STUDY OF $\pi\pi$ ELASTIC SCATTERING USING THE CHEW-LOW EXTRAPOLATION METHOD*

S. Marateck, V. Hagopian, and W. Selove
University of Pennsylvania, Philadelphia, Pennsylvania

and

L. Jacobs
Lawrence Radiation Laboratory, University of California, Berkeley, California, and CERN, Geneva, Switzerland

and

F. Oppenheimer and W. Schultz
University of Colorado, Boulder, Colorado

and

L. J. Gutay and D. H. Miller
Purdue University, Lafyette, Indiana

and

J. Prentice and E. West
University of Toronto, Ontario, Canada

and

W. D. Walker
University of Wisconsin, Madison, Wisconsin
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The cross sections for $\pi^-\pi^0$ and $\pi^-\pi^+$ elastic scattering, and the $I=0$ s-wave and $I=1$ p-wave phase shifts, $\delta_0^{\,0}$ and δ_1, are calculated for $400 < M_{\pi\pi} < 900$ MeV using a Chew-Low extrapolation. The extrapolation information permits resolving a fourfold ambiguity which has been present in other attempts to determine $\delta_0^{\,0}$ from the reaction $\pi^-p \to \pi^+\pi^-n$. The result for $\delta_0^{\,0}$ is consistent with a curve which increases through 90° around $M_{\pi\pi}=720$. The ρ width obtained from the extrapolated p-wave amplitude is 105 ± 15 MeV.

In this paper we present the results of a study of $\pi^-\pi^+$ and $\pi^-\pi^0$ elastic scattering for di-pion masses between 400 and 900 MeV using the Chew-Low extrapolation method.[1] Recently, Barton, Laurens, and Reignier[2] at Saclay reported on their results for a study of $\pi^-\pi^0$ elastic scattering using the extrapolation method.

Our analysis is for a compilation of $\pi^- + p$ data with beam momenta between 1.89 and 3.00 BeV/c and $\Delta^2 \leq 12\mu^2$ for

$$\pi^- + p \to \pi^- + \pi^+ + n, \quad 14\,890 \text{ events}, \tag{1}$$

$$\pi^- + p \to \pi^- + \pi^0 + p, \quad 8124 \text{ events}, \tag{2}$$

where Δ^2 is the square of the four-momentum transfer between the two nucleons and μ is the mass of the charged pion.

The Chew-Low formula relating the $\pi\pi$ elastic cross section $\sigma_{\pi\pi}$ to measurements in the physical region is

$$\sigma_{\pi\pi} = \lim_{\Delta^2 \to -\mu^2} [F/(\Delta^2/\mu^2)], \tag{3}$$

where

$$\frac{F}{\Delta^2/\mu^2} = \frac{2\pi K^2 \mu^2 (\Delta^2 + \mu^2)^2 (\partial^2\sigma/\partial\omega^2\partial\Delta^2)}{[f^2 \Delta^2 \omega (\frac{1}{4}\omega^2 - \mu^2)^{1/2}]}, \tag{4}$$

and ω is the di-pion mass $M_{\pi\pi}$, K is the incident pion momentum in the laboratory system, f^2 is the pion-nucleon coupling constant which equals 0.081 for the $\pi^-\pi^0$ system and 0.162 for the $\pi^-\pi^+$ system, and $\partial^2\sigma/\partial\omega^2\partial\Delta^2$ is the experimental differential cross section which is proportional to $dN/d\Delta^2$,[3] the number of events per unit Δ^2 interval for a fixed $M_{\pi\pi}$ interval. We note that also

$$\sigma_{\pi\pi} = \lim_{\Delta^2 \to -\mu^2} (-F). \tag{5}$$

The method used in the extrapolation is indicated in Fig. 1. Figure 1(a) shows a plot of $dN/d\Delta^2$ vs Δ^2 for Reactions (1) and (2) for $760 \leq M_{\pi\pi} \leq 780$ MeV. The values of F/Δ^2 obtained from the $dN/d\Delta^2$ distribution are plotted as a function of Δ^2 as shown in Fig. 1(b). After fitting a curve

