How to find smooth parts of integers

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Integer-factorization bottleneck:
Given sequence of numbers,
find nonempty subsequence
with square product.
e.g. given $6, 7, 8, 10, 15$,
discover $6 \cdot 10 \cdot 15 = 30^2$.

Discrete-log bottleneck:
Given sequence of numbers,
find 1 as nontrivial
product of powers.
e.g. given $6, 7, 8, 10, 15$,
discover $6^3 7^0 8^{-2} 10^3 15^{-3} = 1$.

More generally: find kth power.

This is a bottom-up talk
aiming at these bottlenecks.

Will focus on integers.
Can use same techniques,
and more, for polynomials
in function-field sieve etc.

Will focus on
conventional architectures:
e.g. multitape Turing machines.
Optimization is very different
for mesh architectures.

Multiplication and division

Given $r, s \in \mathbf{Z}$, can compute rs
in time $\le b(\lg b)^{1+o(1)}$
where b is number of input bits.

(1971 Pollard; independently
1971 Nicholson; independently
1971 Schönhage Strassen)

Also time $\le b(\lg b)^{1+o(1)}$
where b is number of input bits:
Given $r, s \in \mathbf{Z}$ with $s \ne 0$,
compute $\lfloor r/s \rfloor$ and $r \bmod s$.

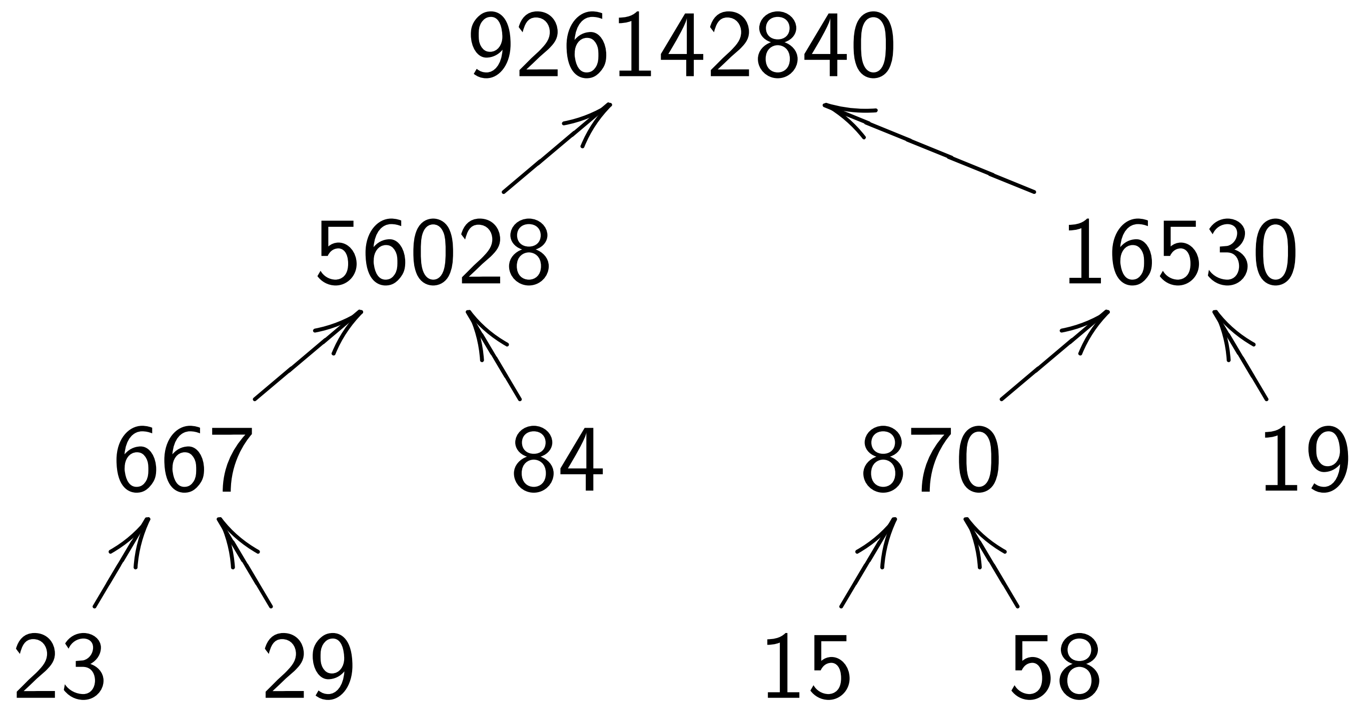
(reduction to product: 1966 Cook)

Product trees

Time $\le b(\lg b)^{2+o(1)}$
where b is number of input bits:
Given $x_1, x_2, \ldots, x_n \in \mathbf{Z}$,
compute $x_1 x_2 \cdots x_n$.

