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ANGULAR MOMENTUM ANALYSIS OF PIONIC NUCLEON CAPTURE

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A B S T R A C T

The pionic capture reaction $A(N,\pi)A+1$ is analyzed in terms of the Mandelstam model for pion production by two nucleons. The results are applied to the proton capture reactions $^{12}C(p,\pi^+)^{13}C$ and $^{4}He(p,\pi^+)^{5}He$, and comparisons are made with experiments.

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1. - INTRODUCTION

There has recently appeared a fair amount of experimental as well as theoretical work on the subject of pionic capture of nucleons, i.e., the reactions in which the target nucleus captures an incident nucleon and emits a pion. An experimental group at CERN [1),2)] has studied the (p,π^+) reaction for a number of target nuclei in the mass region from deuterium to ^{14}N, for 600 MeV protons and at forward angle. For the reactions going to the ground state of the final nucleus the laboratory differential cross-section ranges from some 30 $\mu b/sr$ for deuterium to about 0.4 $\mu b/sr$ for ^{13}C and ^{14}N. A group at Uppsala [3),4)] has measured differential cross-sections for the target nuclei ^9Be and ^{12}C at the proton energy 185 MeV. A number of final nuclear states are observed in the experiment, partly with quite striking angular distributions of pions.

On the theoretical side, the situation is somewhat unclear, and several reaction mechanisms and models have been proposed. The one-nucleon model of Eisenberg et al. [5),6)] is based on the assumption that the incoming nucleon emits a pion and is captured directly into a nuclear orbit. In the absence of rescattering corrections, the cross-section is then directly proportional to the square of the one-nucleon wave function in momentum space, evaluated at the momentum transfer in question. Since the momentum transfer is large the cross-section is very sensitive to the nuclear model [2)]. A model in the same spirit has been studied by Chatwin [7)], the idea being to treat the basic reaction $p \rightarrow n + \pi^+$ as analogous to a nuclear stripping process. On the other hand one has the models in which one considers two nucleons in the basic pion-production process. Of this type is the model of Ingram et al. [8)], where the input is the cross-section for the reaction $pp \rightarrow d\pi^+$. Another such model was considered by the present author [9)], the reaction being assumed to take place by one-pion exchange through the (3,3) resonance.

One of the most characteristic differences between the one and two-nucleon reaction mechanisms is their different predictions concerning the rare (p,π^-) capture mode. In the one-nucleon case this mode is forbidden, whereas in the particular two-nucleon model of Ref. 9) the π^+/π^- ratio is of the order 10. Experimentally [10)] this ratio is 100-200, which indicates that the mechanism for a general (N,π) reaction is not as simple as assumed in any of the models mentioned above.

In the present paper we make an attempt at presenting a phenomenological rather than a dynamic description of the (N,π) reactions. Our treatment is based on the Mandelstam [11] model for the reactions $NN \to NN\pi$. In this model the reaction amplitude is decomposed into various angular momentum modes, with strength parameters that can be adjusted to fit the $NN \to NN\pi$ and $pp \to d\pi^+$ data. The same amplitudes are then used for the nuclear case in an impulse approximation. Our model is thus similar in spirit to that of Ingram et al. [8] ; however, in our case the nuclear reaction is analyzed in terms of several $NN \to NN\pi$ amplitudes rather than in terms of a single $pp \to d\pi^+$ cross-section, and the cross-section for the nuclear case then depends strongly upon the angular momentum properties of the initial and final nucleus.

The present model is a two-nucleon one in the sense that the input is amplitudes connected with pion production by two nucleons. The reaction diagrams that are considered are of the type shown in Fig. 1 of Ref. 9), including exchange diagrams, but with no specific assumptions about the dynamics of the process. In Ref. 9) the contributions from the two diagrams corresponding to the one-nucleon mechanism with pion rescattering were not included, in order to separate the contribution from the two-nucleon mechanism. In the present paper these diagrams are taken into account ; in this sense the <u>reaction mechanism</u> that we consider is therefore a mixture of the one and two-nucleon cases.

2. - THE MANDELSTAM MODEL FOR $NN \to NN\pi$

In our experience the original paper [11] on the Mandelstam model is somewhat difficult to digest. For pedagogical reasons and because we need a generalized version of the model we find it necessary to give an independent and more or less self-contained presentation of it. We use units for which $\hbar = c = 1$.

In terms of the transition amplitude $\mathcal{T}_{fi}$ we can write the centre-of-mass differential cross-section for the reaction $NN \to NN\pi$ as

$$\frac{d^5\sigma}{d\Omega\, d(\cos\Theta)\, d\Phi\, d(\cos\theta)\, d\Psi} = \left(\frac{1}{2\pi}\right)^5 \frac{M^4}{4WP_{in}} \frac{QP}{W-\Omega} \sum_f \bar{\sum_i} |\mathcal{T}_{fi}|^2 . \qquad (1)$$

The kinematic variables are the total centre-of-mass energy W, the incident nucleon momentum $\underset{\sim}{P}_{in}$, the energy Ω and momentum $\underset{\sim}{Q}$ for the pion, and the relative momentum $\underset{\sim}{P}$ for the nucleons in the final state ; M is the nucleon mass. The angles Θ and Φ are the usual direction angles for the pion, and θ is the angle between $\underset{\sim}{P}$ and $\underset{\sim}{Q}$, the nucleon momenta in the final state being $\underset{\sim}{P}_1 = \underset{\sim}{p} = \underset{\sim}{P} - \frac{1}{2}\underset{\sim}{Q}$ and $\underset{\sim}{P}_2 = \underset{\sim}{p}' = -\underset{\sim}{P} - \frac{1}{2}\underset{\sim}{Q}$. The angle Ψ is equivalent but not identical to the ϕ angle for $\underset{\sim}{P}$, see below. We refer to Fig. 1a-b for a graphical explanation of the various quantities above, for the special case in which $\underset{\sim}{P}_{in}$ is chosen as the z axis. We mention that a small term depending on $\cos\theta$ is neglected in the phase space factor of Eq. (1).

The transition amplitude is expressed in terms of antisymmetrized wave functions for both initial and final nucleon states. This means that the transition operator need not be symmetrized, and the operator $\mathcal{T}_{12}$ below by definition produces the pion on nucleon 2. In terms of the initial (unprimed quantum numbers) and final (primed quantum numbers) spin and isospin states the transition amplitude is

$$
\begin{aligned}
\mathcal{T}_{fi} &= \mathcal{T}(\underset{\sim}{P}\,\underset{\sim}{Q}\; S'S'_z\, T'T'_z\, T'_{z\pi} \,|\, \underset{\sim}{P}_{in}\; S S_z\, T T_z) \\
&= \sum_{S'_{z1}} \sum_{S'_{z2}} \sum_{T'_{z1}} \sum_{T'_{z2}} \langle \tfrac{1}{2}\tfrac{1}{2} S'_{z1} S'_{z2} \,|\, S'S'_z \rangle \langle \tfrac{1}{2}\tfrac{1}{2} T'_{z1} T'_{z2} \,|\, T'T'_z \rangle \\
&\qquad \times \mathcal{T}(\underset{\sim}{P}\,\underset{\sim}{Q}\; S' S'_{z1} S'_{z2}\, T' T'_{z1} T'_{z2} T'_{z\pi} \,|\, \underset{\sim}{P}_{in}\; S S_z\, T T_z) ,
\end{aligned}
\tag{2}
$$

where $T'_{z\pi}$ is the isospin component for the pion and the other quantum numbers refer to the nucleons, and where $< j_1 j_2 j_{z1} j_{z2} | j j_z >$ is a Clebsch-Gordan coefficient. In the amplitude appearing on the right-hand side of Eq. (2) the nucleons in the final state have definite z components for spin and isospin, and

$$
\mathcal{T}(\underset{\sim}{P}\,\underset{\sim}{Q}\; S' S'_{z1} S'_{z2}\, T' T'_{z1} T'_{z2} T'_{z\pi} \,|\, \underset{\sim}{P}_{in}\; S S_z T T_z) = \langle \underset{\sim}{P}\,\underset{\sim}{Q}\; S' S'_{z1} S'_{z2}\, T' T'_{z1} T'_{z2} T'_{z\pi} \,|\, \mathcal{T}_{12} \,|\, \underset{\sim}{P}_{in}\; S S_z T T_z \rangle ,
\tag{3}
$$

where the antisymmetrized wave functions are

$$
\begin{aligned}
|\underset{\sim}{P}\,\underset{\sim}{Q}\; S' S'_{z1} S'_{z2}\, T' T'_{z1} T'_{z2} T'_{z\pi} \rangle_{12\pi} &= 2^{-\frac{1}{2}} \{ |\underset{\sim}{p}\rangle_1 |\underset{\sim}{p}'\rangle_2 + (-1)^{S'+T'+1} |\underset{\sim}{p}'\rangle_1 |\underset{\sim}{p}\rangle_2 \} |\underset{\sim}{Q}\rangle_\pi \\
&\qquad \times |\tfrac{1}{2} S'_{z1}\rangle_1 \, |\tfrac{1}{2} S'_{z2}\rangle_2 \, |\tfrac{1}{2} T'_{z1}\rangle_1 \, |\tfrac{1}{2} T'_{z2}\rangle_2 \, |1\, T'_{z\pi}\rangle_\pi , \\
|\underset{\sim}{P}_{in}\; S S_z T T_z \rangle_{12} &= 2^{-\frac{1}{2}} \{ |\underset{\sim}{P}_{in}\rangle_{12} + (-1)^{S+T+1} |-\underset{\sim}{P}_{in}\rangle_{12} \} \, |S S_z\rangle_{12} \, |T T_z\rangle_{12} .
\end{aligned}
\tag{4}
$$

We decompose the initial state into partial waves with definite orbital angular momentum ℓ and total angular momentum J in the usual fashion, i.e.,

$$|\mathbf{p}_{in}\, S S_z T T_z\rangle = 2^{\frac{1}{2}} \sum_J \sum_{J_z} \sum_{\ell} \sum_m \langle \ell S m S_z | J J_z\rangle\, \delta_{S+T+\ell,\,odd} \times |\mathbf{p}_{in}\, \ell S m, J J_z\rangle\, |T T_z\rangle . \tag{5}$$

The decomposition of the final $NN\pi$ state (the nucleons are denoted below by N_1 and N_2) is particular to the Mandelstam model : (i) the orbital angular momentum $\boldsymbol{\lambda}$ for the relative motion of the pion and N_2 is coupled to the spin $\mathbf{S}_2$ of N_2 to a resultant angular momentum $\boldsymbol{\kappa}$, this part of the coupling procedure being the same as in the ordinary partial wave treatment of πN scattering ; (ii) the angular momentum $\boldsymbol{\kappa}$ and the spin $\mathbf{S}_1$ of N_1 are coupled to a resultant $\mathbf{j}$; (iii) the angular momentum $\mathbf{j}$ and the orbital angular momentum $\boldsymbol{\ell}'$ of N_1 are coupled to a total angular momentum $\mathbf{J}$, where $\boldsymbol{\ell}'$ should not be confused with the angular momentum for the relative motion of N_1 and N_2. The isospin is treated in a similar fashion : (i) the isospins $\mathbf{T}_\pi$ and $\mathbf{T}_2$ for π and N_2 are coupled to $\boldsymbol{\tau}$; (ii) the resultant $\boldsymbol{\tau}$ and the isospin $\mathbf{T}_1$ for N_1 are coupled to a total isospin $\mathbf{T}$. We let $\mathbf{q}$ and $\mathbf{q}'$ denote the relative momenta of the pion and the nucleons with momenta $\mathbf{p}'$ and $\mathbf{p}$, and write

$$|\mathbf{P}\mathbf{Q}\, S' S'_{z1} S'_{z2} T' T'_{z1} T'_{z2} T'_{z\pi}\rangle = 2^{-\frac{1}{2}} \sum_J \sum_{J_z} \sum_{\ell'} \sum_{m'} \sum_j \sum_{j_z} \sum_{\kappa} \sum_{\kappa_z} \sum_{\lambda} \sum_{\lambda_z} \sum_T \sum_{T_z} \sum_{\tau} \sum_{\tau_z}$$

$$\times \langle j \ell' j_z m' | J J_z\rangle \langle \kappa \tfrac{1}{2} \kappa_z S'_{z1} | j j_z\rangle \langle \lambda \tfrac{1}{2} \lambda_z S'_{z2} | \kappa \kappa_z\rangle \langle \tau \tfrac{1}{2} \tau_z T'_{z1} | T T_z\rangle \tag{6}$$

$$\times \langle 1 \tfrac{1}{2} T'_{z\pi} T'_{z2} | \tau \tau_z\rangle \{ |\mathbf{p}\mathbf{q}\, \lambda \kappa j \ell' \lambda_z m', J J_z\rangle + (-1)^{S'+T'+1} |\mathbf{p}'\mathbf{q}'\, \lambda \kappa j \ell' \lambda_z m', J J_z\rangle \} |\tau, T T_z\rangle .$$

By writing $\mathbf{p}\,\mathbf{q}$ in the first wave function on the right-hand side of Eq. (6), we indicate that N_1 has momentum $\mathbf{p}$ and that N_2 and the pion have relative momentum $\mathbf{q}$, correspondingly for $\mathbf{p}'\,\mathbf{q}'$.