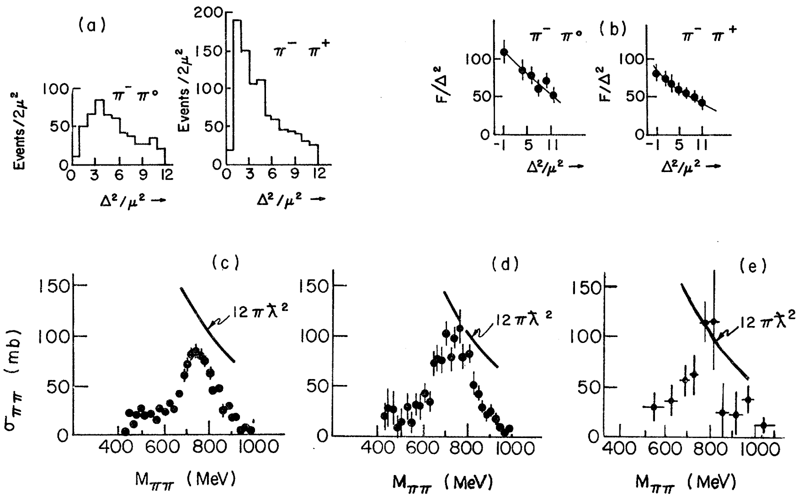

FIG. 1. (a) $dN/d\Delta^2$ vs Δ^2 for Reactions (1) and (2) for $760 \leq M_{\pi\pi} < 780$ MeV. (b) F/Δ^2 vs Δ^2 for Reactions (1) and (2) for $760 \leq M_{\pi\pi} < 780$ MeV. (c) and (d) Extrapolated cross sections as a function of $M_{\pi\pi}$ for (−+) and (−0) systems, respectively, using a linear fit in F/Δ^2. (e) Extrapolated cross section for (−+) system using F (quadratic fit).

(or a straight line) through these points, we extrapolate the curve to $\Delta^2 = -\mu^2$, obtaining $\sigma_{\pi\pi}$ for the $M_{\pi\pi}$ interval. If the Δ^2 dependence of $dN/d\Delta^2$ agrees exactly with that of one-pion exchange (OPE) in the physical region, F/Δ^2 will be independent of Δ^2. Empirically F/Δ^2 varies with Δ^2. With our statistics and $(\Delta^2)_{\min}$ cutoff [typically $(\Delta^2)_{\min} \sim \mu^2$ for the $\pi^-\pi^+$ system and $\sim 3\mu^2$ for the $\pi^-\pi^0$ system], we find that a straight-line fit to F/Δ^2 is adequate.

In order for a low-order power series in F/Δ^2 to provide a suitable basis for extrapolation, $dN/d\Delta^2$ must vanish at $\Delta^2 = 0$ as OPE predicts. If it is finite at $\Delta^2 = 0$, then F/Δ^2 will diverge at $\Delta^2 = 0$. However, an extrapolation can be made using F instead of F/Δ^2; even if $dN/d\Delta^2$ is finite at $\Delta^2 = 0$, F will not diverge. The only difficulty in extrapolating F to the pole is that in general it requires much better statistics than an extrapolation of F/Δ^2. E.g., a linear extrapolation of F/Δ^2 corresponds to having a precise constrained value of $F = 0$ at $\Delta^2 = 0$ (which is a value of Δ^2 "very close" to the pole at $\Delta^2 = -\mu^2$) in a quadratic extrapolation of F.

In Fig. 1(a), one sees evidence that there is a bias present in the experimental data for Reaction (2) which is not present in (1)—as Δ^2 decreases, $dN(+-)/d\Delta^2$ continues to rise until $\Delta^2 = \mu^2$ or so, whereas $dN(-0)/d\Delta^2$ begins to fall at $\Delta^2 = 3\mu^2$. This bias is due to the difficulty of seeing short protons and is completely clear in a study of the elastic π^-p reaction as is shown by Baton.[4] In fact, Baton's analysis shows that his $dN(-0)/d\Delta^2$ distribution is relatively free of biases above $\Delta^2 = 3\mu^2$. Thus, for Reaction (2) we use only events with $\Delta^2 \geq 3\mu^2$. Figure 1(b) shows a plot of F/Δ^2 vs Δ^2 for $760 < M_{\pi\pi} \leq 780$ MeV—the same di-pion mass interval used in Fig. 1(a)—for both charged states.

