Twisted Hessian curves

cr.yp.to/papers.html#hessian

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1986 Chudnovsky–Chudnovsky,
"Sequences of numbers
generated by addition
in formal groups
and new primality
and factorization tests":

"The crucial problem becomes
the choice of the model
of an algebraic group variety,
where computations mod p
are the least time consuming."

Most important computations:
ADD is $P, Q \mapsto P + Q$.
DBL is $P \mapsto 2P$.

"It is preferable to use models of elliptic curves lying in low-dimensional spaces, for otherwise the number of coordinates and operations is increasing. This limits us . . . to 4 basic models of elliptic curves."

Short Weierstrass:
$y^2 = x^3 + ax + b$.

Jacobi intersection:
$s^2 + c^2 = 1$, $as^2 + d^2 = 1$.

Jacobi quartic: $y^2 = x^4 + 2ax^2 + 1$.

Hessian: $x^3 + y^3 + 1 = 3dxy$.

"Our experience shows that the expression of the law of addition on the cubic Hessian form (d) of an elliptic curve is by far the best and the prettiest."

$$X_3 = Y_1 X_2 \cdot Y_1 Z_2 - Z_1 Y_2 \cdot X_1 Y_2,$$
$$Y_3 = X_1 Z_2 \cdot X_1 Y_2 - Y_1 X_2 \cdot Z_1 X_2,$$
$$Z_3 = Z_1 Y_2 \cdot Z_1 X_2 - X_1 Z_2 \cdot Y_1 Z_2.$$

12**M** for ADD, where **M** is the cost of multiplication in the field.

8.4**M** for DBL, assuming 0.8**M** for the cost of squaring in the field.

1990s: ECC standards instead use short Weierstrass curves in Jacobian coordinates for “the fastest arithmetic”.

15.2**M** for ADD, much slower than Hessian.

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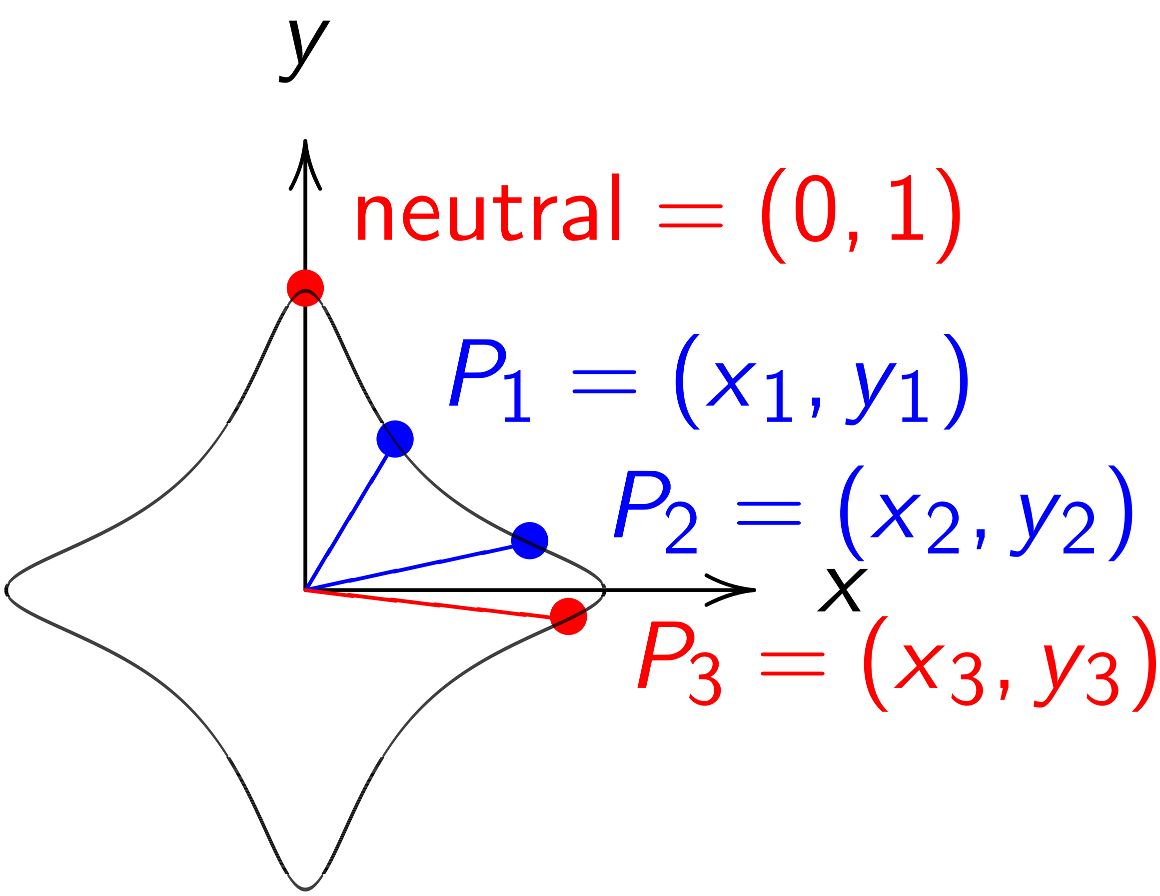
Compared to Hessian, Weierstrass saves 4**M** in typical DBL-DBL-DBL-DBL-DBL-ADD.

2007 Edwards: new curve shape.
2007 Bernstein–Lange: generalize, analyze speed, completeness.



Example: $x^2 + y^2 = 1 - 30x^2y^2$.
Sum of (x_1, y_1) and (x_2, y_2) is
$((x_1y_2+y_1x_2)/(1-30x_1x_2y_1y_2),$
$(y_1y_2-x_1x_2)/(1+30x_1x_2y_1y_2))$.

2007 Bernstein–Lange:

10.8**M** for ADD, 6.2**M** for DBL.

