Batch NFS

D. J. Bernstein
University of Illinois at Chicago &
Technische Universiteit Eindhoven

Tanja Lange
Technische Universiteit Eindhoven

In this talk $\log L$ means $(1+o(1))(\log N)^{1/3}(\log\log N)^{2/3}$. L is often written "$L_N(1/3)$" or "$L_N(1/3)^{1+o(1)}$".

Exponents of L in this talk are limited to $10^{-6}\mathbf{Z}$.

Rigorously proven? Ha ha ha.

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Best result known for *one* key
time $L^{1.185632}$
using chip area $L^{0.790420}$;
AT is $L^{1.976052}$.

Our main result for
a batch of $L^{0.5}$ keys:
time $L^{1.022400}$
using chip area $L^{1.181600}$;
AT per key is $L^{1.704000}$.

This paper also looks more closely at $L^{o(1)}$, analyzing asymptotic speedup from early-abort ECM. Results are not what one would guess from 1982 Pomerance.

Asymptotic consequences:

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Eratosthenes for s

Sieving small integ

using primes $2, 3,$

1								
2	2							
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Eratosthenes for smoothness

Sieving small integers $i > 0$ using primes $2, 3, 5, 7$:

1								
2	2							
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6	2				3			
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8	2	2	2					
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14	2							7
15					3		5	
16	2	2	2	2				
17								
18	2				3	3		
19								
20	2	2					5	

etc.

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2	2							
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7								7
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9					3	3		
10	2						5	
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12	2	2			3			
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14	2							7
15					3		5	
16	2	2	2	2				
17								
18	2				3	3		
19								
20	2	2					5	

etc.

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7								7
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etc.

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16	2	2	2	2	
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18	2				3
19					
20	2	2			

etc.

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Eratosthenes for smoothness

Sieving small integers $i > 0$ using primes $2, 3, 5, 7$:

1	
2	2
3	3
4	2 2
5	5
6	2 3
7	7
8	2 2 2
9	3 3
10	2 5
11	
12	2 2 3
13	
14	2 7
15	3 5
16	2 2 2 2
17	
18	2 3 3
19	
20	2 2 5

etc.

The **Q** sieve

Sieving i and 611
using primes 2, 3, 5

1	
2	2
3	3
4	2 2
5	5
6	2 3
7	7
8	2 2 2
9	3 3
10	2 5
11	
12	2 2 3
13	
14	2 7
15	3 5
16	2 2 2 2
17	
18	2 3 3
19	
20	2 2 5

612	2
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625	
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628	2
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Eratosthenes for smoothness

Sieving small integers $i > 0$ using primes $2, 3, 5, 7$:

1	
2	2
3	3
4	2 2
5	5
6	2 3
7	7
8	2 2 2
9	3 3
10	2 5
11	
12	2 2 3
13	
14	2 7
15	3 5
16	2 2 2 2
17	
18	2 3 3
19	
20	2 2 5

etc.

The **Q** sieve

Sieving i and $611 + i$ for sm using primes $2, 3, 5, 7$:

1	
2	2
3	3
4	2 2
5	5
6	2 3
7	7
8	2 2 2
9	3 3
10	2 5
11	
12	2 2 3
13	
14	2 7
15	3 5
16	2 2 2 2
17	
18	2 3 3
19	
20	2 2 5

612	2 2 3 3
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614	2
615	3 5
616	2 2 2
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618	2 3
619	
620	2 2 5
621	3 3 3
622	2
623	
624	2 2 2 2 3
625	5
626	2
627	3
628	2 2
629	
630	2 3 3 5
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etc.

Eratosthenes for smoothness

Sieving small integers $i > 0$ using primes $2, 3, 5, 7$:

1	
2	2
3	3
4	2 2
5	5
6	2 3
7	7
8	2 2 2
9	3 3
10	2 5
11	
12	2 2 3
13	
14	2 7
15	3 5
16	2 2 2 2
17	
18	2 3 3
19	
20	2 2 5

etc.

The **Q** sieve

Sieving i and $611 + i$ for small i using primes $2, 3, 5, 7$:

1	
2	2
3	3
4	2 2
5	5
6	2 3
7	7
8	2 2 2
9	3 3
10	2 5
11	
12	2 2 3
13	
14	2 7
15	3 5
16	2 2 2 2
17	
18	2 3 3
19	
20	2 2 5

612	2 2 3 3
613	
614	2
615	3 5
616	2 2 2 7
617	
618	2 3
619	
620	2 2 5
621	3 3 3
622	2
623	7
624	2 2 2 2 3
625	5 5 5 5
626	2
627	3
628	2 2
629	
630	2 3 3 5 7
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etc.

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small integers $i > 0$
imes $2, 3, 5, 7$:

	5	
		7
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	5	
		7
	5	
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	5	

The **Q** sieve

Sieving i and $611 + i$ for small i
using primes $2, 3, 5, 7$:

1								
2	2							
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4	2	2						
5							5	
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7								7
8	2	2	2					
9					3	3		
10	2						5	
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14	2							7
15					3		5	
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18	2				3	3		
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612	2	2			3	3						
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614	2											
615					3			5				
616	2	2	2									7
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626	2											
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630	2				3	3		5				7
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etc.