Figures 1(c) and 1(d) show the extrapolated cross section obtained by a linear extrapolation of F/Δ^2 for Reactions (1) and (2), respectively, plotted as a function of $M_{\pi\pi}$. The unitarity limit for a $J = 1$ resonance is also plotted here. $\sigma_{\pi\pi}(-0)$ agrees with the unitarity limit at the ρ mass, but $\sigma_{\pi\pi}(-+)$ is 20% lower than the unitarity limit. The fact that ω exchange can occur for the (−0) mode may account for this difference. At this time we know of no extrapolation procedure for F/Δ^2 practical for the present statistics which will yield the same $\sigma_{\pi\pi}$ for both the (−0) and (−+) systems.

Recent results of Selove, Yuta, and Forman[5] for $\pi^- + p \to \pi^- + \pi^+ + n$ at a beam momentum of 8 BeV/c indicate that as Δ^2 decreases below μ^2 the experimental distribution does not fall as OPE predicts but instead is either level or rising. Thus, there is no a priori reason to believe that $dN/d\Delta^2$ should vanish at $\Delta^2 = 0$. Even if it does not, we can extrapolate F, of course. The result, for the $\pi^-\pi^+$ data, is shown in Fig. 1(e). The statistical accuracy is now quite poor, but the results are not inconsistent with the unitarity limit.

The Saclay group,[2] because they were analyzing only one experiment (at a beam momentum of

2.77 BeV/c), could evaluate their biases for low Δ^2 in Reaction (2), and thus could cut off at $\Delta^2 = 2\mu^2$ for most of their data. We could not evaluate the cumulative biases at low Δ^2 in the $\pi^-\pi^0$ system since we do not have the comparable elastic data; moreover, the statistical accuracy for each contributing laboratory is relatively low. This forced us to cut at $\Delta^2 = 3\mu^2$. The resulting lack of statistical accuracy prevented us from extrapolating F for the (−0) system.

We now discuss the determination of the $\pi\pi$ phase shifts by extrapolation. The lack of statistical accuracy for Reaction (2), due to the Δ^2 cutoff at $3\mu^2$ previously discussed, also prevented us from doing a detailed analysis of the phase shifts for Reaction (2). A rough analysis of the angular distribution for Reaction (2), however, does yield results consistent with those from Saclay. We now go on to discuss the phase-shift analysis for Reaction (1).

The differential form of the Chew-Low formula is

$$\frac{d\sigma}{d\cos\theta} = \lim_{\Delta^2 \to -\mu^2} F'(\omega^2, \Delta^2, \cos\theta), \tag{6}$$

where

$$\frac{F'}{\Delta^2/\mu^2} = \frac{2\pi K^2 \mu^2 (\Delta^2+\mu^2)^2 (\partial^3\sigma/\partial\omega^2\partial\Delta^2\partial\cos\theta)}{[f^2\Delta^2\omega(\tfrac{1}{4}\omega^2-\mu^2)^{1/2}]}, \tag{7}$$

and θ is the $\pi\pi$ scattering angle in the di-pion center-of-mass system. Fixing an $M_{\pi\pi}$ interval, and noting that $\partial^3\sigma/\partial\omega^2\partial\Delta^2\partial\cos\theta$ is proportional to $\partial^2 N/\partial\Delta^2\partial\cos\theta$, which we abbreviate as $dN/d\cos\theta$, we expand in a power series in $\cos\theta$, keeping only terms corresponding to s and p waves:

$$dN/d\cos\theta = a_0 + a_1\cos\theta + a_2\cos^2\theta. \tag{8}$$

Figure 2(a) shows $dN/d\cos\theta$ as a function of Δ^2 for $740 \le M_{\pi\pi} < 780$. In order to carry out the extrapolation, we first fit these distributions with Eq. (8) to find the $a_l(\Delta^2)$. We then multiply each a_l by $(\Delta^2+\mu^2)^2/\Delta^2$, which is the functional dependence of F'/Δ^2 on Δ^2 prescribed by the "pure OPE" formula (which assumes $F'=0$ at $\Delta^2=0$). We thus obtain coefficients A_l, defined by