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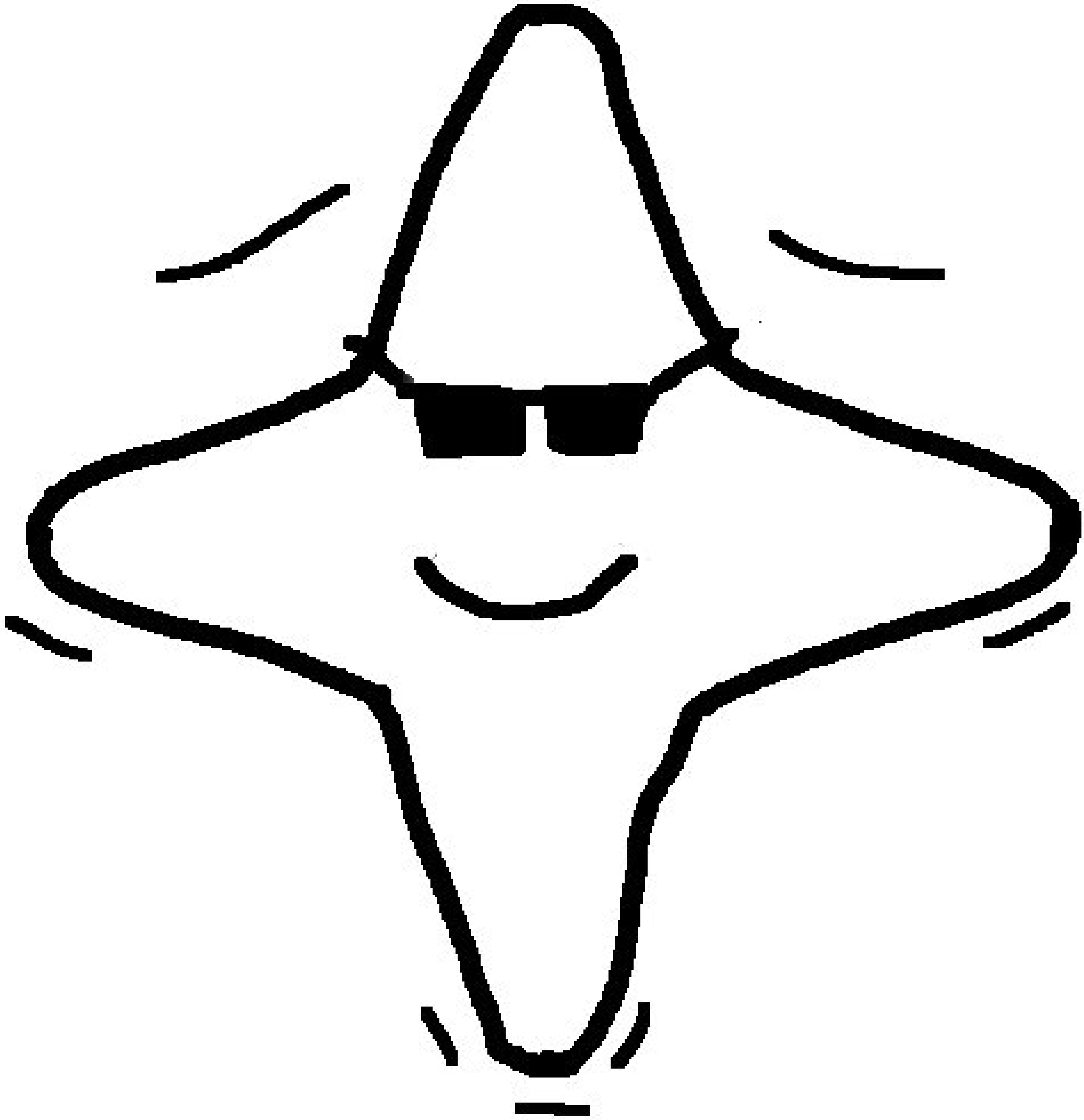