The original version of the Mandelstam model applies to the (3,3) resonance contribution and isospin $T = 1$ only. In our generalized version of the model the matrix element of $\mathcal{T}_{12}$ taken between the states on the right-hand sides of Eqs. (5) and (6) is assumed to have the form

$$\langle \tau, T T_z | \langle \underset{\sim}{p} \underset{\sim}{q} \lambda \kappa j \ell' \lambda_z m', J J_z | \mathcal{T}_{12} | \underset{\sim}{p}_{in} \ell S m, J J_z \rangle | T T_z \rangle$$
$$= F [\lambda \ell' \ell]^{\frac{1}{2}} i^{\ell - \ell' - \lambda} \delta_{\ell' + \ell + \lambda, odd} \, A_{\lambda \kappa j \ell' \ell J \tau T}$$
$$\times (p/\mu)^{\ell'} f_{\lambda \kappa \tau}(q) \hat{Y}_{\lambda}^{\lambda_z}(\hat{\underset{\sim}{q}}) \hat{Y}_{\ell'}^{m'}(\hat{\underset{\sim}{p}}) \hat{Y}_{\ell}^{m *}(\hat{\underset{\sim}{p}}_{in}), \qquad (7)$$

where μ is the pion mass and

$$F = 4\pi^{\frac{3}{2}} M^{-2} \mu^{-2} (W_0 P^0_{in})^{\frac{1}{2}},$$
$$W_0 = 2M + \mu, \quad P^{0\,2}_{in} = (\tfrac{1}{2} W_0)^2 - M^2,$$
$$\hat{Y}_{\ell}^{m}(\theta, \phi) = P_{\ell}^{m}(\theta) e^{im\phi} = (4\pi)^{\frac{1}{2}} [\ell]^{-\frac{1}{2}} Y_{\ell}^{m}(\theta, \phi),$$
$$[a] = 2a + 1, \quad [ab/c]^n = \{(2a+1)(2b+1)/(2c+1)\}^n, \text{ etc.} \qquad (8)$$

The factor F is introduced just for later convenience, and Y_{ℓ}^{m} is the usual normalized spherical harmonic (we use Condon-Shortley phase conventions throughout). The factor $\delta_{\ell'+\ell+\lambda, odd}$ is due to the negative intrinsic parity of the pion. Apart from the factor $(p/\mu)^{\ell'}$ and the spherical harmonics, the dynamics of the problem is contained in the strength parameters A, which depend upon the various quantum numbers that are indicated by subscripts, and in the amplitudes $f_{\lambda\kappa\tau}$. We shall later limit ourselves to s waves ($\lambda = 0$, $\kappa = \frac{1}{2}$, $\tau = \frac{1}{2}, \frac{3}{2}$) and the resonant p wave ($\lambda = 1$, $\kappa = \tau = \frac{3}{2}$), and we take

$$f_{0\frac{1}{2}\tau}(q) = f_0(q) = 1,$$
$$f_{1\frac{3}{2}\frac{3}{2}}(q) = f_1(q) = f(q) = \frac{q}{\omega(1 - r_a \omega) - \frac{4}{3} i (f_{\pi N}/\mu)^2 q^3},$$
$$\omega^2 = \mu^2 + q^2, \quad r_a = 0.518 \mu^{-1}, \quad f^2_{\pi N} = 0.08, \qquad (9)$$

where we have used the same effective range formula as Mandelstam [11] for f(q). In the Appendix, we have shown how the expression (7) for the matrix element follows in a special simple model.

It is, at this point, convenient to recouple the angular momenta in such a way that the pairs $\ell'\lambda$, S'S and T'T of quantum numbers appear in the same C-G coefficient. The quantum numbers for the vector sums corresponding to the first two of these pairs are denoted by g and h, whereas $\mathbf{T}'$ and $\mathbf{T}$ of course add up to the isospin $\mathbf{1}$ of the pion. After summation over the redundant z components the matrix element (2) becomes

$$
\begin{aligned}
\mathcal{J}(\mathbf{p}\,\mathbf{Q}\,S'S_z'T'T_z'T_{z\pi}'|\mathbf{p}_{in}\,S\,S_z\,T\,T_z) &= i\,F\,(-1)^{T'+1}\,[T'S'/S]^{\frac{1}{2}}\,\langle 1\,T'\,T_{z\pi}'\,T_z'\,|\,T\,T_z\rangle \\
&\times \sum_J \sum_{\ell'} \sum_\lambda \sum_\ell \sum_h \sum_g (-1)^{J+h}\,[Jh]\,[\ell]^{\frac{1}{2}}\,W(S'Sg\ell:hJ)\,B(S'T'STJg\lambda\ell'\ell) \\
&\times \sum_m \sum_\eta \sum_{\varrho'} \langle S'h\,S_z'\eta\,|\,S\,S_z\rangle\,\langle h\ell\eta m\,|\,g\varrho'\rangle\,\hat{Y}_\ell^{m*}(\hat{\mathbf{p}}_{in}) \\
&\times \left\{ y_{\lambda\ell' g}^{\ \ \varrho'}(\mathbf{p},\mathbf{q}) + (-1)^{S'+T'+1}\,y_{\lambda\ell' g}^{\ \ \varrho'}(\mathbf{p}',\mathbf{q}') \right\},
\end{aligned}
\tag{10}
$$

where

$$
\begin{aligned}
B(S'T'STJg\lambda\ell'\ell) &= [\lambda\ell']^{\frac{1}{2}}\,i^{1-\ell'-\ell-\lambda}\,\delta_{S+T+\ell,\,odd}\,\delta_{\ell'+\ell+\lambda,\,odd}\,\delta_{\ell SJ} \\
&\times \sum_\tau \sum_j \sum_k [\tau j k]^{\frac{1}{2}}\,W(1\tfrac{1}{2}T\tfrac{1}{2}:\tau T')\,W(S'jg\ell':\lambda J)\,W(\lambda\tfrac{1}{2}j\tfrac{1}{2}:kS')\,A_{\lambda k j\ell'\ell J\tau T},
\end{aligned}
$$

$$
y_{\lambda\ell' g}^{\ \ \varrho'}(\mathbf{p},\mathbf{q}) = (p/\mu)^{\ell'}\,f_\lambda(q)\sum_{\lambda_z}\sum_{m'}\langle\lambda\ell'\lambda_z m'\,|\,g\varrho'\rangle\,\hat{Y}_\lambda^{\lambda_z}(\hat{\mathbf{q}})\,\hat{Y}_{\ell'}^{m'}(\hat{\mathbf{p}}).
\tag{11}
$$

In accordance with Eq. (9) we have at this point assumed that it is sufficient to use the index λ rather than the set $\lambda k \tau$ of indices to define the amplitudes f(q). The notation $\delta_{\ell SJ}$ indicates that the triangle condition should be fulfilled for $\boldsymbol{\ell}$, $\mathbf{S}$ and $\mathbf{J}$.

Since the quantities $y_{\lambda\ell' g}^{\ \ \varrho'}$ of Eq. (11) transform like angular momentum eigenfunctions of rank g we are at liberty to rotate them to a system in which $\mathbf{Q}$ is the z axis and $\mathbf{P}$ (therefore also $\mathbf{p}$, $\mathbf{q}$, $\mathbf{p}'$ and $\mathbf{q}'$) are in the xz plane. The corresponding Euler angles are Θ, Φ and Ψ, where $\Psi + \pi$ is the ϕ angle for the original (arbitrary) z axis in the new system. With the notation (see Fig. 1c) $\alpha = \sphericalangle\, \mathbf{p}, \mathbf{q}$, $\beta = \sphericalangle\, \mathbf{p}, \mathbf{Q}$, $\gamma = \beta - \alpha = \sphericalangle\, \mathbf{q}, \mathbf{Q}$ and using the definition of Ref. 12) for the rotation matrices we have

$$y_{\lambda \ell' q}^{\;\;\varrho'}(\underset{\sim}{p}, \underset{\sim}{q}) = \sum_{\varrho} i^{\varrho - \varrho'} D^{q}_{\varrho' \varrho}(\Theta, \Phi, \Psi)\, \bar{y}_{\lambda \ell' q}^{\;\;\varrho}(\underset{\sim}{p}, \underset{\sim}{q}),$$

$$\bar{y}_{\lambda \ell' q}^{\;\;\varrho}(\underset{\sim}{p}, \underset{\sim}{q}) = (p/\mu)^{\ell'} f_{\lambda}(q) \sum_{\lambda_2} \sum_{m'} \langle \lambda \ell' \lambda_2 m' | q \varrho \rangle P_{\lambda}^{\lambda_2}(\gamma) P_{\ell'}^{m'}(\beta). \tag{12}$$

So far no consideration has been given to the interaction between the nucleons in the initial and final states. As in Mandelstam's paper [11] we take this interaction into account in relative s states only. As far as the initial state is concerned we shall only consider one transition that involves an s state, and since the energy for the initial relative motion is quite high even at threshold we simply neglect the interaction between the initial nucleons. For the final state the relative s state part of the matrix element is that which is independent of θ and Ψ, i.e., the part which is obtained by taking $\varrho = 0$ and averaging over θ. This part of the matrix element is then modified by an effective range method, which means multiplying it by a spin dependent function $R_{S'}(P) = 1 + g_{S'}(P)$ defined by Mandelstam [11]. The matrix element (10) then becomes

$$\mathcal{T}(\underset{\sim}{P}\underset{\sim}{Q}\, S' S_z' T' T_z' T_{z\pi}' | \underset{\sim}{P}_{in} S S_z T T_z) = 2iF(-1)^{T'+1}[T'S'/S]^{\frac{1}{2}} \langle 1 T' T_{z\pi}' T_z' | T T_z \rangle$$

$$\times \sum_{J} \sum_{\ell} \sum_{h} \sum_{q} (-1)^{J+h} [Jh][\ell]^{\frac{1}{2}} W(S'Sq\ell : hJ) \sum_{m} \sum_{\eta} \sum_{\varrho'} \langle S' h S_z' \eta | S S_z \rangle$$

$$\times \langle h \ell \eta m | q \varrho' \rangle \hat{Y}_{\ell}^{m*}(\hat{\underset{\sim}{P}}_{in}) \sum_{\varrho} i^{\varrho - \varrho'} J_{S'T'STJ\ell q}^{\;\;\varrho}(Q, \cos\theta) D^{q}_{\varrho' \varrho}(\Theta, \Phi, \Psi), \tag{13}$$

where

$$J_{S'T'ST\ell J q}^{\;\;\varrho}(Q, \cos\theta) = \sum_{\lambda} \sum_{\ell'} B(S'T'STJq\lambda\ell'\ell)\, J_{S'T'\lambda\ell' q}^{\;\;\varrho}(Q, \cos\theta),$$

$$J_{S'T'\lambda\ell' q}^{\;\;\varrho}(Q, \cos\theta) = \sum_{\lambda_2} \sum_{m'} \langle \lambda \ell' \lambda_2 m' | q \varrho \rangle J_{S'T'\lambda \ell'}^{\lambda_2 m'}(Q, \cos\theta),$$

$$J_{S'T'\lambda\ell'}^{\lambda_2 m'}(Q, \cos\theta) = J_{\lambda \ell'}^{\lambda_2 m'}(Q, \cos\theta) + (-1)^{S'+T'+1} J_{\lambda \ell'}^{-\lambda_2 m'}(Q, -\cos\theta)$$

$$+ \delta_{S'+T',\,odd}\, \delta_{\lambda_2, -m'}\, g_{S'}(P)\, M_{\lambda \ell'}^{\lambda_2}(Q),$$

$$J_{\lambda\ell'}^{\lambda_2 m'}(Q, \cos\theta) = \tfrac{1}{2} f_{\lambda}(q) (p/\mu)^{\ell'} P_{\lambda}^{\lambda_2}(\gamma) P_{\ell'}^{m'}(\beta),$$

$$M_{\lambda\ell'}^{\lambda_2}(Q) = \int_{-1}^{1} d(\cos\theta)\, J_{\lambda\ell'}^{\lambda_2 \, -\lambda_2}(Q, \cos\theta). \tag{14}$$

At this place some remarks should be inserted concerning the relationship between the various kinematic variables that enter into the calculations. When comparing the cross-sections for the various initial and final charge states we keep the maximum pion energy Ω_{max} fixed and use exact kinematics to find the relationship between Ω_{max} and the corresponding laboratory kinetic energy T_{lab} in the initial state. With initial and final masses M_1 M_2 (M_2 stationary in the laboratory) and M_1' M_2' μ' this means that

$$T_{lab} = \tfrac{1}{2} M_2^{-1} (W'^2 - 4M_i^2), \quad W' = (4M_f^2 + \Omega_{max}^2 - \mu'^2)^{\frac{1}{2}} + \Omega_{max} ,$$

$$M_i = \tfrac{1}{2}(M_1 + M_2), \quad M_f = \tfrac{1}{2}(M_1' + M_2'). \tag{15}$$

However, when calculating the cross-sections from the previous expressions we use an average nucleon mass $M = \frac{1}{2}(M_n + M_p)$ and an average pion mass $\mu = \frac{1}{2}$ $(\mu_{\pi\pm} + \mu_{\pi^0})$ regardless of the actual charge states, with $W = (4M^2 + \Omega_{max}^2 - \mu^2)^{\frac{1}{2}} + \Omega_{max}$ and $P_{in}^2 = (\frac{1}{2}W)^2 - M^2$. Furthermore, we use the approximate, θ independent expression,

$$P^2(Q) = \tfrac{1}{2} W (\Omega_{max} - \Omega) , \quad Q^2 = \Omega^2 - \mu^2 \tag{16}$$

for the relative nucleon momentum in the final state, the exact expression being the above multiplied by $\{1 - Q^2 \cos^2\theta \, (W - \Omega)^{-2}\}^{-1}$. The final state nucleon momentum p, the pion-nucleon relative momentum q and the angles α , β , γ are functions of Q and θ, and are given by the expressions

$$\begin{aligned}
&p^2 = P^2 - PQ\cos\theta + \tfrac{1}{4}Q^2 , \quad q^2 = \tfrac{1}{4} s'^{-1} \{ (s' - M^2 - \mu^2)^2 - 4M^2\mu^2 \} , \\
&\cos\beta = p^{-1}(P\cos\theta - \tfrac{1}{2}Q) , \quad \cos\gamma = c^{-1} \{ (a + \tfrac{1}{2}b) Q + bP\cos\theta \} , \quad \alpha = \beta - \gamma , \\
&a = s' + 2E'W' + M^2 - \mu^2 , \quad b = s' + 2\Omega W' - M^2 + \mu^2 , \quad c = 2qW'(W' + E' + \Omega) , \\
&s' = (E' + \Omega)^2 - p^2 , \quad W' = s'^{\frac{1}{2}} , \quad E'^2 = M^2 + p^2 + 2PQ\cos\theta .
\end{aligned} \tag{17}$$

That part of the matrix element which depends upon $\underset{\sim}{p}'$ and $\underset{\sim}{q}'$ rather than upon p and q is obtained from the above expressions by changing the sign of the $\cos\theta$ terms, as has already been indicated in Eq. (14).

