Heavy quark fragmentation

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Abstract
The fragmentation of heavy quarks into hadrons is a key non-perturbative ingredient for the heavy quark production calculations. The formalism is reviewed, and the extraction of non-perturbative parameters from e^+e^- and from ep data is discussed.

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1 Introduction[1]

When we try to describe in QCD the production of a hadron we are always faced with the necessity to take into account the non-perturbative hadronization phase, i.e. the processes which transform perturbative objects (quarks and gluons) into real particles. In the case of light hadrons the QCD factorization theorem [1–6] allows to factorize these non-perturbative effects into universal (but factorization-scheme dependent) *fragmentation functions* (FF):

$$\frac{d\sigma_h}{dp_T}(p_T) = \sum_i \int \frac{dx}{x} \frac{d\sigma_i}{dp_T}\left(\frac{p_T}{x};\mu\right) D_{i\to h}(x;\mu) + \mathcal{O}\left(\frac{\Lambda}{p_T}\right) . \tag{1}$$

In this equation, valid up to higher twist corrections of order Λ/p_T (Λ being a hadronic scale of the order of a few hundred MeV and p_T for instance a transverse momentum), the partonic cross sections $d\sigma_i/dp_T$ for production of the parton i are calculated in perturbative QCD, while the fragmentation functions $D_{i\to h}(x;\mu)$ are usually extracted from fits to experimental data. Thanks to their universality they can be used for predictions in different processes. The artificial factorization scale μ is a reminder of the non-physical character of both the partonic cross sections and the fragmentation functions: it is usually taken of the order of the hard scale p_T of the process, and the fragmentation functions are evolved from a low scale up to μ by means of the DGLAP evolution equations.

This general picture becomes somewhat different when we want to calculate the production of heavy-flavoured mesons. In fact, thanks to the large mass of the charm and the bottom quark, acting as a cutoff for the collinear singularities which appear in higher orders in perturbative calculations, one can calculate the perturbative prediction for heavy *quark* production. Still, of course, the quark $\to$ hadron transition must be described. Mimicking the factorization theorem given above, it has become customary to complement the perturbative calculation for heavy quark production with a non-perturbative fragmentation function accounting for its hadronization into a meson:

$$\frac{d\sigma_H}{dp_T}(p_T) = \int \frac{dx}{x} \frac{d\sigma_Q^{pert}}{dp_T}\left(\frac{p_T}{x},m\right) D^{np}_{Q\to H}(x) . \tag{2}$$

It is worth noting that at this stage this formula is not given by a rigorous theorem, but rather by some sensible assumptions. Moreover, it will in general fail (or at least be subject to large uncertainties) in the region where the mass m of the heavy quark is not much larger than its transverse momentum p_T, since the choice of the scaling variable is not unique any more, and $\mathcal{O}(m/p_T)$ corrections cannot be neglected.

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Basic arguments in QCD allow to identify the main characteristics of the non-perturbative fragmentation function $D^{np}_{Q\to H}(x)$. In 1977 J.D. Bjorken [7] and M. Suzuki [8] independently argued that the average fraction of momentum lost by the heavy quark when hadronizing into a heavy-flavoured hadron is given by

$$\langle x\rangle^{np} \simeq 1 - \frac{\Lambda}{m} \,. \tag{3}$$

Since (by definition) the mass of a heavy quark is much larger than a hadronic scale Λ, this amounts to saying that the non-perturbative FF for a heavy quark is very hard, i.e. the quark loses very little momentum when hadronizing. This can also be seen with a very simplistic argument: a fast-traveling massive quark will lose very little speed (and hence momentum) when picking up from the vacuum a light quark of mass Λ to form a heavy meson[2].

This basic behaviour is to be found as a common trait in all the non-perturbative heavy quark FFs derived from various phenomenological models. Among the most commonly used ones we can cite the Kartvelishvili-Likhoded-Petrov [12], Bowler [13], Peterson-Schlatter-Schmitt-Zerwas [14] and Collins-Spiller [15] fragmentation functions. These models all provide some functional form for the $D^{np}_{Q\to H}(x)$ function, and one or more free parameters which control its hardness. Such parameters are usually not predicted by the models (except perhaps on an order-of-magnitude basis), and must be fitted to the experimental data.

During the '80s many such fits were performed, and these and similar functions were also included in many Monte Carlo event generators. Eventually, some 'best' set of parameter values (for instance for the PSSZ form) was determined [16] and subsequently widely used.

These first applications, given the limited accuracy of the available data, tended to overlook two aspects which have become more important in recent years, when the accuracy of the data has vastly improved:

- A non-perturbative FF is designed to describe the heavy quark $\to$ hadron transition, dealing with events mainly populated by soft gluons of energies of a few hundred MeV. However, if a heavy quark is produced in a high energy event it will initially be far off shell: perturbative hard gluons will be emitted to bring it on-shell, reducing the heavy quark momentum and yielding in the process large collinear logarithms (for instance of the form $\alpha_s^n \log^p(p_T/m)$ in a transverse momentum differential cross section). Of course, the amount of gluon radiation is related to the distance between the heavy quark mass scale and the hard scale of the interaction, and is therefore process-dependent. One can (and it was indeed done) either fit different free parameters at different centre-of-mass energies (or transverse momenta), or try to evolve directly the non-perturbative FF by means of the DGLAP equations, hence including into it the perturbative collinear logarithms. However, this is not what non-perturbative fragmentation functions are meant for, and doing so spoils the validity of the relation in Eq. (3).
- Since only the final heavy hadron is observed, both the non-perturbative FF and the perturbative cross section for producing the heavy quark must be regarded as non-physical objects. The details of the fitted non-perturbative FF (e.g. the precise value(s) of its free parameter(s)) will depend on those of the perturbative cross sections: different perturbative calculations (leading order, next-to-leading order, Monte Carlo, ...) and different perturbative parameters (heavy quark masses, strong coupling, ...) will lead to different non-perturbative FFs. These in turn will have to be used only with a perturbative description similar to the one they have been determined with. Hence the limited accuracy (and hence usefulness) of a 'standard' determination of the parameters [17].

The first point was addressed by Mele and Nason in a paper [18] which deeply changed the field of heavy quark fragmentation, and essentially propelled it into the modern era. Mele and Nason observed

[2] More modern and more rigorous derivations of this result can be found in [9–11].

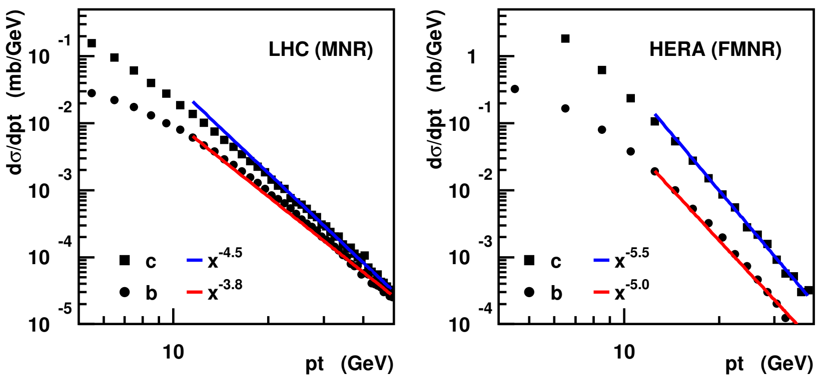

Fig. 1: Power-law fits to the heavy quark p_T distributions at LHC (left) and HERA (right) obtained with the NLO programs MNR and FMNR. The resulting exponents are $N = 4.5/3.8$ for charm/beauty at LHC and $N = 5.5/5.0$ for *c*/*b* at HERA.