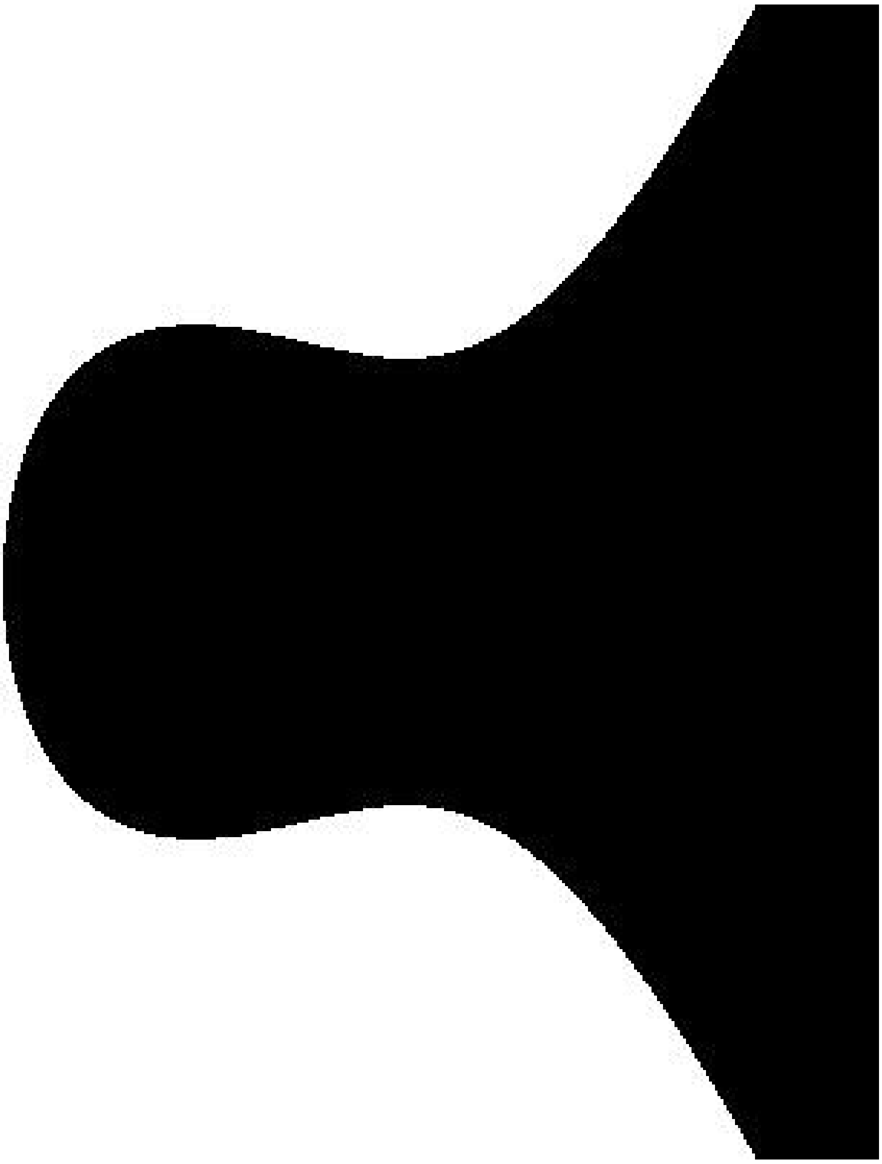
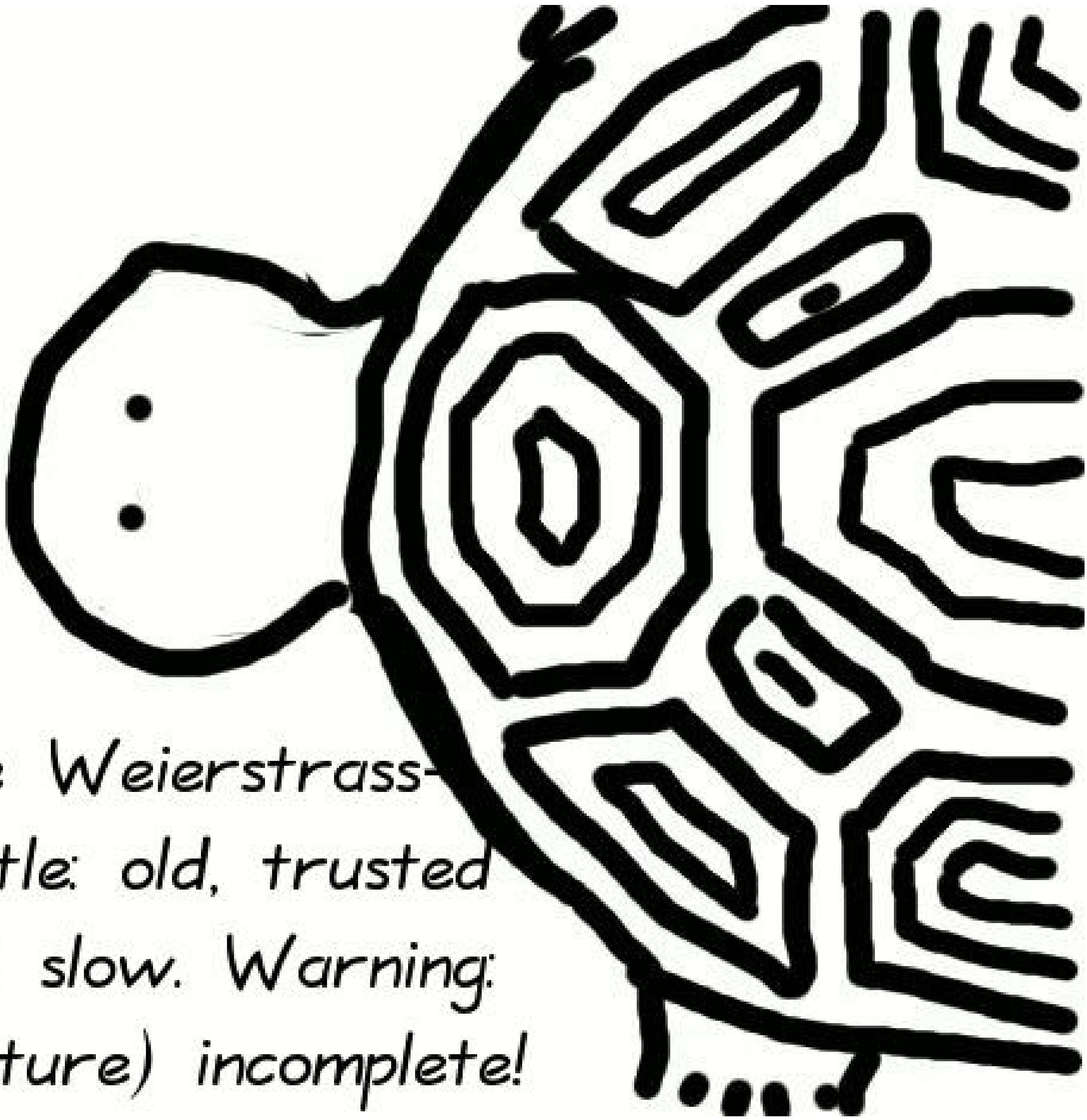
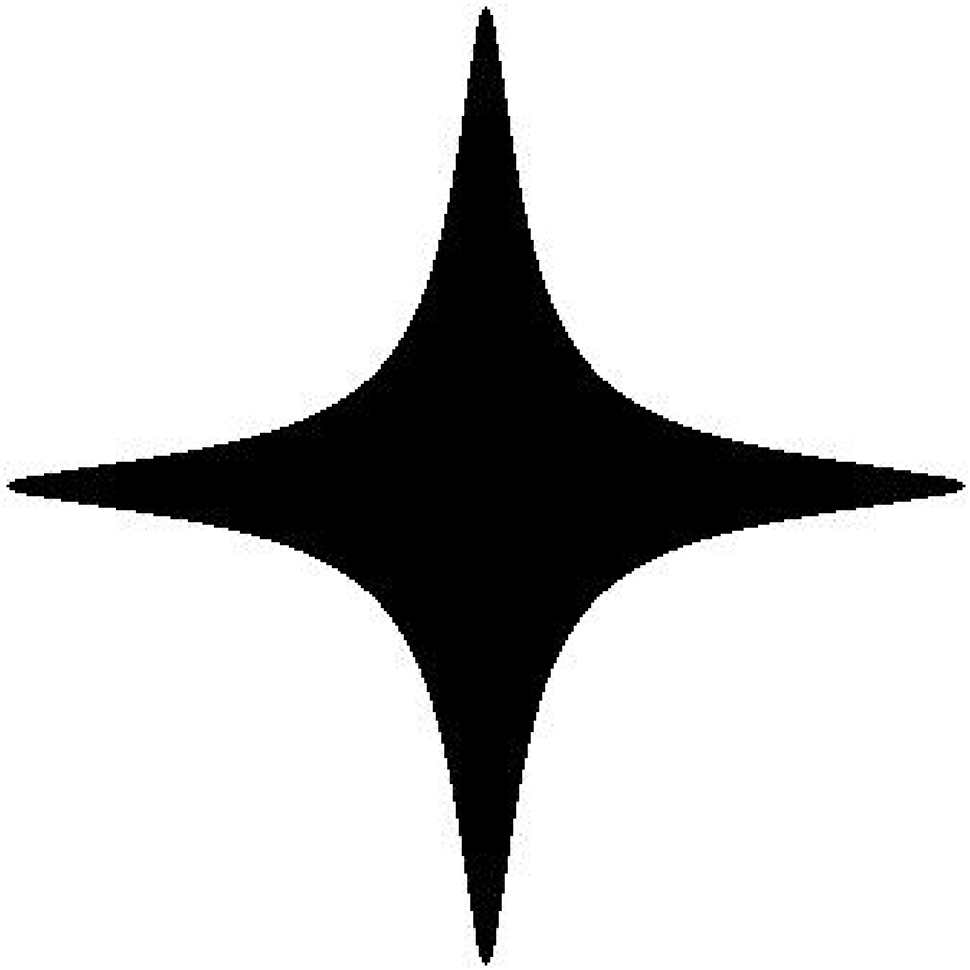