Actually compute
product tree of $x_1, x_2, \ldots, x_n$.
Root is $x_1 x_2 \cdots x_n$.
Has left subtree if $n \ge 2$:
product tree of $x_1, \ldots, x_{\lceil n/2 \rceil}$.
Also right subtree if $n \ge 2$:
product tree of $x_{\lceil n/2 \rceil + 1}, \ldots, x_n$.

e.g. tree for 23, 29, 84, 15, 58, 19:



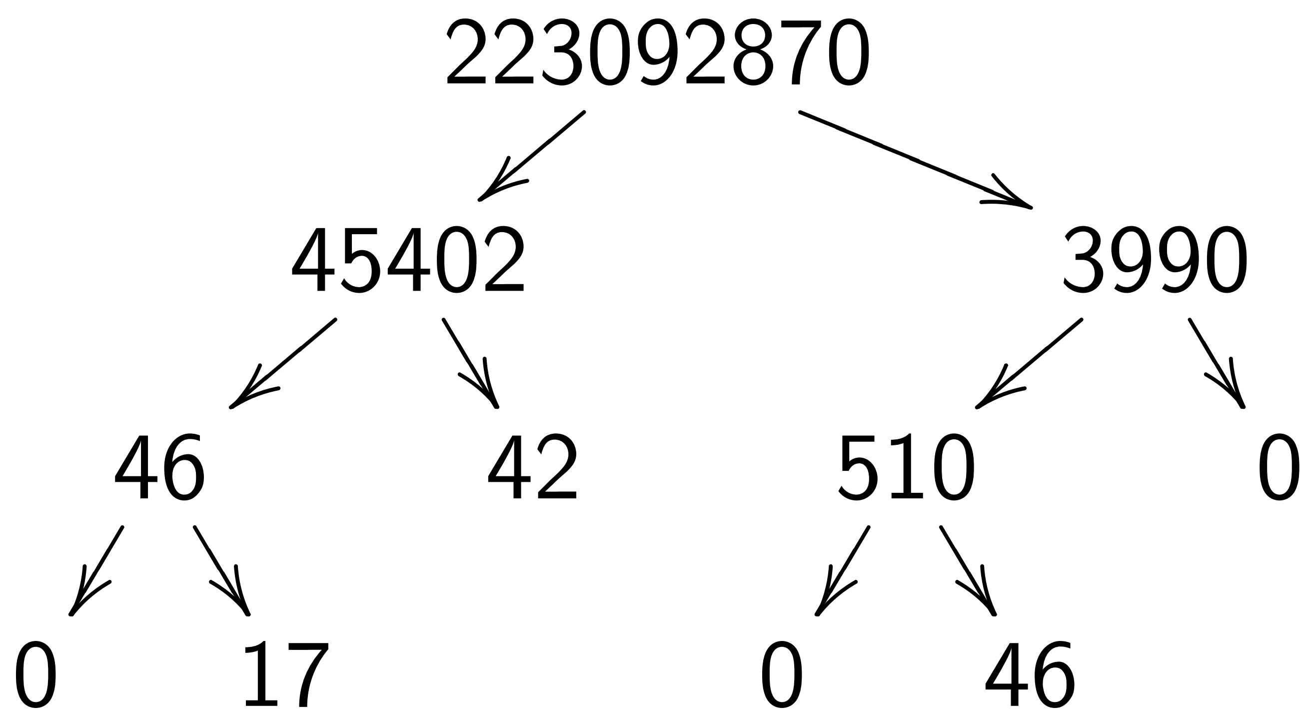
Tree has $\leq (\lg b)^{1+o(1)}$ levels.
Each level has $\leq b(\lg b)^{0+o(1)}$ bits.

