Bulk fields in AdS_5 from probe D7 branes

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We relate bulk fields in Randall-Sundrum AdS_5 phenomenological models to the world-volume fields of probe D7 branes in the Klebanov-Witten background of type IIB string theory. The string constructions are described by $AdS_5 \times T^{1,1}$ in their near-horizon geometry, with $T^{1,1}$ a 5d compact internal manifold that yields $\mathcal{N} = 1$ supersymmetry in the dual 4d gauge theory. The effective 5d Lagrangian description derived from the explicit string construction leads to additional features that are not usually encountered in phenomenological model building.

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I. INTRODUCTION

Phenomenological models in a slice of AdS_5 have recently provided a new alternative to the usual four-dimensional (4d) supersymmetric models in addressing the hierarchy problem. These models generalize the Randall-Sundrum model (RS1) [1] by allowing for the presence of bulk fermion and gauge fields. A striking feature of these models is that they are conjectured to be dual to 4d gauge theories with a composite Higgs and top quark [2]. In all these bottom-up constructions the bulk fields are put in by hand. However one would like to seek a more fundamental description of these fields from a string construction where, in particular, the "preons" of the composite states are identified. In this article we begin a study to determine the possible top-down constructions.

From the top-down there are various well-defined string constructions that, in the near-horizon geometry, are well-described by an $AdS_5 \times X_5$ background, with X_5 a five-dimensional (5d) compact manifold. The simplest example is the one that appeared in the seminal works on the AdS/CFT correspondence, where $X_5 = S^5$ [3–5]. However, in the $X_5 = S^5$ case the dual 4d gauge theory has $\mathcal{N} = 4$ supersymmetry, which is too restrictive for phenomenological purposes.

More interesting for our present purposes is the Klebanov-Witten (KW) construction [6], where $X_5 = T^{1,1} \simeq (SU(2) \times SU(2))/U(1)$. In this case, the dual description is a 4d $\mathcal{N} = 1$ superconformal gauge theory. The theory is unregulated in the infrared (IR), corresponding to a naked singularity in the geometric picture. The resolution of this singularity leads to the more refined, Klebanov-Strassler (KS) construction [7]. In both the KW and KS backgrounds, the geometry is noncompact, extending to infinity in the direction associated with the AdS_5 radius. Correspondingly, the ultraviolet (UV) of the dual gauge theory lacks a regulator. On the gravity side of the duality, this can be addressed if the geometry is completed by a compact Calabi-Yau (CY) manifold in the region far from the "throat". When this is done in the KS construction, as has been considered by Giddings, Kachru, and Polchinski (GKP) [8], the spectrum is normalizable and discrete. This is very much like the RS1 setup: the tip of the KS throat represents the IR brane, and the compact CY represents the UV brane. The general picture is sketched in Fig. 1.

However, it is expected that the effective 5d action describing the dimensional reduction of the string model on X_5 will differ in a variety of ways from typical phenomenological AdS_5 actions. Ultimately, we want to compare the two classes of effective actions, to delineate the

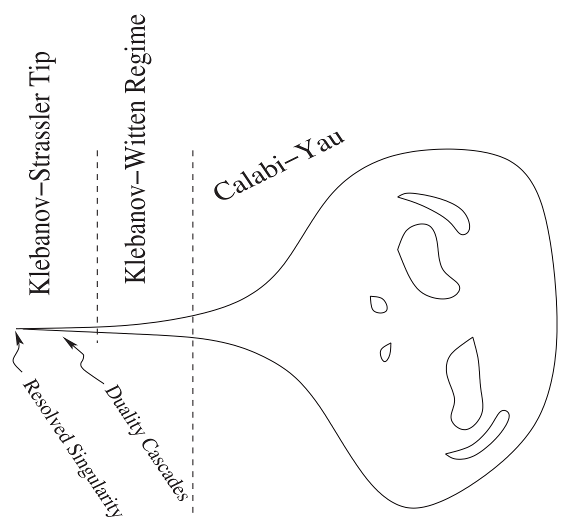

FIG. 1. A KS throat with a CY glued on to regulate the UV of the dual gauge theory. The KS tip serves to regulate the IR of the dual gauge theory. A section of AdS_5 is asymptotically contained in the throat. The CY and the KS tip are, respectively, to be thought of as refinements of the UV and IR branes of RS1. Much of the throat can be approximated by the significantly simpler KW solution that we study in the text. Furthermore, for the construction that we study, the end of the D7 brane provides an alternative IR boundary, as far as quark fields are concerned.

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extent to which they differ, and to better understand the circumstances in which they agree. Furthermore, we would like to obtain string-inspired constraints on the 5d phenomenological models. For instance, what 5d bulk masses are possible, what are the boundary conditions at the branes, and what is the possible field content? In this article we address some of these questions.

The construction that is studied here involves embedding a D7 probe brane in the KW geometry. In this way bulk matter and gauge fields are introduced; i.e., fields that are not already contained in the $\mathcal{N} = 2$ AdS_5 supergravity multiplet associated with type IIB compactification on $T^{1,1}$ [9–12]. In the dual 4d gauge theory, this corresponds to the introduction of "quark" flavors. The D7 brane wraps an internal cycle of the internal 5d compact space $T^{1,1}$ in such a way that it "disappears" at some distance from the tip of the throat, much as in the $AdS_5 \times S^5$ models with probe branes [13–15]. This ending of the D7 brane leads to an IR boundary for the theory. In the dual 4d gauge theory, it has the effect of bare masses for the quarks. The IR cutoff of the KS background corresponds to the confinement scale of the dual gauge theory. We will work in the heavy quark limit, so that the flavors are not propagating degrees of freedom near the confinement scale. Thus, for the investigation that follows, the IR cutoff of KS is not a detail that we will need. For this reason, we will utilize the simpler KW geometry.

Next we give a summary of the remainder of this article:

(i) In Sec. II we outline the bosonic fluctuations associated with embedding a probe D7 brane in the KW background, following Levi and Ouyang [16]. We introduce a reparameterization of the embedding coordinates that allows us to describe the action of the embedding scalars as a function of the AdS_5 radius. We extract the effective 5d action that describes the corresponding modes that are independent of the angular coordinates of the internal 5d space $T^{1,1}$. We briefly address the boundary terms that appear for these scalars. We find that a certain convenient simplification occurs. Finally, we summarize the action of the D7 brane world-volume vector boson, and make some remarks on the effective 5d compactification.

(ii) In Sec. III, we isolate the regime in which the scalar action can be mapped into conventional AdS_5 formulations, such as phenomenological models in a slice of AdS_5. We show that both bulk and boundary masses arise. The bulk mass-squared is negative, but satisfies the Breitenlohner-Freedman bound [17,18], just as in the cases of probe branes in $AdS_5 \times S^5$ geometries [13–15]. We describe how effects near the IR boundary where the D7 brane ends can be translated into an effective description that resembles what is typically considered in model-building applications. In particular, we show how Cauchy boundary conditions at the IR boundary of the effective AdS_5 regime are generic, and how they emerge from Dirichlet boundary conditions that occur at the radius where the D7 brane ends.

(iii) In Sec. IV we make concluding remarks and outline some outstanding issues. In particular, our present findings indicate the need for a thorough understanding of the harmonics on the somewhat complicated geometry of the internal 3-manifold of the embedding, and of the detailed effects near the end of the D7 brane—items that we have touched on, but mostly left to future work.