that, in the limit where one neglects heavy quark mass terms suppressed by a large energy scale, a heavy quark cross section can be factored into a massless, $\overline{MS}$-subtracted cross section for producing a light parton, and a process-independent[3], *perturbative* heavy quark fragmentation function describing the transition of the massless parton into the heavy quark:

$$\frac{d\sigma_Q^{pert,\,res}}{dp_T}(p_T, m) = \sum_i \int \frac{dx}{x}\, \frac{d\sigma_i}{dp_T}\left(\frac{p_T}{x};\mu\right)\; D_{i\to Q}(x;\mu,m) + \mathcal{O}\left(\frac{m}{p_T}\right)\,. \tag{4}$$

The key feature of this equation is that it is entirely perturbative: every term can be calculated in perturbative QCD. The perturbative fragmentation functions $D_{i\to Q}(x;\mu,m)$ (not to be confused with the non-perturbative one $D^{np}_{Q\to H}(x)$) can be evolved via DGLAP equations from an initial scale of the order of the heavy quark mass up to the large scale of the order of p_T. This resums to all orders in the strong coupling the collinear logarithms generated by the gluon emissions which bring the heavy quark on its mass shell, leading to a more accurate theoretical prediction for $d\sigma_Q/dp_T$.

Once a reliable perturbative cross section for the production of a heavy quark is established, one is simply left with the need to account for its hadronization. For this purpose one of the functional forms listed above can be used for the non-perturbative FF, and implemented as in Eq. (2), but using the improved, resummed cross section given by Eq. (4). Since most of the the scaling-violation logarithms are accounted for by the evolution of the perturbative FF, the non-perturbative one can now be scale-independent and only contain the physics related to the hadronization of the heavy quark. It will always, however, depend on the details of the perturbative picture used.

2 Extraction of heavy quark fragmentation parameters from e^+e^- and their impact on HERA and LHC[4]

2.1 Importance of $\langle x\rangle^{\rm np}$

According to the factorization of the fragmentation functions (FF), the differential cross section $d\sigma/dp_T$ for the production of a heavy hadron H can be written as the convolution of the perturbative heavy quark differential cross section $d\sigma^{\rm pert}/dp_T$ and the non-perturbative fragmentation function $D^{\rm np}(x)$:

$$\frac{d\sigma}{dp_T}(p_T) = \int \frac{dx}{x}\, D^{\rm np}(x)\, \frac{d\sigma^{\rm pert}}{dp_T}\left(\frac{p_T}{x}\right). \tag{5}$$

[3] Mele and Nason extracted this function from the e^+e^- cross section, convincingly conjecturing its process independence, which was successively established on more general grounds in [19]

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Table 1: Test functions used in Fig. 2. The functions assume a value different from zero in the range given by the third column.

Function	$D(x)$	parameters	x range
Kartvelishvili	$(1-x)x^{\alpha}$	$\alpha = 2/\delta - 3$	$[0,1]$
Peterson	$\frac{1}{x}\left(1-\frac{1}{x}-\frac{\epsilon}{(1-x)}\right)^{-2}$	ϵ	$[0,1]$
Gauss	$\exp(-(\frac{x-\mu}{2\sigma})^2)$	$\mu=1-\delta$ $\sigma=\delta/2$	$[-\infty, \infty]$
Box	const.	–	$[1-2\delta, 1]$
Triangular:	$x - x_0$	$x_0 = 1 - 3\delta$	$[1-3\delta, 1]$

This convolution neglects mass terms $\mathcal{O}(m_Q/p_T)$ and non-perturbative terms $\mathcal{O}(\Lambda_{\rm qcd}/m_Q)$.

The heavy quark p_T distribution behaves at large p_T like a power law $d\sigma^{\rm pert}/dp_T = Cp_T^{-N}$. Figure 1 shows power-law fits to the p_T distributions of heavy quarks at LHC and in photoproduction at HERA as obtained from the NLO programs MNR [20] and FMNR [21]. For $p_T > 10$ GeV N was found to range from 3.8 (b at LHC) to 5.5 (c at HERA). Combining this power-law behavior with Eq. (5), the hadron p_T distribution is given by

$$\frac{d\sigma}{dp_T}(p_T) = \int dx\, x^{N-1}\, D^{\rm np}(x)\, Cp_T^{-N} = \frac{d\sigma^{\rm pert}}{dp_T}\hat{D}_N^{\rm np}, \tag{6}$$

where $\hat{D}_N^{\rm np} = \int dx\, x^{N-1}\, D^{\rm np}(x)$ is the N^{th} Mellin moment of the non-perturbative FF.

The hadron distribution is therefore governed by the 4^{th} - 5^{th} Mellin moments of $D^{\rm np}(x)$. It is interesting to translate the Mellin moments into more intuitive central moments

$$\mu_n = \int dx\, (x - \langle x\rangle)^n\, D^{\rm np}(x) \quad \text{for} \quad n \geq 2 \tag{7}$$

where $\langle x\rangle = \int dx\, xD^{\rm np}(x)$ is the mean value. The first Mellin moments, written in terms of $\langle x\rangle$ and μ_n, are: $\hat{D}_1 = 1$, $\hat{D}_2 = \langle x\rangle$, $\hat{D}_3 = \langle x\rangle^2 + \mu_2$, $\hat{D}_4 = \langle x\rangle^3 + 3\mu_2\langle x\rangle + \mu_3$.

In heavy quark fragmentation, the mean value of $D^{\rm np}(x)$ can be written as $\langle x\rangle = 1 - \delta$ where $\delta = \mathcal{O}(\Lambda_{\rm qcd}/m_Q)$ is small [11]. For any positive function with $\langle x\rangle = 1 - \delta$, defined in the interval $[0,1]$, the central moments are limited by δ, $|\mu_n| \leq \delta$. In practice, reasonable heavy quark fragmentation functions are concentrated in a small region around $1 - \delta$ and therefore the higher central moments are small. To be specific, if the function is different from zero in a region of size $\pm K\delta$ (with $K = \mathcal{O}(1)$) around $1 - \delta$ then $|\mu_n| \leq (K\delta)^n$. This means that the Mellin moments of reasonable FFs are given, to a good approximation, by the mean value to the $N - 1$ power:

$$\hat{D}_N = \langle x\rangle^{N-1} + \mathcal{O}(\delta^2). \tag{8}$$

The expansion to δ^2 involves the second central moment μ_2: $\hat{D}_N = \langle x\rangle^{N-1} + \frac{(N-1)!}{2(N-3)!}\mu_2\langle x\rangle^{N-3} + \mathcal{O}(\delta^3)$. For a reasonable FF and a perturbative distribution falling with the power $-N$, Eq. 6 and 8 give

$$\frac{d\sigma}{dp_T}(p_T) = \frac{d\sigma^{\rm pert}}{dp_T^Q}(p_T)\ (\langle x\rangle^{\rm np})^{N-1} + \mathcal{O}(\delta^2). \tag{9}$$

Therefore the effect of the non-perturbative FF is to introduce a shift in the normalisation that depends on the average x, while the details of the shape of $D(x)$ have negligible effect. To check that this reasoning