Have co
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for some

$14 \cdot 625$

$64 \cdot 675$

$75 \cdot 686$

$14 \cdot 64 \cdot$
$= 2^8 3^4 5$

$\gcd\{611$
$= 47.$

$611 = 4$

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gers $i > 0$

5, 7:

The **Q** sieve

Sieving i and $611 + i$ for small i using primes $2, 3, 5, 7$:

1								
2	2							
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6	2				3			
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8	2	2	2					
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14	2							7
15					3		5	
16	2	2	2	2				
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18	2				3	3		
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612	2	2			3	3						
613												
614	2											
615					3			5				
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626	2											
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630	2				3	3		5				7
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etc.

Have complete fac

the congruences i

for some i's.

$14 \cdot 625 = 2^1 3^0 5^4$

$64 \cdot 675 = 2^6 3^3 5^2$

$75 \cdot 686 = 2^1 3^1 5^2$

$14 \cdot 64 \cdot 75 \cdot 625 \cdot$

$= 2^8 3^4 5^8 7^4 = (2^4$

gcd$\{611, 14 \cdot 64 \cdot$

$= 47.$

$611 = 47 \cdot 13.$

s

The **Q** sieve

Sieving i and $611 + i$ for small i using primes $2, 3, 5, 7$:

1				
2	2			
3		3		
4	2 2			
5			5	
6	2	3		
7				7
8	2 2 2			
9		3 3		
10	2		5	
11				
12	2 2	3		
13				
14	2			7
15		3	5	
16	2 2 2 2			
17				
18	2	3 3		
19				
20	2 2		5	

612	2 2	3 3		
613				
614	2			
615		3	5	
616	2 2 2			7
617				
618	2	3		
619				
620	2 2		5	
621		3 3 3		
622	2			
623				7
624	2 2 2 2	3		
625			5 5 5 5	
626	2			
627		3		
628	2 2			
629				
630	2	3 3	5	7
631				

etc.

Have complete factorization

the congruences $i \equiv 611 +$

for some i's.

$14 \cdot 625 = 2^1 3^0 5^4 7^1$.

$64 \cdot 675 = 2^6 3^3 5^2 7^0$.

$75 \cdot 686 = 2^1 3^1 5^2 7^3$.

$14 \cdot 64 \cdot 75 \cdot 625 \cdot 675 \cdot 686$

$= 2^8 3^4 5^8 7^4 = (2^4 3^2 5^4 7^2)^2$.

$\gcd\{611, 14 \cdot 64 \cdot 75 - 2^4 3^2$

$= 47$.

$611 = 47 \cdot 13$.

The $\mathbf{Q}$ sieve

Sieving i and $611 + i$ for small i using primes $2, 3, 5, 7$:

1								
2	2							
3					3			
4	2	2						
5							5	
6	2				3			
7								7
8	2	2	2					
9					3	3		
10	2						5	
11								
12	2	2			3			
13								
14	2							7
15					3		5	
16	2	2	2	2				
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18	2				3	3		
19								
20	2	2					5	

612	2	2			3	3						
613												
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615					3			5				
616	2	2	2									7
617												
618	2				3							
619												
620	2	2						5				
621					3	3	3					
622	2											
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624	2	2	2	2	3							
625								5	5	5	5	
626	2											
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628	2	2										
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630	2				3	3		5				7
631												

etc.

Have complete factorization of the congruences $i \equiv 611 + i$ for some i's.

$14 \cdot 625 = 2^1 3^0 5^4 7^1$.
$64 \cdot 675 = 2^6 3^3 5^2 7^0$.
$75 \cdot 686 = 2^1 3^1 5^2 7^3$.

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$= 2^8 3^4 5^8 7^4 = (2^4 3^2 5^4 7^2)^2$.

$\gcd\{611, 14 \cdot 64 \cdot 75 - 2^4 3^2 5^4 7^2\}$
$= 47$.

$611 = 47 \cdot 13$.

sieve

i and $611+i$ for small i

imes $2, 3, 5, 7$:

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612	2	2			3	3						
613												
614	2											
615					3			5				
616	2	2	2									7
617												
618	2				3							
619												
620	2	2						5				
621					3	3	3					
622	2											
623												7
624	2	2	2	2	3							
625								5	5	5	5	
626	2											
627					3							
628	2	2										
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630	2				3	3		5				7
631												

Have complete factorization of the congruences $i \equiv 611+i$ for some i's.

$14 \cdot 625 = 2^1 3^0 5^4 7^1$.
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$75 \cdot 686 = 2^1 3^1 5^2 7^3$.

$14 \cdot 64 \cdot 75 \cdot 625 \cdot 675 \cdot 686$
$= 2^8 3^4 5^8 7^4 = (2^4 3^2 5^4 7^2)^2$.

$\gcd\{611, 14 \cdot 64 \cdot 75 - 2^4 3^2 5^4 7^2\}$
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$611 = 47 \cdot 13$.

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→ $a-b$
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Optimal
$(\mu + o(1$

$+ i$ for small i

$5, 7$:

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			3				5				
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2											
			3	3			5				7

Have complete factorization of the congruences $i \equiv 611 + i$ for some i's.

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$75 \cdot 686 = 2^1 3^1 5^2 7^3$.

$14 \cdot 64 \cdot 75 \cdot 625 \cdot 675 \cdot 686$
$= 2^8 3^4 5^8 7^4 = (2^4 3^2 5^4 7^2)^2$.