(iv) In the Appendix we discuss the geometry and topology of the D7 embedding. We show that the 3-manifold wrapped by the D7 brane at fixed AdS_5 radius is topologically equivalent to S^3. This justifies an angle-independent assumption that is made in the main text. Of course this assumption is only valid for fields that can be expanded on scalar harmonics of the 3-manifold. We also identify the isometry group of the embedding.

II. D7 BRANES IN THE KLEBANOV-WITTEN BACKGROUND

Let us begin by reviewing the KW background [6]. The ten-dimensional (10d) metric is given by

$$ds_{10}^2 = H(r)^{-1/2}\eta_{\mu\nu}dx^\mu dx^\nu + H(r)^{1/2}ds_6^2,$$
$$ds_6^2 = dr^2 + r^2 ds_{T^{1,1}}^2, \qquad H(r) = 1 + \frac{L^4}{r^4}, \tag{2.1}$$

where L is the curvature length and $ds^2_{T^{1,1}}$ is the metric for $T^{1,1}$,

$$ds^2_{T^{1,1}} = \frac{1}{9}\Big(d\psi + \sum_{i=1,2}\cos\theta_i d\phi_i\Big)^2 + \frac{1}{6}\sum_{i=1,2}(d\theta_i^2 + \sin^2\theta_i d\phi_i^2). \tag{2.2}$$

The base of the conifold is determined by

$$z_1 z_2 = z_3 z_4, \qquad z_i \in \mathbf{C}, \tag{2.3}$$

where by definition, $z_i \sim r^{3/2}$. The angular dependence of the z_i that give rise to (2.2) can be found in [6]. We work in the near-horizon limit $r \ll L$ and approximate the "warp factor" $H(r) \approx (L/r)^4$. Unless otherwise stated, we will work in $L = 1$ units (i.e., dimensions are restored in what follows via $r \to Lr$, $x^\mu \to Lx^\mu$, etc.).

A. Scalar fluctuations of the D7 brane embedding

Following Levi and Ouyang [16], the embedding of the D7 brane is characterized by

$$z_1 = r^{3/2}\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}e^{(i/2)(\psi-\phi_1-\phi_2)} \equiv \mu > 0. \tag{2.4}$$

Equivalently, $r = r_0(\theta_i)$ and $\psi = \psi_0(\phi_i)$ with

$$r_0^{3/2} \sin\frac{\theta_1}{2} \sin\frac{\theta_2}{2} = \mu, \qquad \psi_0 = \phi_1 + \phi_2. \tag{2.5}$$

Thus the minimum radius is $r = \mu^{2/3}$. Since the AdS_5 regime is in the near-horizon limit $r \ll 1$, in order for the flavors on the D7 brane to behave as fields in AdS_5 in some regime it is necessary that $\mu^{2/3} \ll 1$. We assume that this is true in what follows. The embedding studied here is among those that have been shown to be consistent in the analysis of [19].

The fluctuations of the brane in the two orthogonal directions are given by the scalar modes[1]

$$r = r_0(1 + \chi), \qquad \psi = \psi_0 + 3\eta, \tag{2.6}$$

where χ, η are generally functions of the eight world-volume coordinates of the D7 brane. The induced metric on the D7 brane is

$$(g_0)_{ab} = \mathrm{diag}(r_0^2\eta_{\mu\nu}, g_{\theta_i\theta_j}, g_{\phi_i\phi_j}), \qquad g_{\theta_i\theta_j} = \begin{pmatrix} \frac{1}{6} + \frac{1}{9}\cot^2\frac{\theta_1}{2} & \frac{1}{9}\cot\frac{\theta_1}{2}\cot\frac{\theta_2}{2} \\ \frac{1}{9}\cot\frac{\theta_1}{2}\cot\frac{\theta_2}{2} & \frac{1}{6} + \frac{1}{9}\cot^2\frac{\theta_2}{2} \end{pmatrix},$$
$$g_{\phi_i\phi_j} = \begin{pmatrix} \frac{1}{6}\sin^2\theta_1 + \frac{1}{9}(1+\cos\theta_1)^2 & \frac{1}{9}(1+\cos\theta_1)(1+\cos\theta_2) \\ \frac{1}{9}(1+\cos\theta_1)(1+\cos\theta_2) & \frac{1}{6}\sin^2\theta_2 + \frac{1}{9}(1+\cos\theta_2)^2 \end{pmatrix}, \tag{2.7}$$

where "diag" denotes a block diagonal matrix. The Dirac-Born-Infeld (DBI) action,

$$S_{\mathrm{DBI}} = -\tau_7 \int d^4x d^2\theta d^2\phi \sqrt{\varphi^\star(g) + \varphi^\star(B) + 2\pi\alpha' F}, \tag{2.8}$$

with $\tau_7 = (2\pi)^{-7}\alpha'^4 g_s^{-1}$, contains the scalar fluctuations (2.6) through perturbations of the pullback $\varphi^\star(g) = g_0 + \delta g_0(\chi, \eta)$ of the 10d metric to the D7 brane world-volume. We will return to the world-volume 2-form $F = dA$ in Sec. II D below. The pullback of the NS-NS 2-form $\varphi^\star(B)$ will be neglected in our analysis, since we are interested in fields that carry flavor quantum numbers. Note that the B-field vanishes in the KW background. To quadratic order, the action for the scalar fluctuations (2.6) is:

$$S = -\tau_7 \int d^4x d^2\theta d^2\phi \bigg\{ \sqrt{-g_0}\bigg[\frac{g_0^{ab}}{2C}(\partial_a\chi\partial_b\chi + \partial_a\eta\partial_b\eta) + \frac{4}{C}\Big(\sin^2\frac{\theta_i}{2}\Big)^{-1}\chi\partial_{\phi_i}\eta - \frac{2}{C^2}\Big(\sin^2\frac{\theta_i}{2}\Big)^{-1}\cot\frac{\theta_j}{2}\partial_{\theta_j}(\chi\partial_{\phi_i}\eta)\bigg]$$
$$- \partial_{\theta_i}\bigg[\frac{\sqrt{-g_0}}{C}\cot\frac{\theta_i}{2}(3\chi^2 + 2\chi)\bigg]\bigg\}, \tag{2.9}$$

where it has been convenient to define

$$C = 1 + \frac{2}{3}\cot^2\frac{\theta_1}{2} + \frac{2}{3}\cot^2\frac{\theta_2}{2}, \tag{2.10}$$

and implicit sums over $a, b \in \{x^\mu, \theta_i, \phi_i\}$, with $\mu \in \{0, \ldots, 3\}$, and $i, j \in \{1, 2\}$. Thus we agree with Eq. (29) of Levi and Ouyang [16]; note, however, that we have explicitly included the boundary terms that occur in the simplifications (integration by parts) that lead to (2.9).

