SONIFICATION OF THE RIEMANN ZETA FUNCTION

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ABSTRACT

The Riemann zeta function is one of the great wonders of mathematics, with a deep and still not fully solved connection to the prime numbers. It is defined via an infinite sum analogous to Fourier additive synthesis, and can be calculated in various ways. It was Riemann who extended the consideration of the series to complex number arguments, and the famous Riemann hypothesis states that the non-trivial zeroes of the function all occur on the critical line $0.5 + ti$, and what is more, hold a deep correspondence with the prime numbers. For the purposes of sonification, the rich set of mathematical ideas to analyse the zeta function provide strong resources for sonic experimentation. The positions of the zeroes on the critical line can be directly sonified, as can values of the zeta function in the complex plane, approximations to the prime spectrum of prime powers and the Riemann spectrum of the zeroes rendered; more abstract ideas concerning the function also provide interesting scope.

1. INTRODUCTION

The Riemann zeta function [1, 2] is a construction in analytic number theory of great beauty and wide scope, with a central assertion in its theory, the Riemann Hypothesis (RH), that has remained unsolved since its original statement in 1859. Musical analogies have often been made in referring to the problem, with Fourier analysis a tool in the analysis of the equation, and the dual structure of the non-trivial zeroes of the function and the prime numbers analogous to the spectral and time domain viewpoints of a sound signal [3, p. 89]. There is a great deal of interesting mathematics surrounding the zeta function, commensurate with the efforts of mathematicians for centuries to gain handles on the RH that all the non-trivial zeroes of the function appear only along one 'critical line' in the complex number plane. The fuller exploitation of equations and data relating to RH for musical purposes is the subject of this present paper, and we treat direct synthesis ('audification'), as well as sonification of rhythms and pitch structures.

This paper does not present the first ever sonification of the zeta function. Multiple authors have synthesized the zeroes of the function in particular, including Jeffrey Stopple (http://web.math.ucsb.edu/~stopple/explicit.html), Robert Munafo (https://mrob.com/pub/ries/zeta.html) and Andrey Kulsha (http://empslocal.ex.ac.uk/people/staff/mrwatkin/zeta/kulsha.htm). Such sonifications tend to be based on sinusoidal resynthesis following the gradual journey along the critical line where all known zeroes have been found, incorporating the contribution of each zero as it arises. In perhaps the most developed precedent, the distinguished physicist Michael V Berry explores a number of sonifications [4], including a sum of sinusoids corresponding to the Riemann zeroes, and direct synthesis of the zeta function along the critical line based on the Riemann-Siegel formula. We differ from this prior work in considering direct synthesis based on the naive approach of summing the zeta function, on exploring rhythm and scales, and in a greater willingness to accept any 'noisy' outputs as acceptable within the wider space of sound available in computer music. We also provide SuperCollider code to accompany the paper, providing immediate sound examples and realtime interactive synthesis capability.

This work is in the spirit of composers who have integrated mathematics into the core of their music compositions, perhaps foremost of which was Iannis Xenakis, who adapted such content as hyperbolic curves, probability theory and statistical functions, group theory and game theory [5, 6]. The inter-relationship of music and mathematics is a wider topic than we have space to fully survey here [7], but composers have demonstrated a number of approaches to the incorporation of algorithms into their practice, from strict observance of algorithmic output data to taking various liberties [8, 9, 10]. Prime numbers have often appeared in composer's work, from just intonation theory using small integer ratios often favouring primes, to sonification of the sequence of prime numbers.[1] In terms of the model of Vickers and Hogg [11] the present work is more abstract, as pertaining to a Platonic space of mathematics rather than real world data, and central within the continuum between music and scientific sonification. We are interested in new sonic resources [12, 13], and do not harbour illusions that sonifications of the zeta function rather than hard mathematics will somehow resolve the RH. However, the consideration of such mathematics does widen the appreciation of beautiful ideas and human ingenuity, genuinely inspiring for new musical creation, and illuminating with respect to the audibility (or otherwise) of transplanted advanced mathematics.