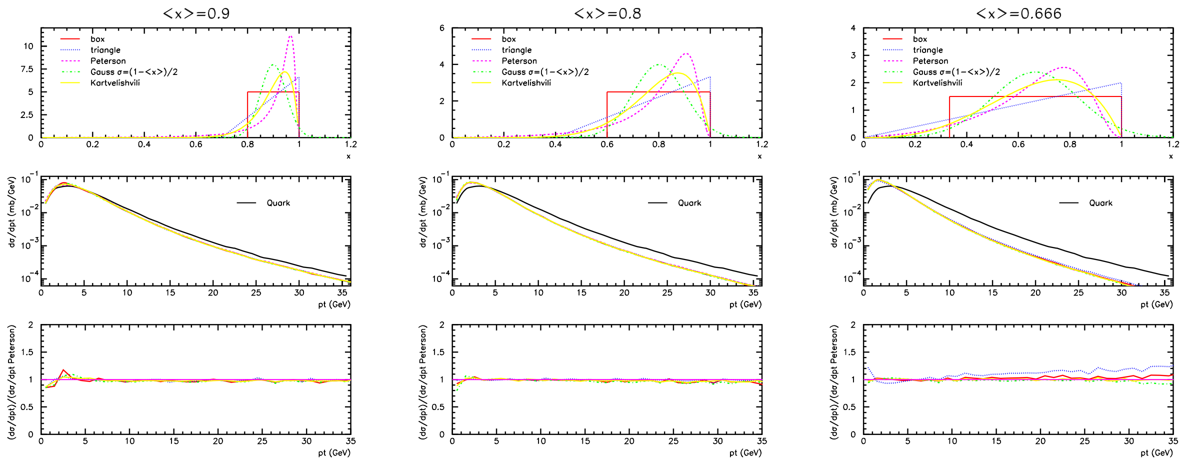

Fig. 2: Effect of the convolution of the heavy quark transverse momentum distribution with different test functions for different values of the non-perturbative FF $\langle x \rangle = 0.9$ (left), 0.8 (center), 0.666 (right). For each $\langle x \rangle$, the upper plot shows the test functions, the middle plot shows the perturbative p_T distribution obtained with the MNR program for beauty at LHC and the hadron p_T distributions after the convolution with the test functions. The lower plot shows the ratio of the different hadron p_T distributions to the result obtained with the Peterson one.

works with realistic fragmentation functions and realistic perturbative p_T distributions, various functions with the same $\langle x \rangle$ but different shapes have been tested in convolution with the perturbative p_T spectrum for b production at LHC obtained with the NLO program MNR. The test functions considered are the Peterson [14] and Kartvelishvili [12] fragmentation functions, a Gaussian distribution with $\sigma = 1 - \mu$, a flat and a triangular distribution. Table 1 gives more detail about these functions. Three average values were chosen: $\langle x \rangle = 0.9, 0.8, 0.666$. Figure 2 shows the result of this test. For each average value, the convolutions are very similar, even if the test functions are very different. For $\langle x \rangle = 0.9, 0.8$, which are typical beauty or charm values, the hadron spectra agree within few %. For the extreme value of $\langle x \rangle = 0.666$, the results for the Peterson and Kartvelishvili functions give very similar hadron spectra while the less realistic Gaussian and Box shapes differ at most 10% from Peterson at large p_T and the extreme Triangular function shows deviations up to $\sim 20\%$.

In conclusion the relevant fragmentation parameter for the inclusive hadron spectra at pp and ep colliders is the mean value $\langle x \rangle^{\rm np}$ of the non-perturbative FF. The next part will discuss, on the basis of e^+e^- data, what values of $\langle x \rangle^{\rm np}$ are relevant for different calculations.

2.2 Extraction of $\langle x \rangle^{\rm np}$ from e^+e^- data

In e^+e^- interactions it is convenient to express the factorization ansatz, given for the heavy-hadron p_T in Eq. (5), in terms of the heavy-hadron momentum normalized to the maximum available momentum: $x_p = p^H/p^H_{\rm max}$, where $p^H_{\rm max} = \sqrt{(\frac{1}{2}E_{\rm cms})^2 - m_H^2}$:

$$\frac{d\sigma}{dx_p}(x_p) \;=\; \int \frac{dx}{x}\, D^{\rm np}(x)\, \frac{d\sigma^{\rm pert}}{dx_p}(\frac{x_p}{x})$$

which corresponds to the following relation for the mean values: $\langle x_p \rangle = \langle x \rangle^{\rm np} \langle x \rangle^{\rm pert}$ where $\langle x_p \rangle$ is the mean hadron x_p, $\langle x \rangle^{\rm np}$ is the mean value of the non-perturbative FF and $\langle x \rangle^{\rm pert} = \int\, dx\; x\, \frac{d\sigma^{\rm pert}}{dx_p}$ is the mean value of the perturbative distribution. Then, taking $\langle x_p \rangle$ from experimental data and $\langle x \rangle^{\rm pert}$ from a particular perturbative calculation, it is possible to extract the value of $\langle x \rangle^{\rm np}$ valid for that calculation as

$$\langle x \rangle^{\rm np} = \langle x_p \rangle / \langle x \rangle^{\rm pert}. \tag{10}$$

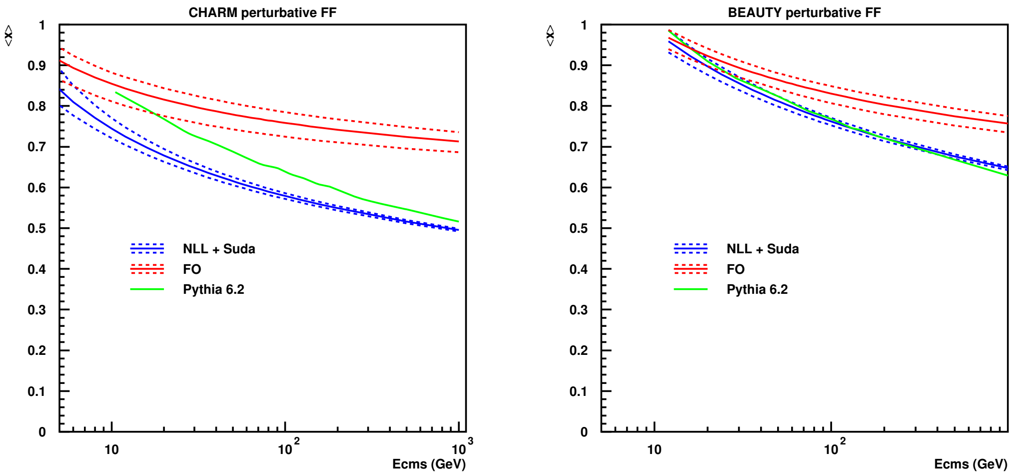

Fig. 3: Average fragmentation function from the perturbative calculations for charm (left) and beauty (right) as a function of the e^+e^- center of mass energy.

Two perturbative calculations will be considered to extract $\langle x\rangle^{\mathrm{pert}}$: a fixed-order (FO) next-to-leading order (i.e. $\mathcal{O}(\alpha_S)$) calculation and a calculation that includes also the resummation of next-to-leading logarithms (NLL) and Sudakov resummation, both obtained with the HVQF program [19]. From the point of view of fragmentation, the FO calculation only considers the emission of a gluon from one of the two heavy quarks generated in the e^+e^- collision while the NLL calculation includes the evolution of the FF from the hard interaction scale down to the scale given by the heavy quark mass. The parameters used for the FO and NLL models are $m_c = 1.5$ GeV, $m_b = 4.75$ GeV, $\Lambda_{\mathrm{QCD}} = 0.226$ GeV and the renormalisation and factorization scales $\mu_R = \mu_F = E_{\mathrm{cms}}$. The starting scale for FF evolution in the NLL model was chosen to be m_Q. The theoretical uncertainty was obtained by varying independently the normalisation and factorization scales by a factor 2 and $1/2$ and taking the largest positive and negative variations as the uncertainty.