Our interest is in the lightest states, which are not excited modes on the internal compact space $T^{1,1}$. Setting $\partial_{\phi_i} = 0$ and integrating over the ϕ_i coordinates we find:

$$S = -4\pi^2\tau_7 \int d^4x d^2\theta \bigg(\sqrt{-g_0}\bigg[\frac{g_0^{mn}}{2C}(\partial_m\chi\partial_n\chi + \partial_m\eta\partial_n\eta)\bigg] - \partial_{\theta_i}\bigg[\frac{\sqrt{-g_0}}{C}\cot\frac{\theta_i}{2}(3\chi^2 + 2\chi)\bigg]\bigg), \tag{2.11}$$

where we now have indices $m, n \in \{x^\mu, \theta_i\}$. To proceed further, we must extract the radial dependence that has been hidden in the angular variables θ_i; cf. (2.5).

B. A radial reparametrization

It is convenient to introduce a scaled radius: $r_0 = \mu^{2/3}\hat{r}$. We can express the D7 brane embedding (2.5) in the equivalent form

$$2 = \hat{r}^{3/2}(\cos\theta_- - \cos\theta_+), \qquad \theta_\pm \equiv \frac{1}{2}(\theta_1 \pm \theta_2). \tag{2.12}$$

We will eliminate θ_+ in favor of the coordinates $\hat{r}$, θ_-. The domain of θ_- depends on $\hat{r}$, and is given by

$$\theta_- \in [-\theta_0(\hat{r}), \theta_0(\hat{r})], \qquad \theta_0(\hat{r}) \equiv \frac{\pi}{2} - \sin^{-1}\hat{r}^{-3/2}. \tag{2.13}$$

Taking into account the Jacobian of the transformation, we have

[1]This is the definition of χ that was actually used in [16], although version 1 of the preprint had a typo [20].

$$\int_0^\pi d\theta_1 \int_0^\pi d\theta_2(\cdots) = \int_1^\infty d\hat{r} \int_{-\theta_0(\hat{r})}^{\theta_0(\hat{r})} d\theta_- \times \frac{6}{\hat{r}^{5/2}\sin\theta_+(\hat{r},\theta_-)}(\cdots), \quad (2.14)$$

where

$$\sin\theta_+(\hat{r},\theta_-) = [1 - (\cos\theta_- - 2\hat{r}^{-3/2})^2]^{1/2}. \quad (2.15)$$

We will reduce to a 5d effective theory by imposing θ_- independence, corresponding to no excitation in this compact coordinate. The validity of this for the eight-dimensional (8d) scalars χ, η rests on the fact that at fixed $\hat{r}$ the D7 brane wraps a 3-manifold that is topologically equivalent to S^3, as shown in the Appendix. In the reduction to 5d, the 8d scalars should be expanded on scalar harmonics of this 3-manifold, which will include the constant "$\ell = 0$" mode. This translates into a θ_-, ϕ_i independent mode in the coordinates that are used here.

Note that we retain the parameterization (2.6) of the D7 embedding fluctuations, although we have now taken r as a parameter of the D7 brane world-volume. In fact, it is not difficult to show that χ can be reinterpreted as a fluctuation of θ_+ away from $\theta_+ = \theta_+(\hat{r},\theta_-)$ determined from (2.12).

The quantity C that appears in (2.10) can be written as:

$$C = \frac{1}{3}[-1 + 4\hat{r}^{3/2}\cos\theta_- + \hat{r}^3(1-\cos 2\theta_-)]. \quad (2.16)$$

We also find that the metric density of the old coordinates takes the form

$$\sqrt{-g_0} \equiv \mu^{8/3}\sqrt{-g(\hat{r},\theta_-)}, \qquad \sqrt{-g(\hat{r},\theta_-)} = \frac{C}{9}\hat{r}(\hat{r}^{3/2}\cos\theta_- - 1). \quad (2.17)$$

For the metric $\hat{g}_{mn}$ in the new coordinates $m, n \in \{x^\mu, \hat{r}, \theta_-, \phi_i\}$, we have

$$\sqrt{-\hat{g}} = \frac{6\mu^{8/3}\sqrt{-g(\hat{r},\theta_-)}}{\hat{r}^{5/2}\sin\theta_+(\hat{r},\theta_-)} = \frac{2\mu^{8/3}C(\hat{r}^{3/2}\cos\theta_- - 1)}{3\hat{r}^{3/2}\sin\theta_+(\hat{r},\theta_-)}, \quad (2.18)$$

in accordance with (2.14). The components of the new metric $\hat{g}$ are given in (2.27) below. In the new coordinates we obtain the action:

$$S = -4\pi^2\tau_7 \int d^4x \int_1^\infty d\hat{r} \int_{-\theta_0(\hat{r})}^{\theta_0(\hat{r})} d\theta_- \left\{\sqrt{-\hat{g}}\left[\frac{\hat{g}^{ij}(\hat{r},\theta_-)}{2C(\hat{r},\theta_-)} \times (\partial_i\chi\partial_j\chi + \partial_i\eta\partial_j\eta)\right] + \text{t.d.}\right\}, \quad (2.19)$$

where "t.d." = total derivatives, $\hat{r}$, θ_- dependence has been made explicit, and now $i, j \in \{\mu, \hat{r}\}$. The transformed inverse metric $\hat{g}^{ij}$ is just the one that follows from the change of coordinates. We only need[2]:

$$\hat{g}^{ij} = \begin{pmatrix} \hat{g}^{\mu\nu} & 0 \\ 0 & \hat{g}^{\hat{r}\hat{r}} \end{pmatrix} = \begin{pmatrix} \mu^{-4/3}\hat{r}^{-2}\eta^{\mu\nu} & 0 \\ 0 & \left(1-\frac{1}{C}\right)\hat{r}^2 \end{pmatrix}. \quad (2.20)$$

We still must integrate over the θ_- dependence that appears explicitly in the Lagrangian. To this end we define the following $\hat{r}$-dependent functions[3]:

$$F_1(\hat{r}) \equiv \int_{-\theta_0(\hat{r})}^{\theta_0(\hat{r})} d\theta_- \frac{\sqrt{-g}}{\sin\theta_+ C}(\hat{r},\theta_-), \qquad \tilde{F}_1(\hat{r}) \equiv \int_{-\theta_0(\hat{r})}^{\theta_0(\hat{r})} d\theta_- \frac{\sqrt{-g}(C-1)}{\sin\theta_+ C^2}(\hat{r},\theta_-). \quad (2.21)$$

Then the effective 5d action for θ_-, ϕ_i independent modes is:

$$S = -24\pi^2\mu^{8/3}\tau_7 \int d^4x \int_1^\infty d\hat{r}\left\{\frac{1}{2}\mu^{-4/3}\hat{r}^{-9/2}F_1(\hat{r}) \times \eta^{\mu\nu}(\partial_\mu\chi\partial_\nu\chi + \partial_\mu\eta\partial_\nu\eta) + \frac{1}{2}\hat{r}^{-1/2}\tilde{F}_1(\hat{r})[(\partial_{\hat{r}}\chi)^2 + (\partial_{\hat{r}}\eta)^2] + \text{t.d.}\right\}. \quad (2.22)$$

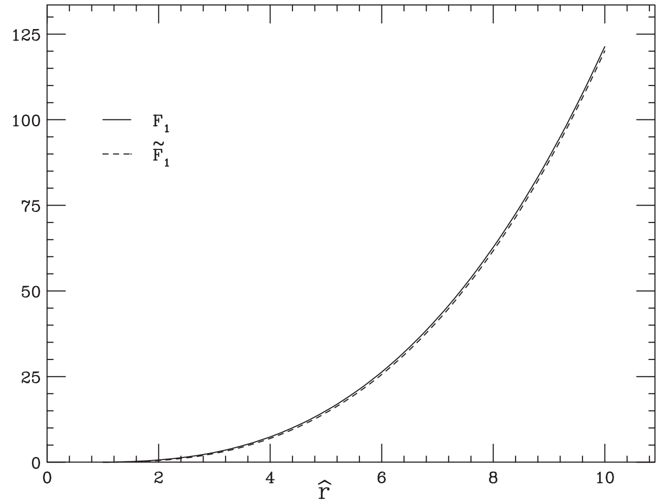

FIG. 2. The functions $F_1(\hat{r})$ and $\tilde{F}_1(\hat{r})$, defined in (2.21). Both functions are well-described by the approximation (3.4).