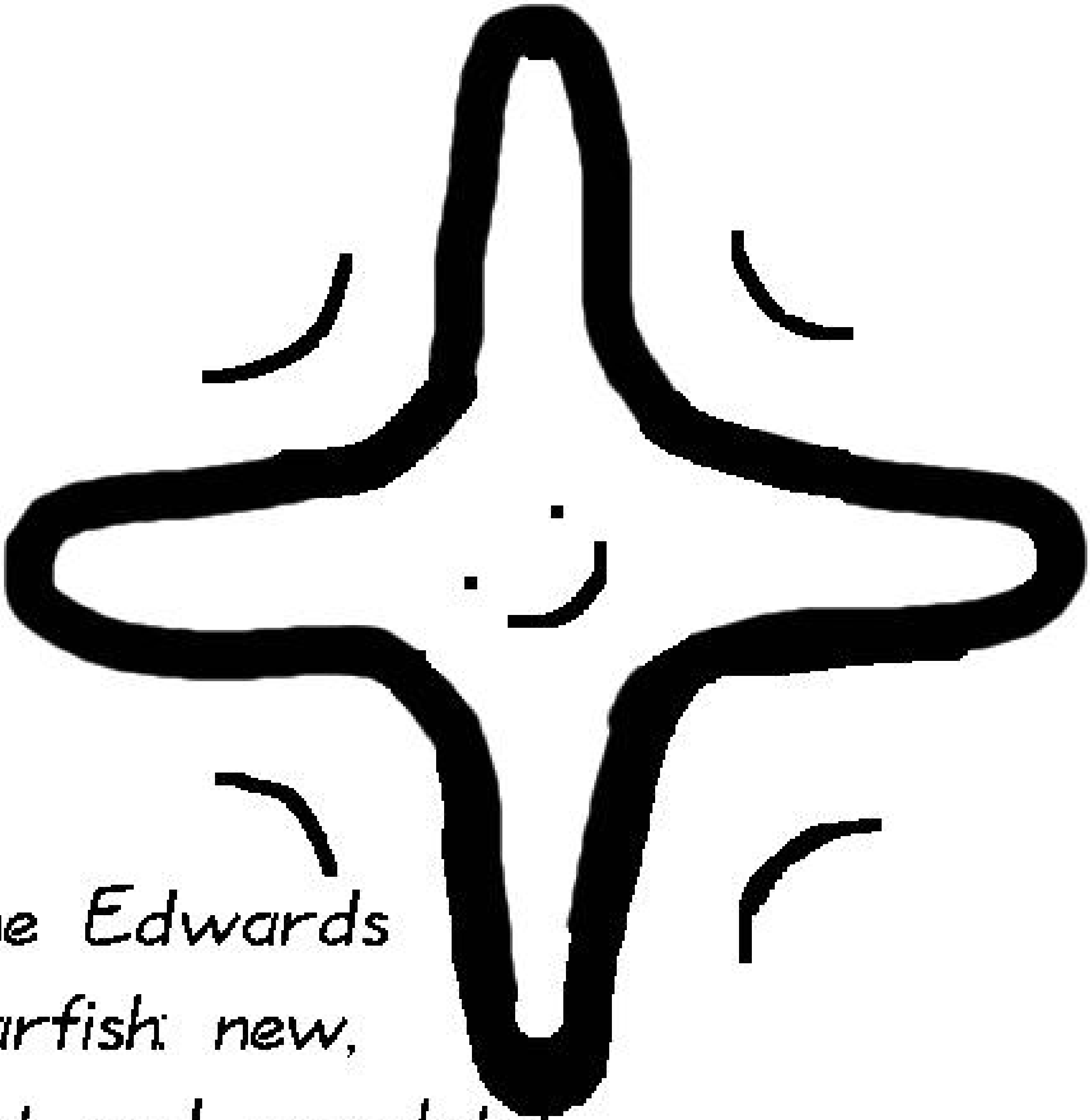
$$y^2 = x^3 - 0.4x + 0.7$$



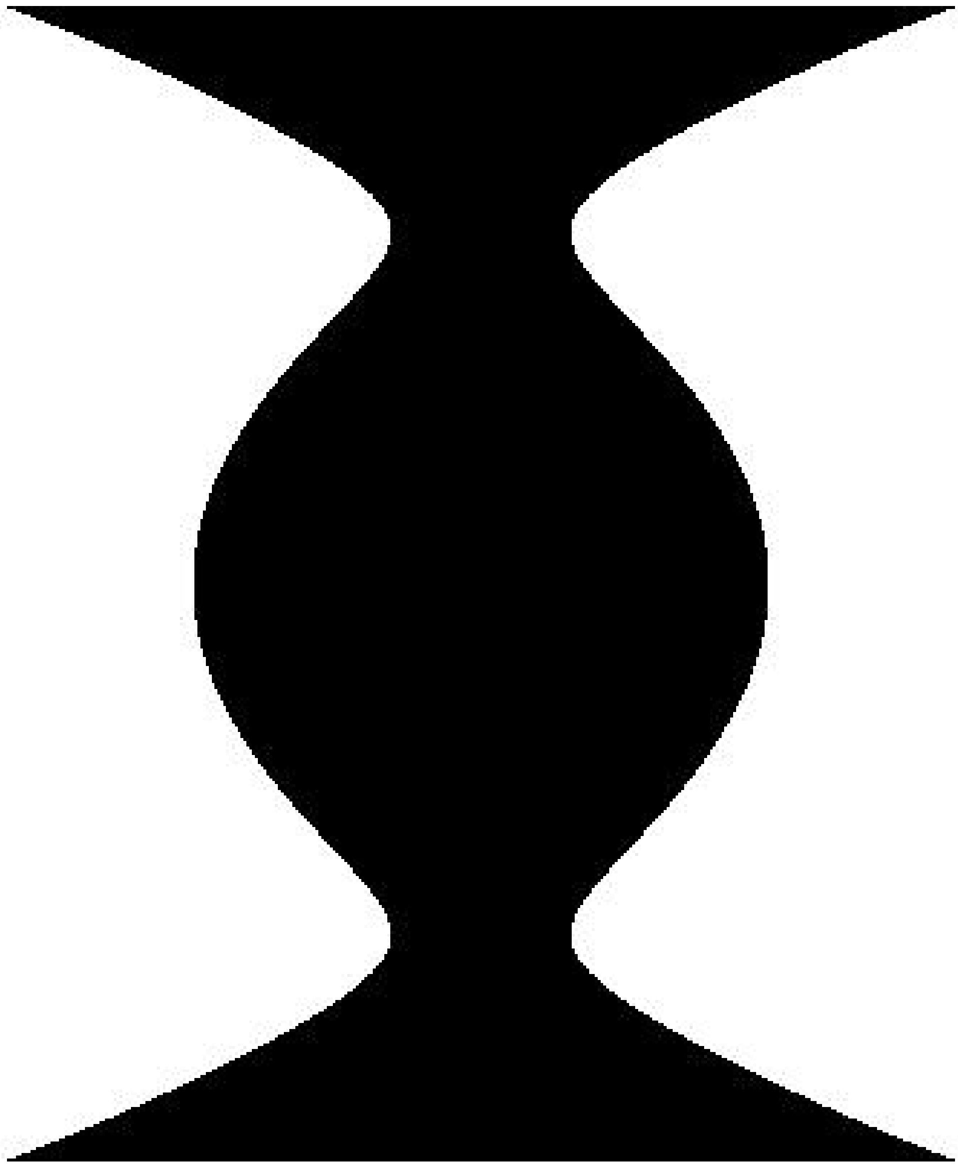
The Weierstrass-turtle: old, trusted and slow. Warning: (picture) incomplete!