The matrix element (13) depends on Θ , Φ and Ψ only through the rotation matrices D, which means that one can easily integrate over these variables ; to find the total cross-section, one then has to integrate numerically over Ω (or Q) and θ. By keeping the Θ dependence we can find the angular distribution for the pion, which after summation over S_z' and averaging over S_z is found to be given by the expression

$$\frac{d^3\sigma(S'T'T_3'T_{3\pi}'|STT_3)}{d\Omega\, d(\cos\Theta)\, d(\cos\theta)} = 4\, D_N(\Omega)\,[S'T'/S]\,|\langle 1T'T_{3\pi}'T_3'|TT_3\rangle|^2$$

$$\times \sum_{h}\sum_{\rho'}\sum_{\rho}\Big|\sum_{J}\sum_{l}\sum_{q}(-1)^J[J][hl]^{\frac{1}{2}}\,W(S'Sql:hJ)\,\langle hl\rho'0|q\rho'\rangle$$

$$\times J_{S'T'STlJ}{}^{\rho}_{q}(Q,\cos\theta)\, d^{q}_{\rho'\rho}(\Theta)\Big|^2, \qquad (18)$$

$$D_N(\Omega) = (W_0/W)(P_{in}^0/P_{in})\{2M/(W-\Omega)\}(Q/\mu)(P/M)\,\mu^{-3},$$

where $\underset{\sim}{P}_{in}$ has been chosen as the z axis and where the d functions are defined as in Ref. 12). Integration over Θ, and summation and averaging over S' and S gives

$$\frac{d^2\sigma(T'T_3'T_{3\pi}'|TT_3)}{d\Omega\, d(\cos\theta)} = 2\,|\langle 1T'T_{3\pi}'T_3'|TT_3\rangle|^2\,\frac{d^2\sigma_{T'T}}{d\Omega\, d(\cos\theta)},$$

$$\frac{d^2\sigma_{T'T}}{d\Omega\, d(\cos\theta)} = D_N(\Omega)\, I_{T'T}(Q,\cos\theta), \qquad (19)$$

$$I_{T'T}(Q,\cos\theta) = [T']\sum_{S'}\sum_{S}\sum_{J}\sum_{l}\sum_{q}\sum_{\rho}[S'J]\,\Big|J_{S'T'STlJ}{}^{\rho}_{q}(Q,\cos\theta)\Big|^2 .$$

In terms of the cross-sections $\sigma_{T'T}$ the cross-sections for the reactions induced by pp and pn initial states are

$$\sigma_{pp\to pp\pi^0} = \sigma_1 = \sigma_{11},$$

$$\sigma_{pp\to pn\pi^+} = \sigma_2 = \sigma_{11} + 2\sigma_{01}, \qquad (20)$$

$$\sigma_{pn\to pn\pi^0} = \sigma_3 = \tfrac{3}{2}\sigma_{01} + \tfrac{1}{6}\sigma_{10},$$

$$\sigma_{pn\to nn\pi^+} = \sigma_{pn\to pp\pi^-} = \sigma_4 = \tfrac{3}{4}\sigma_{11} + \tfrac{1}{6}\sigma_{10}.$$

After having calculated the cross-sections (20) by the method described above, it is necessary to introduce a correction which ensures that unitarity is conserved. As described by Mandelstam, this amounts to multiplying each matrix element $J^{\varrho}_{S'T'STJ\ell g}$ by a factor $\{1 + (\sigma_{J,u}/\sigma_{J,max})\}^{-1}$, where $\sigma_{J,max} = 4\pi\,[J]/P^2_{in}$ and where $\sigma_{J,u}$ is the previously calculated, uncorrected total cross-section for the given J and charge state. For the pp initial state we therefore take $\sigma_{J,u} = \sigma_{J,1} + \sigma_{J,2} + \sigma_{J,d}$, where σ_d is the cross-section for $pp \rightarrow d\,\pi^+$ (see the next section), and for the pn initial state $\sigma_{J,u} = \sigma_{J,3} + 2\,\sigma_{J,4}$.

In the actual calculations we shall, as Mandelstam [11], limit ourselves to S and P state production, i.e., to transitions with $\ell' = 0$ and 1. However, in addition to the Mandelstam resonant wave $\lambda = 1$, $\kappa = \tau = \frac{3}{2}$ we take into account the s waves $\lambda = 0$, $\kappa = \frac{1}{2}$, $\tau = \frac{1}{2}, \frac{3}{2}$, since we are interested in using the model also at energies near threshold. In order to keep the number of phenomenological parameters down we neglect the small p waves. Furthermore, for s wave, P state production, we include the initial relative s state ($\ell = 0$) only and neglect the $\ell = 2$ contribution, which is also allowed by the selection rules. In addition to the six Mandelstam parameters a, b_{01}, b_{1a}, b_{2a}, b_{1b} and b_{2b} (defined here with a different normalization) we get the s wave parameters A_0, A_1 and B, the connection between the previous general parameters $A_{\lambda\kappa j\ell'\ell J\tau T}$ and the present ones being

$$A_0 = 2^{\frac{1}{2}} A_{0\frac{1}{2}0010\frac{1}{2}1} + A_{0\frac{1}{2}0010\frac{3}{2}1}, \quad s \text{ wave}, \; S \text{ state}, \; J = 0,$$

$$A_1 = -A_{0\frac{1}{2}1011\frac{1}{2}1} + 2^{\frac{1}{2}} A_{0\frac{1}{2}1011\frac{3}{2}1}, \quad s \text{ wave}, \; S \text{ state}, \; J = 1,$$

$$B = A_{0\frac{1}{2}1101\frac{1}{2}0}, \quad s \text{ wave}, \; P \text{ state}, \; J = 1,$$

$$a = A_{1\frac{3}{2}2022\frac{3}{2}1}, \quad p \text{ wave}, \; S \text{ state}, \; J = 2,$$

$$b_{Jj} = A_{1\frac{3}{2}j11J\frac{3}{2}1}, \quad p \text{ wave}, \; P \text{ state}, \; Jj = 01, 11, 12, 21, 22,$$

$$b_{1a} = 6^{-\frac{1}{2}}(5^{\frac{1}{2}} b_{11} - b_{12}), \quad b_{1b} = 6^{-\frac{1}{2}}(b_{11} + 5^{\frac{1}{2}} b_{12}),$$

$$b_{2a} = 2^{-\frac{1}{2}}(b_{21} + b_{22}), \quad b_{2b} = 2^{-\frac{1}{2}}(b_{21} - b_{22}). \qquad (21)$$

We mention already here that whereas Mandelstam [11] puts $b_{1b} = b_{2b} = 0$ we shall keep these parameters and instead take $b_{1a} = b_{2a} = 0$. We justify this by using the simplified model in the Appendix ; in that model b_{2a} is strictly zero, whereas b_{1a} is smaller in magnitude than b_{01}, b_{1b} and b_{2b}.

With all b terms still included we find the non-zero matrix elements shown in the Table, the quantities that enter being defined as

$$J_{\lambda l' q}^{|s|\pm}(Q,\cos\theta) = \tfrac{1}{2}\{J_{\lambda l' q}^{|s|}(Q,\cos\theta) \pm J_{\lambda l' q}^{|s|}(Q,-\cos\theta)\}\ , \quad M_{\lambda l' q}(Q) = \tfrac{1}{2}\int_{-1}^{1} d(\cos\theta)\, J_{\lambda l' q}^{0}(Q,\cos\theta),$$

$$J_{011}^{0}(Q,\cos\theta) = (p/\mu)\cos\beta\ , \quad J_{011}^{1}(Q,\cos\theta) = 2^{-\frac{1}{2}}(p/\mu)\sin\beta\,,$$

$$J_{110}^{0}(Q,\cos\theta) = f(q)\,(p/\mu)\cos\alpha\ ,$$

$$J_{101}^{0}(Q,\cos\theta) = f(q)\cos\gamma\ , \quad J_{101}^{1}(Q,\cos\theta) = 2^{-\frac{1}{2}} f(q)\sin\gamma\,,$$

$$J_{111}^{1}(Q,\cos\theta) = f(q)\,(p/\mu)\sin\alpha\ ,$$

$$J_{112}^{0}(Q,\cos\theta) = \tfrac{1}{4} f(q)\,(p/\mu)\{3\cos(\beta+\gamma) + \cos\alpha\}\,, \tag{22}$$

$$J_{112}^{1}(Q,\cos\theta) = f(q)\,(p/\mu)\sin(\beta+\gamma)\ , \quad J_{112}^{2}(Q,\cos\theta) = f(q)\,(p/\mu)\sin\beta\sin\gamma\,.$$

By writing the parameters as $A_0 = |A_0|\exp(i\phi_{A_0})$, etc., and now taking $b_{1a} = b_{2a} = 0$ we find that the quantities $I_{T'T}$ of Eq. (19) are

$$I_{11} = |A_0|^2 I_{11A} + |a|^2 I_{11a} + |A_0||b_{01}|\{\cos(\phi_{b_{01}} - \phi_{A_0})\, I_{11Ab+} + \sin(\phi_{b_{01}} - \phi_{A_0})\, I_{11Ab-}\}$$
$$+ |b_{01}|^2 I_{110} + |b_{1b}|^2 I_{111} + |b_{2b}|^2 I_{112}\,,$$

$$I_{10} = |B|^2 I_{10B}\,,$$

$$I_{01} = |A_1|^2 I_{01A} + |a|^2 I_{01a} + |A_1||b_{1b}|\{\cos(\phi_{b_{1b}} - \phi_{A_1})\, I_{01Ab+} + \sin(\phi_{b_{1b}} - \phi_{A_1})\, I_{01Ab-}\}$$
$$+ |b_{01}|^2 I_{010} + |b_{1b}|^2 I_{011} + |b_{2b}|^2 I_{012}\,, \tag{23}$$

where

$$I_{11A} = \tfrac{1}{3}\,|R_0(p)|^2 \;, \quad I_{11a} = \tfrac{5}{3}\left\{ |J_{101}^{0-}|^2 + 2\,|J_{101}^{1-}|^2 \right\} ,$$

$$I_{11Ab+} = \tfrac{1}{2}\left(I_{11Ab} + I^{*}_{11Ab}\right) , \quad I_{11Ab-} = \tfrac{1}{2}\,i\,\left(I_{11Ab} - I^{*}_{11Ab}\right) , \quad I_{11Ab} = \tfrac{1}{3}\,2^{\frac{3}{2}}\,R_0^{*}(p)\left\{J_{110}^{0+} + q_0(p)\,M_{110}\right\} ,$$

$$I_{110} = \tfrac{1}{6}\,|J_{111}^{1-}|^2 + \tfrac{2}{3}\,|J_{110}^{0+} + q_0(p)\,M_{110}|^2 \;,$$

$$I_{111} = 2\,|J_{110}^{0-}|^2 + \tfrac{1}{3}\,|J_{111}^{1-}|^2 + \tfrac{1}{6}\,|J_{111}^{1+}|^2 \;,$$

$$I_{112} = \tfrac{5}{12}\,|J_{111}^{1-}|^2 + |J_{112}^{0-}|^2 + \tfrac{3}{4}\,|J_{112}^{1-}|^2 + 3\,|J_{112}^{2-}|^2$$

$$+ \tfrac{2}{3}\left\{ |J_{112}^{0+} + q_0(p)\,M_{112}|^2 + \tfrac{3}{4}\,|J_{112}^{1+}|^2 + 3\,|J_{112}^{2+}|^2 \right\} ,$$

$$I_{10B} = 3\left\{ |J_{011}^{0-}|^2 + 2\,|J_{011}^{1-}|^2 \right\} ,$$

$$I_{01A} = |R_1(p)|^2 \;, \quad I_{01a} = \tfrac{10}{3}\left\{ |J_{101}^{0+} + q_1(p)\,M_{101}|^2 + 2\,|J_{101}^{1+}|^2 \right\} ,$$

$$I_{01Ab+} = \tfrac{1}{2}\left(I_{01Ab} + I^{*}_{01Ab}\right) , \quad I_{01Ab-} = \tfrac{1}{2}\,i\,\left(I_{01Ab} - I^{*}_{01Ab}\right) , \quad I_{01Ab} = 4\,R_1^{*}(p)\left\{J_{110}^{0+} + q_1(p)\,M_{110}\right\} ,$$

$$I_{010} = 2\left\{ \tfrac{1}{6}\,|J_{111}^{1+}|^2 + \tfrac{2}{3}\,|J_{110}^{0-}|^2 \right\} ,$$

$$I_{011} = 2\left\{ 2\,|J_{110}^{0+} + q_1(p)\,M_{110}|^2 + \tfrac{1}{3}\,|J_{111}^{1+}|^2 + \tfrac{1}{6}\,|J_{111}^{1-}|^2 \right\} ,$$

$$I_{012} = 2\left(\tfrac{5}{12}\,|J_{111}^{1+}|^2 + |J_{112}^{0+} + q_1(p)\,M_{112}|^2 + \tfrac{3}{4}\,|J_{112}^{1+}|^2 + 3\,|J_{112}^{2+}|^2 \right.$$

$$\left. + \tfrac{2}{3}\left\{ |J_{112}^{0-}|^2 + \tfrac{3}{4}\,|J_{112}^{1-}|^2 + 3\,|J_{112}^{2-}|^2 \right\} \right) . \qquad (24)$$

3. - THE REACTION $pp \to d\pi^+$

The $pp \to d\pi^+$ reaction will serve as an example of the application of the Mandelstam model to a nuclear final state ; furthermore, this reaction is used in the determination of the Mandelstam parameters. Apart from the fact that we include more parameters than in the original work [11] our treatment is also different in that we have found it necessary to take the deuteron Fermi motion into account, at least in an approximative manner. In the numerical calculations we treat the deuteron as a pure s state in the np system. This is not to say that the d state contribution is negligible, the point being, however, that in the nuclear cases to be considered later we use pure harmonic oscillator wave functions without tensor force admixture ; it then seems more consistent to treat the deuteron in the same fashion.