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$= 47$.

$611 = 47 \cdot 13$.

The number-field s

Generalize $i \equiv i +$
$\to a \equiv a + bN \quad ($
$\to a - bm \equiv a - bc$
for root $\alpha \in \mathbf{C}$
of nonzero integer

For any m can fin
so that factoring r
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Optimal choice of
$(\mu + o(1))(\log N)^2$

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Have complete factorization of the congruences $i \equiv 611 + i$ for some i's.

$14 \cdot 625 = 2^1 3^0 5^4 7^1$.
$64 \cdot 675 = 2^6 3^3 5^2 7^0$.
$75 \cdot 686 = 2^1 3^1 5^2 7^3$.

$14 \cdot 64 \cdot 75 \cdot 625 \cdot 675 \cdot 686$
$= 2^8 3^4 5^8 7^4 = (2^4 3^2 5^4 7^2)^2$.

$\gcd\{611, 14 \cdot 64 \cdot 75 - 2^4 3^2 5^4 7^2\}$
$= 47$.

$611 = 47 \cdot 13$.

The number-field sieve

Generalize $i \equiv i + N \quad$ (moc
$\to a \equiv a + bN \quad (\mathrm{mod}\ N)$
$\to a - bm \equiv a - b\alpha \quad$ (mod
for root $\alpha \in \mathbf{C}$
of nonzero integer poly.

For any m can find α
so that factoring $m - \alpha$
produces factorization of N.

Optimal choice of $\log m$ is
$(\mu + o(1))(\log N)^{2/3}(\log \log$

Have complete factorization of the congruences $i \equiv 611 + i$ for some i's.

$14 \cdot 625 = 2^1 3^0 5^4 7^1$.
$64 \cdot 675 = 2^6 3^3 5^2 7^0$.
$75 \cdot 686 = 2^1 3^1 5^2 7^3$.

$14 \cdot 64 \cdot 75 \cdot 625 \cdot 675 \cdot 686$
$= 2^8 3^4 5^8 7^4 = (2^4 3^2 5^4 7^2)^2$.

$\gcd\{611, 14 \cdot 64 \cdot 75 - 2^4 3^2 5^4 7^2\}$
$= 47$.

$611 = 47 \cdot 13$.

The number-field sieve

Generalize $i \equiv i + N \pmod{N}$
$\rightarrow a \equiv a + bN \pmod{N}$
$\rightarrow a - bm \equiv a - b\alpha \pmod{m - \alpha}$
for root $\alpha \in \mathbf{C}$
of nonzero integer poly.

For any m can find α
so that factoring $m - \alpha$
produces factorization of N.

Optimal choice of $\log m$ is
$(\mu + o(1))(\log N)^{2/3}(\log \log N)^{1/3}$.

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gruences $i \equiv 611 + i$
e i's.

$= 2^1 3^0 5^4 7^1$.
$= 2^6 3^3 5^2 7^0$.
$= 2^1 3^1 5^2 7^3$.

$75 \cdot 625 \cdot 675 \cdot 686$
$^8 7^4 = (2^4 3^2 5^4 7^2)^2$.

$, 14 \cdot 64 \cdot 75 - 2^4 3^2 5^4 7^2\}$

$7 \cdot 13$.

The number-field sieve

Generalize $i \equiv i + N \pmod{N}$
$\rightarrow a \equiv a + bN \pmod{N}$
$\rightarrow a - bm \equiv a - b\alpha \pmod{m - \alpha}$
for root $\alpha \in \mathbf{C}$
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so that factoring $m - \alpha$
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$(\mu + o(1))(\log N)^{2/3}(\log\log N)^{1/3}$.

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Sieve L^1
Find $L^{0.}$
with $a -$
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$\equiv 611 + i$

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7^3.

$675 \cdot 686$

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$75 - 2^4 3^2 5^4 7^2\}$

The number-field sieve

Generalize $i \equiv i + N \pmod{N}$
$\to a \equiv a + bN \pmod{N}$
$\to a - bm \equiv a - b\alpha \pmod{m - \alpha}$
for root $\alpha \in \mathbf{C}$
of nonzero integer poly.

For any m can find α
so that factoring $m - \alpha$
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Optimal choice of $\log m$ is
$(\mu + o(1))(\log N)^{2/3}(\log\log N)^{1/3}$.

RAM cost analysis

1993 Buhler–Lenst
Smoothness bound
Sieve $L^{1.923000}$ pai
Find $L^{0.961500}$ pair
with $a - bm$ and
Total RAM time L

1993 Coppersmith
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Generalize $i \equiv i + N \pmod{N}$
$\to a \equiv a + bN \pmod{N}$
$\to a - bm \equiv a - b\alpha \pmod{m - \alpha}$
for root $\alpha \in \mathbf{C}$
of nonzero integer poly.

For any m can find α
so that factoring $m - \alpha$
produces factorization of N.

Optimal choice of $\log m$ is
$(\mu + o(1))(\log N)^{2/3}(\log\log N)^{1/3}$.

