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# THE KINETICS OF ELECTRON COOLING OF BEAMS IN HEAVY PARTICLE STORAGE RINGS†

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A study is made of the kinetic features of the method of electron cooling of beams in heavy particle storage rings. In the first part of this work, Landau's collision integral for Coulomb interactions is used to obtain a kinetic equation for a beam of particles passing periodically through an accompanying stream of electrons. Apart from collisions with electrons, account is taken of scattering on coherent fluctuations (of nonthermodynamic origin) of the space charge of the electron beam, and on atoms of residual gas. The final kinetic equation is the Fokker–Plank equation in action variables.

In the second part, an investigation is made on this basis into the effect on the kinetic process of deviations in the electron stream from a state of thermodynamic equilibrium (in the accompanying system). A qualitative investigation is made into the dependence of attenuation speed and of the stabilized values of angular and energy spread on the velocity distribution of electrons and on the spatial inhomogeneity of the electron beam in the radial direction. An evaluation is made of the permissible level of coherent noise in the electron stream. A solution is obtained for the kinetic equation in the region of small angles ($\theta_p < \theta_e$). An important theorem is established concerning the sum of decrements of the oscillations. An investigation is also made into the nature of the solution in the region of large amplitude.

## INTRODUCTION

As is known, synchrotron radiation, which is used in storage rings of electrons and positrons for obtaining dense beams, is practically absent in the case of heavy particles. G. Budker has proposed the method of "electron cooling" for particle beams in storage rings of heavy particles,[1] based on the heat-energy transfer from the beam to an electron stream with lower temperature. In the first approximation, the kinetic process can be described as the usual relaxation of a two-component plasma.[1–3] If the electron current is continuously renewed, then at complete equalization of temperatures the angular and the relative energy spread for protons (antiprotons) is decreased by $(M/m)^{1/2}$ times with respect to the electron spread. The "relaxation" time at electron density $n = 10^8\ \mathrm{cm}^{-3}$, proton energy $W = 1$ GeV, initial angular spread $\theta_p \lesssim \theta_e = 3{\cdot}10^{-3}$ and the orbit filling factor for the electron beam $\eta = 0{\cdot}1$ and for a Maxwellian distribution of free particles is about 100 sec.

The kinetics of electron cooling is actually much more complex since one cannot consider the focused proton beam moving in the vicinity of a closed orbit as a free particle gas. On the other hand, an electron beam cannot, strictly speaking, be considered as a thermostat, as the possible (and partially inevitable) deviations from the thermodynamic equilibrium state in an electron current will not damp (in contrast with the closed system) and affect the kinetic process.

The kinetic equation enabling a correct account of the features of the electron-cooling kinetics is derived in the first part of the present work. In Section 1 on the basis of the Landau collision integral[2] and its relativistic generalisation,[4] an expression is obtained for the collision term of pair collisions of protons with the beam electrons and atoms of the residual gas in terms of phase-action variables. In Section 2 the derivation is performed for the "coherent" part of a collision term describing the particle diffusion from fluctuations of nonthermodynamical origin in the electron stream. In Section 3, a basis is given for averaging the equation coefficients over the phases; after that the kinetic process is described by the Fokker–Plank equations in the space of the action variables.

In the second part of this work, a study is made, on this basis, of the effect of "nonideal" features of the electron stream on the damping rate of the momentum spread and its stabilized width. In Section 1, a qualitative study is made of the part played by deviations in electron distribution in the accompanying system from the Maxwellian. It is

† The work was performed in 1968 (Preprint INP No. 255 (1968); CERN Trans. 69-18).

shown that Coulomb collisions may cause anti-damping in the oscillations of individual degrees of freedom if the energy of orderly motion of the electrons exceeds the thermal energy. As a result of the coupling of radial and longitudinal motion in accelerators, the kinetic process is also quite sensitive to spatial inhomogeneity of the beam of electrons. In Section 2, an evaluation is made of the permissible values of the gradients of the average velocity, density and temperature of the electrons, above which the proton oscillations become unstable. In Section 3, an evaluation is made of the permissible level of coherent noise in the electron current and an examination is made of other questions. The solution of the kinetic equation in the range of small angles $\theta_p < \theta_e$ ($\theta_e$ is the electron angular spread) is obtained in Section 4. In addition, an important theorem is established concerning the positiveness of the sum of decrements of the oscillations, and its nondependence on the gradients of electron distribution in the phase space ($\mathbf{p}, \mathbf{r}$). Finally, an investigation is made in Section 5 into the nature of the solution in the range $\theta_p > \theta_e$ and the dependence of the damping rate on the cross section of the electron beam.

