16$

$000001000 = 8 \leftarrow$

$010010010 = 146$

$001001101 = 77$

$000000010 = 2$

$000000001 = 1$

plus 2 additions.

Example of Bos–Coster:

$000100000 = 32$

$000010000 = 16$

$100101100 = 300$

$010010010 = 146$

$001001101 = 77$

$000000010 = 2$

$000000001 = 1$

Goal: Compute $32P$, $16P$, $300P$, $146P$, $77P$, $2P$, $1P$.

Reduce largest row:

$000100000 = 32$

$000010000 = 16$

$000001000 = 8$

$001000101 = 69 \leftarrow$

$001001101 = 77$

$000000010 = 2$

$000000001 = 1$

plus 3 additions.

Example of Bos–Coster:

$000100000 = 32$

$000010000 = 16$

$100101100 = 300$

$010010010 = 146$

$001001101 = 77$

$000000010 = 2$

$000000001 = 1$

Goal: Compute $32P$, $16P$, $300P$, $146P$, $77P$, $2P$, $1P$.

Reduce largest row:

$000100000 = 32$

$000010000 = 16$

$000001000 = 8$

$001000101 = 69$

$000001000 = 8 \leftarrow$

$000000010 = 2$

$000000001 = 1$

plus 4 additions.

Example of Bos–Coster:

$000100000 = 32$

$000010000 = 16$

$100101100 = 300$

$010010010 = 146$

$001001101 = 77$

$000000010 = 2$

$000000001 = 1$

Goal: Compute $32P$, $16P$, $300P$, $146P$, $77P$, $2P$, $1P$.

Reduce largest row:

$000100000 = 32$

$000010000 = 16$

$000001000 = 8$

$000100101 = 37 \leftarrow$

$000001000 = 8$

$000000010 = 2$

$000000001 = 1$

plus 5 additions.

Example of Bos–Coster:

$000100000 = 32$

$000010000 = 16$

$100101100 = 300$

$010010010 = 146$

$001001101 = 77$

$000000010 = 2$

$000000001 = 1$

Goal: Compute $32P$, $16P$, $300P$, $146P$, $77P$, $2P$, $1P$.

Reduce largest row:

$000100000 = 32$

$000010000 = 16$

$000001000 = 8$

$000000101 = 5 \leftarrow$

$000001000 = 8$

$000000010 = 2$

$000000001 = 1$

plus 6 additions.

Example of Bos–Coster:

$000100000 = 32$

$000010000 = 16$

$100101100 = 300$

$010010010 = 146$

$001001101 = 77$

$000000010 = 2$

$000000001 = 1$

Goal: Compute $32P$, $16P$, $300P$, $146P$, $77P$, $2P$, $1P$.

Reduce largest row:

$000010000 = 16 \leftarrow$

$000010000 = 16$

$000001000 = 8$

$000000101 = 5$

$000001000 = 8$

$000000010 = 2$

$000000001 = 1$

plus 7 additions.

Example of Bos–Coster:

$000100000 = 32$

$000010000 = 16$

$100101100 = 300$

$010010010 = 146$

$001001101 = 77$

$000000010 = 2$

$000000001 = 1$

Goal: Compute $32P$, $16P$, $300P$, $146P$, $77P$, $2P$, $1P$.

Reduce largest row:

$000000000 = 0$

$000010000 = 16$

$000001000 = 8$

$000000101 = 5$

$000001000 = 8$

$000000010 = 2$

$000000001 = 1$

plus 7 additions.

Example of Bos–Coster:

$000100000 = 32$

$000010000 = 16$

$100101100 = 300$

$010010010 = 146$

$001001101 = 77$

$000000010 = 2$

$000000001 = 1$

Goal: Compute $32P$, $16P$, $300P$, $146P$, $77P$, $2P$, $1P$.

Reduce largest row:

$000000000 = 0$

$000001000 = 8 \leftarrow$

$000001000 = 8$

$000000101 = 5$

$000001000 = 8$

$000000010 = 2$

$000000001 = 1$

plus 8 additions.