The experimental data are also compared to the PYTHIA 6.2 Monte Carlo program [37] which contains an effective resummation of leading-logarithms based on a parton-shower algorithm and which is interfaced to the Lund fragmentation model. In this case the MC model gives directly $\langle x_p\rangle$, while $\langle x\rangle^{\mathrm{pert}}$ has been obtained taking the heavy quark at the end of the parton shower phase. The quark masses have been set to $m_c = 1.5$ GeV and $m_b = 4.75$ GeV, and all the parameters were set to the default values except for specific fragmentation parameters explained below. Three sets of fragmentation parameters were chosen for charm: the default fragmentation (Lund-Bowler), a longitudinal string fragmentation of the Peterson form with $\epsilon = 0.06$ (`MSTJ(11)=3, PARJ(54)=-0.06`) and the Lund-Bowler fragmentation with parameters re-tuned by the CLEO collaboration [28] (`PARJ(41)=0.178, PARJ(42)=0.393, PARJ(13)=0.627`). The two sets chosen for beauty are the default Lund-Bowler fragmentation and the Peterson fragmentation with $\epsilon = 0.002$ (`MSTJ(11)=3, PARJ(55)=-0.002`). Figure 3 shows $\langle x\rangle^{\mathrm{pert}}$ from the perturbative calculations as a function of the centre of mass energy E_{cms} for charm and bottom.

2.3 Charm

Charm fragmentation data are available from various e^+e^- experiments. The most precise are those at the Z^0 pole at LEP (ALEPH [23], OPAL [22], DELPHI [24]) and near the $\Upsilon(4s)$ (ARGUS [27], CLEO [28], BELLE [29]). Less precise data are available in the intermediate continuum region from DELCO [26] at PEP and TASSO [25] at PETRA. Measurements in which the beauty component was not subtracted have been discarded [38–40]. The experimental data are reported in Table 2. Only measurements relative to the $D^{*\pm}(2010)$ meson are considered, to avoid the complications due to cascade decays that are

Table 2: Experimental results on the average fragmentation function in e^+e^- collisions for D^* mesons and weakly decaying beauty hadrons. The table reports, for each experiment, the published variable and the corrections applied to obtain $\langle x_p \rangle^{\mathrm{corr}}$. All the measurement have been corrected for initial state radiation (ISR). Measurements reported in terms of $\langle x_E \rangle$ have been corrected to $\langle x_p \rangle$. In the case of ARGUS the average has been calculated from the full distribution. In the case of TASSO the error on the average was re-evaluated using the full distribution since the published error seems incompatible with the data. DELCO reports a fit with a Peterson distribution that has been translated into $\langle x_p \rangle^{\mathrm{corr}}$. Systematical and statistical uncertainties, where reported separately, have been added in quadrature. The ALEPH beauty measurement refers to B^+ and B^0 mesons only (i.e. excluding B_s and Λ_b), a MC study shows that this correspond to underestimating $\langle x_p \rangle^{\mathrm{corr}}$ by $\sim 0.1\%$ only, which is negligible.

Charm (D^*) measurement	E_{cms} (GeV)	Measured variable	Value	ISR corr. (%)	$x_E \to x_p$ (%)	$\langle x_p \rangle^{\mathrm{corr}}$
OPAL [22]	92	$\langle x_E \rangle$	$0.516^{+0.008}_{-0.005} \pm 0.010$	$+0.4$	-0.4	0.516 ± 0.012
ALEPH [23]	92	$\langle x_E \rangle$	$0.4878 \pm 0.0046 \pm 0.0061$	$+0.4$	-0.4	0.488 ± 0.008
DELPHI [24]	92	$\langle x_E \rangle$	$0.487 \pm 0.015 \pm 0.005$	$+0.4$	-0.4	0.487 ± 0.016
TASSO [25]	36.2	$\langle x_E \rangle$	0.58 ± 0.02	$+6.7$	-1.8	0.61 ± 0.02
DELCO [26]	29	$\epsilon^*_{\mathrm{Pet.}}$	$0.31^{+0.10}_{-0.08}$	$+6.3$	-	0.55 ± 0.03
ARGUS [27]	10.5	$\langle x_p \rangle$	0.64 ± 0.03	$+4.2$	-	0.67 ± 0.03
CLEO [28]	10.5	$\langle x_p \rangle$	$0.611 \pm 0.007 \pm 0.004$	$+4.2$	-	0.637 ± 0.008
BELLE [29]	10.58	$\langle x_p \rangle$	$0.61217 \pm 0.00036 \pm 0.00143$	$+4.2$	-	0.6379 ± 0.0016
Beauty (B^{wd}) measurement	**E_{cms} (GeV)**	**Measured variable**	**Value (%)**	**ISR corr. (%)**	**$x_E \to x_p$**	**$\langle x_p \rangle^{\mathrm{corr}}$**
OPAL [30]	92	$\langle x_E \rangle$	$0.7193 \pm 0.0016 \pm 0.0038$	$+0.3$	-0.9	0.715 ± 0.004
SLD [31]	92	$\langle x_E \rangle$	$0.709 \pm 0.003 \pm 0.005$	$+0.3$	-0.9	0.705 ± 0.006
ALEPH [32]	92	$\langle x_E \rangle$	$0.716 \pm 0.006 \pm 0.006$	$+0.3$	-0.9	0.712 ± 0.008
DELPHI [33]	92	$\langle x_E \rangle$	$0.7153 \pm 0.0007 \pm 0.0050$	$+0.3$	-0.9	0.711 ± 0.005
JADE [34]	36.2	$\langle x_E \rangle$	$0.76 \pm 0.03 \pm 0.04$	$+5.4$	-3.5	0.77 ± 0.06
DELCO [35]	29	$\langle x_E \rangle$	0.72 ± 0.05	$+4.8$	-4.7	0.72 ± 0.05
PEP4-TPC [36]	29	$\langle x_E \rangle$	$0.77 \pm 0.04 \pm 0.03$	$+4.8$	-4.7	0.77 ± 0.07

present for ground state mesons. Charm quarks originating from gluon splitting rather than from the virtual boson from e^+e^- annihilation may be relevant at LEP energies. This contribution is anyway already subtracted in the published data considered here, and it is consistently not considered in the perturbative calculations. Most of the experiments published the mean value of the x distribution. The only exception is ARGUS, for which the mean value was computed from the published distribution. Some of the experiments give the results directly in terms of x_p, others in terms of the energy fraction $x_E = 2E_H/E_{\mathrm{cms}}$. The latter has been corrected to x_p using the PYTHIA MC. The difference between $\langle x_p \rangle$ and $\langle x_E \rangle$ can be as large as 12% at $E_{\mathrm{cms}} = 10.5$ GeV and reduces to less than 1% at $E_{\mathrm{cms}} = 92$ GeV. Since the low-mass measurements are already given in terms of x_p, the applied corrections from x_E to x_p was always small. QED corrections are also needed to compare the experimental data to the QCD predictions. The initial state radiation (ISR) from the electrons has the effect of reducing the energy available for the e^+e^- annihilation and therefore to reduce the observed value of $\langle x_p \rangle$. A correction, obtained by comparing the PYTHIA MC with and without ISR, was applied to the data to obtain $\langle x_p \rangle^{\mathrm{corr}}$. The correction is $\sim 4\%$ at $E_{\mathrm{cms}} = 10.5$ GeV, is largest in the intermediate region and is negligible at $E_{\mathrm{cms}} = 92$ GeV.