[1] For example, the links at http://empslocal.ex.ac.uk/people/staff/mrwatkin//zeta/curiosities.htm provide some online projects.

2. CALCULATION AND DIRECT SYNTHESIS

Whilst the zeta function for complex argument s is written

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{1}$$

it is far simpler for computation to consider the related eta function:

$$\eta(s) = \sum_{n=1}^{\infty} \frac{-1^{n-1}}{n^s} \tag{2}$$

where $\eta(s) = (1-2^{1-s})\zeta(s)$.[2]

Since we are considering complex argument s we write $s = \alpha + ti$ and use standard identities

$$\begin{aligned} n^s &= \\ n^{\alpha+it} &= \\ \exp\log(n^{\alpha+it})) &= \\ \exp((\alpha+it)\log(n)) &= \\ \exp(\alpha\log(n))\exp(it\log(n)) &= \\ n^{\alpha}(\cos(t\log n) + i\sin(t\log n)) \end{aligned} \tag{3}$$

Thus in the last step using the Euler form of a complex exponential as cosine and sine terms, and rewriting with $-s = -\alpha - it$:

$$\eta(\alpha+it) = \sum_{n=1}^{\infty} -1^{n-1} n^{-\alpha}(\cos(t\log n) - i\sin(t\log n)) \tag{4}$$

we end up with a Fourier-like representation. The complex number result can be expressed as two infinite sums, one over cosines and one over sines (which suggests immediate additive synthesis rendering):

$$\begin{aligned} &\eta(\alpha+it) = \\ &\sum_{n=1}^{\infty} -1^{n-1} n^{-\alpha}\cos(t\log n) - i\sum_{n=1}^{\infty} -1^{n-1} n^{-\alpha}\sin(t\log n) \end{aligned} \tag{5}$$

The recurring term $t\log(n)$ expresses a scaling by t of $\log(n)$, which when passed as argument to a trigonometric function of period 2π, pushes the terms more or less far in phase. The zeroes of the Riemann zeta function occur in the remarkable situation that both the sum of cosines and of sines cancel out.

For additive sound synthesis, brute force summation can be carried out for s in regions of convergence of the η function ($\alpha > 0$), though less efficiently than some series acceleration methods allow. Euler-Maclaurin summation or the Riemann-Siegel formula on the critical line [1], the Borwein method [14] or the FastZeta algorithm [15] are possible improvements, though best convergence still requires on the order of $t^{1/2}$ summands, and complexity of calculation is thus highly dependent on the height of t. In the naive sum, the $n^{-\alpha}$ coefficients in combination with the sinusoids

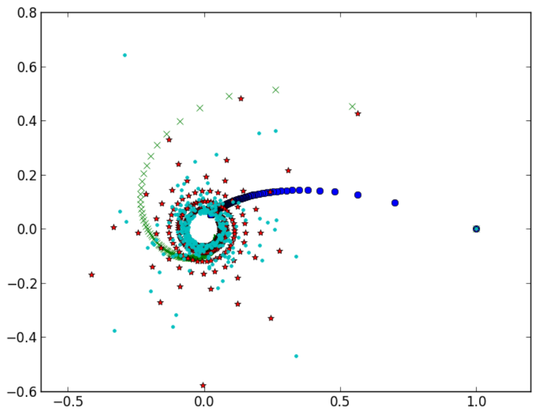

Figure 1: Example sequences of $n^{-0.5}(\cos(t\log n) + i\sin(t\log n))$ for $n = 1..100$ and $t = 0.2$, 1, 10 and 1000

cause a spiralling in of the magnitude of the complex numbers, though this is often a slow process (see Figure 1). Indeed, in perceptual terms, the $n^{-\alpha}$ prefix does not drop off at all quickly for $0 < \alpha < 1$ in the region of most interest to studies of the zeta function. A -60 dB drop requires $-dbamp(60)^{\frac{-1}{\alpha}} = 0.001^{\frac{-1}{\alpha}}$ terms, so for $\alpha = 0.5$ on the critical line, one million terms. Note though that the cosine and sine components will periodically drop to zero, and that depending on t and n, the summands can enter long runs at particular phases corresponding to positions near trigonometric zeroes, due to the slowing of $\log(n)$.