We compute as before the cross-section in the centre-of-mass system, where the angular distribution of the pion is approximately given by

$$\frac{d^2\sigma}{d(\cos\Theta)\, d\Phi} = \left(\frac{1}{2\pi}\right)^2 \frac{M^2 Q}{W^2 P_{in}} \sum_{f} \bar{\bar{\sum_{i}}} |\mathcal{T}_{fi}|^2 ,$$

$$Q^2 = \tfrac{1}{4} W^{-2} \{ (W^2 - \mu^2 - M_d^2)^2 - 4\mu^2 M_d^2 \} . \tag{25}$$

Here M_d is the mass of the deuteron, whereas Q, W and P_{in} are as before the pion momentum, the total energy of the system and the initial nucleon momentum.

With the d state still included the deuteron wave function in co-ordinate space is

$$\psi_d^{J_z}(\underset{\sim}{r}) = \sum_{L=0,2} U_L(r)\, \mathcal{Y}_{L11}^{J_z}(\hat{\underset{\sim}{r}}) \;, \quad \int_0^\infty r^2 dr\, |U_L(r)|^2 = 1 \;, \tag{26}$$

where $\mathcal{Y}_{LSJ}^{J_z}$ is the usual spin angle function. We introduce the dimensionless momentum space wave function

$$\varphi_L(P) = (2/\pi)^{\frac{1}{2}} \mu^{\frac{3}{2}} \int_0^\infty r^2 dr\, U_L(r)\, j_L(Pr) \tag{27}$$

and write the matrix element as

$$\mathcal{T}_{fi} = \mathcal{T}_d(J_z | S S_z) = (2\pi)^{-\frac{3}{2}} (4\pi)^{-\frac{1}{2}} \mu^{\frac{3}{2}} \sum_{L=0,2} i^L [L]^{\frac{1}{2}} \sum_{M} \sum_{S_z'} \langle L1MS_z' | 1J_z \rangle$$
$$\times \int d^3(P/\mu)\, \varphi_L(P)\, \hat{Y}_L^{M*}(\hat{\underset{\sim}{P}})\, \mathcal{T}(\underset{\sim}{P}\underset{\sim}{Q}\, 1S_z'\, 001 | \underset{\sim}{P}_{in} S S_z 11) , \tag{28}$$

where the matrix element $\mathcal{T}$ is given by Eq. (13) and $d^3P = P^2 dP d(\cos\theta) d\Psi$. In analogy with the $NN \to NN\pi$ case we transform $\hat{Y}_L^M(\hat{\underset{\sim}{P}})$ to the system with Q as z axis and $\underset{\sim}{P}$ in the xz plane. We couple L and the quantum number g appearing in $\mathcal{T}$ to a resultant f and integrate over Ψ. The external quantum numbers $1J_z$ and SS_z are recoupled so that they appear in the same C-G coefficient, the resultant being denoted by b. The matrix element then becomes

$$\mathcal{T}_d(J_z|SS_z) = -(3/2)^{\frac{1}{2}}\pi^{-1} i\,\mu^{\frac{3}{2}} F \sum_b \sum_f \sum_J \sum_\ell \sum_g \sum_L (-1)^b [fJ][b\ell]^{\frac{1}{2}} W(1Jb\ell:fS)$$
$$\times \sum_\beta \sum_\varrho (-1)^\beta \langle bS-\beta S_z|1J_z\rangle \langle \ell f 0\beta|b\beta\rangle \hat{Y}_f^\beta(\hat{\underset{\sim}{Q}}) K_{LSf\ell Jg}^{\varrho}(Q)\,, \qquad (29)$$

where, as before, $\underset{\sim}{P}_{in}$ has been chosen as the original z axis, and where

$$K_{LSf\ell Jg}^{\varrho}(Q) = i^L [g]^{\frac{1}{2}} W(1JLg:f1) \langle fg0\varrho|L\varrho\rangle$$
$$\times \mu^{-3} \int P^2 dP\, d(\cos\theta)\, \varphi_L(P)\, P_L^\varrho(\theta)\, J_{10S1\ell Jg}^{\varrho}(Q,P,\cos\theta). \qquad (30)$$

The quantities $J_{S'T'ST\ell Jg}^{\varrho}$ have been dealt with in Section 2 ; however, in the present case the variables are P and θ rather than Q and θ. In the $NN \to NN\pi$ case Q varies, and P is determined from Q by Eq. (16), i.e., the quantities (17) which determine $J_{S'T'ST\ell Jg}^{\varrho}$ are functions of Q and θ. In the $pp \to d\pi^+$ case Q is fixed by Eq. (25), and the contributions from the various values of P are integrated over the deuteron momentum distribution.

We sum over the z components J_z of the deuteron angular momentum, average over initial spin states and integrate over Φ. The angular distribution for the pion then becomes

$$\frac{d\sigma_d}{d(\cos\Theta)} = \frac{9}{8} D_d(\Omega) \sum_S \sum_b \sum_\beta \Big| \sum_f \sum_J \sum_\ell \sum_g \sum_L [fJ][\ell]^{\frac{1}{2}} W(1Jb\ell:fS)$$
$$\times \langle \ell f 0\beta|b\beta\rangle P_f^\beta(\Theta) \sum_\varrho K_{LSf\ell Jg}^{\varrho}(Q) \Big|^2, \qquad (31)$$

$$D_d(\Omega) = 4(W_0/W)(P_{in}^0/P_{in})(2M/W)(Q/\mu)M^{-2},$$

and the total cross-section is

$$\sigma_d = \frac{9}{4} D_d(\Omega) \sum_s \sum_f \sum_J \sum_\ell [fJ] \left| \sum_L \sum_q \sum_g K_{Lsf\ell Jq}^{g}(Q) \right|^2 . \tag{32}$$

Including the deuteron s state only (L = 0 and f = g) and the same parameters as for the NN → NNπ case we have

$$\sigma_d = D_d(\Omega) \{ |A_1 K_0 + 2 b_{1b} K_{110}|^2 + \tfrac{10}{3}|a|^2 |K_{101}|^2 + 2|b_{2b}|^2 |K_{112}|^2 \} ,$$
$$K_0 = \mu^{-3} \int_0^\infty P^2 dP\, g_0(P)\, R_1(P) , \quad K_{\lambda\ell' q}(Q) = \mu^{-3} \int_0^\infty P^2 dP\, g_0(P)\, R_1(P)\, M_{\lambda\ell' q}(Q,P), \tag{33}$$

where $M_{\lambda\ell' g}$ is defined in Eq. (22).

4. - THE REACTION A(p, π^+)A+1

The nuclear pionic capture cross-section is calculated in the laboratory system. We neglect a small angle dependent term in the phase space factor and write the differential cross-section as

$$\frac{d^2\sigma}{d(\cos\Theta')\, d\Phi'} = \left(\frac{1}{2\pi}\right)^2 \frac{M Q'}{2 P_i} \sum_f \overline{\sum_i} |\mathcal{T}_{fi}|^2 , \tag{34}$$

where $\underset{\sim}{Q}'$ and $\underset{\sim}{P}_i$ are the momenta of the final pion and initial proton, the corresponding pion energy being $\Omega' = (\mu^2 + Q'^2)^{\frac{1}{2}}$. In terms of the proton kinetic energy T_{lab} and the angle Θ' between $\underset{\sim}{P}_i$ and $\underset{\sim}{Q}'$ we have

$$P_i^2 = 2 M_p T_{lab} \{ 1 + \tfrac{1}{2}(T_{lab}/M_p) \} , \quad Q' = a + (a^2 + b)^{\frac{1}{2}} ,$$
$$a = d/c , \quad b = e/c , \quad c = T^2 - P_i^2 \cos\Theta' ,$$
$$d = \tfrac{1}{2} P_i \cos\Theta' (T^2 - m^2 + \mu^2) , \quad e = \tfrac{1}{4}\{T^2 - (m+\mu)^2\}\{T^2 - (m-\mu)^2\} ,$$
$$T = T_{lab} + M_A + M_n , \quad m^2 = M_{A'}^2 + P_i^2 , \tag{35}$$

M_A and $M_{A'}$ being the masses of the initial and final nuclei. In the actual calculations we use the value of Q' corresponding to $\Theta' = 0$.

We have here taken the nuclear recoil into consideration (in an approximate manner) as far as the phase space factor is concerned, and we also make another correction for the finite mass of the nucleus by adapting a procedure from Ref. 13). This amounts to computing the matrix element in the absence of recoil and then multiplying the result by a factor $R(\Delta^2) = \exp(\frac{1}{4}A''^{-1}\Delta^2/\alpha^2)$, where A'' is the mean mass number for the initial and final nuclei and $\mathbf{\Delta} = \mathbf{Q}' - \mathbf{P}_i$. It is here assumed that the nuclear states are described by harmonic oscillator wave functions, with the shell model parameter α normalized so that the one-nucleon wave functions are proportional to $\exp(-\frac{1}{2}\alpha^2 r^2)$. We do not consider recoil corrections in the internal dynamics [8], these being less important in the present model than in the one-nucleon case.

We consider the matrix element corresponding to the initial nucleon states $\zeta_1 = N_1 L_1 J_1 J_{z1} T_{z1}$ and $\zeta_2 = S_{z2} T_{z2}$, and the final nucleon states $\zeta'_j = N'_j L'_j J'_j J'_{zj} T'_{zj}$, $j = 1,2$. Here ζ_2 indicates the incoming nucleon, the other states are shell model states. The nucleon in state ζ_1 has momentum $\mathbf{q}_i$, corresponding to a relative momentum $\mathbf{P}_{in} = \frac{1}{2}(\mathbf{q}_i - \mathbf{P}_i)$ for the initial nucleons. The orbital motion of the final nucleons is separated into relative and centre-of-mass motion, with quantum numbers NL and $N'L'$, and momenta $\mathbf{P}$ and $\mathbf{K} = \mathbf{K}_i - \mathbf{Q}'$, respectively, where $\mathbf{K}_i = \mathbf{P}_i + \mathbf{q}_i$. This introduces a Brody-Moshinsky [14] transformation bracket $\langle NL, N'L', \Lambda | N'_1 L'_1, N'_2 L'_2, \Lambda \rangle$, where $\mathbf{\Lambda} = \mathbf{L}'_1 + \mathbf{L}'_2 = \mathbf{L} + \mathbf{L}'$. The initial and final two-nucleon states are expressed in terms of states with definite total spins S and S', and isospins T and T'. The antisymmetrized nuclear matrix element can then be written as

$$
\begin{aligned}
\mathcal{T}_{fi} &= \mathcal{T}_A(\mathbf{Q}'\zeta'_1\zeta'_2 T'_{z\pi} | \mathbf{P}_i \zeta_1 \zeta_2) = (2\pi)^{-\frac{3}{2}}(4\pi)^{-\frac{3}{2}}\mu^{\frac{3}{2}} \sum_{S'}\sum_{T'}\sum_{S}\sum_{T}\sum_{N}\sum_{L}\sum_{N'}\sum_{L'}\sum_{\Lambda} \\
&\times i^{L'+L-L_1}[L'LL_1]^{\frac{1}{2}} \langle NL, N'L', \Lambda | N'_1 L'_1, N'_2 L'_2, \Lambda \rangle \sum_{S'_z}\sum_{T'_z}\sum_{S_z}\sum_{T_z}\sum_{S'_{z1}}\sum_{S'_{z2}}\sum_{S_{z1}} \\
&\times \langle \tfrac{1}{2}\tfrac{1}{2} S'_{z1} S'_{z2} | S' S'_z \rangle \langle \tfrac{1}{2}\tfrac{1}{2} T'_{z1} T'_{z2} | T' T'_z \rangle \langle \tfrac{1}{2}\tfrac{1}{2} S_{z1} S_{z2} | S S_z \rangle \langle \tfrac{1}{2}\tfrac{1}{2} T_{z1} T_{z2} | T T_z \rangle \\
&\times \sum_{M'_1}\sum_{M'_2}\sum_{M_1}\sum_{\nu}\sum_{M'}\sum_{M} \langle L'_1 \tfrac{1}{2} M'_1 S'_{z1} | J'_1 J'_{z1} \rangle \langle L'_2 \tfrac{1}{2} M'_2 S'_{z2} | J'_2 J'_{z2} \rangle \langle L_1 \tfrac{1}{2} M_1 S_{z1} | J_1 J_{z1} \rangle \\
&\times \langle L'_1 L'_2 M'_1 M'_2 | \Lambda \nu \rangle \langle L L' M M' | \Lambda \nu \rangle \int d^3(q_i/\mu)\, d^3(P/\mu)\, \varphi_{N_1 L_1}(q_i)\, \varphi_{N'L'}(2^{-\frac{1}{2}}K)\, \varphi_{NL}(2^{\frac{1}{2}}P) \\
&\times \hat{Y}^{M_1}_{L_1}(\hat{\mathbf{q}}_i)\, \hat{Y}^{M'*}_{L'}(\hat{\mathbf{K}})\, \hat{Y}^{M*}_{L}(\hat{\mathbf{P}})\, \mathcal{T}(\mathbf{P}\mathbf{Q}\, S' S'_z\, T' T'_z T'_{z\pi} | \mathbf{P}_{in}\, S S_z\, T T_z),
\end{aligned}
\tag{36}
$$

where the matrix element $\mathcal{T}$ for the $NN \to NN\pi$ reaction is treated in Section 2. The dimensionless harmonic oscillator wave functions $\varphi_{n\ell}(p)$ are proportional to $\exp(-\frac{1}{2}p^2/\alpha^2)$ and are normalized so that $\int_0^\infty |\varphi_{n\ell}(p)|^2\, p^2 dp = \mu^3$.