Example of Bos–Coster:

$000100000 = 32$

$000010000 = 16$

$100101100 = 300$

$010010010 = 146$

$001001101 = 77$

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$000000000 = 0$

$000000000 = 0 \leftarrow$

$000001000 = 8$

$000000101 = 5$

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$000000010 = 2$

$000000001 = 1$

plus 8 additions.

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Reduce largest row:

$000000000 = 0$

$000000000 = 0$

$000000000 = 0 \leftarrow$

$000000101 = 5$

$000001000 = 8$

$000000010 = 2$

$000000001 = 1$

plus 8 additions.

Example of Bos–Coster:

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$000010000 = 16$

$100101100 = 300$

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$000000010 = 2$

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Reduce largest row:

$000000000 = 0$

$000000000 = 0$

$000000000 = 0$

$000000101 = 5$

$000000011 = 3 \leftarrow$

$000000010 = 2$

$000000001 = 1$

plus 9 additions.

Example of Bos–Coster:

$000100000 = 32$

$000010000 = 16$

$100101100 = 300$

$010010010 = 146$

$001001101 = 77$

$000000010 = 2$

$000000001 = 1$

Goal: Compute $32P$, $16P$, $300P$, $146P$, $77P$, $2P$, $1P$.

Reduce largest row:

$000000000 = 0$

$000000000 = 0$

$000000000 = 0$

$000000010 = 2 \leftarrow$

$000000011 = 3$

$000000010 = 2$

$000000001 = 1$

plus 10 additions.

Example of Bos–Coster:

$000100000 = 32$

$000010000 = 16$

$100101100 = 300$

$010010010 = 146$

$001001101 = 77$

$000000010 = 2$

$000000001 = 1$

Goal: Compute $32P$, $16P$, $300P$, $146P$, $77P$, $2P$, $1P$.

Reduce largest row:

$000000000 = 0$

$000000000 = 0$

$000000000 = 0$

$000000010 = 2$

$000000001 = 1 \leftarrow$

$000000010 = 2$

$000000001 = 1$

plus 11 additions.

Example of Bos–Coster:

$000100000 = 32$

$000010000 = 16$

$100101100 = 300$

$010010010 = 146$

$001001101 = 77$

$000000010 = 2$

$000000001 = 1$

Goal: Compute $32P$, $16P$, $300P$, $146P$, $77P$, $2P$, $1P$.

Reduce largest row:

$000000000 = 0$

$000000000 = 0$

$000000000 = 0$

$000000000 = 0 \leftarrow$

$000000001 = 1$

$000000010 = 2$

$000000001 = 1$

plus 11 additions.

Example of Bos–Coster:

$000100000 = 32$

$000010000 = 16$

$100101100 = 300$

$010010010 = 146$

$001001101 = 77$

$000000010 = 2$

$000000001 = 1$

Goal: Compute $32P$, $16P$, $300P$, $146P$, $77P$, $2P$, $1P$.

Reduce largest row:

$000000000 = 0$

$000000000 = 0$

$000000000 = 0$

$000000000 = 0$

$000000001 = 1$

$000000001 = 1 \leftarrow$

$000000001 = 1$

plus 12 additions.

Example of Bos–Coster:

$000100000 = 32$

$000010000 = 16$

$100101100 = 300$

$010010010 = 146$

$001001101 = 77$

$000000010 = 2$

$000000001 = 1$

Goal: Compute $32P$, $16P$, $300P$, $146P$, $77P$, $2P$, $1P$.

Reduce largest row:

$000000000 = 0$

$000000000 = 0$

$000000000 = 0$

$000000000 = 0$

$000000000 = 0 \leftarrow$

$000000001 = 1$

$000000001 = 1$

plus 12 additions.

Example of Bos–Coster:

$000100000 = 32$

$000010000 = 16$

$100101100 = 300$

$010010010 = 146$

$001001101 = 77$

$000000010 = 2$

$000000001 = 1$

Goal: Compute $32P$, $16P$, $300P$, $146P$, $77P$, $2P$, $1P$.

Reduce largest row:

$000000000 = 0$

$000000000 = 0$

$000000000 = 0$

$000000000 = 0$

$000000000 = 0$

$000000000 = 0 \leftarrow$

$000000001 = 1$

plus 12 additions.