If synthesis of the whole zeta function sum is carried out for increasing t on the critical line $\alpha = 0.5$, zero amplitude of the function will be heard at the famous zeroes.

For high t on the critical line, the sinusoidal components of the zeta function sum revolve incredibly quickly until $\log(n)$ is changing slowly enough to offset large t; this will drop to under one cycle per n at $2\pi = t\log(n+1) - t\log(n) = t\log\frac{n+1}{n}$ so $\exp\frac{2\pi}{t} = 1 + \frac{1}{n}$ and therefore $n = \frac{1}{\exp\frac{2\pi}{t} - 1}$. Close movement in phase may also accompany higher multiples of 2π so this is just the point past which every update is under a cycle in difference; the formula is quickly adjusted for under ϵ in difference. Under 1 unit in difference has the approximate solution $n = t$ since a rough approximation, especially applicable for higher t as $\log(n)$ changes more and more slowly, follows from the derivative $\frac{d\log(n)}{dn} = 1/n$ so that a change of 1 corresponds approximately to the difference $1/n = \log(n+1) - \log(n)$.

If a zero s off the critical line was ever found with $0 < \alpha < 1$, the functional equation of the zeta function, and the fact that complex conjugates of zeroes are also zeroes implies that four different equations are true, namely $0 = \zeta(\alpha + ti) = \zeta(\alpha - ti) = \zeta(1 - \alpha + ti) = \zeta(1 - \alpha - ti)$ and four versions of the sums in cos and sin above sum to zero (there aren't eight because equating complex parts to zero, negative or positive versions of the real and imaginary sums are both zero), so the two sums already in (5) and further $\sum_{n=1}^{\infty} -1^{n-1} n^{-1+\alpha}\cos(t\log n)$ and $\sum_{n=1}^{\infty} -1^{n-1} n^{-1+\alpha}\sin(t\log n)$. The sonification of positions off the critical line could consider rendering these sums and thus illuminating their difference from each other and zero.

[2] Zeroes due to the term $1 - 2^{1-s}$ only occur for $\alpha = 1$ and $t = \frac{k2\pi}{\log 2}$, for integer k thus outside the ordinary area of interest of the function

As an alternative sound synthesis resource, expressions of the form $\cos(t\log(n))$ are actually quite productive, including for real rather than integer n. Conversion of the sum over n to an approximating integral and error term, as in the derivation of the Euler-Maclaurin formula, is a precedent to consider continuous n.

3. SUPERCOLLIDER SYNTHESIS EXAMPLES

The domain specific audio programming language SuperCollider [16] was used for sonifications; it has the great advantage that it is designed for realtime sound synthesis with interactive coding allowing for fast prototyping [17]. As an example of coding in the language, the first code block presented here generates the correct shape on the critical line for the eta function (here, $0 \leq t \leq 30$). Sound and code examples are available along with the release of this paper.[3]

```
(0,0.1..30.0).collect{|t|
var real ,imag;
var signal = Array.fill(100,{|i|
      var n = i+1;
      var phase = t * log(n) + (pi*i);
      //+0.5pi to make cosine, pi*i is (-1)
          **(n-1) when put through cosine

      (n**(-0.5))*[sin(phase+0.5pi),sin(
          phase)];
});

signal = signal.sum;

real = signal[0];
imag = signal[1];

((real*real) + (imag*imag)).sqrt;
}.plot
```