We choose the z axis along the direction of $\underset{\sim}{P}_i$, and the initial relative momentum $\underset{\sim}{P}_{in}$ appearing in the matrix element (10) for the elementary reaction now varies with $\underset{\sim}{q}_i$. As in the deuteron case the quantities J of Eqs. (13), (14) depend upon the relative momentum $\underset{\sim}{P}$ for the nucleons in the final state, and we have

$$\int d^3(P/\mu)\, \varphi_{NL}(2^{\frac12}P)\, \hat{Y}_L^{M*}(\hat{\underset{\sim}{P}})\, i^{\rho-\rho'}\, J^{\rho}_{S'T'ST\ell Jq}(\underset{\sim}{Q}', \underset{\sim}{q}_i, \underset{\sim}{P})\, D^{q}_{\rho'\rho}(\Theta, \Phi, \Psi)$$

$$= 4\pi K^{\rho}_{S'T'STNL\ell Jq}(\underset{\sim}{Q}', \underset{\sim}{q}_i) \sum_f \sum_\phi [f/L]^{\frac12} \langle L\, q\, {-M}\, \rho' | f\, \phi \rangle \langle f\, q\, 0\, \rho | L\, \rho \rangle\, \hat{Y}_f^{\phi}(\hat{\underset{\sim}{Q}}), \qquad (37)$$

$$K^{\rho}_{S'T'STNL\ell Jq}(\underset{\sim}{Q}', \underset{\sim}{q}_i) = \tfrac12 \mu^{-3} \int P^2 dP\, d(\cos\theta)\, \varphi_{NL}(2^{\frac12}P)\, P_L^{\rho}(\theta)$$
$$\times J^{\rho}_{S'T'ST\ell Jq}(\underset{\sim}{Q}', \underset{\sim}{q}_i, P, \cos\theta).$$

The cross-section then depends on the integrals

$$K^{M'\; M_1\; m\; \rho\; \phi}_{S'T'STNLJN'L'N_1L_1\ell q f}(\underset{\sim}{Q}') = \int d^3(q_i/\mu)\, \varphi_{N_1L_1}(q_i)\, \varphi_{N'L'}(2^{-\frac12}K)$$
$$\times \hat{Y}^{M_1}_{L_1}(\hat{\underset{\sim}{q}}_i)\, \hat{Y}^{M'*}_{L'}(\hat{\underset{\sim}{K}})\, \hat{Y}^{\phi}_{f}(\hat{\underset{\sim}{Q}})\, \hat{Y}^{m*}_{\ell}(\hat{\underset{\sim}{P}}_{in})\, K^{\rho}_{S'T'STNL\ell Jq}(\underset{\sim}{Q}', \underset{\sim}{q}_i). \qquad (38)$$

By introducing the matrix element (13) in Eq. (36) and summing over z components where possible, we can obtain a, rather cumbersome, expression for the general matrix element $\mathcal{T}_A(\underset{\sim}{Q}'\, \mathcal{S}_1'\, \mathcal{S}_2' T_{z\pi}' | \underset{\sim}{P}_i\, \mathcal{S}_1\, \mathcal{S}_2)$. We give the cross-section explicitly only for the case of a (p, π^+) reaction on nuclei with closed J shells in the initial state, where the final state is assumed to be well described by the same closed shells plus one outside neutron in the state $N_2L_2J_2$. We have

$$\sum_f \overline{\sum_i} |\mathcal{T}_{fi}|^2 = \tfrac12 \sum_{J_{z2}} \sum_{S_{z2}} \Big| \sum_{\mathcal{S}_1} \mathcal{T}_A(\underset{\sim}{Q}'\, \mathcal{S}_1\, \mathcal{S}_2'\, 1 | \underset{\sim}{P}_i\, \mathcal{S}_1\, \mathcal{S}_2) \Big|^2$$

$$= \tfrac12 \sum_{J_{z2}} \sum_{S_{z2}} \Big| \sum_{N_1L_1J_1} \sum_{J_{z1}} \sum_{T_{z1}} \mathcal{T}_A(\underset{\sim}{Q}', N_1L_1J_1J_{z1}T_{z1}, N_2L_2J_2J_{z2} {-\tfrac12}, 1 | \underset{\sim}{P}_i, N_1L_1J_1J_{z1}T_{z1}, S_{z2}\tfrac12) \Big|^2, \qquad (39)$$

in which the summation over $N_1L_1J_1$ goes over the closed shells. Since polarizations are not considered the cross-section is independent of Φ', and we have

$$\frac{d\sigma}{d(\cos\Theta')} = D_A(\Omega')\, R(\Delta^2)\, [J_2] \sum_d \sum_\delta \Big| \sum_{N_1L_1J_1} \sum_{N'L'} \sum_{NL} \sum_{S'} \sum_{T'} \sum_{S} \sum_{T} \sum_{J} \sum_{\ell} \sum_{q} \sum_{f} \sum_{b} \sum_{c}$$

$$\times E(T'T)\, D(N_2L_2J_2 \;\; N_1L_1J_1 \;\; N'L' \;\; NL \;\; S'SJ\ell q \;\; fbcd)$$

$$\times \sum_{\varsigma} \sum_{M'} \sum_{M_1} \sum_{m} \langle f q 0 \varsigma | L \varsigma \rangle \langle f b \phi \beta | \ell m \rangle \langle c L_1 \gamma M_1 | b \beta \rangle \langle c L' \gamma M' | d \delta \rangle$$

$$\times K^{M'\;\;M_1\;\;m\;\;\varsigma\;\;\phi}_{S'T'STNLJN'L'\;N_1L_1\;\ell\;q\;f}(\underset{\sim}{Q}') \Big|^2 , \tag{40}$$

where

$$D_A(\Omega') = \tfrac{2}{3}(2\pi)^{-2}(W_0/2M)(P^0_{in}/P_i)(Q'/\mu)\, M^{-2}, \quad E(T'T) = (-1)^T [T'T]\, W(\tfrac{1}{2}\tfrac{1}{2}1T:T'\tfrac{1}{2}),$$

$$D(N_2L_2J_2 \;\; N_1L_1J_1 \;\; N'L' \;\; NL \;\; S'SJ\ell q \;\; fbcd) = (-1)^{S+b}\, i^{L'+L+L_1}\, [J_1 S' J f c]$$

$$\times [L_1 L' S q b]^{\frac{1}{2}} \sum_{\Lambda} (-1)^{\Lambda} [\Lambda] \langle NL, N'L', \Lambda | N_1L_1, N_2L_2, \Lambda \rangle \sum_{J'} \sum_{\Lambda'} (-1)^{J'}$$

$$\times [J'\Lambda']\, W(J'Jb\ell : fS)\, W(J'JLq : fS')\, W(L'L\Lambda'S' : \Lambda J')$$

$$\times X\begin{pmatrix} \frac{1}{2} & J_1 & L_1 \\ \frac{1}{2} & J_2 & L_2 \\ S' & \Lambda' & \Lambda \end{pmatrix} \begin{Bmatrix} \frac{1}{2} & J_1 & L_1 & c \\ \frac{1}{2} & J_2 & d & J' \\ S & \Lambda' & b & L' \end{Bmatrix} , \tag{41}$$

$$\gamma = \delta - M', \quad \beta = \gamma + M_1, \quad \phi = m - \beta .$$

The X coefficients are defined as in Ref. 15), and the 12j symbol is defined as

$$\begin{Bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & \ell \end{Bmatrix} = \sum_m [m]\, W(k\ell cg : md)\, W(k\ell ij : mh)\, X\begin{pmatrix} i & j & m \\ a & b & c \\ e & f & g \end{pmatrix} . \tag{42}$$

For given N_1L_1 and N_2L_2 the possible values of NL, $N'L'$ and Λ follow from the Brody-Moshinsky tables 14), and the limits on the other quantum numbers are given by the truncated Mandelstam expansion and by the triangle conditions imposed by the various coefficients of Eqs. (40) and (41).

5. - NUMERICAL RESULTS AND DISCUSSION

The total cross-section for the reaction $pp \rightarrow d\pi^+$ at low energies is experimentally given by the expression $\sigma_d = \alpha (Q/\mu) + \beta (Q/\mu)^3$, where the most recent analysis [16] gives $\alpha = 0.188$ mb and $\beta = 0.90$ mb. The α and β terms are associated with the $|A_1|^2$ and $|a|^2$ contributions of Eq. (33), respectively, and we find $|A_1| = 0.11$ and $|a| = 0.069$. Near threshold the cross-section for the $pp \rightarrow pp\pi^0$ reaction is $\sigma_1 = 0.032(Q/\mu)^2$ mb [17] and we use this to obtain $|A_0| = 0.12$. The experimental information is not good enough to warrant a separate determination of b_{01}, b_{1b} and b_{2b}. However, we assume the relationships $|b_{1b}| = 1.12|b_{01}|$ and $|b_{2b}| = 0.707|b_{01}|$ from the Appendix to hold true and require as Mandelstam [11] that the cross-section for the $pp \rightarrow pn\pi^+$ reaction at 660 MeV be $\sigma_2 = 11.4$ mb ; this gives $|b_{01}| = 0.032$. By taking the $pn \rightarrow pp\pi^-$ cross-section at 405 MeV to be $\sigma_4 = 0.22$ mb [18] we finally obtain $|B| = 0.96$. If the relative phases of the various parameters are taken to be those of the Appendix, the Mandelstam parameters then become

$$A_0 = 0.12, \quad A_1 = -0.11, \quad B = -0.96, \quad a = -0.069,$$
$$b_{01} = 0.032, \quad b_{1b} = -0.035, \quad b_{2b} = 0.022. \tag{43}$$

Before going on to the nuclear case we should like to make a few remarks about the relative merits of the Mandelstam choice $b_{1b} = b_{2b} = 0$ and the present assumption $b_{1a} = b_{2a} = 0$, the latter being based on the simple model in the Appendix. We find that in practice it makes little difference which choice is made, in that the fit to the experimental data can be made with essentially the same accuracy in both cases. Mandelstam's main reason for choosing $b_{1b} = b_{2b} = 0$ was the fact that these parameters, if sizeable, can have a rather strong influence on the angular distribution of pions in the reaction $pp \rightarrow d\pi^+$ and cause a marked deviation from the approximate $\frac{1}{3} + \cos^2\Theta$ distribution which is observed experimentally in the 400-600 MeV region, and reproduced theoretically if $b_{1b} = b_{2b} = 0$. We find, however, that as long as the relative phases between A_1, b_{1b} and b_{2b} are those of Eq. (43), the angular distribution in actual fact remains essentially unchanged by inclusion of the b_{1b} and b_{2b} terms, the point being that the interference terms counteract the effects of the terms proportional to $|b_{1b}|^2$ and $|b_{2b}|^2$. We take this to indicate that at least as far as the parameters A_1, b_{1b} and b_{2b} are concerned, the relative signs given by Eq. (43) are indeed the right ones.

We apply the results of Section 4 to the reactions $p + {}^{12}C \rightarrow \pi^+ + {}^{13}C$ and $p + {}^{4}He \rightarrow \pi^+ + {}^{5}He$, using harmonic oscillator wave functions with the same size parameters as in Ref. 9). The differential cross-section obtained for the carbon case at the incident proton energy 185 MeV is shown in Fig. 2 and Fig. 3, for the reactions leading to the ground state and the 3.09 MeV excited state in ${}^{13}C$, respectively. It has been assumed that ${}^{12}C$ can be described as a pure $(1s_{\frac{1}{2}})^4(1p_{\frac{3}{2}})^8$ state, and that the two levels of ${}^{13}C$ that are considered have one neutron in pure $1p_{\frac{1}{2}}$ and $2s_{\frac{1}{2}}$ states outside an unexcited ${}^{12}C$ core. The curves marked (a) correspond to the Mandelstam parameters (43), and should be compared with the experimental curves from Ref. 4).

The agreement between theory and experiment is perhaps as good as that in most calculations of absolute cross-sections for nuclear reactions, but it is nevertheless clear that some improvements in the theoretical predictions would be welcome. We note first of all that the results are quite sensitive not only to the absolute values but also to the relative phases of the Mandelstam parameters, both of which are subject to uncertainties. As an example we have in the curves (b) of Figs. 2 and 3 shown the effect of increasing the absolute values of A_0, A_1 and a by 25% and simultaneously changing the signs of A_0 and B, this in order to obtain a closer resemblance between the theoretical and experimental curves of Fig. 3.

As far as the experimental curve of Fig. 3 is concerned, it actually contains contributions not only from the 3.09 MeV state in ${}^{13}C$, but also from the states at 3.68 and 3.85 MeV. We have tried to include the latter in our calculations on the assumption that it is a pure $1d_{\frac{5}{2}}$ one-particle state, but find that in that model its contribution tends to fill the dip in the theoretical curve and make it less resemblant to the experimental one. The 3.68 MeV state definitely involves core excitations and has not been considered.

The actual structure of the states that have been considered is of course much more complicated than we have assumed here, and as a comparison of Figs. 2 and 3 will indicate, the theoretical cross-section depends strongly upon the shell structure of the state under consideration. In an improved theory, including, e.g., the small p waves and perhaps higher waves in the πN amplitude and rescattering corrections, it should at least in principle be possible to use the measured cross-sections to study the shell structure of nuclear states in some detail.

Figure 4 shows the theoretical differential cross-section for the helium reaction at various energies for the incoming proton. We have considered the reaction leading to the ground state of ^{5}He, described as a pure $1p_{\frac{3}{2}}$ one-particle state, and the Mandelstam parameters are those of Eq. (43). One sees clearly how the cross-section becomes more peaked in the forward direction as the energy increases, and this trend will of course survive even with other absolute values and phases of the Mandelstam parameters.

Finally, Fig. 5 shows the energy dependence of the forward cross-section for both the carbon and the helium reactions, again for the parameters (43). Curves 1, 2 and 3 correspond to the $1p_{\frac{1}{2}}$, $2s_{\frac{1}{2}}$ and $1d_{\frac{5}{2}}$ states in ^{13}C, and curves 4 and 5 apply to the $1p_{\frac{3}{2}}$ and $1p_{\frac{3}{2}}$ states in ^{5}He, the latter being identified with the broad state at 2.6 MeV. At 600 MeV curves 1 and 2 give the forward cross-sections 0.44 and 0.48 μb/sr, which compares quite favourably with the experimental results (0.75 ± 0.30) and (0.77 ± 0.30) μb/sr [1]; a reanalysis of the experimental data actually shows that the measured cross-sections are about a factor 3 smaller than indicated above [10]. For the ^{5}He case the experimental cross-section is (26 ± 3) μb/sr [2], and the sum of the cross-sections from the curves 4 and 5 is only 5.8 μb/sr. However, it does not take much of a change in the parameters to increase the theoretical cross-section considerably.