In Fig. 2 we show the functions F_1 and $\tilde{F}_1$, each of which vanishes at $\hat{r} = 1$. We will have more to say about these functions below.

[2]There are also $\hat{g}^{\hat{r}\theta_-}$ and $\hat{g}^{\phi_i\phi_j}$ components that we are able to ignore because of our angular independence assumption. These components of course make an implicit appearance in the overall measure that appears in (2.19). cf. (2.27) below.

[3]Note that these definitions are expressed in terms of the old metric density with $\mu^{8/3}$ scaled out, the quantity $\sqrt{-g(\hat{r},\theta_-)}$ defined in (2.17), rather than the new metric density (2.18).

C. Boundary terms

Next, we briefly describe the total derivative terms that appear in (2.11). These yield boundary terms at $\theta_{1,2} = 0$ and π. To elucidate the relation of these boundaries to the $\hat{r}, \theta_-$ coordinates, we have mapped $\hat{r}$ into a finite domain in Fig. 3. This figure takes into account the $\hat{r}$-dependent domain of θ_-, Eq. (2.13). Along the $\theta_1 = \pi$ boundary, we can write $\theta_2 = \theta_2(\hat{r}) = \pi - 2\theta_0(\hat{r})$. A similar statement holds along the $\theta_2 = \pi$ boundary, with $\theta_1 = \pi - 2\theta_0(\hat{r})$. It is therefore straightforward to express these two boundary terms as integrals over $\hat{r}$, with $\theta_- = \pm\theta_0(\hat{r})$. (It will be seen below that these boundary terms vanish.) At the $\theta_2 = 0$ boundary, we can write $\theta_1 = 2\theta_-$. A similar statement holds on the $\theta_1 = 0$ boundary, where $\theta_2 = -2\theta_-$. These two boundary terms can therefore be expressed as integrals over $\theta_- \in [0, \pm\frac{\pi}{2}]$ with $\hat{r} \to \infty$. The latter limit will require some care. Corresponding to the total derivative terms in (2.11), we define

$$H_i^{(1)}(\theta_1, \theta_2) = 2\sqrt{-g_0}C^{-1}\cot\frac{\theta_i}{2}\chi,$$
$$H_i^{(2)}(\theta_1, \theta_2) = 3\sqrt{-g_0}C^{-1}\cot\frac{\theta_i}{2}\chi^2. \quad (2.23)$$

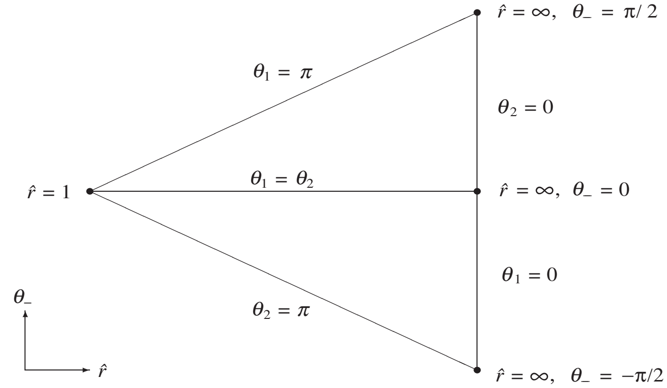

FIG. 3. Relation between coordinates $\theta_{1,2}$ and $\hat{r}$, θ_-, taking into account the $\hat{r}$-dependent domain of θ_- for the D7 embedding, given in (2.13). The boundaries at $\theta_{1,2} = \pi$ correspond to $\theta_- = \pm\theta_0(\hat{r})$, where $\theta_0(\hat{r})$ is monotonically increasing in $\hat{r}$, from $\theta_0(1) = 0$ to $\theta_0(\infty) = \pi/2$. The boundary at $\hat{r} = \infty$ consists of two segments, distinguished by positive and negative θ_-. These correspond to $\theta_{2,1} = 0$. The $\hat{r} = 1$ boundary is just a point in this 2d subspace.

Then it is easy to show that the total derivatives can formally be written as the following boundary action:

$$S_b = -8\pi^2\tau_7\int d^4x\sum_{i,\alpha=1,2}\bigg\{\int_1^\infty d\hat{r}\left|\frac{\partial\theta_0}{\partial\hat{r}}\right|[H_i^{(\alpha)}(\pi, \pi - 2\theta_0) + H_i^{(\alpha)}(\pi - 2\theta_0, \pi)] + \int_0^{\pi/2} d\theta_- H_i^{(\alpha)}(2\theta_-, 0)$$
$$+ \int_{-\pi/2}^0 d\theta_- H_i^{(\alpha)}(0, -2\theta_-)\bigg\}$$
$$= -8\pi^2\tau_7\int d^4x\sum_{i,\alpha=1,2}\bigg\{\int_0^{\pi/2} d\theta_- H_i^{(\alpha)}(2\theta_-, 0) + \int_{-\pi/2}^0 d\theta_- H_i^{(\alpha)}(0, -2\theta_-)\bigg\}. \quad (2.24)$$

Here we provide an intermediate expression in order to emphasize that the boundaries with $\theta_- = \pm\theta_0$ (first line) give vanishing contributions. This is fortunate, since they would otherwise give bulk contributions as far as the 5d reduction is concerned. The final expression is just the $\hat{r} \to \infty$ boundary terms, which should be interpreted in terms of limits. Boundary conditions on χ will have to be imposed such that the result is well-defined.

D. Gauge fields

Under dimensional reduction, the 8d vector boson A_a of the world-volume D7 brane $U(1)$ gauge theory decomposes into a 5d gauge field A_μ and three real scalars A_{θ_-,ϕ_i}. We now extract the quadratic action and equations of motion for these fields. We will then make some brief comments regarding these modes. We will point out the difficulties that arise from the vector harmonic analysis of the A_{θ_-,ϕ_i} modes due to the nontrivial 3-manifold that they are compactified on.