Example of Bos–Coster:

$000100000 = 32$

$000010000 = 16$

$100101100 = 300$

$010010010 = 146$

$001001101 = 77$

$000000010 = 2$

$000000001 = 1$

Goal: Compute $32P$, $16P$, $300P$, $146P$, $77P$, $2P$, $1P$.

Reduce largest row:

$000000000 = 0$

$000000000 = 0$

$000000000 = 0$

$000000000 = 0$

$000000000 = 0$

$000000000 = 0$

$000000000 = 0 \leftarrow$

plus 12 additions.

Final addition chain: 1, 2, 3, 5, 8, 16, 32, 37, 69, 77, 146, 154, 300.

Short, no temporary storage, low two-operand complexity.

e of Bos–Coster:

$00 = 32$

$00 = 16$

$00 = 300$

$10 = 146$

$01 = 77$

$10 = 2$

$01 = 1$

ompute $32P$, $16P$,

$46P$, $77P$, $2P$, $1P$.

Reduce largest row:

$000000000 = 0$

$000000000 = 0$

$000000000 = 0$

$000000000 = 0$

$000000000 = 0$

$000000000 = 0$

$000000000 = 0 \leftarrow$

plus 12 additions.

Final addition chain: 1, 2, 3, 5, 8, 16, 32, 37, 69, 77, 146, 154, 300.

Short, no temporary storage, low two-operand complexity.

Revised

$32P_1 +$

$77P_5 +$

First co

and the

$32P_1 +$

$77P_5 +$

Same sc

Ed2551

verify b

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Coster:

$2P$, $16P$,

$2P$, $1P$.

Reduce largest row:

$000000000 = 0$

$000000000 = 0$

$000000000 = 0$

$000000000 = 0$

$000000000 = 0$

$000000000 = 0$

$000000000 = 0 \leftarrow$

plus 12 additions.

Final addition chain: 1, 2, 3, 5, 8, 16, 32, 37, 69, 77, 146, 154, 300.

Short, no temporary storage, low two-operand complexity.

Revised goal: Com

$32P_1 + 16P_2 + 30$

$77P_5 + 2P_6 + 1P_7$

First compute $P'_4 =$

and then recursive

$32P_1 + 16P_2 + 15$

$77P_5 + 2P_6 + 1P_7$

Same scalars show

Ed25519 batch ve

verify batch of 64

about twice as fas

verifying each sepa

Reduce largest row:

000000000 = 0
000000000 = 0
000000000 = 0
000000000 = 0
000000000 = 0
000000000 = 0
000000000 = 0 ←

plus 12 additions.

Final addition chain: 1, 2, 3, 5, 8, 16, 32, 37, 69, 77, 146, 154, 300.

Short, no temporary storage, low two-operand complexity.

Revised goal: Compute
$32P_1 + 16P_2 + 300P_3 + 146$
$77P_5 + 2P_6 + 1P_7$.

First compute $P_4' = P_4 + P_3$
and then recursively comput
$32P_1 + 16P_2 + 154P_3 + 146$
$77P_5 + 2P_6 + 1P_7$.

Same scalars show up as bef

Ed25519 batch verification:
verify batch of 64 signatures
about twice as fast as
verifying each separately.

Reduce largest row:

$000000000 = 0$
$000000000 = 0$
$000000000 = 0$
$000000000 = 0$
$000000000 = 0$
$000000000 = 0$
$000000000 = 0 \leftarrow$

plus 12 additions.

Final addition chain: 1, 2, 3, 5, 8, 16, 32, 37, 69, 77, 146, 154, 300.

Short, no temporary storage, low two-operand complexity.

Revised goal: Compute $32P_1 + 16P_2 + 300P_3 + 146P_4 + 77P_5 + 2P_6 + 1P_7$.

First compute $P_4' = P_4 + P_3$ and then recursively compute $32P_1 + 16P_2 + 154P_3 + 146P_4' + 77P_5 + 2P_6 + 1P_7$.

Same scalars show up as before.

Ed25519 batch verification: verify batch of 64 signatures about twice as fast as verifying each separately.