Only LEP data at $E_{\mathrm{cms}} = 92$ GeV were used to extract $\langle x \rangle^{\mathrm{np}}$ since the factorization of the non-perturbative FF could be spoiled by large $\mathcal{O}(m_Q/E_{\mathrm{cms}})$ terms at lower energies. Table 3 reports the LEP average $\langle x_p \rangle^{\mathrm{corr}}$, the perturbative results at 92 GeV and the resulting $\langle x \rangle^{\mathrm{np}}$ for NLL and FO calculations as well as $\langle x \rangle$ and $\langle x \rangle^{\mathrm{pert}}$ from PYTHIA. Figure 4 (left) shows $\langle x_p \rangle$ obtained by multiplying the perturbative calculations with the corresponding $\langle x \rangle^{\mathrm{np}}$, compared to the experimental data and to the PYTHIA MC with different fragmentation parameters.

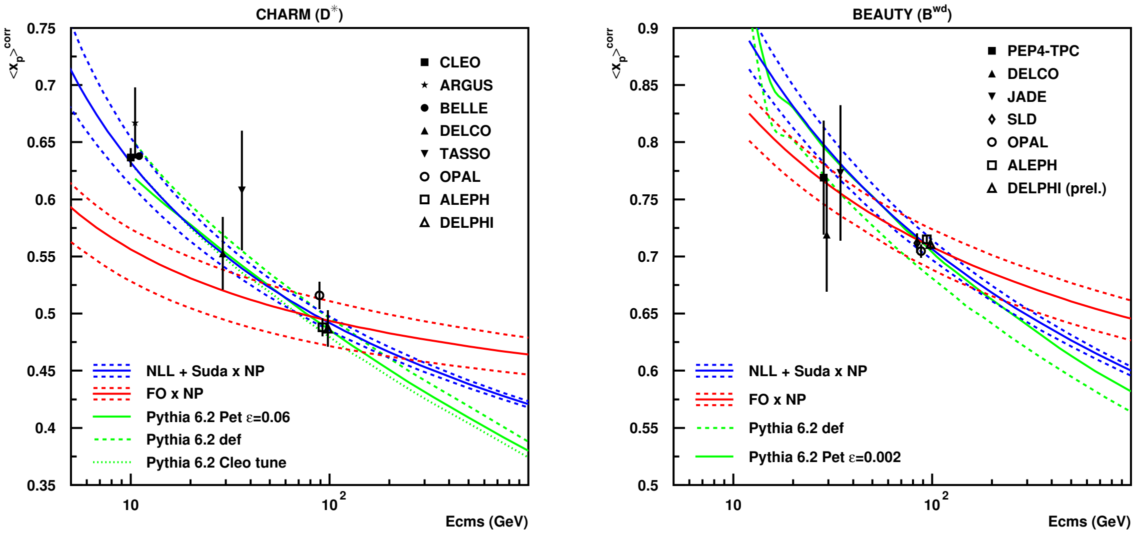

Fig. 4: Average fragmentation function as a function of the center of mass energy for charm (left) and beauty (right). The plots show the experimental results and the curves from NLL and FO theory with a non-perturbative fragmentation obtained using the data at the Z^0 energy. The curves from PYTHIA 6.2 with different fragmentation choices are also shown. The experimental points at the $\Upsilon(4s)$ and Z^0 resonances are shown slightly displaced in the horizontal axis for better legibility.

With the non-perturbative $\langle x\rangle^{\rm np} = 0.849 \pm 0.018$ obtained at LEP energies, the NLL calculation can reproduce all the data within a quite small theoretical uncertainty. The FO calculation is instead too flat to reproduce the data even considering its large theoretical uncertainty band. The non-perturbative fragmentation $\langle x\rangle^{\rm np}$ obtained at LEP energy for the FO calculation is quite small (0.65 ± 0.04) since it compensates the effect of the FF evolution that is missing in the perturbative part. Therefore FO calculations with $\langle x\rangle^{\rm np}$ extracted at LEP energy undershoot drastically the data at the $\Upsilon(4s)$.

The PYTHIA MC with the Lund-Bowler fragmentation reproduces the data reasonably well. The result with default parameters is slightly above the data while the result with the parameters tuned by the CLEO collaboration is slightly below. Both are compatible within the experimental uncertainty with all the experimental values with the exception of the very precise measurement from Belle from which they differ anyway by less than 2%. PYTHIA with the Peterson fragmentation with $\epsilon = 0.06$ reproduces well the LEP data but is too low at lower energies.

2.4 Beauty

In the case of beauty we consider fragmentation measurements for the mix of weakly decaying hadrons $B^{\rm wd}$. Precise measurements are available only at the Z^0 peak (SLD [31], ALEPH [32], OPAL [30], DELPHI [33]). Lower energy measurements from PEP (PEP4-TPC [36], DELCO [35]) and PETRA (JADE [34]) have larger uncertainties. As for charm, corrections have been applied for ISR and to convert $\langle x_E\rangle$ to $\langle x_p\rangle$. The data are shown in Table 2 and the results in Table 3 and Figure 4 (right). Since precise data are available only at a single energy, it is impossible to test the energy beaviour of the theoretical predictions. As in the charm case, the energy dependence of PYTHIA and NLL theory are similar, while the FO prediction is much more flat, suggesting that also for beauty the non-perturbative fragmentation obtained for FO at the Z^0 could not be applied at lower energy. PYTHIA with Peterson fragmentation with $\epsilon = 0.002$ reproduces the data, while the default Lund-Bowler fragmentation is too soft.

Table 3: Average fragmentation functions at the Z^0 resonance for charm (top) and beauty. The table shows the average of the experimental data, the results from the NLL and FO calculations and from the PYTHIA MC with different fragmentation parameters. For the NLL and FO calculations $\langle x_p\rangle^{\mathrm{np}}$ is obtained by dividing the average from the experimental data by the perturbative result $\langle x_p\rangle^{\mathrm{np}} = \langle x_p\rangle^{\mathrm{corr}}/\langle x_p\rangle^{\mathrm{pert}}$.

Charm (D^*) @ 92 GeV	$\langle x_p\rangle^{\mathrm{corr}}$	$\langle x_p\rangle^{\mathrm{pert}}$	$\langle x_p\rangle^{\mathrm{np}}$
Data	0.495 ± 0.006	–	–
NLL	–	0.583 ± 0.007	0.849 ± 0.018
FO	–	0.76 ± 0.03	0.65 ± 0.04
PYTHIA Def.	0.500	0.640	–
PYTHIA CLEO	0.484	0.640	–
PYTHIA Pet. $\epsilon = 0.06$	0.490	0.640	–
Beauty (B^{wd}) @ 92 GeV	$\langle x_p\rangle^{\mathrm{corr}}$	$\langle x_p\rangle^{\mathrm{pert}}$	$\langle x_p\rangle^{\mathrm{np}}$
Data	0.7114 ± 0.0026	–	–
NLL	–	0.768 ± 0.010	0.927 ± 0.013
FO	–	0.83 ± 0.02	0.85 ± 0.02
PYTHIA Def.	0.686	0.773	–
PYTHIA Pet. $\epsilon = 0.002$	0.710	0.773	–

2.5 Effect on predictions for heavy quark production at HERA and LHC

Going back to the heavy-hadron production in ep and pp collisions, Eq. 9 shows that the uncertainty on the differential heavy-hadron cross section $d\sigma/dp_T$ is related to the uncertainty on the average non-perturbative fragmentation by

$$\Delta(d\sigma/dp_T) = N\Delta(\langle x\rangle^{\mathrm{np}}),$$

where $-N$ is the exponent of the differential cross section.