This static generation can be turned into live synthesis. There is a limit to the number of sinusoids summed within a single SynthDef, which can be overcome by writing a new UGen specific to the synthesis capability desired. The EtaFunction UGen utilises a pre-calculated listing of the natural logs of the first 5000 positive integers, as part of strategies for sample by sample rendering of the naive sum.

```
({
var t = Phasor.ar(0,MouseX.kr(1,1000)/
    SampleRate.ir,10,30);
var n = K2A.ar(MouseY.kr(1,200));
var zeta = EtaFunction.ar(DC.ar(0.5),t,n);
var real = zeta[0]; var imag = zeta[1];
Limiter.ar(((real*real) + (imag*imag)).sqrt
     *0.5);
}.scope
)
```

Berry [4] builds up a picture of the primes from the Riemann zeta function zeroes (the equation is further discussed by Mazur and Stein [3, p.110]). In this sense, the prime numbers (due to technicalities, along with the prime powers p^n) have a spectrum defined by the zeta function zeroes ρ, and vice versa.

[3]https://composerprogrammer.com/research/ICAD2019examples.zip

$$primeapproximation_N(x) = -\sum_{n=1}^{N} \cos(\rho_n \log x) \quad (6)$$

$$riemannapproximation_N(\theta) = 2\sum_{p^n<N} p^{-n/2} \log(p) \cos(\theta n \log p) \quad (7)$$

Taking the first of these, the approximation of a prime spectrum from zeta zeroes, in SuperCollider code this might be calculated in the language assuming ˜riemannzeroes is an array of the first 1000 non-trivial zeroes:

```
var primesignal = Array.fill(1000.min(~
    riemannzeroes.size),{|i|
      cos((~riemannzeroes[i]) * log(x))
});
primesignal.sum.neg.plot;
```

The equation can also be rendered via UGens for direct synthesis:

```
var primesignal = Mix.fill(1000.min(~
    riemannzeroes.size),{|i|
      cos((~riemannzeroes[i]) * log(K2A.ar(
          MouseX.kr(2,100)).lag(0.1)))
});
```

In order to explore this further live, a UGen *PrimeSpectrum* was created, with critical input parameters x (for position along the real axis) and N (for the number of zeta zeroes utilised in the sum):

```
{
var x = SinOsc.ar(MouseX.kr(1,100)).range
    (2,100);
var n = K2A.ar(MouseY.kr(1,1000));
//limit and scale down due to larger
    outputs well outside -1 to 1
Limiter.ar(PrimeSpectrum.ar(x,n)*0.1);
}.scope
```

The live modulation of the number of zeroes made available in the approximation is an interesting effect; the clarity of the image of primes and their powers is enhanced the more zeroes are committed. As well as losing their sharp spikes at primes and prime powers, the y values fall off to the negative as more and more zeroes are missed off. The UGen has a maximum of 5000 zeroes based on its internal database of zeroes and the viability of calculation. Very high x can be calculated without sufficient zeroes in the calculation, and the resulting noisy output is still an attractive synthesis resource. This openness to noise in results differs from Berry's strait-laced and rather tonal-centric earlier conception. Similarly, a RiemannSpectrum UGen was built, through a precalculated table of prime powers and with arguments for the position along the spectrum and the number of components in the sum.

Expressions of the form $\cos(t\log(n))$ are easily deployed in SuperCollider unit generator graphs to create novel sound timbres.

The log acts to compress larger numbers more, that is, provide a nonlinear waveshaping (requiring strictly positive input; ≥ 1 is used in the examples here). The t scales the output within the phase input for a cosine function. There is a relationship here to the nonlinearities possible through frequency modulation synthesis [18]. Unlike calculating full sums, use of this formula for an individual summand is very low cost on a modern CPU, with synthesis of these patches just a matter of percentage points.