It is straightforward to expand the DBI action (2.8) to quadratic order in the 8d field strength F_{ab}. In addition, the Wess-Zumino (WZ) term needs to be considered, due to the nontrivial 4-form background that is present in the KW construction

$$S_{\rm WZ} = \frac{1}{2}(2\pi\alpha')^2\tau_7\int C_4\wedge F\wedge F,$$
$$C_4 = \mu^{8/3}\hat{r}^4 dx^0\wedge\cdots\wedge dx^3. \quad (2.25)$$

The $F\wedge F$ that appears in the WZ action can only have "legs" along the $\hat{r}$, θ_-, ϕ_i directions, which we will label collectively by α, β etc. Altogether, we have

$$S_{F^2} = (2\pi\alpha')^2\tau_7\int d^4x d\hat{r} d\theta_- d^2\phi\bigg\{-\frac{1}{4}\sqrt{-\hat{g}}F_{ab}F^{ab}$$
$$+ \frac{1}{8}\mu^{8/3}\hat{r}^4\epsilon^{\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta}\bigg\}, \quad (2.26)$$

where the metric is given by

$$\begin{aligned} \hat{g}^{\mu\nu} &= \mu^{-4/3}\hat{r}^{-2}\eta^{\mu\nu}, \\ \hat{g}^{\hat{r}\hat{r}} &= (1-1/C)\hat{r}^2, \\ \hat{g}^{\hat{r}\theta_-} &= \hat{r}^{5/2}C^{-1}\sin\theta_-, \\ \hat{g}^{\theta_-\theta_-} &= (4\hat{r}^{3/2}\cos\theta_- + 3C - 1)/2C, \\ \hat{g}^{\phi_i\phi_j} &= \frac{\hat{r}^3}{4C}\begin{pmatrix} \frac{5-\cos(\theta_- -\theta_+)}{\cos^2\frac{1}{2}(\theta_- +\theta_+)} & -4 \\ -4 & \frac{5-\cos(\theta_- +\theta_+)}{\cos^2\frac{1}{2}(\theta_- -\theta_+)} \end{pmatrix}, \end{aligned} \tag{2.27}$$

with $\theta_\pm$ defined in (2.12), and $\sqrt{-\hat{g}}$, given in (2.18), contributing the same coefficient $\mu^{8/3}$ as appears in the WZ term. The equations of motion are

$$0 = \partial_a(\sqrt{-\hat{g}}F^{ab}) - 4\mu^{8/3}\hat{r}^3\epsilon^{bjk}\partial_j A_k, \tag{2.28}$$

where $\epsilon^{bjk} = 0$ unless $b \in \{\theta_-, \phi_i\}$, and by definition j, $k \in \{\theta_-, \phi_i\}$, the coordinates of the internal 3-manifold X_3 at fixed radius $\hat{r}$ (see the Appendix).

Since the fields A_k are vectors on X_3, it is necessary to expand them on vector harmonics of X_3. This analysis is far from trivial, for a couple of reasons. First, the metric of X_3 depends on $\hat{r}$. Second, even at $\hat{r} \to \infty$, where the metric of X_3 becomes independent of $\hat{r}$, the geometry of the 3-manifold is not simple, as is discussed in the Appendix. To determine the vector harmonics requires an analysis comparable to that done in Refs. [9–12] for the 5-manifold $T^{1,1}$.

It is easy to check that the 5d vector boson $(A_\mu, A_{\hat{r}})$ has vanishing bulk mass. Note that this mode corresponds to the constant scalar harmonic on the compact 3-manifold X_3. From the AdS_5 supersymmetry that is present in the model, we know that there must be a 5d real scalar partner with bulk mass-squared $m^2 = -4/L^2$. This must emerge from the analysis of the modes A_{θ_-}, A_{ϕ_i}, and would be a nontrivial check of the supersymmetry that is beyond the scope of the present work. In addition there will also be a massless (Dirac) fermion corresponding to the gaugino.

III. RELATION TO 5D EFFECTIVE THEORIES

In this section we relate the above string construction to the sort of 5d effective theories that are generally contemplated in phenomenological applications [2]. Our first task is to show that, in an appropriate limit the scalars described above behave like fields in AdS_5.

A. The AdS_5 regime

The action for a massive real scalar field in a semi-infinite slice of AdS_5 is given by

$$S = -\frac{1}{2}\int d^4x \int_R^\infty dr\left\{\frac{r}{L}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{r^5}{L^5}\partial_r\phi\partial_r\phi + \frac{r^3}{L^3}m^2\phi^2\right\}, \tag{3.1}$$

where our coordinate conventions are summarized by the metric:

$$ds_5^2 = \frac{r^2}{L^2}\eta_{\mu\nu}dx^\mu dx^\nu + \frac{L^2}{r^2}dr^2. \tag{3.2}$$

We would like to find a regime where the action (3.1) is a good approximation to the action of χ, η (2.22). We expect this to be possible since in the near-horizon regime ($r \ll L$) the supergravity background is $AdS_5 \times T^{1,1}$. The complication is that the D7 brane is embedded into this space in a way that constrains angles in $T^{1,1}$ to be related to the AdS_5 radius r via (2.5). This complicates the radial dependence of the χ, η action, as can be seen from the various expressions in the previous sections.

However, from (2.12) it is easy to see that in the $\hat{r} \to \infty$ limit, the embedding approaches $\theta_- \equiv \theta_+$, which is independent of the radius $\hat{r}$. Recall that since $\hat{r} = r/\mu^{2/3}$, $\hat{r} \gg 1$ corresponds to $r \gg \mu^{2/3}$ and therefore to remain in the near-horizon limit, we also require $r \ll L$. Thus we want to examine the above expressions in the regime $\mu^{2/3} \ll r \ll L$. In the $L = 1$ units used above, this is equivalent to

$$\hat{r} \gg 1, \qquad \mu^{2/3} \ll 1. \tag{3.3}$$

We will extract the leading order Lagrangian under these assumptions and compare to (3.1). Our finding is that the usual, conformally coupled scalar action is recovered. Deviations from this action due to subleading terms (logarithmic in $\hat{r}$) are related to the breakdown of conformal invariance near the mass threshold of the flavors of the dual gauge theory, corresponding to the end of the D7 brane probe at $r = \mu^{2/3}$.

After a careful numeric and analytical study of the integrals (2.21), we find that

$$F_1 \approx \frac{1}{6}\hat{r}^{5/2}\ln\hat{r}, \tag{3.4}$$

to an approximation that is good to five significant digits at all values of $\hat{r}$. It is possible to obtain an exact expression for F_1 in terms of elliptic functions, which gives (3.4), corrected by subleading logs. Also, $\tilde{F}_1 \approx F_1$ in an approximation that becomes exact in the $\hat{r} \to \infty$ limit; in fact, it can be seen from Fig. 2 that the two functions are nearly equal for all values of $\hat{r}$. However, for $\hat{r} = \mathcal{O}(1)$, in relative terms the right-hand side of (3.4) is a poor approximation to $\tilde{F}_1$.