Obtain each level
in time $\leq b(\lg b)^{1+o(1)}$
by multiplying lower-level pairs.

Remainder trees

Remainder tree
of $r, x_1, x_2, \ldots, x_n$ has
one node $r \bmod t$ for each node t
in product tree of $x_1, x_2, \ldots, x_n$.

e.g. remainder tree of
$223092870, 23, 29, 84, 15, 58, 19$:



Time $\le b(\lg b)^{2+o(1)}$:
Given $r \in \mathbf{Z}$ and nonzero $x_1, \ldots, x_n \in \mathbf{Z}$, compute remainder tree of $r, x_1, \ldots, x_n$.

In particular, compute $r \bmod x_1, \ldots, r \bmod x_n$.

In particular, see which of $x_1, \ldots, x_n$ divide r.

(1972 Moenck Borodin, for "single precision" x_i's, whatever exactly that means)

Small primes, union

Time $\leq b(\lg b)^{2+o(1)}$:
Given $x_1, x_2, \ldots, x_n \in \mathbf{Z}$ and finite set $Q \subseteq \mathbf{Z} - \{0\}$, compute $\{p \in Q : x_1 x_2 \cdots x_n \bmod p = 0\}$.

In particular, when p is prime, see whether p divides any of $x_1, x_2, \ldots, x_n$.

Algorithm:

1. Use a product tree to compute $r = x_1 x_2 \cdots x_n$.
2. Use a remainder tree to see which $p \in Q$ divide r.

Small primes, separately

Time $\le b(\lg b)^{3+o(1)}$:
Given $x_1, x_2, \ldots, x_n \in \mathbf{Z}$ and finite set Q of primes, compute $\{p \in Q : x_1 \bmod p = 0\}$, $\ldots$, $\{p \in Q : x_n \bmod p = 0\}$.
(2000 Bernstein)

Algorithm for $n \ge 1$:

1. Replace Q with $\{p \in Q : x_1 \cdots x_n \bmod p = 0\}$.
2. If $n = 1$, print Q and stop.
3. Recurse on $x_1, \ldots, x_{\lceil n/2 \rceil}, Q$.
4. Recurse on $x_{\lceil n/2 \rceil + 1}, \ldots, x_n, Q$.

Factor $2543, 6766, 8967, 7598$
over $\{\underline{2}, \underline{3}, 5, \underline{7}, 11, 13, \underline{17}\}$

$2543, 6766$
over
$\underline{2}, 3, 7, \underline{17}$

$8967, 7598$
over
$\underline{2}, \underline{3}, \underline{7}, 17$

2543
over
$2, 17$

6766
over
$\underline{2}, \underline{17}$

8967
over
$2, \underline{3}, \underline{7}$

7598
over
$\underline{2}, 3, 7$

Each level has $\leq b(\lg b)^{0+o(1)}$ bits.

Exponents of a small prime

Time $\le b(\lg b)^{2+o(1)}$:
Given nonzero $p, x \in \mathbf{Z}$,
find $e, p^e, x/p^e$ with maximal e.

Algorithm:

1. If $x \bmod p \ne 0$:
   Print $0, 1, x$ and stop.
2. Find $f, (p^2)^f, r = (x/p)/(p^2)^f$
   with maximal f.
3. If $r \bmod p = 0$: Print
   $2f + 2, (p^2)^f p^2, r/p$ and stop.
4. Print $2f + 1, (p^2)^f p, r$.

Exponents of small primes

Time $\le b(\lg b)^{3+o(1)}$:
Given finite set Q of primes and nonzero $x \in \mathbf{Z}$, find maximal $e, \prod_{p \in Q} p^{e(p)}, x / \prod_{p \in Q} p^{e(p)}$.

Algorithm:

1. Replace Q with $\{p \in Q : x \bmod p = 0\}$.
2. Find maximal f, s, r with $s = \prod (p^2)^{f(p^2)}$, $r = (x / \prod p)/s$.
3. Find $T = \{p \in Q : r \bmod p = 0\}$.
4. Answer is $e, s \prod_{p \in T} p, r / \prod_{p \in T} p$ where $e(p) = 2f(p^2) + [p \in T]$.