In the first SuperCollider example, a sine oscillator is used to sweep up and down in n, with the range of sweep under mouse control on the Y axis, and a further mouse control on the X axis for t:

```
{cos(MouseX.kr(1,100) * log( abs(SinOsc.ar(
    MouseY.kr(1,1000,'exponential'))*10) +
    1))}.play
```

The abs function causes full wave rectification, which has the spectral effect of creating many harmonics of the base frequency with amplitude fall off as per $\frac{1}{1-4k^2}$ for harmonic multiple k. The log function can be analysed as follows:

$$\begin{aligned} \log(10|\cos(x)|+1) &= \\ \log 10 + \log(|\cos(x)| + \frac{1}{10}) &= \\ \log 10 + \log(|\cos(x)| - \frac{9}{10} + 1) &= \\ \log 10 + \sum_{k=1}^{\infty} \frac{(|\cos(x)| - \frac{9}{10})^k}{k} & \end{aligned} \tag{8}$$

using the power series formula for $\log(1+x)$ since $||\cos(x)| - \frac{9}{10}| < 1$. The final expression illustrates the complexity of harmonics that will arise from the composition of absolute and log operators here; the powers by k of a sum of harmonics will lead to a further set of ring modulated components at sum and difference frequencies (at further multiples of the harmonics, reinforcing in a complex way the amplitude of the original absolute of sinusoid harmonics). The final expression is then the modulator for frequency modulation, leading to complicated sidebands from modulations of modulations, with amplitudes following a product of Bessel functions [19]. Since all the input sinusoids forming the modulator are harmonically related, the output of FM will be itself at harmonic frequencies, with a very complicated distribution of energy. The t scale factor will scale the indices of modulation, increasing the audibility of sidebands (harmonics) for larger t.

The second SuperCollider interactive example demonstrates a nice combination, two trigonometric expressions of the argument $t\log(n)$ fluttering against each other, with user control via mouse of t and the contrasting rate of sweep on n between the two components:

```
( {
var y = MouseY.kr(1,1000,'exponential');
var n = SinOsc.ar(4,0,y).abs +1;
var m = SinOsc.ar(7.9,0,y).abs +1;
var t = MouseX.kr(1,100);

(n.squared.reciprocal)* cos(t * log(n) ) +
(m.squared.reciprocal) * sin(t * log(m) )
}.play
)
```

The last example is a variant of the main naive sum formula for complex output, here sonified to left and right stereo positions, with 40 summands, and a phasor continually incrementing t in a sawtooth rise, the rate of progress controlled by the user via mouse X position on screen. The tanh function is used to keep the final output within bounds and provide a little distortion edge:

```
({
var alpha = 0.5;
var t = Phasor.ar(0,MouseX.kr(1,10000,'
    exponential')/SampleRate.ir,1,100);

tanh(
      Mix.fill(40,{|i|
      var n = i+1;
            var temp = t * log(n);
            (n**(alpha.neg)) * ((-1)**i) *
                [cos ( temp), sin ( temp)] ;
}));

}.play
)
```

4. SPACINGS BY ZEROES

The set of zeroes ρ of the zeta function are an interesting resource for the spacing of events.

Figure 2 presents the spacing of the first seventeen zeros of the zeta function,[4] as a rhythm.

To four decimal places, the rhythmic events are at:

14.1347, 21.022, 25.0109, 30.4249, 32.9351, 37.5862, 40.9187, 43.3271, 48.0052, 49.7738, 52.9703, 56.4462, 59.347, 60.8318, 65.1125, 67.0798

The inter-onset interval (the gaps) corresponding to these, including the rest at the start before the first event, are:

14.1347, 6.8873, 3.9888, 5.414, 2.5102, 4.6511, 3.3325, 2.4084, 4.6781, 1.7687, 3.1965, 3.4759, 2.9008, 1.4847, 4.2808, 1.9673, 2.4666

An optimal re-scaling was sought by exhaustive grid search to bring this set of event times as close as possible to uniform 24[th] notes (0.16666 of a beat), leading to the quantised solution:

1, 0.5, 0.3333, 0.3333, 0.1667, 0.3333, 0.1667, 0.1667, 0.3333, 0.1667, 0.1667, 0.1667, 0.1667, 0.1667, 0.3333, 0.1667, 0.1667

Listening to the original zero spacing, versus this quantisation, the two are clearly distinct, though the general shape of the former is captured by the approximation.