The state of the art calculations for photo- and hadro-production (FONLL [41, 42]) include NLO matrix elements and the resummations of next-to-leading logarithms. The appropriate non-perturbative fragmentation for FONLL is therefore obtained with the NLL theory which has the same kind of perturbative accuracy [43]. Since the NLL calculation gives a good description of e^+e^- data, it seems appropriate to use the value and the uncertainty of $\langle x\rangle^{\mathrm{np}}$ as obtained from e^+e^- data at the Z^0 peak. The relative error for the D^* fragmentation is $\Delta\langle x\rangle^{\mathrm{np}}/\langle x\rangle^{\mathrm{np}} = 2\%$ which translates into an uncertainty of 9% on charm production at large p_T at LHC ($N = 4.5$) of 9%. For beauty, the relative uncertainty $\Delta\langle x\rangle^{\mathrm{np}}/\langle x\rangle^{\mathrm{np}} = 1.4\%$ translates into an uncertainty on large-p_T B-hadron production at LHC ($N = 3.8$) of 5.3%. These uncertainty are smaller or of the order of the perturbative uncertainties of the calculation. Nevertheless, it should be noted that this approach is only valid for large transverse momenta. At small transverse momenta the factorization ansatz breaks down and large corrections of order m_Q/p_T may appear. Therefore, for the low-p_T region, the uncertainty on the p_T distribution is large and difficult to evaluate.

For processes such as DIS and for particular observables FONLL calculations are not available. The best theory available in this case is the fixed order NLO theory. In this case the situation is complex since the equivalent FO calculation for e^+e^- does not reproduce the experimental data. The proposed solution is to vary $\langle x\rangle^{\mathrm{np}}$ from the same value obtained in the NLL case (that would be correct at low p_T, where the FF evolution is irrelevant) to the value obtained at the Z^0 energy (that would be valid at $p_T \sim 100$ GeV). Therefore we consider for charm $\langle x\rangle^{\mathrm{np}} = 0.075 \pm 0.010$ and for beauty $\langle x\rangle^{\mathrm{np}} = 0.089 \pm 0.004$. When these values are transported to heavy-hadron production at LHC, the corresponding uncertainties on $d\sigma/dp_T$ at large p_T are 60% for charm and 20% for beauty. Therefore the NLO fixed order calculations cannot be used for precise predictions of the charm (and to a lesser extent beauty) production at pp and ep colliders.

Table 4: Proposed value and uncertainty on $\langle x \rangle^{\rm np}$ to be used with FO-NLO and FONLL programs for photo- and hadro-production of D^* mesons and weakly decaying B hadrons. The corresponding value and range for the Peterson ϵ and for the Kartvelishvili α parameters are also reported. The last columns show the corresponding relative uncertainty on $d\sigma/dp_T$ at LHC (assuming a negative power $N = 4.5/3.8$ for charm/beauty) and HERA ($N = 5.5/5.0$ for c/b).

	$\langle x^{\rm np} \rangle$	$\epsilon(\min:\max)$	$\alpha(\min:\max)$	$\Delta\langle x^{\rm np}\rangle/\langle x^{\rm np}\rangle$	$\Delta\sigma/\sigma$ (LHC)	$\Delta\sigma/\sigma$ (HERA)
FONLL D^*	0.849 ± 0.018	$0.0040(0.0027:0.0057)$	$10(9:12)$	2.1%	9%	12%
FONLL B^{wd}	0.927 ± 0.013	$0.00045(0.00026:0.00072)$	$24(20:30)$	1.4%	5%	7%
FO-NLO D^*	0.75 ± 0.10	$0.02(0.004:0.08)$	$5(3:10)$	13%	60%	70%
FO-NLO B^{wd}	0.89 ± 0.04	$0.0015(0.0004:0.004)$	$15(10:25)$	4.5%	20%	22%

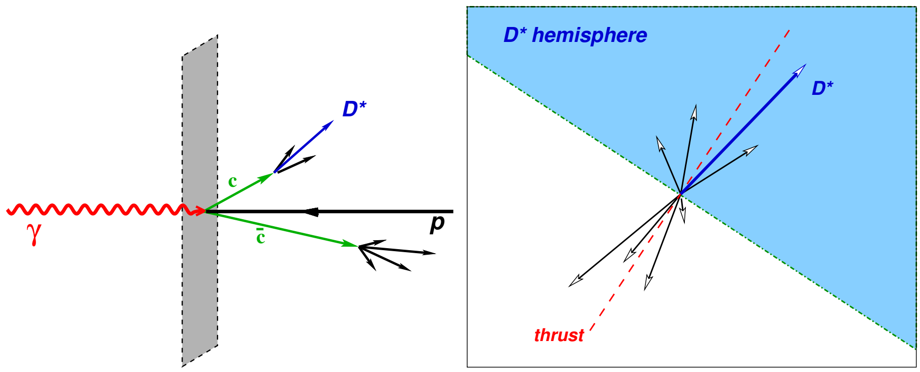

Fig. 5: Hemisphere method

In the FO-NLO and FONLL programs the hadron distributions are obtained by reducing the quark momenta according to a given fragmentation functions. Typical fragmentation functions used in these programs are the Peterson and Kartvelishvili forms. Table 4 summarises the proposed values and uncertainties for $\langle x \rangle^{\rm np}$ to be used with FO-NLO and FONLL calculations and reports the corresponding values and ranges for the Peterson and Kartvelishvili parameters. Similar ranges are used in the calculations presented in the section on “Benchmark cross sections” in these proceedings.

3 Measurements of the charm quark fragmentation function at HERA[5]

The differential cross section for the inclusive production of a heavy hadron H from a heavy quark h can be computed in perturbative QCD (pQCD) as a convolution of a short-distance cross section $\hat\sigma(p_h)$ with a fragmentation function $D_H^h(z)$:

$$d\sigma(p_H) = \int dz dp_h d\hat\sigma(p_h) D_H^h(z)\delta(p - zp_h) \tag{11}$$

The quantity z is the fractional momentum of the heavy quark h which is transferred to the heavy hadron H, and the normalized fragmentation function $D_H^h(z)$ gives the probability to observe the hadron H with a momentum fraction z.

The precise definition of $D_{\rm H}^{(h)}(z)$ is in some sense arbitrary. Due to the short and long-distance processes involved, the fragmentation function contains a perturbative and a non-perturbative component.

[5]Authors: J. Bracinı́k and G. Grindhammer

Since the former can be calculated only up to some order in the strong coupling, the non-perturbative component in practice will have to absorb some of the missing higher order corrections. The calculable perturbative part can be absorbed into the definition of $\hat{\sigma}(p_h)$. Since for heavy quarks perturbative gluon emission do not lead to collinear divergencies, the perturbative evolution is well defined, and it is possible to absorb them into $\hat{\sigma}(p_h)$ and to perform perturbative evolution down to a scale of the heavy quark mass m_h. In this case the non-perturbative fragmentation function $D_H^h(z)$ accounts for the transition of an almost on-shell quark h into a heavy hadron H.