We have not investigated the π^- production processes in any detail. By a very rough estimate we find that the present model can give π^+/π^- ratios which typically are around 50. We should like to point out, however, that this number is very uncertain and subject to great sensitivity to the input parameters and the angular momentum properties of the states in question.

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APPENDIX

MANDELSTAM PARAMETERS IN THE OPE MODEL WITHOUT RECOIL

The purpose of this appendix is to show how the Mandelstam parametrization of the $NN \rightarrow NN\pi$ reaction can be introduced in a simple model for the dynamics of the process, namely the non-relativistic, static one-pion exchange model. The reason for going through this exercise is twofold ; firstly, we feel that our understanding of the physics of the Mandelstam model benefits from such an explicit comparison, and secondly, it may be hoped that in this way we can arrive at some quantitative predictions which will survive in a more adequate dynamical picture.

In the static limit the pion momenta $\mathbf{q}$ and $\mathbf{Q}$ (see Section 2) are identical, and the momenta that are involved in the process are shown in Fig. 6. Since we are only interested in the relationship between the various amplitudes we neglect all common factors, including the propagator for the exchanged pion (this should in reality contain corrections for off-shell effects, different for the various states). In the non-relativistic limit the interaction operator in the lower vertex is

$$\mathcal{H}_1 = C(1)\, \boldsymbol{\sigma}_1 \cdot (\mathbf{p}_1 - \mathbf{k}_1) , \tag{A.1}$$

where C is the appropriate operator in isospin space. The interaction operator in the upper vertex is

$$\mathcal{H}_2 = \sum_{\lambda} \sum_{\kappa} \sum_{\tau} \mathcal{H}_{\lambda\kappa\tau} , \tag{A.2}$$

where, for s waves ($\lambda = 0$, $\kappa = \frac{1}{2}$, $\tau = \frac{1}{2}, \frac{3}{2}$) and resonant p waves ($\lambda = 1$, $\kappa = \tau = \frac{3}{2}$), we have

$$\begin{aligned}
&\mathcal{H}_{0\frac{1}{2}\tau} = q^{-1} F_\tau(q)\, D_\tau(2) , \quad F_\tau(q) = F_{0\frac{1}{2}\tau}(q) , \\
&\mathcal{H}_{1\frac{3}{2}\frac{3}{2}} = q^{-1} F(q)\, D_{\frac{3}{2}}(2)\, (qq')^{-1} \{ 2\mathbf{q}'\cdot\mathbf{q} + i\boldsymbol{\sigma}_2 \cdot (\mathbf{q}' \times \mathbf{q}) \} , \quad F(q) = F_{1\frac{3}{2}\frac{3}{2}}(q), \\
&F_{\lambda\kappa\tau}(q) = \sin\delta_{\lambda\kappa\tau}(q)\, \exp\{ i\delta_{\lambda\kappa\tau}(q) \} ,
\end{aligned} \tag{A.3}$$

in which D is the isospin operator. The operator $\mathcal{H} = \mathcal{H}_2\mathcal{H}_1$ is applied between the final state $|S'_{z1}S'_{z2}T'_{z1}T'_{z2}T'_{z\pi}\rangle$ and the antisymmetrized initial state $|SS_zTT_z\rangle$ in spin isospin space, and the transition matrix element is proportional to

$$\langle \underset{\sim}{p}\, \underset{\sim}{\ell}\, S'_{z1} S'_{z2} T'_{z1} T'_{z2} T'_{z\pi} | \mathcal{T}'_{12} | \underset{\sim}{p}_{in}\, S S_z T T_z \rangle = \sum_{\lambda} \sum_{\kappa} \sum_{\tau} \sum_{S_{z1}} \sum_{S_{z2}} \sum_{T_{z1}} \sum_{T_{z2}}$$

$$\times \langle \tfrac{1}{2} \tfrac{1}{2} S_{z1} S_{z2} | S S_z \rangle \langle \tfrac{1}{2} \tfrac{1}{2} T_{z1} T_{z2} | T T_z \rangle \langle S'_{z1} S'_{z2} T'_{z1} T'_{z2} T'_{z\pi} | \mathcal{H}^{ST}_{\lambda\kappa\tau} | S_{z1} S_{z2} T_{z1} T_{z2} \rangle, \tag{A.4}$$

where

$$\mathcal{H}^{ST}_{0\frac{1}{2}\tau} = q^{-1} F_\tau(q) D_\tau(2) C(1) \left(\delta_{S+T,odd}\, \underset{\sim}{\sigma}_1 \cdot \underset{\sim}{p} - \delta_{S+T,even}\, \underset{\sim}{\sigma}_1 \cdot \underset{\sim}{p}_{in} \right),$$

$$\mathcal{H}^{ST}_{1\frac{3}{2}\frac{3}{2}} = q^{-1} F(q) D_{\frac{3}{2}}(2) C(1) (q'q)^{-1}$$

$$\times \Big(\{ 2\underset{\sim}{p}_{in} \cdot \underset{\sim}{q} + i\, \underset{\sim}{\sigma}_2 \cdot (\underset{\sim}{p}_{in} \times \underset{\sim}{q}) \} (\delta_{S+T,even}\, \underset{\sim}{\sigma}_1 \cdot \underset{\sim}{p} - \delta_{S+T,odd}\, \underset{\sim}{\sigma}_1 \cdot \underset{\sim}{p}_{in}) \tag{A.5}$$

$$+ \{ 2\underset{\sim}{p} \cdot \underset{\sim}{q} + i\, \underset{\sim}{\sigma}_2 \cdot (\underset{\sim}{p} \times \underset{\sim}{q}) \} (\delta_{S+T,even}\, \underset{\sim}{\sigma}_1 \cdot \underset{\sim}{p}_{in} - \delta_{S+T,odd}\, \underset{\sim}{\sigma}_1 \cdot \underset{\sim}{p}) \Big).$$

By expanding the momentum vectors in spherical waves and recoupling the angular momenta in such a way that the matrix element appears in a Mandelstam form we find

$$\langle \underset{\sim}{p}\, \underset{\sim}{\ell}\, S'_{z1} S'_{z2} T'_{z1} T'_{z2} T'_{z\pi} | \mathcal{T}'_{12} | \underset{\sim}{p}_{in}\, S S_z T T_z \rangle = -3^{-\frac{1}{2}} i \sum_{J} \sum_{J_z} \sum_{\ell} \sum_{m} \sum_{\ell'} \sum_{m'}$$

$$\times \sum_{j} \sum_{j_z} \sum_{\kappa} \sum_{\kappa_z} \sum_{\lambda} \sum_{\lambda_z} \sum_{\tau} \sum_{\tau_z} [\lambda \ell' \ell]^{\frac{1}{2}}\, i^{\ell-\ell'-\lambda}\, \delta_{S+T+\ell,odd}\, \delta_{\ell'+\ell+\lambda,odd}\, \delta_{\ell S J}$$

$$\times (p/\mu)^{\ell'}\, \hat{Y}^{\lambda_z}_{\lambda}(\hat{\underset{\sim}{q}})\, \hat{Y}^{m'}_{\ell'}(\hat{\underset{\sim}{p}})\, \hat{Y}^{m*}_{\ell}(\hat{\underset{\sim}{p}}_{in}) \langle j \ell' j_z m' | J J_z \rangle \langle \kappa \tfrac{1}{2} \kappa_z S'_{z1} | j j_z \rangle$$

$$\times \langle \lambda \tfrac{1}{2} \lambda_z S'_{z2} | \kappa \kappa_z \rangle \langle \ell S m S_z | J J_z \rangle \langle \tau \tfrac{1}{2} \tau_z T'_{z1} | T T_z \rangle \langle 1 \tfrac{1}{2} T'_{z\pi} T'_{z2} | \tau \tau_z \rangle \tag{A.6}$$

$$\times A'_{\lambda\kappa j \ell' \ell J \tau T}(p_{in}, q, q'),$$

where

$$
\begin{aligned}
A'_{\lambda\kappa j\ell'\ell J\tau T}(P_{in}, q, q') = {} & (P_{in}/\mu)^{\ell}\, \delta_{S+T+\ell,\,\mathrm{odd}}\, (\delta_{\tau\frac{1}{2}}\delta_{T1} + 3\delta_{\tau\frac{1}{2}}\delta_{T0} + 2^{\frac{3}{2}}\delta_{\tau\frac{3}{2}}\delta_{T1}) \\
& \times \Big[2^{\frac{1}{2}} \delta_{\lambda 0}\delta_{\kappa\frac{1}{2}} (q/\mu)^{-1} F_{\tau}(q)(-1)^{S} [S]^{\frac{1}{2}} W(1\tfrac{1}{2}S\tfrac{1}{2} : \tfrac{1}{2}j) \\
& \qquad \times (\delta_{\ell'0}\delta_{\ell 1}\delta_{Jj} - [j/S]^{\frac{1}{2}} \delta_{\ell'1}\delta_{\ell 0}\delta_{JS}) \\
& + 2\cdot 3^{-\frac{1}{2}} \delta_{\lambda 1}\delta_{\kappa\frac{3}{2}} (q/q')(q/\mu)^{-2} F(q)\, [S]^{\frac{1}{2}} \\
& \times \Big(\{1 + 2\cdot 3^{\frac{1}{2}}(-1)^{S}\} \delta_{\ell'0}\delta_{\ell 2}\delta_{Jj}\, X\begin{pmatrix} \frac{1}{2} & J & \frac{3}{2} \\ 1 & 2 & 1 \\ \frac{1}{2} & S & \frac{1}{2} \end{pmatrix} \\
& + 2^{\frac{1}{2}}(1 + \tfrac{1}{2}3^{-\frac{1}{2}}) \delta_{\ell'1}\delta_{\ell 1} (-1)^{J} [j]^{\frac{1}{2}} \{ W(\tfrac{3}{2}1\tfrac{1}{2}S : \tfrac{1}{2}J)\, W(J\tfrac{3}{2}1\tfrac{1}{2} : \tfrac{1}{2}j) \\
& \qquad + (-1)^{j+S} W(\tfrac{1}{2}1\tfrac{1}{2}S : \tfrac{1}{2}J)\, W(J\tfrac{1}{2}1\tfrac{3}{2} : \tfrac{1}{2}j) \} \Big) \Big] .
\end{aligned}
\tag{A.7}
$$

By comparison with the corresponding Mandelstam expression we have

$$
f_{\lambda}(q) A_{\lambda\kappa j\ell'\ell J\tau T} \propto A'_{\lambda\kappa j\ell'\ell J\tau T}(P_{in}, q, q'). \tag{A.8}
$$

To arrive at an expansion where the Mandelstam parameters are constants we can then replace $(q/\mu)^{-1}F_{\tau}(q)$ by the s wave scattering lengths $a_{(2\tau)}$ and use constant (effective) values for (q/q') and (P_{in}/μ), whereas $f(q)$ in Section 2 is the same as $\frac{4}{3}f^{2}_{\pi N}(q/\mu)^{-2}\,F(q)$.

In this way we find the phase relationships given by Eq. (43). For the Mandelstam parameters b, which only differ in j and J, we also use the results from the OPE model to find the ratios between the absolute values, which are given by

$$
\begin{aligned}
& b_{1a} : b_{2a} : b_{1b} : b_{2b} : b_{01} \\
& = \tfrac{1}{6} 5^{\frac{1}{2}} (4\cdot 6^{-\frac{1}{2}} - 1) : 0 : -\tfrac{1}{6}(14\cdot 6^{-\frac{1}{2}} + 1) : 2^{-\frac{1}{2}} : 1 .
\end{aligned}
\tag{A.9}
$$

Since b_{2a} is zero and b_{1a} is considerably smaller than the other non-zero parameters, we have disregarded both b_{1a} and b_{2a} in the previous calculations.

TABLE : Matrix elements $J_{S'T'ST\ell Jg}$ for the reaction NN → NNπ.