To obtain an action corresponding to the form (3.1), the following field redefinitions must be made:

$$\chi = \hat{r}^{3/2}(\ln\hat{r})^{-1/2}\chi' \qquad \eta = \hat{r}^{3/2}(\ln\hat{r})^{-1/2}\eta'. \tag{3.5}$$

Note that to an excellent approximation, this is just a rescaling by $F_1^{-1/2}$ and an appropriate power of $\hat{r}$. Taking into account the powers of $\hat{r}$ that arise from (2.22) and (3.4), the following radial gradient term occurs in the Lagrangian:

$$\hat{r}^2 \ln\hat{r}(\partial_{\hat{r}}\chi)^2 = \hat{r}^5(\partial_{\hat{r}}\chi')^2 - \left(\frac{15}{4} - \frac{1}{2\ln\hat{r}} + \frac{1}{4(\ln\hat{r})^2}\right)\hat{r}^3\chi'^2 + \text{t.d.}, \tag{3.6}$$

with an identical equation for η. Substitution of (3.5) and (3.6) into (2.22) yields the bulk action

$$S(\chi') \approx -2\pi^2\mu^{8/3}\tau_7 \int d^4x \int_{\hat{R}}^{\infty} d\hat{r}\left\{\frac{\hat{r}}{\mu^{4/3}}\eta^{\mu\nu}\partial_\mu\chi'\partial_\nu\chi' + f(\hat{r})[\hat{r}^5(\partial_{\hat{r}}\chi')^2 + \hat{r}^3 m^2(\hat{r})\chi'^2]\right\}, \tag{3.7}$$

and an action for η' that is the same. We have introduced the ratio

$$f(\hat{r}) = \tilde{F}_1(\hat{r})/F_1(\hat{r}) \approx 1 \quad \text{for } \hat{r} \gg 1. \tag{3.8}$$

In (3.7), a cutoff $\hat{R} \gg 1$ on the radial integration has been introduced. The effects of integration over $\hat{r} \in [1, \hat{R}]$, as well as total derivative terms could, for instance, be incorporated into an effective boundary action. This is discussed in Sec. III B below. For $\hat{R} \gg 1$, it is a good approximation to take $f(\hat{r}) = 1$ in (3.7), which is what we do in the following.

The $\hat{r}$-dependent mass is

$$m^2(\hat{r}) = -\frac{15}{4} + \frac{1}{2\ln\hat{r}} - \frac{1}{4(\ln\hat{r})^2}. \tag{3.9}$$

The terms proportional to $1/\ln\hat{r}$ or its square are subleading in the $\hat{r} \gg 1$ regime. In Fig. 4 we display the mass (3.9). In the $\hat{r} \to 1$ limit, $m^2(\hat{r})$ becomes infinitely negative. As a consequence, physical solutions must satisfy Dirichlet boundary conditions:

$$\lim_{\hat{r}\to 1}\chi'(\hat{r}) = \lim_{\hat{r}\to 1}\eta'(\hat{r}) = 0. \tag{3.10}$$

The approach to zero at $\hat{r} = 1$ must be stronger than $\ln\hat{r}$. In Ref. [16], it is stated that for regularity the original fields χ, η should satisfy Neumann boundary conditions at $\theta_1 = \theta_2 = 1$, equivalent to $\hat{r} = 1$. Taking into account the factor $\sqrt{\ln\hat{r}}$ that appears in (3.5), it is clear that a finite χ, η at $\hat{r} = 1$ implies a vanishing χ', η'. Thus the two findings on boundary conditions at $\hat{r} = 1$ are consistent.

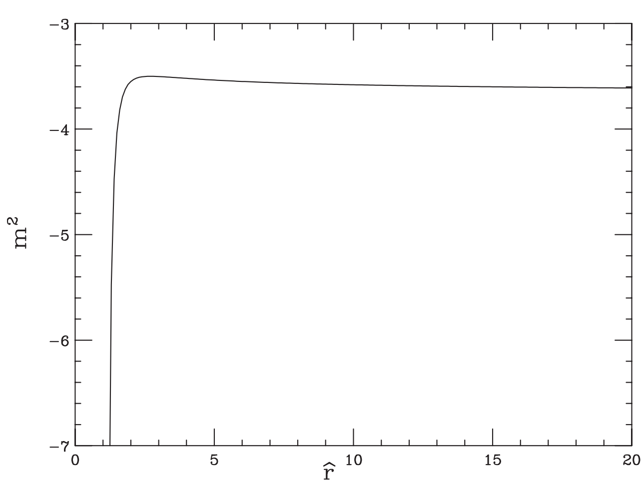

FIG. 4. The mass that is defined in (3.9). It approaches the constant value $-15/4$ at large $\hat{r}$. The infinitely negative value at $\hat{r} \to 1$ gives rise to the Dirichlet boundary conditions (3.10).

The mass degeneracy for χ', η' is a consequence of 5d supersymmetry in the AdS_5 regime: the real scalars χ', η' combine into a complex scalar of a 5d supersymmetry hypermultiplet.

If we return to the variables $r = \hat{r}\mu^{2/3}$ and $R = \hat{R}\mu^{2/3}$, and reintroduce the AdS_5 radius L explicitly by scaling the fields $\chi' \to \chi'/L$, $\eta' \to \eta'/L$, we obtain

$$S(\chi') \approx -2\pi^2 L^{-5}\tau_7 \int d^4x \int_R^{\infty} dr\left\{\frac{r}{L}\eta^{\mu\nu}\partial_\mu\chi'\partial_\nu\chi' + \frac{r^5}{L^5}(\partial_r\chi')^2 - \frac{15}{4L^2}\frac{r^3}{L^3}\chi'^2\right\}, \tag{3.11}$$

and similarly for η'. Here we have neglected the subleading logs in the mass terms and the prefactor (3.8).

For χ', η' we obtain a negative mass-squared:

$$m^2 = -\frac{15}{4L^2} > -\frac{4}{L^2}, \tag{3.12}$$

where the inequality expresses the fact that the mass-squared satisfies the Breitenlohner-Freedman bound [17,18]. The explanation of the negative mass-squared is similar to that given for probe branes in $AdS_5 \times S^5$ geometries [13–15]. We show in the Appendix that at fixed $\hat{r}$ the D7 brane wraps a 3-manifold that is topologically equivalent to S^3. The radius of this S^3 shrinks to zero as $\hat{r} \to 1$. It is therefore a topologically trivial 3-cycle in the conifold. The negative mass-squared corresponds to a "slipping mode."

The bulk mass is independent of the scale $\mu^{2/3}$, as it should be, since that is an IR boundary scale. The μ dependence has also disappeared from the overall factor in front of the action, which also makes sense from this perspective. The purely numerical value $-15/4$ arises from the radial independence of the embedding $\theta_- \equiv \theta_+$ that occurs in the $\hat{r} \to \infty$ limit. In fact, $-15/4$ is interesting because it is nothing but the conformally coupled scalar in AdS_5 [21,22], and corresponds to the Laplacian eigenvalue for the lowest mode of scalar harmonics on $T^{1,1}$ [9–12]. Since we are considering only modes that are independent of the $T^{1,1}$ coordinates, they do not "know" that the D7 brane is actually restricted to a submanifold of $T^{1,1}$. This explains the equivalence to the lowest $T^{1,1}$ scalar harmonic.

The $\mathcal{O}(1/\ln\hat{r})$ mass terms in (3.9) represent the leading effect of the breaking of conformal symmetry due to $\mu \neq 0$. In the dual gauge theory this parameter is related to the Yukawa and mass parameters of massive "flavor probes" that have been added to the original KW construction. At scales where the mass of these flavors is noticeable, the conformal symmetry is broken. The threshold for these flavors corresponds to $\hat{r} = 1$, where the D7 brane ends. Far away from this tip, at $\hat{r} \gg 1$, the dual gauge theory is at energy scales far above the threshold, where universal behavior dominates and scaling dimensions become apparent.