Results on the number of zeroes up to height T tend towards:

[4]Tables of zeroes are available online, see for instance `http://www.dtc.umn.edu/~odlyzko/zeta_tables/index.html` as well as multiple associated pages across the On-Line Encyclopedia of Integer Sequences `https://oeis.org/A013629`. There are web pages that allow access to higher runs of the zeroes `http://www.lmfdb.org/zeros/zeta/?limit=100&t=100000`

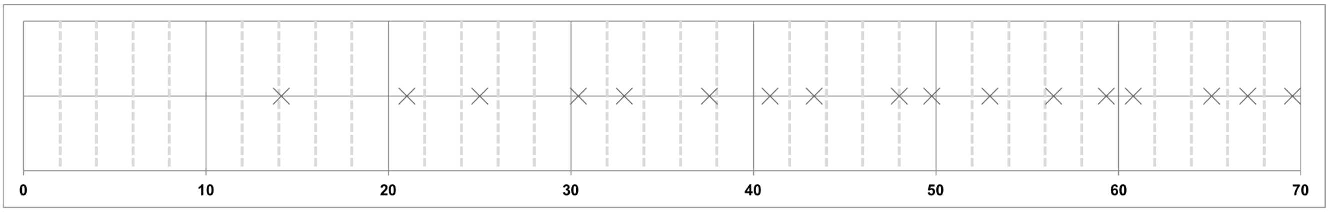

Figure 2: A Riemann zeta function rhythm: the spacing of the first seventeen zeroes

$$N(T) \approx \frac{T}{2\pi} \log \frac{T}{2\pi e} \quad (9)$$

The spacing of the zeroes is closer over increasing t, the difference being approximately $\frac{2\pi}{\log n}$,[5] asymptotically the n^{th} zero appearing at $\frac{2\pi n}{\log n}$. This is actually an inverse dual to the primes, which are approximately distributed such that the number to n are $\frac{n}{\log(n)}$ and the n^{th} prime appears around $n\log(n)$.[6] High up the critical line Odlyzko demonstrates three consecutive zeroes separated by only around 0.05 for t proximate to 10^{22} [20]. Synthesis for low $n = 1..10$ for the approximate equation (and cumulative sum) leads to the values

9.065, 14.784, 19.316, 23.22, 26.727, 29.956, 32.977, 35.837, 38.566, 41.186

We can synthesize rhythms using this approximation equation, or quantised approximations to the approximation, and scale them as needed. Due to the slowing of the log function, which can be thought of as only really increasing by 1 on the order of each additional digit in the decimal number[7], short runs in n at high t will be approximately linear. So the most interesting part of this formula for rhythms is the early part. There are studies which show that for high t the spacing of the zeta function takes on a random aspect relating to the eigenvalues of certain random matrices studied in physics [21, 22]; in a related publication, Michael Berry has sonified various categories of random matrix, including in comparison to Riemann zero data, using frequency modulation [23].

Following the dual viewpoint of rhythmic grids and tuning systems [24], the data set of the zeroes is also immediately applicable to construct pitch systems.[8] A scale based on the spacing of the first 10 zeroes, if constrained proportionally within one octave and with initial gap from 0 preserved, would be mapped to the interval [0,1] as follows to 4 d.p.:

0, 0.284, 0.4224, 0.5025, 0.6113, 0.6617, 0.7551, 0.8221, 0.8705, 0.9645, 1

Adding 1 to all values would shift these to be ratios from unison 1 to standard octave 2.