$$A_l = a_l(\Delta^2+\mu^2)^2/\Delta^2. \tag{8}$$

We then do an extrapolation of the A_l as shown in Fig. 2(b). This is analogous to an extrapolation of F/Δ^2. Unfortunately, we do not have sufficient statistical accuracy to perform an extrapolation of $a_l(\Delta^2+\mu^2)^2$, which would correspond to an extrapolation of F. We will explain below why we feel that this procedure is justified even though the extrapolation of F/Δ^2 defined by Eq. (4) yielded a result 20% lower than the unitarity limit.

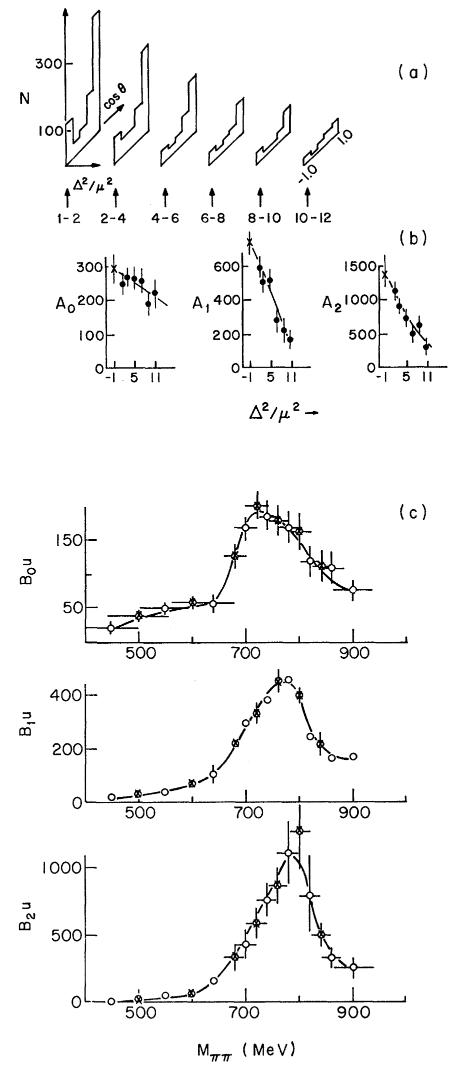

FIG. 2. (a) $dN/d\cos\theta$ vs Δ^2 for (−+) system for $740 \le M_{\pi\pi} < 780$ MeV. The ordinate is the number of events in a bin for which $\delta\Delta^2 = 2\mu^2$ and $\delta\cos\theta = 0.4$. (b) A_l vs Δ^2 for $740 \le M_{\pi\pi} < 780$ MeV, where $A_l = a_l(\Delta^2+\mu^2)^2/\Delta^2$; the a_l are obtained from (a). The point at −1 is the extrapolated point. (c) B_0u, B_1u, B_2u vs $M_{\pi\pi}$ where $u = (\frac{1}{4}\omega^2 - \mu^2)^{1/2}/\omega^2$. The crosses and circles show the results for two overlapping choices of mass intervals.

A linear fit is used to extrapolate the A_0 and A_1 and for most mass intervals a quadratic fit is used to extrapolate A_2. The choice of linear and quadratic fits is made on the basis of a detailed study of the data for all $M_{\pi\pi}$ intervals.

In Fig. 2(c) the extrapolated values of the A_l

multiplied by a phase-space factor $u=(\frac{1}{4}\omega^2-\mu^2)^{1/2}/\omega$ are plotted. We define

$$B_l = A_l(\Delta^2 = -\mu^2). \tag{10}$$

The factor u relates the B_l directly to the phase shifts as follows

$$B_0 u = \{4/9\sin^2\delta_0^0 + \tfrac{1}{9}\sin^2\delta_0^2 + 4/9\cos(\delta_0^0-\delta_0^2)\sin\delta_0^0\sin\delta_0^2\}D,$$
$$B_1 u = \{4\cos(\delta_0^0-\delta_1)\sin\delta_0^0\sin\delta_1 + 2\cos(\delta_0^2-\delta_1)\sin\delta_0^2\}D,$$
$$B_2 u = \{9\sin^2\delta_1\}D, \tag{11}$$