S'	T'	S	T	ℓ	J	g	ρ	$J_{S'T'ST\ell Jg}^{\rho}$
1	1	1	1	1	0	1	± 1	$\frac{1}{6}\, 3^{-\frac{1}{2}}\, b_{01}\, J_{111}^{1-}$
					1	0	0	$\frac{1}{9}\, 6^{\frac{1}{2}}\, b_{1b}\, J_{110}^{0-}$
						1	± 1	$-\frac{1}{18}\, 2^{-\frac{1}{2}}\, (5^{\frac{1}{2}}\, b_{1a} - 2\, b_{1b})\, J_{111}^{1-}$
						2	0	$\frac{1}{3}\, (15)^{-\frac{1}{2}}\, b_{1a}\, J_{112}^{0-}$
							± 1	$\mp \frac{1}{6}\, (10)^{-\frac{1}{2}}\, b_{1a}\, J_{112}^{1-}$
							± 2	$\frac{1}{3}\, (10)^{-\frac{1}{2}}\, b_{1a}\, J_{112}^{2-}$
					2	1	± 1	$-\frac{1}{6}\, 6^{-\frac{1}{2}}\, (2\, b_{2a} - b_{2b})$
						2	0	$-\frac{1}{3}\, 5^{-\frac{1}{2}}\, b_{2b}\, J_{112}^{0-}$
							± 1	$\pm \frac{1}{2}\, (30)^{-\frac{1}{2}}\, b_{2b}\, J_{112}^{1-}$
							± 2	$-(30)^{-\frac{1}{2}}\, b_{2b}\, J_{112}^{2-}$
1	1	1	0	0	1	1	0	$\frac{1}{3}\, B\, J_{011}^{0-}$
							± 1	$\mp \frac{1}{3}\, B\, J_{011}^{1-}$
1	1	0	1	2	2	1	0	$-\frac{1}{3}\, 3^{-\frac{1}{2}}\, a\, J_{101}^{0-}$
							± 1	$\pm \frac{1}{3}\, 3^{-\frac{1}{2}}\, a\, J_{101}^{1-}$

1	0	1	1	1	0	1	±1	$\frac{1}{3} 2^{-\frac{1}{2}} b_{01} J_{111}^{1+}$
					1	0	0	$\frac{1}{3} (A_1 R_1(P) + 2 b_{1b} \{J_{110}^{0+} + q_1(P) M_{110}\})$
						1	±1	$-\frac{1}{6} 3^{-\frac{1}{2}} (5^{\frac{1}{2}} b_{1a} - 2 b_{1b}) J_{111}^{1+}$
						2	0	$\frac{2}{3} (10)^{-\frac{1}{2}} b_{1a} \{J_{112}^{0+} + q_1(P) M_{112}\}$
							±1	$\mp \frac{1}{2} (15)^{-\frac{1}{2}} b_{1a} J_{112}^{1+}$
							±2	$(15)^{-\frac{1}{2}} b_{1a} J_{112}^{2+}$
					2	1	±1	$-\frac{1}{6} (2 b_{2a} - b_{2b}) J_{111}^{1+}$
						2	0	$-2 (30)^{-\frac{1}{2}} b_{2b} \{J_{112}^{0+} + q_1(P) M_{112}\}$
							±1	$\pm \frac{1}{2} 5^{-\frac{1}{2}} b_{2b} J_{112}^{1+}$
							±2	$-5^{-\frac{1}{2}} b_{2b} J_{112}^{2+}$
1	0	0	1	2	2	1	0	$-\frac{1}{3} 2^{\frac{1}{2}} a \{J_{101}^{0+} + q_1(P) M_{101}\}$
							±1	$\pm \frac{1}{3} 2^{\frac{1}{2}} a J_{101}^{1+}$
0	1	1	1	1	0	0	0	$\frac{1}{3} (A_0 R_0(P) + 2^{\frac{1}{2}} b_{01} \{J_{110}^{0+} + q_0(P) M_{110}\})$
					1	1	±1	$-\frac{1}{6} 3^{-\frac{1}{2}} (5^{\frac{1}{2}} b_{1a} + b_{1b}) J_{111}^{1+}$
					2	2	0	$-\frac{2}{3} (10)^{-\frac{1}{2}} (b_{2a} + b_{2b}) \{J_{112}^{0+} + q_0(P) M_{112}\}$
							±1	$\pm \frac{1}{2} (15)^{-\frac{1}{2}} (b_{2a} + b_{2b}) J_{112}^{1+}$
							±2	$-(15)^{-\frac{1}{2}} (b_{2a} + b_{2b}) J_{112}^{2+}$
0	0	1	1	1	0	0	0	$2 \cdot 3^{-\frac{1}{2}} b_{01} J_{110}^{0-}$
					1	1	±1	$-\frac{1}{3} 2^{-\frac{1}{2}} (5^{\frac{1}{2}} b_{1a} + b_{1b}) J_{111}^{1-}$
					2	2	0	$-2 (15)^{-\frac{1}{2}} (b_{2a} + b_{2b}) J_{112}^{0-}$
							±1	$\pm (10)^{-\frac{1}{2}} (b_{2a} + b_{2b}) J_{112}^{1-}$
							±2	$-2 (10)^{-\frac{1}{2}} (b_{2a} + b_{2b}) J_{112}^{2-}$

REFERENCES

1) J.J. Domingo, V.W. Allardyce, C.H.Q. Ingram, S. Rohlin, N.W. Tanner, J. Rohlin, E.M. Rimmer, G. Jones and J.P. Girardeau-Montaut - Phys.Letters 32B, 309 (1970).

2) K. Gabathuler, J. Rohlin, J.J. Domingo, C.H.Q. Ingram, S. Rohlin and N.W. Tanner - Nuclear Phys. B40, 32 (1972).

3) S. Dahlgren, B. Höistad and P. Grafström - Phys.Letters 35B, 219 (1971).

4) S. Dahlgren, P. Grafström, B. Höistad and A. Åsberg - Uppsala Report GWI-PH 4/71 (1971).

5) J. Letourneux and J.M. Eisenberg - Nuclear Phys. 87, 331 (1966).

6) W.B. Jones and J.M. Eisenberg - Nuclear Phys. A154, 49 (1970).

7) R. Chatwin - to be published.

8) C.H.Q. Ingram, N.W. Tanner, J.J. Domingo and J. Rohlin - Nuclear Phys. B31, 331 (1971).

9) A. Reitan - Nuclear Phys. B29, 525 (1971).

10) J. Rohlin, K. Gabathuler, N.W. Tanner, C.R. Cox and J.J. Domingo - to be published.

11) S. Mandelstam - Proc.Roy.Soc. A244, 491 (1958).

12) M.A. Preston - "Physics of the Nucleus", Addison-Wesley, Reading (1962).

13) R.H. Bassel and C. Wilkin - Phys.Rev. 174, 1179 (1968).

14) T.A. Brody and M. Moshinsky - "Tables of Transformation Brackets", Gordon and Breach, New York (1967).

15) D.M. Brink and G.R. Satchler - "Angular Momentum", Clarendon, Oxford (1962).

16) C. Richard-Serre, W. Hirt, D.F. Measday, E.G. Michaelis, M.J. Saltmarsh and P. Skarek - Nuclear Phys. B20, 413 (1970).

17) A.F. Dunaitsev and Yu.D. Prokoshkin - Soviet Phys.JETP (USA) 9, 1179 (1959).

18) M. Gell-Mann and K.M. Watson - Ann.Rev.Nucl.Sci. 4, 219 (1954).

FIGURE CAPTIONS

Figure 1 Kinematics for the $NN \to NN\pi$ reaction.

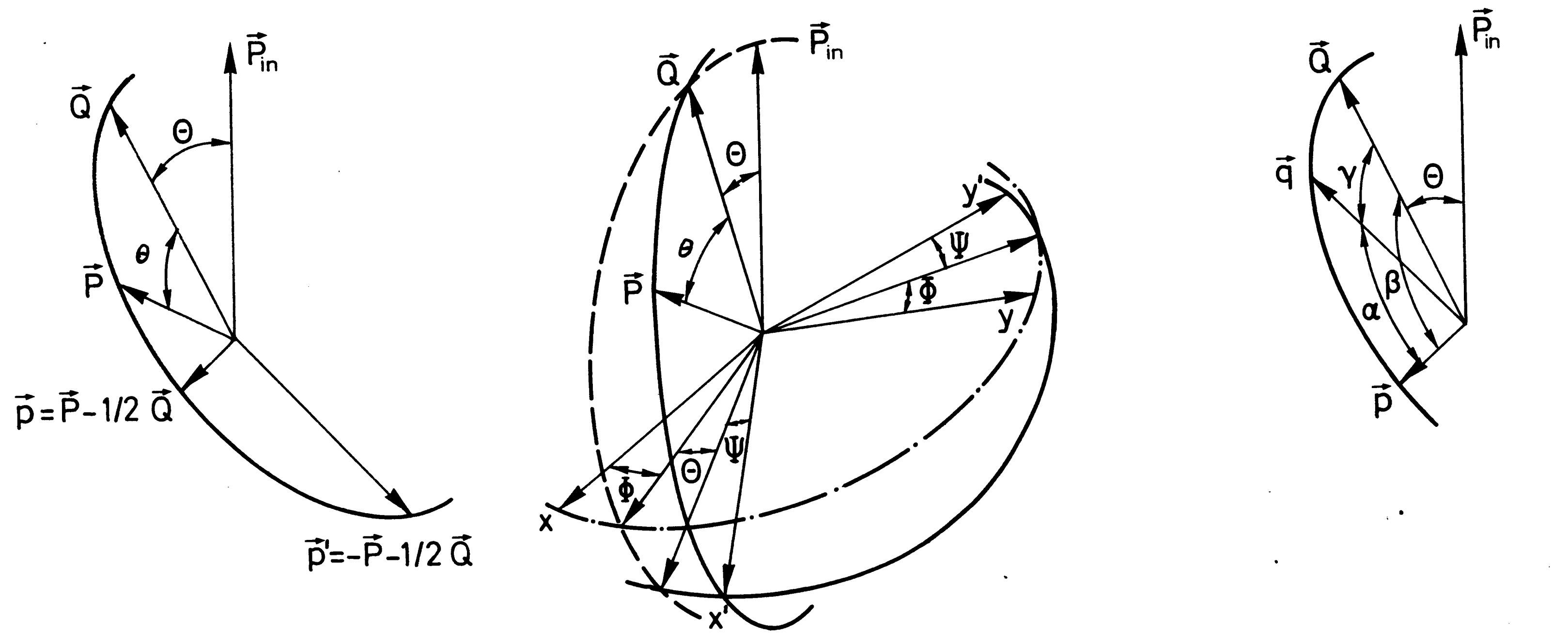
Figure 2 Experimental [4] cross-section for the reaction $p + {}^{12}C \to \pi^+ + {}^{13}C(g.s.)$, and theoretical cross-sections for a $(0^+, 1p_{\frac{1}{2}})$ state in ^{13}C. Curve (a) corresponds to the Mandelstam parameters (43). In curve (b) the absolute values of A_0, A_1 and a have been increased by 25%, and the signs have been changed for A_0 and B.

Figure 3 Experimental cross-section for the reaction $p + {}^{12}C \to \pi^+ + {}^{13}C$ (3.09 - 3.85 MeV), and theoretical cross-sections for a $(0^+, 2s_{\frac{1}{2}})$ state in ^{13}C. For the notation (a) and (b), see the caption for Fig. 2.

Figure 4 Theoretical cross-sections for the reaction $p + {}^{4}He \to \pi^+ + {}^{5}He(0^+, 1p_{\frac{3}{2}})$. The laboratory kinetic energy for the incident proton is given in MeV.

Figure 5 Theoretical forward cross-sections as functions of the laboratory kinetic energy for the incident proton.
1) $^{13}C(0^+, 1p_{\frac{1}{2}})$; 2) $^{13}C(0^+, 2s_{\frac{1}{2}})$; 3) $^{13}C(0^+, 1d_{\frac{5}{2}})$
4) $^{5}He(0^+, 1p_{\frac{3}{2}})$; 5) $^{5}He(0^+, 1p_{\frac{1}{2}})$.

Figure 6 The one-pion exchange model.


$\vec{P}_{in}$
$\vec{Q}$
Θ
θ
$\vec{P}$
$\vec{p}=\vec{P}-1/2\,\vec{Q}$
$\vec{p}'=-\vec{P}-1/2\,\vec{Q}$
$\vec{Q}$
$\vec{P}_{in}$
Θ
θ
$\vec{P}$
y'
Ψ
Φ
y
x
Φ
Θ
Ψ
x'
$\vec{Q}$
$\vec{P}_{in}$
$\vec{q}$
γ
Θ
β
α
$\vec{p}$

a) b) c)