RAM cost analysis

1993 Buhler–Lenstra–Pomer
Smoothness bound $L^{0.96150}$
Sieve $L^{1.923000}$ pairs (a, b).
Find $L^{0.961500}$ pairs
with $a - bm$ and $a - b\alpha$ sm
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The factorization factory

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There *exists* an algorithm
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Is $a - b\alpha_9$ smooth? If so, store. Send (a, b). right. Repeat.
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Generate (a,b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a,b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a,b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a,b). Is $a - bm$ smooth? If so, store. Repeat.
Generate (a,b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a,b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a,b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a,b). Is $a - bm$ smooth? If so, store. Repeat.
Generate (a,b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a,b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a,b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a,b). Is $a - bm$ smooth? If so, store. Repeat.
Generate (a,b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a,b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a,b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a,b). Is $a - bm$ smooth? If so, store. Repeat.

Is $a - b\alpha_1$ smooth? If so, store. Send (a,b). right. Repeat.	Is $a - b\alpha_2$ smooth? If so, store. Send (a,b). right. Repeat.
Is $a - b\alpha_5$ smooth? If so, store. Send (a,b). up. Repeat.	Is $a - b\alpha_6$ smooth? If so, store. Send (a,b). left. Repeat.
Is $a - b\alpha_9$ smooth? If so, store. Send (a,b). right. Repeat.	Is $a - b\alpha_{10}$ smooth? If so, store. Send (a,b). right. Repeat.
Is $a - b\alpha_{13}$ smooth? If so, store. Send (a,b). up. Repeat.	Is $a - b\alpha_{14}$ smooth? If so, store. Send (a,b). left. Repeat.

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Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.
Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.
Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.
Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.

Is $a - b\alpha_1$ smooth? If so, store. Send (a, b). right. Repeat.	Is $a - b\alpha_2$ smooth? If so, store. Send (a, b). right. Repeat.	Is $a - b\alpha_3$ smooth? If so, store. Send (a, b). right. Repeat.	I If S do
Is $a - b\alpha_5$ smooth? If so, store. Send (a, b). up. Repeat.	Is $a - b\alpha_6$ smooth? If so, store. Send (a, b). left. Repeat.	Is $a - b\alpha_7$ smooth? If so, store. Send (a, b). left. Repeat.	I If S le
Is $a - b\alpha_9$ smooth? If so, store. Send (a, b). right. Repeat.	Is $a - b\alpha_{10}$ smooth? If so, store. Send (a, b). right. Repeat.	Is $a - b\alpha_{11}$ smooth? If so, store. Send (a, b). right. Repeat.	I If S do
Is $a - b\alpha_{13}$ smooth? If so, store. Send (a, b). up. Repeat.	Is $a - b\alpha_{14}$ smooth? If so, store. Send (a, b). left. Repeat.	Is $a - b\alpha_{15}$ smooth? If so, store. Send (a, b). left. Repeat.	I If S le

Generate (a,b). Is $a-bm$ smooth? If so, store. Repeat.	Generate (a,b). Is $a-bm$ smooth? If so, store. Repeat.	Generate (a,b). Is $a-bm$ smooth? If so, store. Repeat.	Generate (a,b). Is $a-bm$ smooth? If so, store. Repeat.
Generate (a,b). Is $a-bm$ smooth? If so, store. Repeat.	Generate (a,b). Is $a-bm$ smooth? If so, store. Repeat.	Generate (a,b). Is $a-bm$ smooth? If so, store. Repeat.	Generate (a,b). Is $a-bm$ smooth? If so, store. Repeat.
Generate (a,b). Is $a-bm$ smooth? If so, store. Repeat.	Generate (a,b). Is $a-bm$ smooth? If so, store. Repeat.	Generate (a,b). Is $a-bm$ smooth? If so, store. Repeat.	Generate (a,b). Is $a-bm$ smooth? If so, store. Repeat.
Generate (a,b). Is $a-bm$ smooth? If so, store. Repeat.	Generate (a,b). Is $a-bm$ smooth? If so, store. Repeat.	Generate (a,b). Is $a-bm$ smooth? If so, store. Repeat.	Generate (a,b). Is $a-bm$ smooth? If so, store. Repeat.

Is $a-b\alpha_1$ smooth? If so, store. Send (a,b). right. Repeat.	Is $a-b\alpha_2$ smooth? If so, store. Send (a,b). right. Repeat.	Is $a-b\alpha_3$ smooth? If so, store. Send (a,b). right. Repeat.	Is $a-b\alpha_4$ smooth? If so, store. Send (a,b). down. Repeat.
Is $a-b\alpha_5$ smooth? If so, store. Send (a,b). up. Repeat.	Is $a-b\alpha_6$ smooth? If so, store. Send (a,b). left. Repeat.	Is $a-b\alpha_7$ smooth? If so, store. Send (a,b). left. Repeat.	Is $a-b\alpha_8$ smooth? If so, store. Send (a,b). left. Repeat.
Is $a-b\alpha_9$ smooth? If so, store. Send (a,b). right. Repeat.	Is $a-b\alpha_{10}$ smooth? If so, store. Send (a,b). right. Repeat.	Is $a-b\alpha_{11}$ smooth? If so, store. Send (a,b). right. Repeat.	Is $a-b\alpha_{12}$ smooth? If so, store. Send (a,b). down. Repeat.
Is $a-b\alpha_{13}$ smooth? If so, store. Send (a,b). up. Repeat.	Is $a-b\alpha_{14}$ smooth? If so, store. Send (a,b). left. Repeat.	Is $a-b\alpha_{15}$ smooth? If so, store. Send (a,b). left. Repeat.	Is $a-b\alpha_{16}$ smooth? If so, store. Send (a,b). left. Repeat.