B. Effective boundary action

On physical grounds, there is one boundary condition (BC) at $\hat{r} = 1$ that must be satisfied: since the fields end there, and this should happen continuously, we have Dirichlet BCs (3.10). It was seen above that this naturally emerges from infinitely negative mass terms. Solving the equations of motion in the region $\hat{r} \in \{1, \hat{R}\}$, we can impose one more BC, generally Cauchy, at $\hat{R}$. Thus, we obtain a discrete set of permissible Cauchy BCs at $\hat{R}$, parametrized by the functional condition:

$$G[\chi'(x, \hat{R}), \partial_{\hat{r}}\chi'(x, \hat{R})] = 0 \quad \forall\ x, \tag{3.13}$$

and similarly for η'. This constraint may then be translated into an effective boundary action involving a Lagrange multiplier ψ:

$$S' = \int d^4x \psi(x) G[\chi'(x, \hat{R}), \partial_{\hat{r}}\chi'(x, \hat{R})]. \tag{3.14}$$

ψ is interpreted as a boundary field; we can give it dynamics on the boundary, provided it still has the effect of setting $G = 0$ to a good approximation.

From Fig. 4, we see that to a first approximation the $\hat{r} \approx 1$ effects just impose Dirichlet BCs at $\hat{r} = \hat{R} = \mathcal{O}(1)$:

$$G[\chi'(x, \hat{R}), \partial_{\hat{r}}\chi'(x, \hat{R})] \approx \chi'(x, \hat{R}). \tag{3.15}$$

That is, Fig. 4 shows that $\hat{R}$ of just "a few" suffices to approach the constant value of $m^2 = -15/4$; the requirement $\hat{R} \gg \hat{r}$ is stronger than is actually needed, in order to render the log corrections in (3.9) negligible. Thus, the leading order behavior is just that of a conformally coupled scalar with Dirichlet BCs at the IR boundary.

C. Auxiliary scalar action

Here we briefly touch on an alternative effective description of the small $\hat{r}$ behavior. The approach here is modeled after what was done in [23]. One introduces two scalars $h_{\chi',\eta'}$ to imitate the effect of the $\hat{r}$-dependent part of the masses (3.9). These auxiliary scalars are static, in the sense that for the modes that couple to χ', η', we can neglect dependence on 4d spacetime coordinates. For this to work, it is necessary to replace the $\hat{r}$-dependent parts of the mass terms for χ', η' with

$$-2\pi^2\mu^{8/3}\tau_7 \int d^4x \int_{\hat{R}}^{\infty} d\hat{r}\Big\{\hat{r}^5[(\partial_{\hat{r}}h_{\chi'})^2 + (\partial_{\hat{r}}h_{\eta'})^2] + \hat{r}^3[V(h_{\chi'}, h_{\eta'}) + h_{\chi'}\chi'^2 + h_{\eta'}\eta'^2]\Big\}. \tag{3.16}$$

The potential V is engineered such that once the equations of motion for $h_{\chi',\eta'}$ are imposed the profile of the auxiliary scalars is just

$$h_{\chi'} = h_{\eta'} = m^2(\hat{r}) + \frac{15}{4}. \tag{3.17}$$

We will not pursue this effective description further, since the "microscopic" description of Sec. III B that is available from the string construction is more fundamental and elegant. The only point that we wish to make is that the unusual small $\hat{r}$ behavior of the scalar action can be mimicked by a coupling to a quasistatic scalar with a nontrivial profile for its lowest mode—something that a low-energy phenomenologist might be more likely to consider.

IV. CONCLUSIONS AND OUTLOOK

The introduction of probe D7 branes in the Klebanov-Witten background provides a more fundamental description of 5d phenomenological models in a slice of AdS_5. In this article we have concentrated on the AdS_5 regime that exists for a single D7 brane embedded into the Klebanov-Witten background and derived the effective 5d action for the scalar fluctuations. Whereas there is a significant departure from the conventional scalar in AdS_5 near the end of the D7 brane, far away from that region the D7 embedding fluctuations become conformally coupled scalars of AdS_5. Furthermore by supersymmetry there are also (massless) 5d bulk fermions.

In addition we have shown that in the AdS_5 regime there are massless gauge fields. These fields mimic the bulk gauge fields considered in 5d phenomenological models in a slice of AdS_5. Again by supersymmetry we then infer that in the 5d bulk there are also massless (Dirac) fermions and a massive scalar with $m^2 = -4/L^2$. Thus, probe D7 branes can provide all the necessary 5d bulk fields required for phenomenological model-building.

The simple setup that we have considered in this article can be generalized to provide a more realistic 5d phenomenological model that incorporates the standard model, although a number of outstanding questions remain.

In particular one would like a full understanding of the 5d supergravity that occurs when the super-D7 brane effective action and type IIB supergravity is reduced on the 8d subspace $AdS_5 \times X_3$, where we recall that X_3 is the $\hat{r}$-dependent D7 embedding into $T^{1,1}$. At $\hat{r} \gg 1$, this would determine the complete multiplet structure for the bulk 5d supergravity and matter fields. Eventually supersymmetry will also need to be broken so that a realistic low-energy spectrum is obtained. One possible way would be to study flux compactifications as in GKP [8].

To construct models with semirealistic gauge groups in the bulk, multiple D7 branes need to be considered—corresponding to a nonabelian gauge group generalization. Standard model matter can then be obtained by studying intersecting D7 brane models, where strings stretched between multiple D7 branes in the internal compact coordinates gives rise to matter with the usual standard model quantum numbers.

Normally 5d phenomenological models are compactified on S^1/Z_2 orbifolds, with corresponding bulk and boundary masses. Thus, a detailed examination of the effective boundary action or auxiliary scalar action, as sketched in Sec. III B and III C, would be necessary. This may also require studying the D7-brane fermion action for the Klebanov-Witten background, following the techniques of [24–26], supplemented by an analysis of spinor harmonics on X_3. However, as noted earlier, information about the bulk fermion masses already follows from the scalar mass-squared analysis, due to AdS_5 supergravity constraints.

The most important aspect of the 5d phenomenological models is their dual holographic interpretation as composite 4d theories [2]. The probe D7 branes introduce fundamental "quarks" in the dual gauge theory. Identifying the corresponding operators in the dual gauge theory, especially with a realistic standard model spectrum, would elucidate the holographic correspondence of composite states, like the top quark and Higgs scalar field. This remains one of the most interesting avenues to study further.

Furthermore various refinements could also be introduced to the simple Klebanov-Witten construction that has been considered in this work. As mentioned in the Introduction, one could introduce an IR cutoff for nonprobe modes by generalizing to the Klebanov-Strassler background [7]. Here, the conifold (2.3) is deformed:

$$z_1 z_2 - z_3 z_4 = \epsilon^2. \tag{4.1}$$

The parameter ϵ determines the IR cutoff, and consistency of the supergravity theory requires a background three-form flux. This is a significant complication for the spectral computation. The D7 brane probes of this background have been studied, for instance, in [27,28]. The embeddings that were chosen are somewhat different from (2.5). For all these D7 embeddings, the main results will be essentially the same: there is an AdS_5 regime far away from where the D7 brane ends; the end of the D7 brane can be replaced by an effective boundary action, or an auxiliary scalar; the small 5d radius regime, near where the D7 brane ends, differs significantly from AdS_5.