where δ_J^T denotes the phase shift for isospin T and spin J, and D is a normalization constant. OPEA (absorption-modified OPE theory) predicts that even though the ρ is a $J=1$ resonance, its $dN/d\cos\theta$ distribution will not be pure $\cos^2\theta$ even at $\Delta^2\to 0$, but instead will also have an isotropic part. Thus one would expect that unless the extrapolation is exactly correct, the $M_{\pi\pi}$ dependence of B_0, the isotropic term, will reflect the shape of B_2. We in fact observe this to be true for the $\pi^-\pi^0$ system as does the Saclay group. For the $\pi^-\pi^+$ system, B_0 does not reflect the shape of B_2 as can be seen from Fig. 2(c); B_0 peaks rather sharply near 720 MeV whereas B_2 peaks at 785 MeV. Other analyses[6-8] (based on much of the same data used here, but carried out by methods other than extrapolation) have suggested that the $T=0$ s-wave phase shift is in the vicinity of 90° near 700 MeV. Our result for B_0 seems to be in agreement with this. To explore the situation further, we investigate the behavior of the phase shifts as a function of $M_{\pi\pi}$.

We normalize the plot of B_2 (which represents the ρ devoid of background if the extrapolation has been done correctly) so that the p-wave phase shift goes through 90° at $M_{\pi\pi}=785$ MeV. We interpret the high point at 800 MeV as a statistical fluctuation. (Perhaps it indicates a residual contribution from $\omega^0\to\pi^-\pi^+$.) The full width at half-maximum, Γ_ρ, is 105 ± 15 MeV. It has recently become apparent that different values of Γ_ρ are obtained from different reactions and that the value also varies with the analysis method. E.g., a compilation by Pisut and Roos[9] in the physical region yields $\Gamma_\rho=147\pm4$ MeV; the e^+e^- colliding beam experiment of Auslander et al. at Novosibirsk[10] yields $\Gamma_\rho=93\pm15$ MeV; and the Saclay group, using two different methods of analysis, has obtained two different values from the same data—$\Gamma_\rho=110\pm9$ MeV using extrapolation,[2] and $\Gamma_\rho=150\pm5$ MeV using a different modification of the physical region data.[4] Our result, $\Gamma_\rho=105\pm15$ MeV, agrees with the Novosibirsk e^+e^- result and with the Saclay extrapolation result.

To calculate δ_0^0 we use the ratio of B_1 to B_2, and do not use B_0 directly. We use the result of Ref. 2 for δ_0^2 (our own $\pi^-\pi^0$ results are in agreement, but less accurate). We then use Eq. (11) and B_1/B_2 to determine δ_0^0. The reason we do not use B_0 for this purpose is that the shape of A_0 as a function of $M_{\pi\pi}$ can be expected—throughout the physical region and even at $\Delta^2=0$—to reflect residual absorption effects from the ρ. The contributions of these effects to A_0 can be estimated theoretically to be large compared with the s-wave contribution. Thus, our extrapolated value of B_0 would be incorrect. The effect of absorption on the ratio A_1/A_2, however, is not as great. Durand,[11] and Bander, Shaw, and Fulco,[12] suggest that the ratio A_1/A_2 can be expected to be relatively free of absorption effects in the physical region. Consequently, we expect that B_1/B_2 is not strongly affected by absorption effects.

As first noted by Gutay et al.,[8] the determination of δ_0^0 using B_1/B_2 in this way produces in general a pair of possible solutions, δ_0^0 and $\delta_0^{0\prime}$, where $\delta_0^{0\prime}=\frac{1}{2}\pi+\delta_1-\delta_0^0$. It should be also noted that δ_0^0 (and all phase shifts) can be determined only modulo π. We shall define all phase shifts as being zero at threshold.