.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.
.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.
.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.
.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.

Is $a - b\alpha_1$ smooth? If so, store. Send (a, b). right. Repeat.	Is $a - b\alpha_2$ smooth? If so, store. Send (a, b). right. Repeat.	Is $a - b\alpha_3$ smooth? If so, store. Send (a, b). right. Repeat.	Is $a - b\alpha_4$ smooth? If so, store. Send (a, b). down. Repeat.
Is $a - b\alpha_5$ smooth? If so, store. Send (a, b). up. Repeat.	Is $a - b\alpha_6$ smooth? If so, store. Send (a, b). left. Repeat.	Is $a - b\alpha_7$ smooth? If so, store. Send (a, b). left. Repeat.	Is $a - b\alpha_8$ smooth? If so, store. Send (a, b). left. Repeat.
Is $a - b\alpha_9$ smooth? If so, store. Send (a, b). right. Repeat.	Is $a - b\alpha_{10}$ smooth? If so, store. Send (a, b). right. Repeat.	Is $a - b\alpha_{11}$ smooth? If so, store. Send (a, b). right. Repeat.	Is $a - b\alpha_{12}$ smooth? If so, store. Send (a, b). down. Repeat.
Is $a - b\alpha_{13}$ smooth? If so, store. Send (a, b). up. Repeat.	Is $a - b\alpha_{14}$ smooth? If so, store. Send (a, b). left. Repeat.	Is $a - b\alpha_{15}$ smooth? If so, store. Send (a, b). left. Repeat.	Is $a - b\alpha_{16}$ smooth? If so, store. Send (a, b). left. Repeat.

$N_1, N_2, N_3,$ $N_5, N_6, N_7,$ N_9, N_{10}, N_{11} N_{13}, N_{14}, N_{15}
$N_1, N_2, N_3,$ $N_5, N_6, N_7,$ N_9, N_{10}, N_{11} N_{13}, N_{14}, N_{15}
$N_1, N_2, N_3,$ $N_5, N_6, N_7,$ N_9, N_{10}, N_{11} N_{13}, N_{14}, N_{15}
$N_1, N_2, N_3,$ $N_5, N_6, N_7,$ N_9, N_{10}, N_{11} N_{13}, N_{14}, N_{15}
$N_1, N_2, N_3,$ $N_5, N_6, N_7,$ N_9, N_{10}, N_{11} N_{13}, N_{14}, N_{15}

Generate (a,b). Is $a-bm$ smooth? If so, store. Repeat.	Generate (a,b). Is $a-bm$ smooth? If so, store. Repeat.
Generate (a,b). Is $a-bm$ smooth? If so, store. Repeat.	Generate (a,b). Is $a-bm$ smooth? If so, store. Repeat.
Generate (a,b). Is $a-bm$ smooth? If so, store. Repeat.	Generate (a,b). Is $a-bm$ smooth? If so, store. Repeat.
Generate (a,b). Is $a-bm$ smooth? If so, store. Repeat.	Generate (a,b). Is $a-bm$ smooth? If so, store. Repeat.

Is $a-b\alpha_1$ smooth? If so, store. Send (a,b). right. Repeat.	Is $a-b\alpha_2$ smooth? If so, store. Send (a,b). right. Repeat.	Is $a-b\alpha_3$ smooth? If so, store. Send (a,b). right. Repeat.	Is $a-b\alpha_4$ smooth? If so, store. Send (a,b). down. Repeat.
Is $a-b\alpha_5$ smooth? If so, store. Send (a,b). up. Repeat.	Is $a-b\alpha_6$ smooth? If so, store. Send (a,b). left. Repeat.	Is $a-b\alpha_7$ smooth? If so, store. Send (a,b). left. Repeat.	Is $a-b\alpha_8$ smooth? If so, store. Send (a,b). left. Repeat.
Is $a-b\alpha_9$ smooth? If so, store. Send (a,b). right. Repeat.	Is $a-b\alpha_{10}$ smooth? If so, store. Send (a,b). right. Repeat.	Is $a-b\alpha_{11}$ smooth? If so, store. Send (a,b). right. Repeat.	Is $a-b\alpha_{12}$ smooth? If so, store. Send (a,b). down. Repeat.
Is $a-b\alpha_{13}$ smooth? If so, store. Send (a,b). up. Repeat.	Is $a-b\alpha_{14}$ smooth? If so, store. Send (a,b). left. Repeat.	Is $a-b\alpha_{15}$ smooth? If so, store. Send (a,b). left. Repeat.	Is $a-b\alpha_{16}$ smooth? If so, store. Send (a,b). left. Repeat.

N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2 N_5, N_6 $N_9, N_{10},$ N_{13}, N_{14}
N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2 N_5, N_6 $N_9, N_{10},$ N_{13}, N_{14}
N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2 N_5, N_6 $N_9, N_{10},$ N_{13}, N_{14}
N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2 N_5, N_6 $N_9, N_{10},$ N_{13}, N_{14}
N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2 N_5, N_6 $N_9, N_{10},$ N_{13}, N_{14}

nerate (a, b). Is $a - bm$ smooth? f so, store. Repeat.
nerate (a, b). Is $a - bm$ smooth? f so, store. Repeat.
nerate (a, b). Is $a - bm$ smooth? f so, store. Repeat.
nerate (a, b). Is $a - bm$ smooth? f so, store. Repeat.