In summary, probe D7 branes in the Klebanov-Witten background provide a more fundamental description of 5d phenomenological models in a slice of AdS that solve the hierarchy problem. This framework allows bulk fields to be introduced and leads to the possibility of explicitly constructing the dual theory.

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APPENDIX: GEOMETRIC DETAILS OF THE D7 BRANE EMBEDDING

Here we provide some brief remarks on the geometry of the D7 embedding relative to the $AdS_5 \times T^{1,1}$ background.

First recall the standard argument that shows that $T^{1,1}$ has isometry group $SU(2) \times SU(2) \times U(1)$. The conifold Eq. (2.3) that defines $T^{1,1}$ may be expressed alternatively in coordinates

$$\begin{aligned} z_1 &= w_1 + i w_2, \qquad z_2 = w_1 - i w_2, \\ z_3 &= w_3 + i w_4, \qquad z_4 = -(w_3 - i w_4), \end{aligned} \tag{A1}$$

yielding

$$\sum_i w_i^2 = \det(w_4 1_2 + i\sigma_a w_a) = 0. \tag{A2}$$

Here we have expressed the constraint in terms of a complex quaternion equation, which has the $SU(2)_1 \times SU(2)_2 \times U(1)$ invariance

$$\begin{aligned} &w_4 1_2 + i\sigma_a w_a \rightarrow e^{i\alpha} U(w_4 1_2 + i\sigma_a w_a)V, \\ &U \in SU(2)_1, \qquad V \in SU(2)_2. \end{aligned} \tag{A3}$$

The $T^{1,1}$ base is the intersection of this with the $S^7 \in \mathbf{C}^4$ of radius $r^{3/2}$:

$$\sum_i |w_i|^2 = r^3 \Leftrightarrow \mathrm{Tr}(w_4 1_2 + i\sigma_a w_a)(w_4 1_2 + i\sigma_a w_a)^\dagger = 2r^3. \tag{A4}$$

This also has the invariance (A3), demonstrating that $SU(2) \times SU(2) \times U(1)$ is an isometry of $T^{1,1}$.

On the other hand, when the embedding $z_1 = \mu$ is imposed, we have a 4d real manifold Y_4 embedded in the $\mathbf{C}^3$ parametrized by z_2, z_3, z_4:

$$\mu z_2 = z_3 z_4. \tag{A5}$$

This has a $U(1) \times U(1)$ invariance with charges (2, 1, 1) and (0, 1, −1) for the three complex coordinates, respectively. There is also a scaling symmetry Γ:

$$\Gamma\colon z_2 \rightarrow \lambda^2 z_2, \qquad z_{3,4} \rightarrow \lambda z_{3,4}, \qquad \lambda \in \mathbf{R}_+. \tag{A6}$$

We declare the base of Y_4 to be $X_3 = Y_4/\Gamma$, since any point in Y_4 can be reached from the application of Γ to a representative in X_3. We can parameterize Y_4 by the pair z_3, z_4, which it is useful to write as

$$z_3 = \rho e^{i\alpha} \cos\frac{\gamma}{2}, \qquad z_4 = \rho e^{i\beta} \sin\frac{\gamma}{2}, \tag{A7}$$

with $\gamma \in [0, \pi]$, α, $\beta \in [0, 2\pi)$, and $\rho \in [0, \infty)$. This is just $\mathbf{C}^2 = \mathbf{R}_+ \times S^3$, or a family of 3-spheres with radii ρ.

Note that (A5) has a solution z_2 for every value of z_3, z_4, so that the entire $\mathbf{R}_+ \times S^3$ is contained in Y_4. Also note that for each value of z_2, there corresponds at least one pair z_3, z_4. Thus the entire Y_4 is parametrized by the $\mathbf{R}_+ \times S^3$ (A7). Alternatively, the base X_3 is the intersection of Y_4 with any $S^3 \in \mathbf{C}^2(z_3, z_4)$ corresponding to:

$$|z_3|^2 + |z_4|^2 = \rho^2. \tag{A8}$$

The Eq. (A5) just tells us how the $\mathbf{R}_+ \times S^3$ parametrized by ρ, γ, α, β is embedded into $\mathbf{C}^3(z_2, z_3, z_4)$.

Next note the homeomorphism determined by the continuous deformation of (A5):

$$\mu z_2 = (1 - s) z_3 z_4, \qquad s \in [0, 1]. \tag{A9}$$

At $s = 1$ the embedding is just $z_2 = 0$ with z_3, z_4 arbitrary. Thus the topology of Y_4 is just $\mathbf{R}_+ \times S^3$, and the topology of X_3 is S^3. The geometry of Y_4 is different, since it is only the projection into $\mathbf{C}^2(z_3, z_4)$ that is geometrically described by $\mathbf{R}_+ \times S^3$, much as an ellipse in 3d can be projected onto a circle in a 2d plane.

It is of interest to relate Y^4 to the conifold geometry, particularly the coordinate r. This relation follows from

$$r^3 = \sum_{i=1}^{4} |z_i|^2 = \mu^2 + \rho^2 + \frac{1}{4\mu^2}\rho^4 \sin^2\gamma. \tag{A10}$$

First, note that as $r \to \mu^{2/3}$, the $X_3 \simeq S^3$ radius ρ shrinks to zero. This is, in detail, how the D7 brane "ends" in the AdS_5 radial dimension. Next note that if we fix r, the $X_3 \simeq S^3$ radius becomes a function of the polar angle γ. The entire domain of γ has a solution, with $\rho(\gamma)$ falling in the range

$$2\mu(r^{3/2} - \mu) \le \rho(\gamma) \le r^3 - \mu^2. \tag{A11}$$

The lower limit is saturated at $\gamma = \pi/2$, whereas the upper limit is saturated at $\gamma = 0$, π. Thus at fixed r the D7 embedding corresponds to a 3d ellipsoid. At $r \to \infty$ the "squishing" disappears and we just have an S^3. This suggests that a harmonic analysis at $r \to \infty$ in terms of the coordinates γ, α, β should be relatively straightforward, involving just the S^3 harmonics.

The $U(1) \times U(1)$ isometry of the 4d manifold Y_4 is also an isometry of the 3d base, as is apparent from (A8). This isometry group will be reflected in the spectrum of eigenmodes and angular dependence. A thorough harmonic analysis on this 3d space X_3 at arbitrary AdS_5 radius r is however, beyond the scope of the present article.

Finally consider the $\hat{r} \to \infty$ embedding in terms of the conifold coordinates. In this limit, the embedding is purely angular and is given by the sum:

$$X_3^\infty \equiv \{\theta_i, \phi_i, \psi | \theta_1 = 0, \psi = \phi_1 + \phi_2\} + \{\theta_i, \phi_i, \psi | \theta_2 = 0, \psi = \phi_1 + \phi_2\}. \tag{A12}$$

The two subspaces intersect at $\theta_1 = \theta_2 = 0$. Each subspace clearly contains an S^2 parametrized by (θ_i, ϕ_i), $i = 1$ or 2. As a consequence, an expansion on spherical harmonics $Y_{\ell m}(\theta_i, \phi_i)$ is valid, $i = 1$ or 2 depending on the subspace. This affords a further justification for our assumption of $\theta_- = (\theta_1 - \theta_2)/2$ independence in the $\hat{r} \to \infty$ limit for the 8d scalars χ, η.

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