According to the QCD factorization theorem, the non-perturbative fragmentation functions (FF) depend neither on the type of the hard process nor on the scale at which the heavy quark h is originally produced. This implies universality of FF and allows - if valid - to extract fragmentation functions from data for one particular reaction (usually e^+e^- annihilation) and to use them to predict cross sections for other reactions (e.g. in pp and ep-collisions). In order to be able to check the reliability of pQCD predictions, it is necessary to check the universality of FF.

In practice, different theoretically motivated functional forms for $D_H^h(z)$ are used, depending on one more free parameters which are fitted to data. Among frequently used expressions are those by Peterson et al. [14] and by Kartvelishvili et al. [12].

From Equation 11 it is clear that $D_H^h(z)$ cannot be measured directly, since all observables are convoluted with the perturbative cross section. In case of ep and pp scattering there are additional convolutions with the parton density functions of one or two interacting hadrons. However, there are some observables which are more sensitive to $D_H^h(z)$ then others.

In e^+e^-, a convenient way to study fragmentation is to study the differential cross section of a heavy meson as a function of a scaled momentum or energy z. A customary experimental definition[6] of z is $\mathrm{z} = E_H/E_{\mathrm{beam}}$, where E_{beam} is the energy of the beams in the center-of-mass system. In leading order, i.e. without gluon emissions, it is also the energy of the charm and anti–charm quark and is equal to $D_H^h(z)$. In contrast to e^+e^- annihilation the choice of a fragmentation observable in ep collisions is more difficult. Two different observables have been used so far, both of them having the feature that in leading order QCD, the z-distributions are equal to $D_H^h(z)$.

In the case of what is called here the jet method, the energy of the charm quark is approximated by the energy of the charm-jet, tagged by a D^*-meson, which is considered to be part of the jet. The scaling variable is then defined as $\mathrm{z_{jet}} = (E+p_\mathrm{L})_{D^*}/(E+p)_{\mathrm{jet}}$.

The idea of the so called hemisphere method (see Figure 5) is to exploit the special kinematics of charm events in ep collisions. The dominant charm production process has been shown to be boson-gluon fusion. If such an event is viewed in the photon-proton center-of-mass frame, the photon puts its full energy into the hard subprocess, while the proton interacts via a gluon, which typically carries only a small fraction of the proton momentum. As the result, both quarks produced, c and $\bar{c}$, move in the direction of the photon. Assuming no initial gluon k_T and no gluon radiation, their transverse momenta are balanced (see Fig. 5, left).

This can be seen best by projecting the quark momenta onto the plane perpendicular to the γ-p axis. In this plane it is possible to distinguish rather efficiently between the products of the fragmentation of the charm quark and its antiquark. The momenta of all particles are projected onto the plane and the thrust axis in this plane is found (see Fig. 5, right). The plane is then divided into two hemispheres by the line perpendicular to the thrust axis. All particles, lying in the hemisphere containing the D^*-meson are marked and their three-momenta and energy are summed-up to give the hemisphere's momentum and energy, which is used to approximate the momentum and energy of the respective charm/anti-charm quark. The scaling variable $\mathrm{z_{hem}}$ is then defined as $\mathrm{z_{hem}} = (E+p_\mathrm{L})_{D^*}/\sum_{\mathrm{hem}}(E+p)$.

The ZEUS collaboration has provided preliminary results [44] on a measurement of normalized differential cross sections of D^*-mesons as a function of $\mathrm{z_{jet}}$. The measurement was done in photopro-

[6]Sometime there are slightly different definitions of z [28] in case of heavy meson production close to threshold.

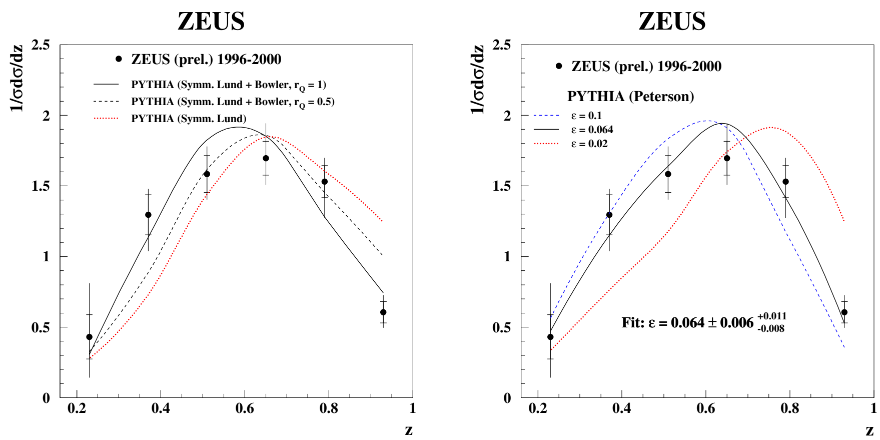

Fig. 6: Normalized differential cross section as a function of $\mathrm{z}_{\mathrm{jet}}$ as measured by ZEUS in photoproduction for jets with an associated D^*-meson with $|\eta_{\mathrm{jet}}| < 2.4$ and $E_{\mathrm{T,jet}} > 9$ GeV.

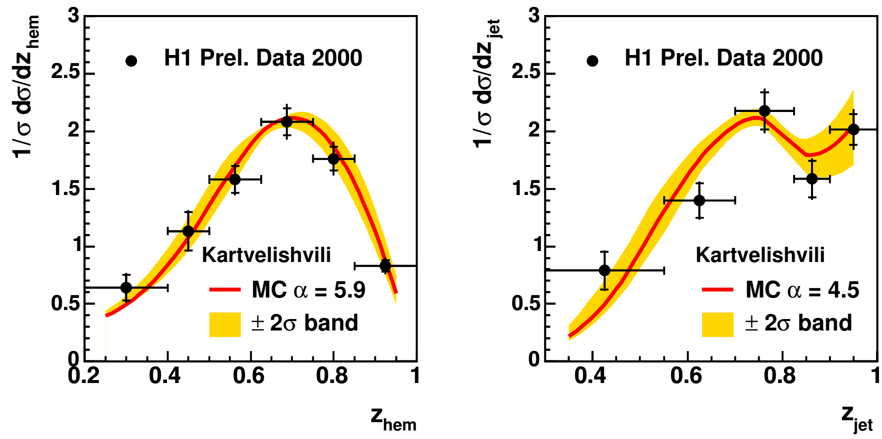

Fig. 7: Normalized differential cross section of D^*-meson as a function of $\mathrm{z}_{\mathrm{jet}}$ and $\mathrm{z}_{\mathrm{hem}}$ in DIS as measured by H1.

duction, in the kinematic range $Q^2 < 1\ \mathrm{GeV}^2$ and $130 < W < 280$ GeV. The D^*-mesons were reconstructed using the 'golden channel' $D^* \to D^0\pi_s \to K\pi\pi_s$ and were required to be in the central rapidity region $|\eta| < 1.5$ and to have $p_T > 2$ GeV. Jets were reconstructed using the inclusive $k_\perp$ algorithm. They fulfill the conditions $|\eta_{\mathrm{jet}}| < 2.4$ and $E_{\mathrm{T,jet}} > 9$ GeV. The jets were reconstructed as massless jets. The beauty contribution to the D^*-meson cross section, which amounts to about 9%, was subtracted using the prediction of PYTHIA. The scaling variable was calculated as $\mathrm{z}_{\mathrm{jet}} = (E + p_L)_{D^*}/(2E_{\mathrm{jet}})$. The cross section as a function of $\mathrm{z}_{\mathrm{jet}}$ is shown in Fig. 6. The uncertainties due to choice of the model used to correct for detector effects, and the subtraction of the beauty component were the largest contributions to the total uncertainty.