Is $a - b\alpha_1$ smooth? If so, store. Send (a, b). right. Repeat.	Is $a - b\alpha_2$ smooth? If so, store. Send (a, b). right. Repeat.	Is $a - b\alpha_3$ smooth? If so, store. Send (a, b). right. Repeat.	Is $a - b\alpha_4$ smooth? If so, store. Send (a, b). down. Repeat.
Is $a - b\alpha_5$ smooth? If so, store. Send (a, b). up. Repeat.	Is $a - b\alpha_6$ smooth? If so, store. Send (a, b). left. Repeat.	Is $a - b\alpha_7$ smooth? If so, store. Send (a, b). left. Repeat.	Is $a - b\alpha_8$ smooth? If so, store. Send (a, b). left. Repeat.
Is $a - b\alpha_9$ smooth? If so, store. Send (a, b). right. Repeat.	Is $a - b\alpha_{10}$ smooth? If so, store. Send (a, b). right. Repeat.	Is $a - b\alpha_{11}$ smooth? If so, store. Send (a, b). right. Repeat.	Is $a - b\alpha_{12}$ smooth? If so, store. Send (a, b). down. Repeat.
Is $a - b\alpha_{13}$ smooth? If so, store. Send (a, b). up. Repeat.	Is $a - b\alpha_{14}$ smooth? If so, store. Send (a, b). left. Repeat.	Is $a - b\alpha_{15}$ smooth? If so, store. Send (a, b). left. Repeat.	Is $a - b\alpha_{16}$ smooth? If so, store. Send (a, b). left. Repeat.

N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1 N_5 N_9, N_{13},
N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1 N_5 N_9, N_{13},
N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1 N_5 N_9, N_{13},
N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1 N_5 N_9, N_{13},
N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1 N_5 N_9, N_{13},

Is $a - b\alpha_1$ smooth? If so, store. Send (a, b). right. Repeat.	Is $a - b\alpha_2$ smooth? If so, store. Send (a, b). right. Repeat.	Is $a - b\alpha_3$ smooth? If so, store. Send (a, b). right. Repeat.	Is $a - b\alpha_4$ smooth? If so, store. Send (a, b). down. Repeat.
Is $a - b\alpha_5$ smooth? If so, store. Send (a, b). up. Repeat.	Is $a - b\alpha_6$ smooth? If so, store. Send (a, b). left. Repeat.	Is $a - b\alpha_7$ smooth? If so, store. Send (a, b). left. Repeat.	Is $a - b\alpha_8$ smooth? If so, store. Send (a, b). left. Repeat.
Is $a - b\alpha_9$ smooth? If so, store. Send (a, b). right. Repeat.	Is $a - b\alpha_{10}$ smooth? If so, store. Send (a, b). right. Repeat.	Is $a - b\alpha_{11}$ smooth? If so, store. Send (a, b). right. Repeat.	Is $a - b\alpha_{12}$ smooth? If so, store. Send (a, b). down. Repeat.
Is $a - b\alpha_{13}$ smooth? If so, store. Send (a, b). up. Repeat.	Is $a - b\alpha_{14}$ smooth? If so, store. Send (a, b). left. Repeat.	Is $a - b\alpha_{15}$ smooth? If so, store. Send (a, b). left. Repeat.	Is $a - b\alpha_{16}$ smooth? If so, store. Send (a, b). left. Repeat.

N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$
N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$
N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$
N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$
N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$

Is $a - b\alpha_2$ smooth? If so, store. Send (a, b). right. Repeat.	Is $a - b\alpha_3$ smooth? If so, store. Send (a, b). right. Repeat.	Is $a - b\alpha_4$ smooth? If so, store. Send (a, b). down. Repeat.
Is $a - b\alpha_6$ smooth? If so, store. Send (a, b). left. Repeat.	Is $a - b\alpha_7$ smooth? If so, store. Send (a, b). left. Repeat.	Is $a - b\alpha_8$ smooth? If so, store. Send (a, b). left. Repeat.
Is $a - b\alpha_{10}$ smooth? If so, store. Send (a, b). right. Repeat.	Is $a - b\alpha_{11}$ smooth? If so, store. Send (a, b). right. Repeat.	Is $a - b\alpha_{12}$ smooth? If so, store. Send (a, b). down. Repeat.
Is $a - b\alpha_{14}$ smooth? If so, store. Send (a, b). left. Repeat.	Is $a - b\alpha_{15}$ smooth? If so, store. Send (a, b). left. Repeat.	Is $a - b\alpha_{16}$ smooth? If so, store. Send (a, b). left. Repeat.

N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$
N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$
N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$
N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$
N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$

Is $a - b\alpha_3$ smooth? If so, store. Send (a, b). right. Repeat.	Is $a - b\alpha_4$ smooth? If so, store. Send (a, b). down. Repeat.
Is $a - b\alpha_7$ smooth? If so, store. Send (a, b). left. Repeat.	Is $a - b\alpha_8$ smooth? If so, store. Send (a, b). left. Repeat.
Is $a - b\alpha_{11}$ smooth? If so, store. Send (a, b). right. Repeat.	Is $a - b\alpha_{12}$ smooth? If so, store. Send (a, b). down. Repeat.
Is $a - b\alpha_{15}$ smooth? If so, store. Send (a, b). left. Repeat.	Is $a - b\alpha_{16}$ smooth? If so, store. Send (a, b). left. Repeat.