Smooth parts, old approach

Time $\le b(\lg b)^{3+o(1)}$:
Given nonzero $x_1, x_2, \ldots, x_n \in \mathbf{Z}$
and finite set Q of primes,
compute Q-smooth part of x_1,
Q-smooth part of x_2, . . . ,
Q-smooth part of x_n.

Q-smooth means
product of powers of elements of Q.

Q-smooth part means
largest Q-smooth divisor.
In particular, see which of
$x_1, x_2, \ldots, x_n$ are smooth.

Algorithm:

1. Find $Q_1 = \{p : x_1 \bmod p = 0\}$, $\ldots$, $Q_n = \{p : x_n \bmod p = 0\}$.
2. For each i separately:
   Find maximal e, s, r with
   $s = \prod_{p \in Q_i} p^{e(p)}$, $r = x_i/s$.
   Print s.

e.g. factoring $2543, 6766, 8967, 7598$ over $\{2, 3, 5, 7, 11, 13, 17\}$:
2543 over $\{\}$, smooth part 1;
6766 over $\{2, 17\}$, smooth part 34;
8967 over $\{3, 7\}$, smooth part 147;
7598 over $\{2\}$, smooth part 2.

Smooth multiplicative dependencies

Recall cryptanalytic bottleneck:
find kth power nontrivially as
product of powers of
$x_1, x_2, \ldots, x_n$.

Choose y; imagine $y = 2^{40}$.
Define Q as set of primes $\le y$.
See which of $x_1, x_2, \ldots, x_n$
are y-smooth, i.e., Q-smooth.
Know their factorizations.
Do linear algebra over $\mathbf{Z}/k$
on the exponent vectors.

Sieving

In linear sieve (1977 Schroeppel), number-field sieve, etc., x's are consecutive values of a low-degree polynomial.

Choose θ; imagine $\theta = 0.5$.
Sieve to discover primes $\le y^\theta$;
say time S per number.
Keep most promising x's.
See which ones are y-smooth;
say time T per number.

Time to find each smooth number is roughly $S^\theta T^{1-\theta}$ after optimization.

Smooth parts, new approach

Given nonzero $x_1, x_2, \ldots, x_n \in \mathbf{Z}$
and finite set Q of primes:
Time typically $\le b(\lg b)^{2+o(1)}$
to obtain smooth parts of x's.
(2004 Franke Kleinjung
Morain Wirth, in ECPP context)

Algorithm:
Compute $r = \prod_{p \in Q} p$.
Compute $r \bmod x_1, \ldots, r \bmod x_n$.
For each i separately:
Replace x_i by $x_i / \gcd\{x_i, r \bmod x_i\}$
repeatedly until gcd is 1.

Slight variant (2004 Bernstein):
Time always $\le b(\lg b)^{2+o(1)}$.

Compute smooth part of x_i as
$\gcd\left\{x_i, (r \bmod x_i)^{2^k} \bmod x_i\right\}$
where $k = \lceil \lg\lg x_i \rceil$.

Subroutine: Computing gcd
takes time $\le b(\lg b)^{2+o(1)}$.
(1971 Schönhage;
core idea: 1938 Lehmer;
$b(\lg b)^{5+o(1)}$: 1971 Knuth)

Or, to see if x_i is smooth,
see if $(r \bmod x_i)^{2^k} \bmod x_i = 0$.

Minor problem: New algorithm finds the smooth numbers but doesn't factor them.

Solution: Feed the smooth numbers to the old algorithm. Very few smooth numbers, so this is very fast.

Bottom line: T, time per number to find and factor smooth numbers, has dropped by $(\lg b)^{1+o(1)}$.

This is big news for cryptanalysis!

Is smooth the right question?

After finding smooth numbers, do first step of linear algebra: Throw away primes that appear only once; throw away numbers with those primes; repeat until stable.

Don't want *all* smooth numbers. Want smooth numbers only if they are built from primes that divide the *other* numbers.

An alternate approach

Given nonzero $x_1, x_2, \ldots, x_n \in \mathbf{Z}$:
Compute $r = x_1 x_2 \cdots x_n$.
Compute $(r/x_1) \bmod x_1$, . . . ,
$(r/x_n) \bmod x_n$.
For each i separately: see if
$((r/x_i) \bmod x_i)^{2^k} \bmod x_i = 0$
where $k = \lceil \lg \lg x_i \rceil$.