Or as cents (value*1200), to 2 d.p. given how a single cent is well under the just noticeable difference, in the absence of beating phenomena from simultaneous presentation:

0, 340.77, 506.82, 602.99, 733.51, 794.03, 906.17, 986.51, 1044.57, 1157.36, 1200

This reveals a proximity to 12TET 4th, tritone, flattened sixth, and sixth. If quantised to bare 12TET MIDI notes the mapping is non-injective:

60, 63, 65, 66, 67, 68, 69, 70, 70, 72, 72

Of course, any number of zeroes can be taken, and the squashing of zeroes together with increasing t will lead to the top part of the scale having increasingly many microtones relative to the initial steps. A scale can be constructed with respect to any enclosing ratio (in the manner of the Bohlen-Pierce 'tritave' of a ratio of 3 or arbitrary ratio r [25]). Scales can also be devised on the primes or prime powers: Though tuning systems are often built using prime powers (for example, 3-limit Pythagorean tuning constructed by rationals of powers of 2 and 3), a tuning system literally lifted from the prime number (or prime powers) spacing is a rarer beast, Roger Dean providing one counter-example by constructing scales based on prime harmonics of a fundamental [26].[9] Given the dual location equations between the primes and the zeroes at $n \log n$ and $\frac{n}{\log n}$ respectively, one attractive potential musical resource is an alternation of expanding and contracting spacings following these formulae.

The signals discussed earlier in the paper, for instance, the approximate prime and Riemann spectra or the eta function rendered over time for changing s, can themselves be the trigger for discrete materials, by the use of such techniques as peak picking and onset detection reacting to extrema in signal or derivative rather than zero crossings. Such discrete sets of values, or the original zeroes, might also be scaled and rounded off to become indices into any set of musical objects, such that aside from the spacing of events and materials for pitch systems, the sequence of positions could control arbitrary parameters in sound synthesis and algorithmic composition.

[5] $\frac{2\pi(n+1)}{\log(n+1)} - \frac{2\pi n}{\log(n)} \approx \frac{2\pi(n+1)}{\log(n)} - \frac{2\pi n}{\log(n)} = \frac{2\pi}{\log n}$

[6] Rosser's theorem states $p_n > \frac{n}{\log(n)}$; the actual prime counting function is a result of Riemann including an effect of the zeroes of the zeta function.

[7] To a scale factor of $\log(10)$, the number of decimal digits in a number is $\log_{10}(n) = \frac{\log(n)}{\log(10)}$

[8] Berry acknowledges that arbitrary enclosing intervals can be taken, but restricts himself in examples to the piano keyboard [4]

[9] A fascinating unpublished manuscript by Peter Buch available online [27] uses the Riemann zeta function as a way to find low integer steps per octave that best approximate pure just intonation ratios; the correspondence that arises with often mentioned steps per octave in the tuning literature (7, 12, 19 et al.) is impressive.

5. CONCLUSIONS

This paper has explored some further mapping possibilities to sound for equations associated with the Riemann zeta function. We have embraced some noisier possibilities and not assumed 12 note equal temperament or any other discrete system is the final aim in rendering to musical sound, though examples have included applications in rhythms and pitch scales. In a number of places we have observed some perceptual limits on the use of such mathematics, and an observer is highly unlikely to recover deep mathematical knowledge of the zeta function or primes from the sonifications. Aesthetic choices in mapping delimit the scientific result [28, 29], and we tend more to the aesthetic potential here.

There remains a huge amount of fascinating mathematics to explore for novel musical mappings. Accompanying the core Riemann Hypothesis are a host of mathematical equivalents, including statements about such mathematical objects as Farey sequences of rationals and permutation groups, of potential applicability in artistic sonification [30]. Indeed, number theory contains many more special kinds of number and number theoretic functions of potential interest to composers.

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