N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	$N_1, N_2, N_3,$ $N_5, N_6, N_7,$ $N_9, N_{10}, N_{11},$ $N_{13}, N_{14}, N_{15},$
N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	$N_1, N_2, N_3,$ $N_5, N_6, N_7,$ $N_9, N_{10}, N_{11},$ $N_{13}, N_{14}, N_{15},$
N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	$N_1, N_2, N_3,$ $N_5, N_6, N_7,$ $N_9, N_{10}, N_{11},$ $N_{13}, N_{14}, N_{15},$
N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	$N_1, N_2, N_3,$ $N_5, N_6, N_7,$ $N_9, N_{10}, N_{11},$ $N_{13}, N_{14}, N_{15},$
N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	$N_1, N_2, N_3,$ $N_5, N_6, N_7,$ $N_9, N_{10}, N_{11},$ $N_{13}, N_{14}, N_{15},$

s $a-b\alpha_4$ smooth? f so, store. Send (a,b). wn. Repeat.
s $a-b\alpha_8$ smooth? f so, store. Send (a,b). ft. Repeat.
s $a-b\alpha_{12}$ smooth? f so, store. Send (a,b). wn. Repeat.
s $a-b\alpha_{16}$ smooth? f so, store. Send (a,b). ft. Repeat.

N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$
N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$
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N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$
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N_4	N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4
N_8	N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8
, N_{12}	$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$
, N_{16}	$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$
N_4	N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4
N_8	N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8
, N_{12}	$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$
, N_{16}	$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$
N_4	N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4
N_8	N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8
, N_{12}	$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$
, N_{16}	$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$
N_4	N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4
N_8	N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8
, N_{12}	$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$
, N_{16}	$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$
N_4	N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4
N_8	N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8
, N_{12}	$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$
, N_{16}	$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$

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$, N_3, N_4$	N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4
$, N_7, N_8$	N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8
N_{11}, N_{12}	$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$
$, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$
$, N_3, N_4$	N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4
$, N_7, N_8$	N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8
N_{11}, N_{12}	$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$
$, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$
$, N_3, N_4$	N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4
$, N_7, N_8$	N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8
N_{11}, N_{12}	$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$
$, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$
$, N_3, N_4$	N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4
$, N_7, N_8$	N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8
N_{11}, N_{12}	$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$
$, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$
$, N_3, N_4$	N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4
$, N_7, N_8$	N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8
N_{11}, N_{12}	$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$
$, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$

Linear algebra for N_1	L
using congruences	
(a, b) (a, b) (a, b)	
(a, b) (a, b) (a, b)	
(a, b) (a, b) (a, b)	
Linear algebra for N_5	L
using congruences	
(a, b) (a, b) (a, b)	
(a, b) (a, b) (a, b)	
(a, b) (a, b) (a, b)	
Linear algebra for N_9	L
using congruences	
(a, b) (a, b) (a, b)	
(a, b) (a, b) (a, b)	
(a, b) (a, b) (a, b)	
Linear algebra for N_{13}	L
using congruences	
(a, b) (a, b) (a, b)	
(a, b) (a, b) (a, b)	
(a, b) (a, b) (a, b)	

$, N_2, N_3, N_4$ $, N_6, N_7, N_8$ N_{10}, N_{11}, N_{12} N_{14}, N_{15}, N_{16}	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$
$, N_2, N_3, N_4$ $, N_6, N_7, N_8$ N_{10}, N_{11}, N_{12} N_{14}, N_{15}, N_{16}	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$
$, N_2, N_3, N_4$ $, N_6, N_7, N_8$ N_{10}, N_{11}, N_{12} N_{14}, N_{15}, N_{16}	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$
$, N_2, N_3, N_4$ $, N_6, N_7, N_8$ N_{10}, N_{11}, N_{12} N_{14}, N_{15}, N_{16}	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$
$, N_2, N_3, N_4$ $, N_6, N_7, N_8$ N_{10}, N_{11}, N_{12} N_{14}, N_{15}, N_{16}	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$

Linear algebra for N_1 using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for using congruences (a, b) (a, b) $(a, b$ (a, b) (a, b) $(a, b$ (a, b) (a, b) $(a, b$
Linear algebra for N_5 using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for using congruences (a, b) (a, b) $(a, b$ (a, b) (a, b) $(a, b$ (a, b) (a, b) $(a, b$
Linear algebra for N_9 using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for using congruences (a, b) (a, b) $(a, b$ (a, b) (a, b) $(a, b$ (a, b) (a, b) $(a, b$
Linear algebra for N_{13} using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for using congruences (a, b) (a, b) $(a, b$ (a, b) (a, b) $(a, b$ (a, b) (a, b) $(a, b$

N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4
N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8
$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$
$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$
N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4
N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8
$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$
$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$
N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4
N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8
$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$
$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$
N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4
N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8
$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$
$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$
N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4
N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8
$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$
$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$