Finds x_i iff all primes in x_i
are divisors of other x's.
Time $\le b(\lg b)^{2+o(1)}$.

(2004 Bernstein)

Compute $(r/x_1) \bmod x_1, \ldots,$
$(r/x_n) \bmod x_n$ by computing
$r \bmod x_1^2, \ldots, r \bmod x_n^2$.
(1972 Moenck Borodin)

Problem: Recognizing the interesting x's is not enough; also need their factorizations.

Solution: Again, very few of them. Have ample time to use rho method (1974 Pollard) or use ECM (1987 Lenstra) or factor into coprimes.

Factoring into coprimes

Time $\le b(\lg b)^{O(1)}$:
Given positive $x_1, x_2, \ldots, x_n$,
find coprime set Q
and complete factorization
of each x_i over Q.

(announced 1995 Bernstein;
now at second-galley stage
for J. Algorithms)

Immediately gives $b(\lg b)^{O(1)}$
for the other factoring problems.
Subsequent research: lg speedups,
constant-factor speedups, etc.

Speedup: aligning roots

Original FFT (1805 Gauss, et al.): $(4.5 + o(1))n \lg n$ operations in **C** to multiply in $\mathbf{C}[x]/(x^n - 1)$; or $(15 + o(1))n \lg n$ operations in **R**.

Split-radix FFT (1968 Yavne; Duhamel, Hollmann, Martens, Stasinski, Vetterli, Nussbaumer): $(4.5 + o(1))n \lg n$ operations in **C** to multiply in $\mathbf{C}[x]/(x^n - 1)$; only $(12 + o(1))n \lg n$ operations in **R**.

Why fewer operations in **R**?

Multiplications in **C** for original FFT:
$1.5n$ by primitive 4th roots of 1,
$1.5n$ by primitive 8th roots of 1,
$1.5n$ by primitive 16th roots of 1,
etc.

For split-radix FFT:
$0.5n \lg n$ by primitive 4th roots of 1,
n by primitive 8th roots of 1,
n by primitive 16th roots of 1,
etc.

Split-radix FFT
aligns many of the roots
to be 4th roots of 1.

In Schönhage-Strassen context, aligning roots produces much larger speedups. (2000 Bernstein)

Consider size-65536 FFT over A where $A = \mathbf{Z}/(2^{16384}+1)$; $2^{12288} - 2^{4096}$ is a square root of 2 in A.

Multiplications by powers of 2 usually mean annoying shifts across word boundaries. Alignment avoids almost all of this. Also sometimes makes slightly larger FFT sizes practical.

Speedup: better caching

Multiply in $\mathbf{Z}/(2^{1048576000} - 1)$ by lifting to $\mathbf{Z}[x]/(x^{65536} - 1)$, mapping to $A[x]/(x^{65536} - 1)$, using FFT. (1971 Schönhage Strassen for negacyclic case)

Reorganize FFT operations to reduce communication costs. (1966 Gentleman Sande, et al.)

Can reduce communication costs even more by aligning roots and violating A operation atomicity. (2004 Bernstein)

Speedup: FFT doubling

(2004 Kramer)

Consider product tree for x_1, x_2, x_3, x_4, each $b/4$ bits.

Compute $x_1 x_2$ as $\mathrm{FFT}_{b/2}^{-1}(\mathrm{FFT}_{b/2}(x_1)\,\mathrm{FFT}_{b/2}(x_2))$.

Compute $x_1 x_2 x_3 x_4$ as $\mathrm{FFT}_b^{-1}(\mathrm{FFT}_b(x_1 x_2)\,\mathrm{FFT}_b(x_3 x_4))$.
First half of $\mathrm{FFT}_b(x_1 x_2)$ is $\mathrm{FFT}_{b/2}(x_1 x_2)$, already known!

For large product trees, $1.5 + o(1)$ speedup.

Some additional speedups

Start Newton for $1/x_1x_2$
at product of approximations
to $1/x_1$ and $1/x_2$.

Remove redundancy in division.

Use 2-adic division.

Eliminate tiny primes.

Further reduce the 2^k
by using powers of small primes.

Balance gcd and powering.