In Fig. 3(a) we plot δ_0^0 vs $M_{\pi\pi}$. For $M_{\pi\pi}$ below the ρ mass, our choice of the modulo-π factor, for both branches shown, is arrived at as follows: (1) We take δ_1 to decrease from $\frac{1}{2}\pi$ at the

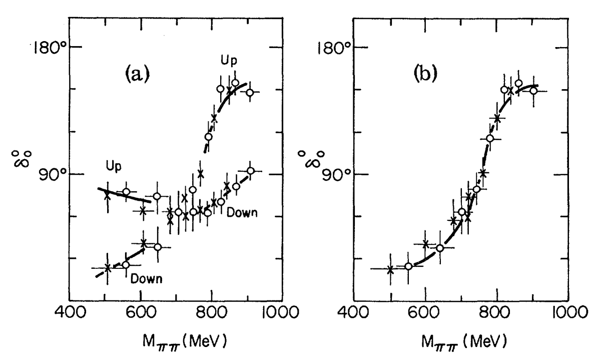

FIG. 3. (a) the s-wave $T=0$ phase shift δ_0^0 vs $M_{\pi\pi}$. (b) The preferred "down-up" solution for δ_0^0. The crosses and circles have the same meaning as for Fig. 2(c).

ρ to zero at threshold without again crossing any resonant value. We estimate δ_1 at 500 MeV to be about 8°. (2) The positive value of B_1 from 700 MeV down to 500 MeV, taken together with the absence of visible sharp resonances in $\sigma_{\pi\pi}$ below 500 MeV in the $\pi^-\pi^+$ system, then indicates that $\delta_0{}^0$ is in the first quadrant—i.e., between 0 and 90°.

For $M_{\pi\pi}$ above the ρ mass, we have plotted those two solutions, $\delta_0{}^0$ and $\delta_0{}^{0\prime}$, which lie between 0° and +180°. Choices differing from these by π, if taken together with either of the two lower $M_{\pi\pi}$ branches, would correspond to large and sudden variations in B_1 in the ρ mass region, which are not observed.

As Fig. 3(a) indicates, $\delta_0{}^0$ and $\delta_0{}^{0\prime}$ become degenerate from 680 to 740 MeV; this degeneracy occurs when $\delta_0{}^0$ happens to equal $\frac{1}{2}\delta_1+\frac{1}{4}\pi$. Within our experimental error, four different smooth curves can be drawn through the points in Fig. 3(a). Following the notation of Malamud and Schlein[7] we can call these the up-up, up-down, down-up, and down-down solutions.

We believe it is possible to make some selection among these on the basis of the shape of $B_0\mathcal{U}$ as seen in Fig. 2(c). The sharp peak in $B_0\mathcal{U}$ near 720 MeV suggests that $\delta_0{}^0$ is passing through 90° more or less rapidly there. This rules out the up-down and down-down solutions. Moreover, the fact that $B_0\mathcal{U}$ falls very sharply as $M_{\pi\pi}$ goes from 720 MeV down to 640 MeV rules out the up-up solution. This leaves only the down-up solution, which we show by itself in Fig. 3(b).

Finally, we remark on an estimate of the theoretical uncertainty associated with our results for $\delta_0{}^0$. We have extrapolated $a_l(\Delta^2+\mu^2)^2/\Delta^2$ rather than $a_l(\Delta^2+\mu^2)^2$. Since the corresponding extrapolation of F/Δ^2 rather than F gives a result for $\sigma_{\pi\pi}(-+)$ which is 20% below the unitarity limit at the ρ peak, we might therefore expect an error of about 20% in B_1 and B_2, and perhaps a larger error in B_0. As mentioned above, absorption-effect theory suggests that the ratio B_1/B_2 should not be affected as much as the B_l themselves. We do not know how to estimate the theoretical uncertainty with any precision. Nevertheless, we believe that the ratio B_1/B_2 is likely to be no more than 20% in error. Such a 20% effect would change $\delta_0{}^0$ at say 600 MeV by only about 5°—this is not large compared with the statistical uncertainty.

Our final results for $\delta_0{}^0$ are consistent with results obtained by previous workers,[6-8] who have found results similar to the four solutions shown in Fig. 3(a). With the additional information provided by the extrapolation result for $B_0\mathcal{U}$, it appears possible to select the single preferred solution of Fig. 3(b) from these four. We note that our Fig. 3(b) agrees well with one of the curves suggested by Lovelace[13] from an analysis of πN backward scattering.

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