FIG. 1

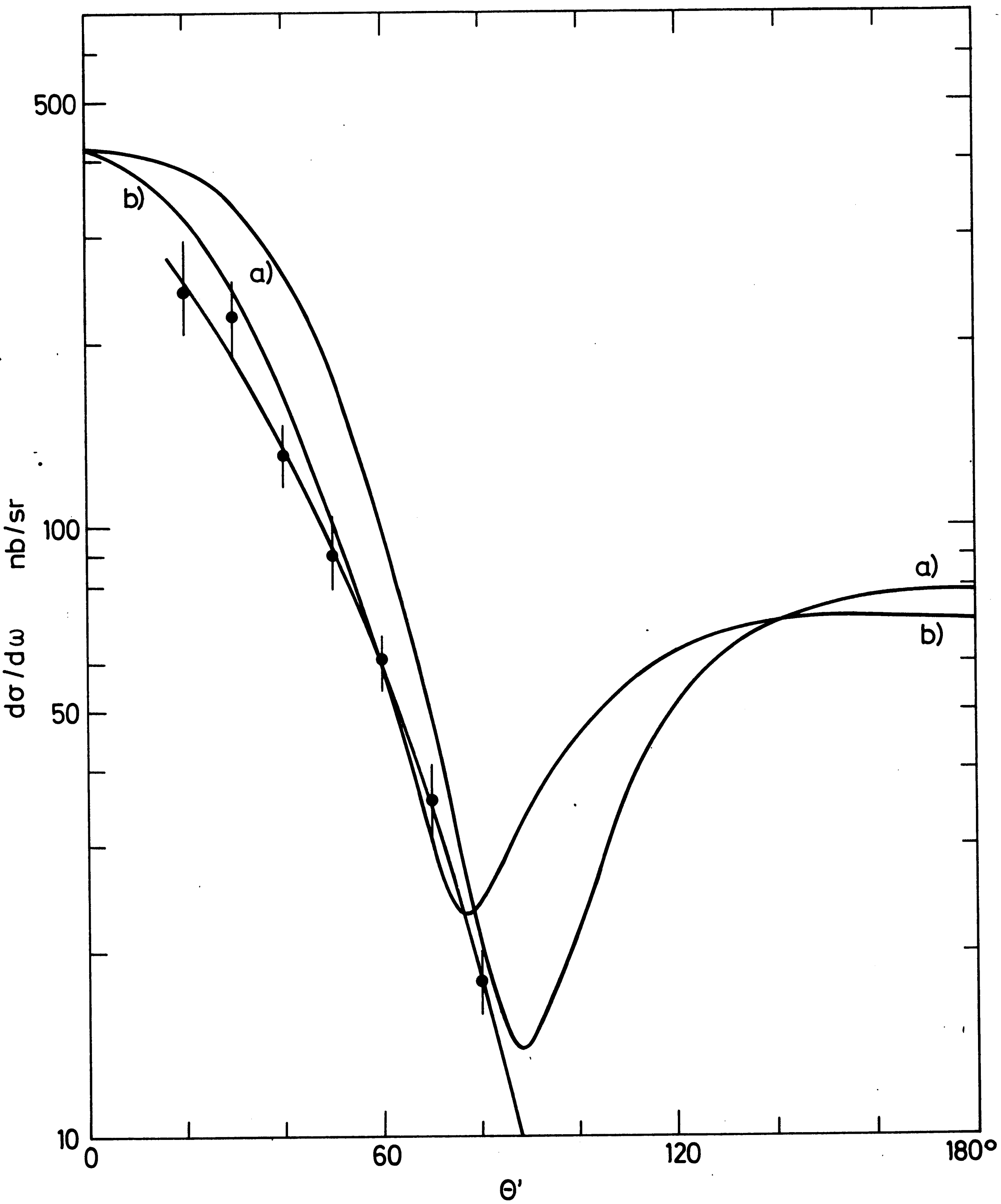


FIG.2

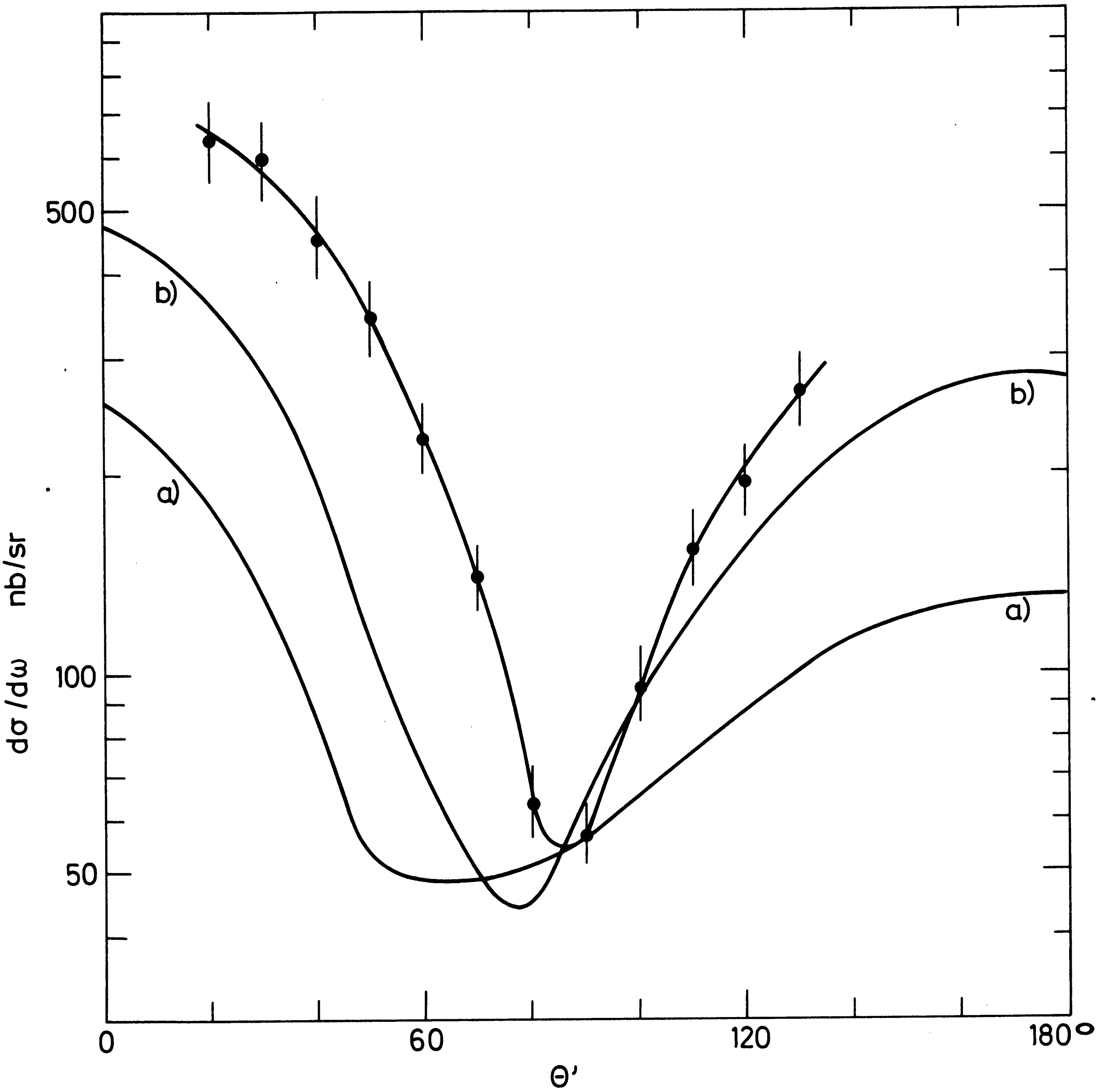


FIG.3

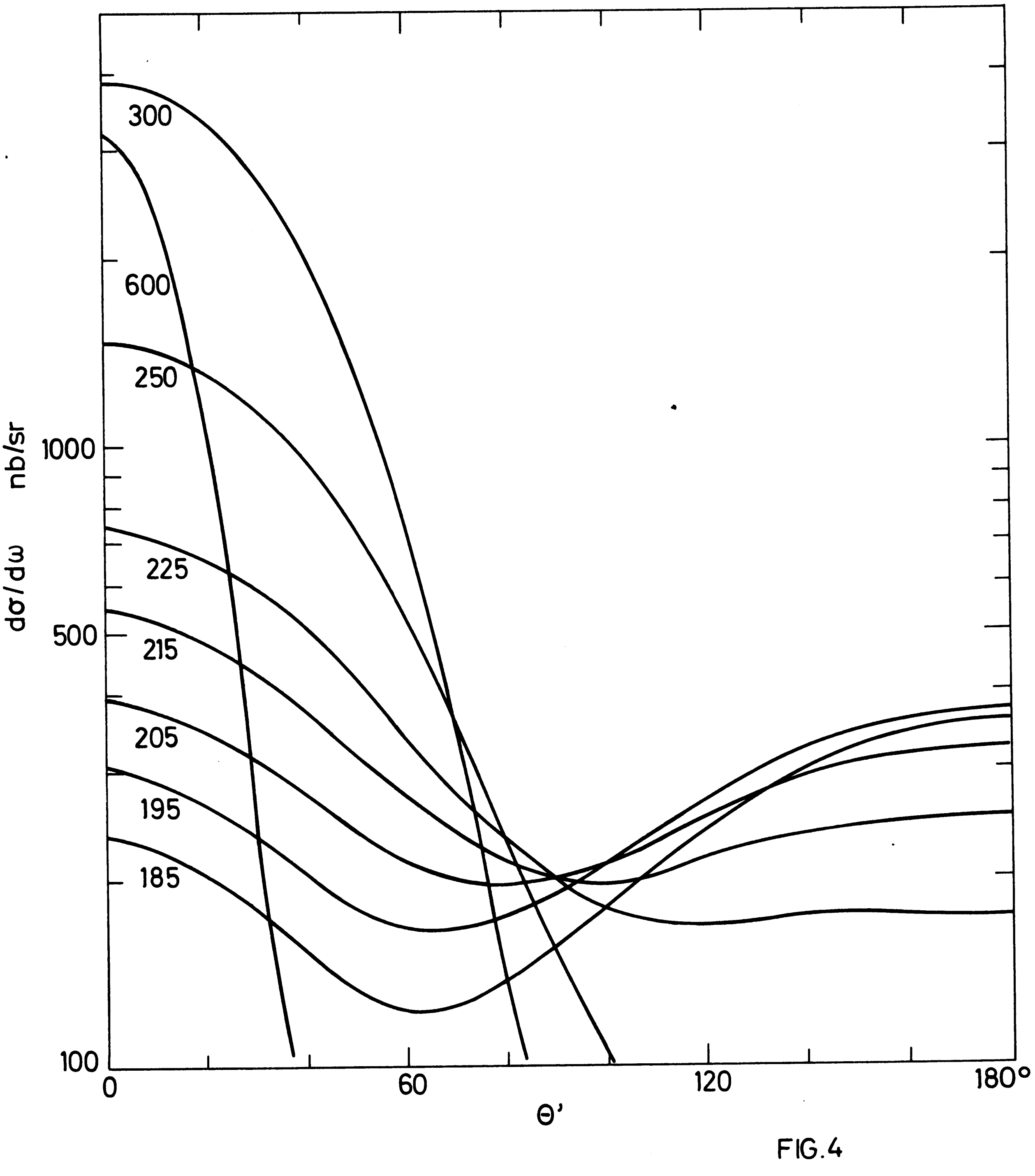

300
600
250
225
215
205
195
185
1000
500
100
dσ/dω nb/sr
0
60
120
180°
Θ'

FIG.4

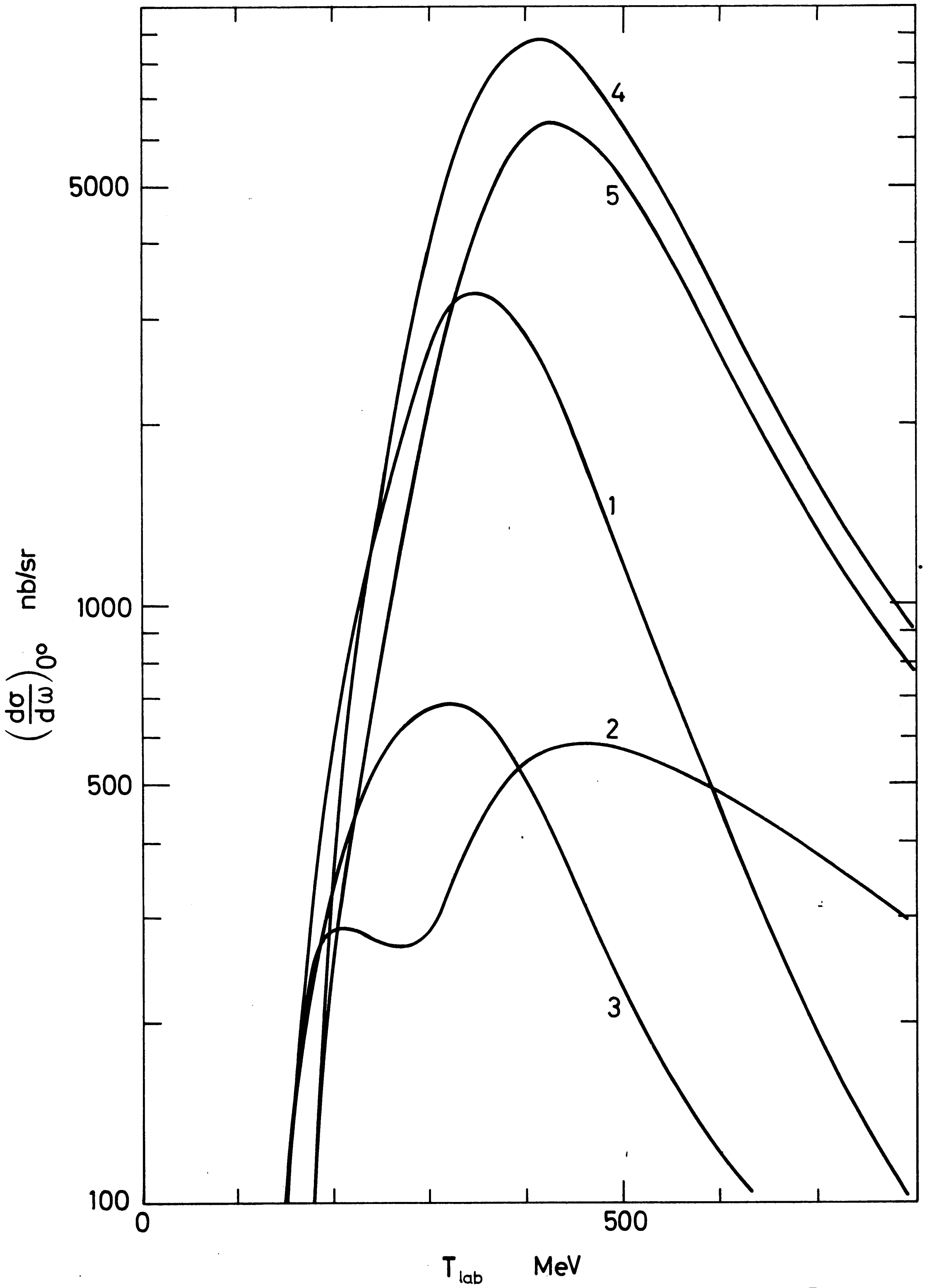

5000
1000
500
100
0
500
$\left(\frac{d\sigma}{d\omega}\right)_{0°}$ nb/sr
T_{lab} MeV
1
2
3
4
5

FIG.5

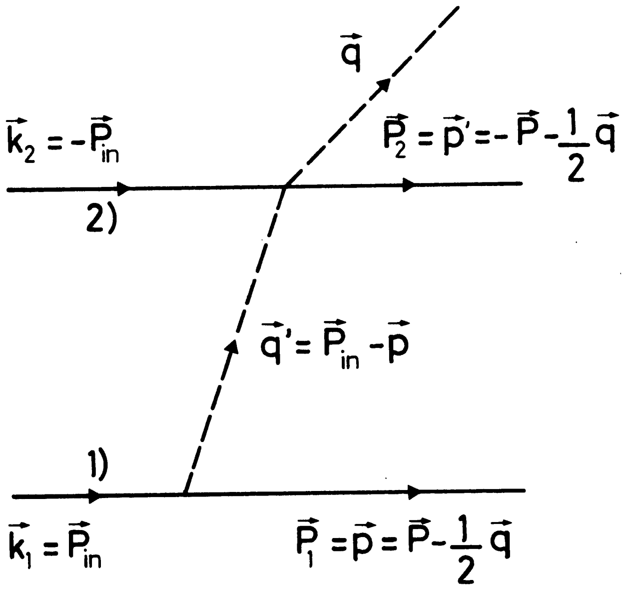

$\vec{q}$
$\vec{k}_2 = -\vec{P}_{in}$
$\vec{P}_2 = \vec{p}' = -\vec{P} - \frac{1}{2}\vec{q}$
2)
$\vec{q}' = \vec{P}_{in} - \vec{p}$
1)
$\vec{k}_1 = \vec{P}_{in}$
$\vec{P}_1 = \vec{p} = \vec{P} - \frac{1}{2}\vec{q}$

FIG.6