$$x^2 + y^2 = 1 - 300x^2y^2$$

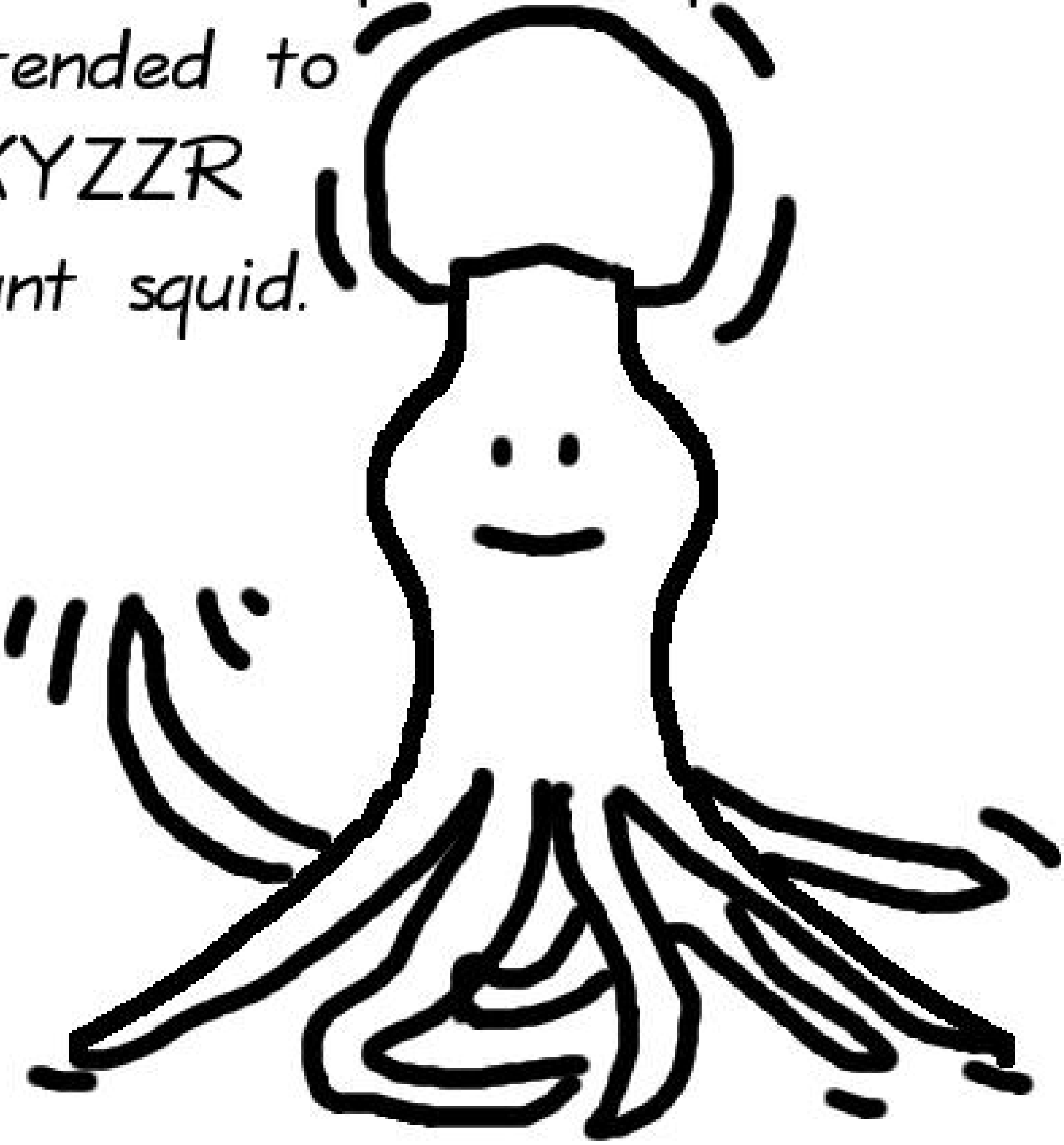


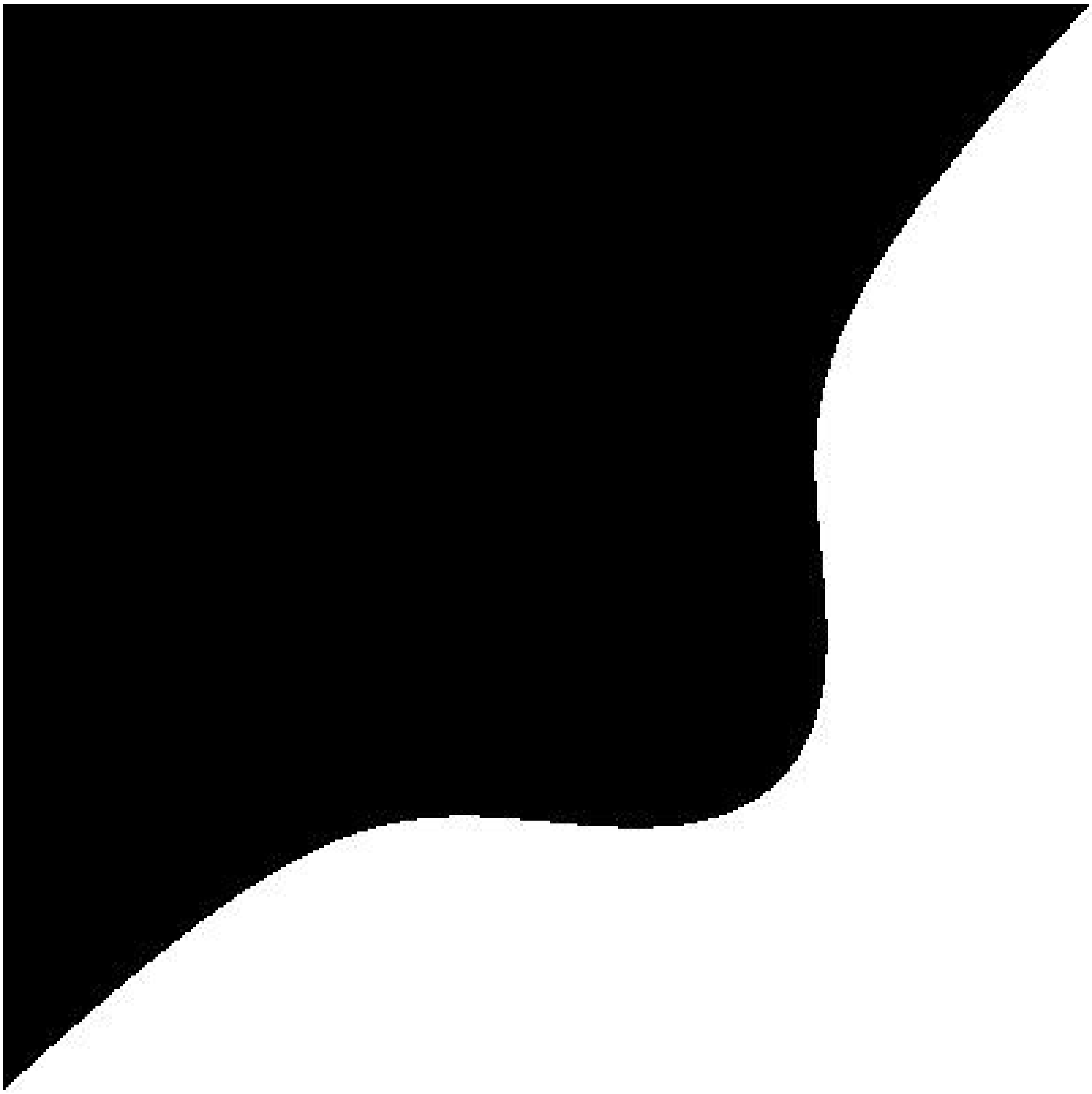
The Edwards
starfish: new,
fast and complete!



$$x^2 = y^4 - 1.9y^2 + 1$$

The Jacobi-quartic squid: can be extended to XXYZZR giant squid.