Linear algebra for N_1	Linear algebra for N_2	Linear al
using congruences	using congruences	using
(a,b) (a,b) (a,b)	(a,b) (a,b) (a,b)	(a,b)
(a,b) (a,b) (a,b)	(a,b) (a,b) (a,b)	(a,b)
(a,b) (a,b) (a,b)	(a,b) (a,b) (a,b)	(a,b)
Linear algebra for N_5	Linear algebra for N_6	Linear al
using congruences	using congruences	using
(a,b) (a,b) (a,b)	(a,b) (a,b) (a,b)	(a,b)
(a,b) (a,b) (a,b)	(a,b) (a,b) (a,b)	(a,b)
(a,b) (a,b) (a,b)	(a,b) (a,b) (a,b)	(a,b)
Linear algebra for N_9	Linear algebra for N_{10}	Linear alg
using congruences	using congruences	using
(a,b) (a,b) (a,b)	(a,b) (a,b) (a,b)	(a,b)
(a,b) (a,b) (a,b)	(a,b) (a,b) (a,b)	(a,b)
(a,b) (a,b) (a,b)	(a,b) (a,b) (a,b)	(a,b)
Linear algebra for N_{13}	Linear algebra for N_{14}	Linear alg
using congruences	using congruences	using
(a,b) (a,b) (a,b)	(a,b) (a,b) (a,b)	(a,b)
(a,b) (a,b) (a,b)	(a,b) (a,b) (a,b)	(a,b)
(a,b) (a,b) (a,b)	(a,b) (a,b) (a,b)	(a,b)

N_1, N_2, N_3, N_4
N_5, N_6, N_7, N_8
$N_9, N_{10}, N_{11}, N_{12}$
$N_{13}, N_{14}, N_{15}, N_{16}$
N_1, N_2, N_3, N_4
N_5, N_6, N_7, N_8
$N_9, N_{10}, N_{11}, N_{12}$
$N_{13}, N_{14}, N_{15}, N_{16}$
N_1, N_2, N_3, N_4
N_5, N_6, N_7, N_8
$N_9, N_{10}, N_{11}, N_{12}$
$N_{13}, N_{14}, N_{15}, N_{16}$
N_1, N_2, N_3, N_4
N_5, N_6, N_7, N_8
$N_9, N_{10}, N_{11}, N_{12}$
$N_{13}, N_{14}, N_{15}, N_{16}$
N_1, N_2, N_3, N_4
N_5, N_6, N_7, N_8
$N_9, N_{10}, N_{11}, N_{12}$
$N_{13}, N_{14}, N_{15}, N_{16}$

Linear algebra for N_1 using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for N_2 using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for N_3 using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)
Linear algebra for N_5 using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for N_6 using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for N_7 using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)
Linear algebra for N_9 using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for N_{10} using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for N_{11} using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)
Linear algebra for N_{13} using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for N_{14} using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for N_{15} using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)

$/_4$
$/_8$
V_{12}
N_{16}
$/_4$
$/_8$
V_{12}
N_{16}
$/_4$
$/_8$
V_{12}
N_{16}
$/_4$
$/_8$
V_{12}
N_{16}
$/_4$
$/_8$
V_{12}
N_{16}

Linear algebra for N_1 using congruences (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b)	Linear algebra for N_2 using congruences (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b)	Linear algebra for N_3 using congruences (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b)	Linear algebra fo using congruer (a,b) (a,b) (a (a,b) (a,b) (a (a,b) (a,b) (a
Linear algebra for N_5 using congruences (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b)	Linear algebra for N_6 using congruences (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b)	Linear algebra for N_7 using congruences (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b)	Linear algebra fo using congruen (a,b) (a,b) (a (a,b) (a,b) (a (a,b) (a,b) (a
Linear algebra for N_9 using congruences (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b)	Linear algebra for N_{10} using congruences (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b)	Linear algebra for N_{11} using congruences (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b)	Linear algebra fo using congruen (a,b) (a,b) (a (a,b) (a,b) (a (a,b) (a,b) (a
Linear algebra for N_{13} using congruences (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b)	Linear algebra for N_{14} using congruences (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b)	Linear algebra for N_{15} using congruences (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b)	Linear algebra fo using congruen (a,b) (a,b) (a (a,b) (a,b) (a (a,b) (a,b) (a

Linear algebra for N_1 using congruences (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b)	Linear algebra for N_2 using congruences (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b)	Linear algebra for N_3 using congruences (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b)	Linear algebra for N_4 using congruences (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b)
Linear algebra for N_5 using congruences (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b)	Linear algebra for N_6 using congruences (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b)	Linear algebra for N_7 using congruences (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b)	Linear algebra for N_8 using congruences (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b)
Linear algebra for N_9 using congruences (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b)	Linear algebra for N_{10} using congruences (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b)	Linear algebra for N_{11} using congruences (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b)	Linear algebra for N_{12} using congruences (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b)
Linear algebra for N_{13} using congruences (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b)	Linear algebra for N_{14} using congruences (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b)	Linear algebra for N_{15} using congruences (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b)	Linear algebra for N_{16} using congruences (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b)

Linear algebra for N_1 using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for N_2 using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for N_3 using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for N_4 using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)
Linear algebra for N_5 using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for N_6 using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for N_7 using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for N_8 using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)
Linear algebra for N_9 using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for N_{10} using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for N_{11} using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for N_{12} using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)
Linear algebra for N_{13} using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for N_{14} using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for N_{15} using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for N_{16} using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)