The H1 collaboration has recently presented preliminary results [45] on the normalized differential cross section also of D^*-mesons as a function of both $\mathrm{z}_{\mathrm{hem}}$ and $\mathrm{z}_{\mathrm{jet}}$. Their measurement was performed in the kinematic range $2 < Q^2 < 100\ \mathrm{GeV}^2$ and $0.05 < y < 0.7$. The D^*-mesons were reconstructed using the 'golden channel' with $|\eta| < 1.5$ in the central rapidity region and $p_T > 1.5$ GeV. The jets were reconstructed using the inclusive $k_\perp$ algorithm in the photon-proton center of mass frame, using the massive recombination scheme. The jets were required to have $E_{\mathrm{T,jet}} > 3$ GeV. The scaling variables were calculated as $\mathrm{z_{jet}} = (E + p_{\mathrm{L}})_{D^*}/(E + p)_{\mathrm{jet}}$ and $\mathrm{z_{hem}} = (E + p_{\mathrm{L}})_{D^*}/\sum_{\mathrm{hem}}(E + p)$ and are shown in Fig. 7. The resolved contribution was varied between 10 and 50% and the beauty contribution as predicted by the model was varied by a factor of two. The resulting uncertainties are part of the systematic error of the data points. For these distributions, the contribution of D^*-mesons coming from the fragmentation of beauty, as predicted by RAPGAP, was subtracted. It amounts to about 1.3% for the hemisphere method and 1.8% for the jet method. The dominant systematic errors are due to the model uncertainty and the signal extraction procedure.

Both collaborations used the normalized z-distributions to extract the best fragmentation parameters for a given QCD model.

In case of ZEUS, PYTHIA was used together with the Peterson fragmentation function. The MC was fit to the data using a χ^2-minimization procedure to determine the best value of ϵ. The result of the fit is $\epsilon = 0.064 \pm 0.006^{+0.011}_{-0.008}$.

The H1 collaboration used RAPGAP 3.1 interfaced with PYTHIA 6.2. The contribution due to D^*-mesons produced in resolved photon processes (in DIS), which amounts to 33% as predicted by the model, has been included in addition to the dominant direct photon contribution. The Peterson and Kartvelishvili parametrizations were both fitted to the data. The results are shown in Table 5.

Table 5: Extracted fragmentation parameters for $\mathrm{z_{jet}}$ and $\mathrm{z_{hem}}$ from H1.

Parametrization		Hemisphere Method	Jet Method	Suggested range
Peterson	ε	$0.018^{+0.004}_{-0.004}$	$0.030^{+0.006}_{-0.005}$	$0.014 < \varepsilon < 0.036$
Kartvelishvili	α	$5.9^{+0.9}_{-0.6}$	$4.5^{+0.5}_{-0.5}$	$4 < \alpha < 6.8$

The parameter of the Peterson fragmentation function as measured by ZEUS and H1 do not agree with each other. This may be due to the different phase-space regions covered by the two measurements (photoproduction versus DIS, $E_{\mathrm{T,jet}} > 9$ GeV versus $E_{\mathrm{T,jet}} > 3$ GeV) and most importantly, the parameters were extracted for two different models[7]. More detailed investigations are needed to resolve this question.

The fragmentation function parameters extracted by H1 with the hemisphere and the jet method differ by less than 3 σ. At the present level of statistical and systematic errors it is not possible to exclude a statistical fluctuation. On the other hand, the potential discrepancy may be a sign of deficiencies in the modelling of the hadronic final state in RAPGAP.

The measured $\mathrm{z_{hem}}$ distribution of H1 is compared to data from the ALEPH [23], OPAL [22] and CLEO [28] collaborations in Fig. 8 (left) and to ZEUS [44] and Belle [29] in Figure 8 (right)[8]. The results of H1 are in rough agreement with recent data from CLEO and Belle, taken at at 10.5 and 10.6 GeV, corresponding roughly to the average energy of the system at H1. Differences beyond the measurement errors can be observed. However, this may be due to the somewhat different definitions used for the fragmentation observable z, different kinematics, different processes, or it may be a sign of the violation of universality.

While the z distributions don't need to agree, the fragmentation parameters, which are extracted from them, should agree. This can be expected only, if a model with consistent parameter settings is used which provides an equally good description of the different processes at their respective scales.

[7]While ZEUS has used the default parameters for PYTHIA, H1 has taken the tuned parameter values of the ALEPH collaboration [46]

[8]Data points were taken from the figure in [29] and [44].

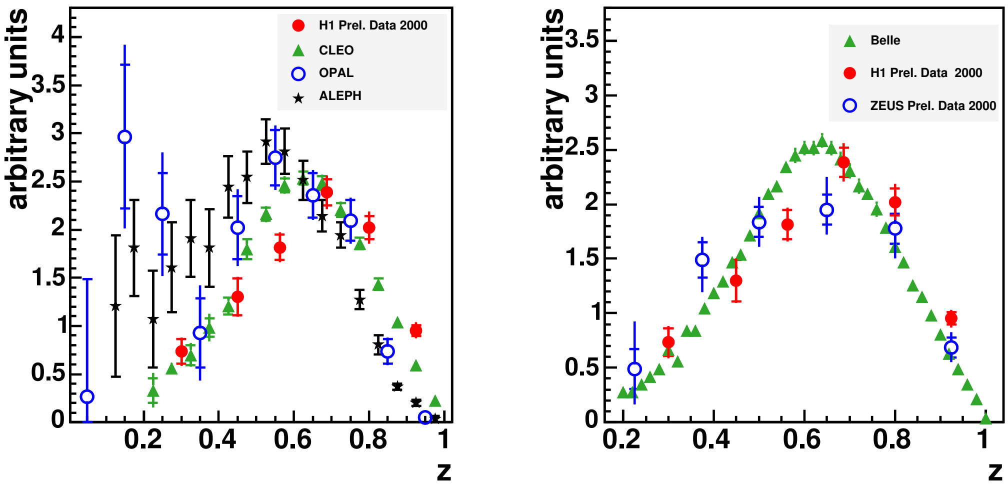

Fig. 8: Comparison of the z-distributions from CLEO, OPAL and ALEPH (left) and Belle and ZEUS (right) with the one from the hemisphere method from H1. All distributions are normalized to unit area from $z = 0.4$ to $z = 1$.

The values of the Peterson fragmentation parameter, as extracted by different experiments within the PYTHIA/JETSET models, are summarized in Table 6.

Table 6: Extracted fragmentation parameters from e^+e^- annihilation data by ALEPH [23], OPAL [22] and BELLE [29] and from *ep* data by ZEUS [44] and H1 [45].

PARAMETRIZATION		ALEPH	OPAL	BELLE	ZEUS	H1: z_{hem}	H1: z_{jet}
Peterson	ε	0.034 ± 0.0037	0.034 ± 0.009	0.054	$0.064^{+0.013}_{-0.010}$	$0.018^{+0.004}_{-0.004}$	$0.030^{+0.006}_{-0.005}$
Kartvelishvili	α	—	4.2 ± 0.6	5.6	—	$5.9^{+0.9}_{-0.6}$	$4.5^{+0.5}_{-0.5}$

Contrary to expectations, discrepancies between various experiments can be seen. A consistent phenomenological analysis of these data is therefore needed in order to resolve the reasons for the discrepancies.

The measurement of the charm fragmentation function at HERA provides an important test of our understanding of heavy quark production. We may hope that HERA II data and a phenomenological analysis of existing data will bring new insights in this area.

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