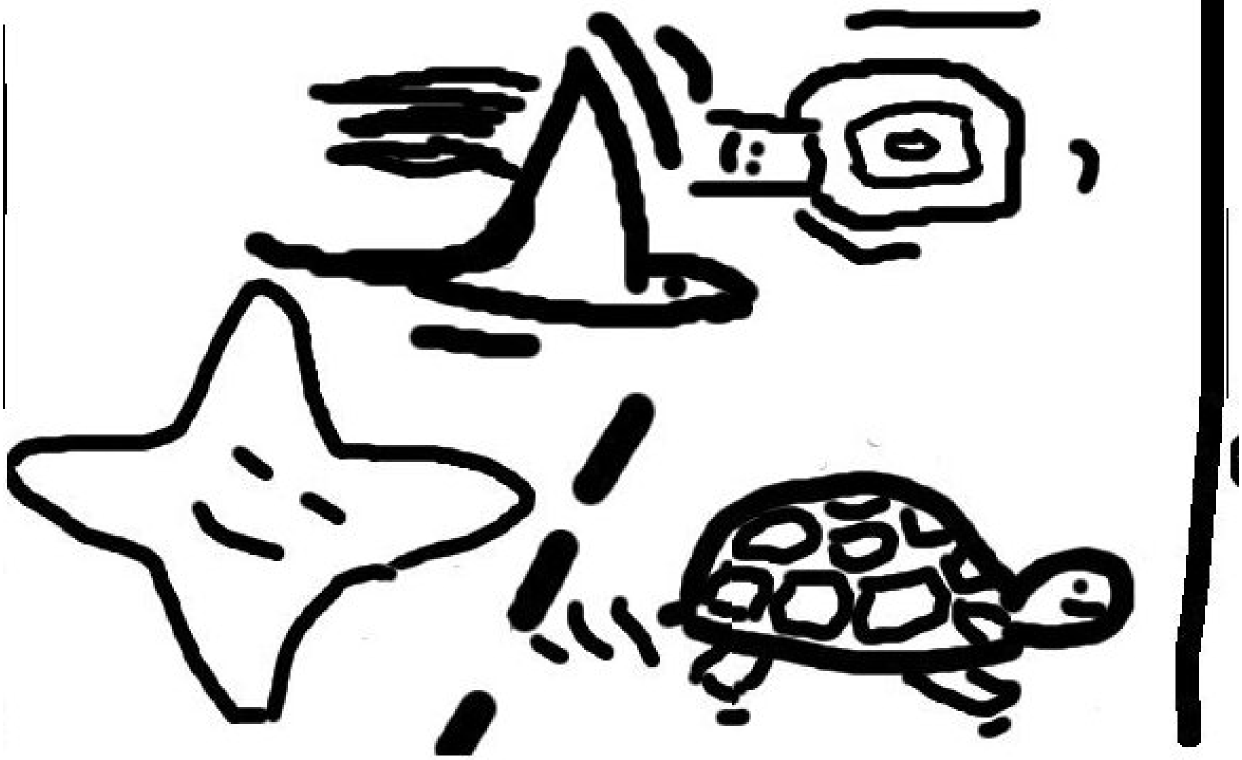


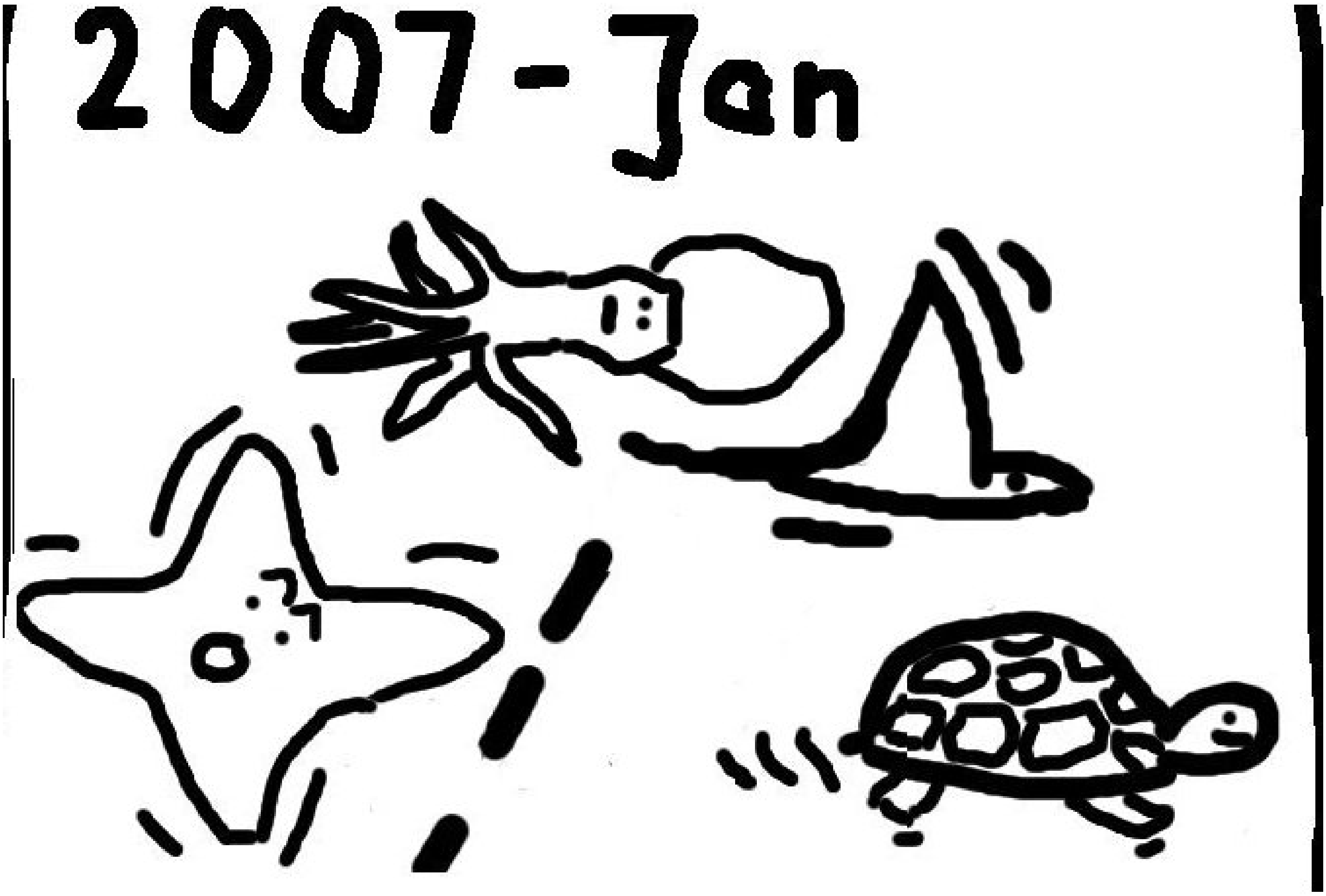
$$x^3 - y^3 + 1 = 0.3xy$$

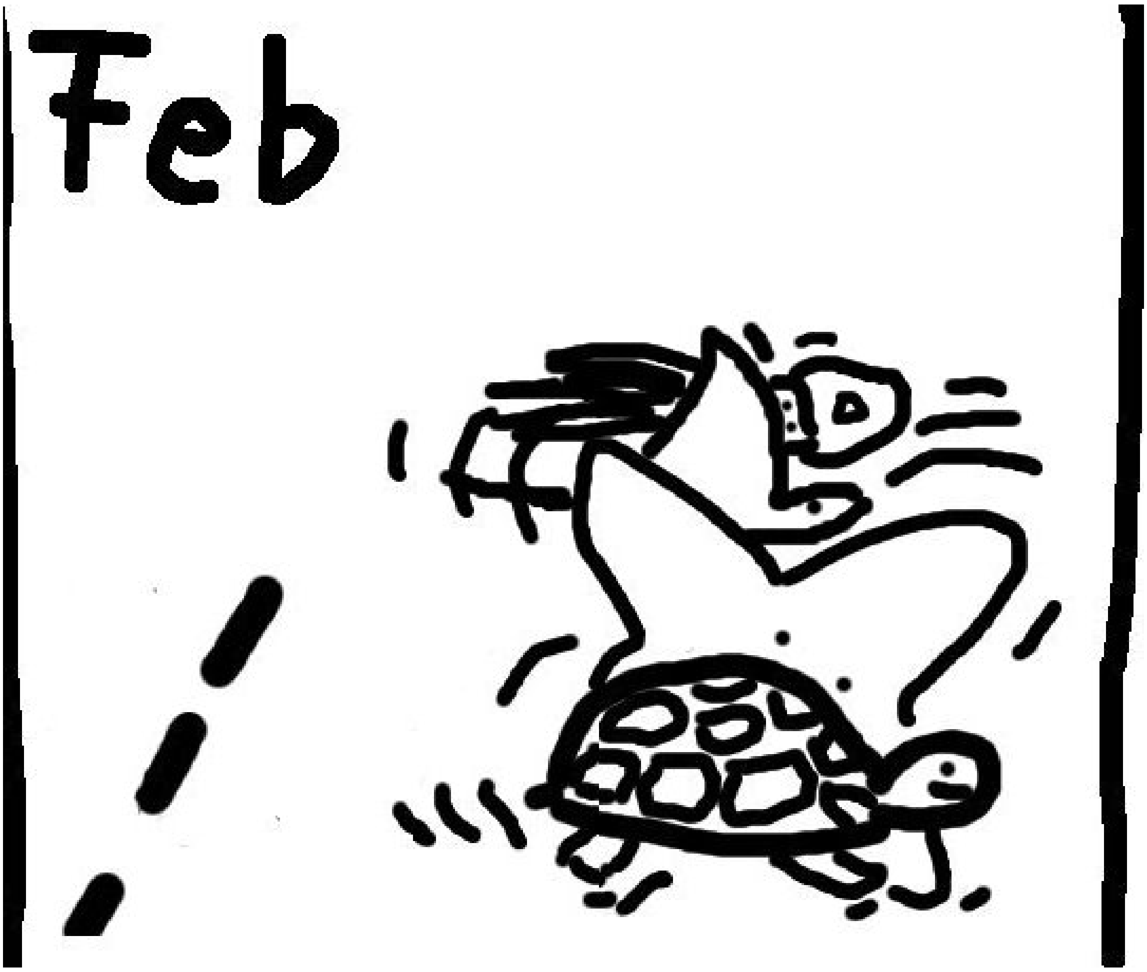

The Hessian-ray: uniform
but
not strongly so


START

1985




2007-Jan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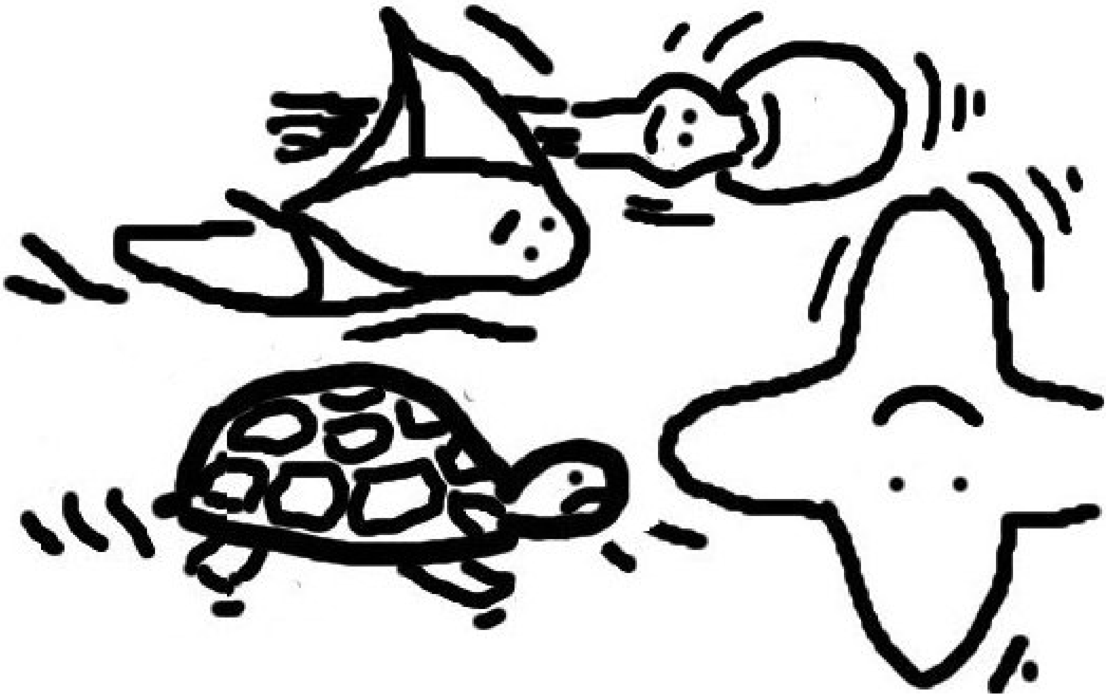
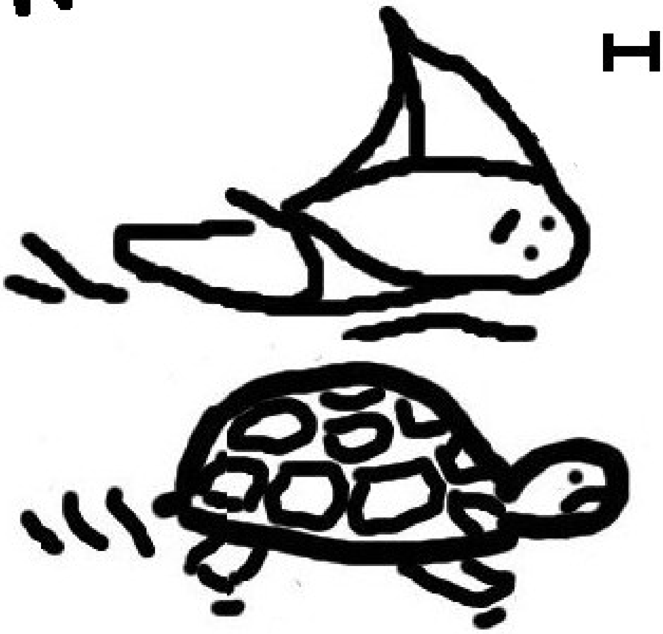


Zoom



Faster Hessian arithmetic

2007 Hisil–Carter–Dawson:
7.8**M** for DBL.

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Hessian tied with Weierstrass for
DBL-DBL-DBL-DBL-DBL-ADD.

Need to zoom in closer:
analyze exact **S**/**M**, overhead
for checking for special cases,
extra DBL, extra ADD, etc.

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Or speed up Hessian more.

New: 7.6**M** for DBL.

New (announced July 2009):

Generalize to more curves:
twisted Hessian curves
$aX^3 + Y^3 + Z^3 = dXYZ$
with $a(27a - d^3) \neq 0$.

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Rotate addition law
so that it also works for DBL;
complete if a is not a cube.
Eliminates special-case overhead,
helps stop side-channel attacks.

Triplings (assuming $d \neq 0$)

TPL is $P \mapsto 3P$.

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12.8**M** for Hessian TPL.

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2015 Kohel: 11.2**M**.

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New: 10.8**M** assuming
field with fast primitive $\sqrt[3]{1}$;
e.g., $\mathbf{F}_q[\omega]/(\omega^2 + \omega + 1)$, or
$\mathbf{F}_p$ with $7p = 2^{298} + 2^{149} + 1$.

(More history in small char.
See paper for details.)

If $aX^3 + Y^3 + Z^3 = dXYZ$
then $VW(V + dU + aW) = U^3$
where
$U = -XYZ, V = Y^3, W = X^3$.

If $VW(V + dU + aW) = U^3$
then $aX_3^3 + Y_3^3 + Z_3^3 = dX_3Y_3Z_3$
where $Q = dU$, $R = aW$,
$S = -(V + Q + R)$,
$dX_3 = R^3 + S^3 + V^3 - 3RSV$,
$Y_3 = RS^2 + SV^2 + VR^2 - 3RSV$,
$Z_3 = RV^2 + SR^2 + VS^2 - 3RSV$.

Compose these 3-isogenies:
$(X_3 : Y_3 : Z_3) = 3(X : Y : Z)$.

To quickly triple $(X : Y : Z)$:

Three cubings for R, S, V.

For three choices of constants (α, β, γ) compute

$$(\alpha R + \beta S + \gamma V) \cdot$$
$$(\alpha S + \beta V + \gamma R) \cdot$$
$$(\alpha V + \beta R + \gamma S)$$
$$= \alpha\beta\gamma d X_3$$
$$+ (\alpha\beta^2 + \beta\gamma^2 + \gamma\alpha^2) Y_3$$
$$+ (\beta\alpha^2 + \gamma\beta^2 + \alpha\gamma^2) Z_3$$
$$+ (\alpha + \beta + \gamma)^3 RSV.$$

Also use $a(R + S + V)^3 = d^3 RSV$.

Solve for dX_3, Y_3, Z_3.

2015 Kohel's 11.2**M**
(4 cubings + 4 mults)
introduced this TPL idea with
$(\alpha, \beta, \gamma) = (1, 1, 1)$,
$(\alpha, \beta, \gamma) = (1, -1, 0)$,
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$(\alpha, \beta, \gamma) = (1, 1, 0)$.

New 10.8**M** (6 cubings)
makes faster choices
assuming fast primitive $\omega = \sqrt[3]{1}$:
$(\alpha, \beta, \gamma) = (1, 1, 1)$,
$(\alpha, \beta, \gamma) = (1, \omega, \omega^2)$,
$(\alpha, \beta, \gamma) = (1, \omega^2, \omega)$.

Are triplings useful?

2005 Dimitrov–Imbert–Mishra "double-base chains": e.g., compute $314159P$ as

$$2^{15}3^2P + 2^{11}3^2P + 2^8 3^1 P + 2^4 3^1 P - 2^0 3^0 P.$$

2TPL, 15DBL, 4ADD.

2006 Doche–Imbert generalized double-base chains: e.g., compute $314159P$ as

$$2^{12}3^3 3P - 2^7 3^3 5P - 2^4 3^1 7P - 2^0 3^0 P$$

after precomputing $3P, 5P, 7P$.

3TPL, 13DBL, 6ADD.

Not good for constant time.
Good for signature verification, factorization, math, etc.

Also need time to compute chain.
Good for scalars used many times.

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Analysis+optimization from 2007
Bernstein–Birkner–Lange–Peters:

Double-base chains speed up
Weierstrass curves slightly:
9.29**M**/bit for 256-bit scalars.

More savings for, e.g., Hessian:
9.65**M**/bit. Still not competitive.

Revisit conclusions
using latest Hessian formulas,
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New: 8.77$\mathbf{M}$/bit for 256 bits.

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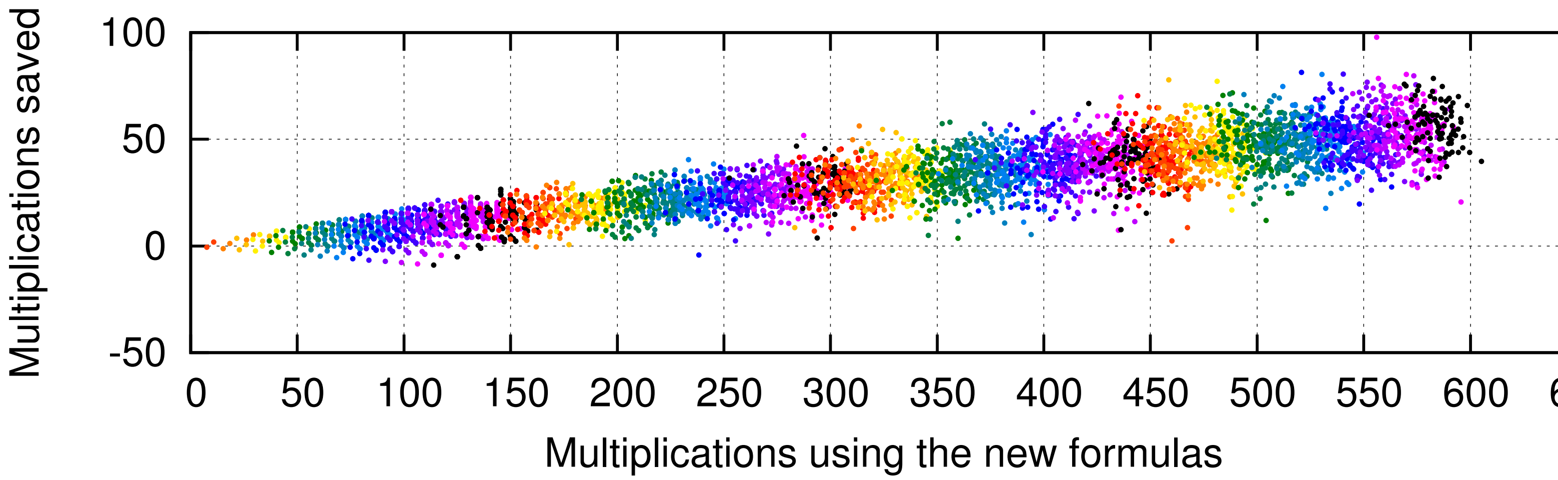
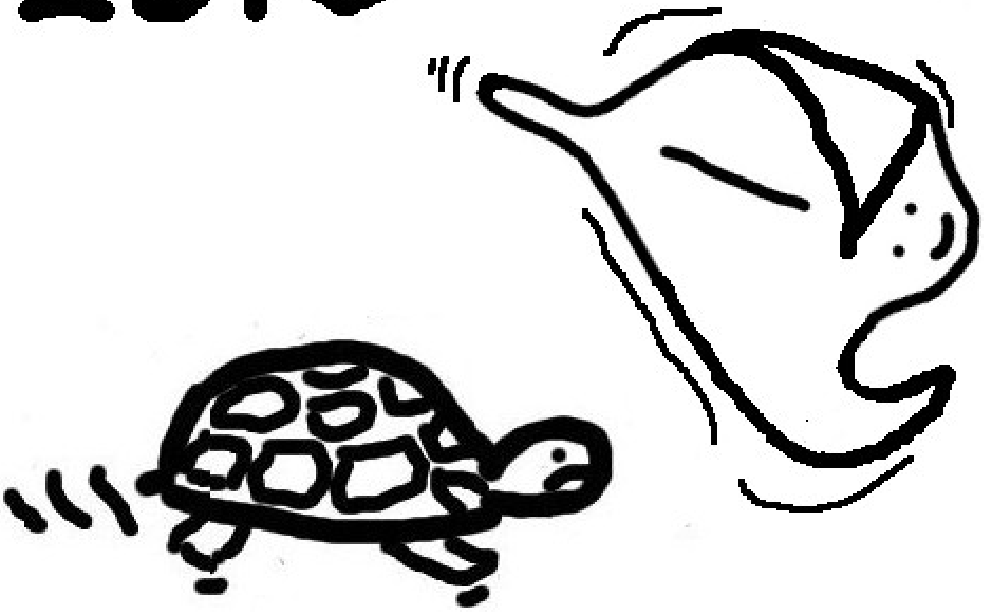
Comparison to Weierstrass for
1-bit, 2-bit, . . . , 64-bit scalars:



Uses 2008 Doche–Habsieger
"tree search" and some new
improvements: e.g., account for
costs of ADD, DBL, TPL.

Mar2015



Summary:
Twisted Hessian curves
solidly beat Weierstrass.

Chuengsatiansup talk tomorrow:
even better double-base chains
from shortest paths in DAG—
and also new Edwards speeds!