## PART I THE KINETIC EQUATION

### 1 *Collision Integral*

The kinetic equation for the distribution function of interacting particles in an external field has the following general form

$$\frac{\partial}{\partial t} f(\mathbf{p}, \mathbf{r}, t) + [\mathscr{H}; f] = -\frac{\partial j_\alpha}{\partial p_\alpha}, \qquad \text{(I.1.1)}$$

where $\mathscr{H}$ is the Hamiltonian of a particle in the external field, $\mathbf{j}$ is the particle current in the phase space resulting from the interaction. If a transformation is made in the equation from the usual momenta and coordinates to canonical integrals of motion in the external field (for example, action-phase) the Poisson bracket drops out of the equation:

$$\frac{\partial}{\partial t} f(C, t) = -\frac{\partial j_i}{\partial C_i}. \qquad \text{(I.1.2)}$$

The components $j_i$ are linked with $j_\alpha$ by the usual rule of vector transformation:

$$j_i = \frac{j_\alpha \, \partial C_i}{\partial p_\alpha}. \qquad \text{(I.1.3)}$$

It is appropriate to rewrite the collision integral in terms of $\mathbf{p}$, $\mathbf{r}$ variables with respect to a moving system, where the interaction of protons and electrons is Coulombian (the relative velocities are nonrelativistic). Since the distribution function is a relativistic invariant,[4] there is no need to include a special designation for it in this system.

The right-hand side (I.1.1) can be represented in the form[5,6]

$$\operatorname{div} \mathbf{j} = [e\varphi_c; f] + \operatorname{div} \mathbf{j}_{st}$$

where $\varphi_c$ is the potential of the self-consistent field of the system of charges and $\mathbf{j}_{st}$ is the collision current proper.

A contribution to $\mathbf{j}_{st}$ is given, not only by the collisions with electrons, but also by the collisions of protons amongst themselves and with atoms of the residual gas. An assessment of the interaction between protons is beyond the scope of the present work. In any case, we shall assume that the density of the beam of protons is sufficiently small for us to be able to ignore their collisions during the time of damping.

Let us first examine collisions with electrons. In the Landau approximation,[2,5,6] in the usual momentum space has the well-known form:

$$j_\alpha = 2\pi e^2 e'^2 L \int d^3p' \, \frac{u^2 \delta_{\alpha\beta} - u_\alpha u_\beta}{u^3} \times \left( \frac{\partial f}{\partial p_\beta} f' - f \frac{\partial f'}{\partial p'_\beta} \right), \qquad \text{(I.1.5)}$$

where $e$ and $e'$ are the particle charges, $f(\mathbf{p}, \mathbf{r}, t)$ and $f'(\mathbf{p}', \mathbf{r}', t)$ are distribution functions, $\mathbf{u} = \mathbf{v} - \mathbf{v}'$ is the relative velocity and $L = \ln(\rho_{\max}/\rho_{\min})$ is the Coulomb logarithm. For the expression (I.1.5) to be correct it is necessary that $L \gg 1$. $\rho_{\min}$ is represented by the impact parameter at which the momentum transfer during the collision is $\Delta p' \sim p' : |ee'|/\rho_{\min} = \mu u^2$ ($\mu$ is the reduced mass). With a relative angular spread in the laboratory system of $10^{-3} - 10^{-2}$, $\rho_{\min} \sim 5(\beta\gamma)^{-2} \cdot (10^{-9} - 10^{-7})$cm. The parameter $\rho_{\max}$ may be represented by the Debye radius of the electron "plasma" d, the cross-sectional dimension of the beam $b$, or the impact parameter $\rho_0 \sim \tau_0 \langle |u| \rangle$, which is determined by the time-of-flight of the proton through the beam $\tau_0$:

$$\rho_{\max} \sim \min\{d, b, \rho_0\}. \qquad \text{(I.1.6)}$$

For typical parameters of the electron beam, $\rho_{\max} \sim 0.1 - 1$ cm. In this case, $L \simeq 20$. The large value of $L$ makes it possible to use (I.1.5) even when spatial inhomogeneity is considerable.

The electron beam can be focused with a longitudinal magnetic field $H$ in order to "compensate" the space charge. If the mean Larmor radius of the electrons $\rho_L$ proves to be much less than $\rho_{\max}$, the expression (I.1.5) will, generally speaking, be unusable, as it is necessary to take into account the Larmor rotation of electrons (with regard to the protons we shall assume that the distortion of their trajectories in the region of beam interactions can be ignored). In this case, it is necessary to use the general expression $\mathbf{j}_{st}$, which takes into account the effect of the external field on the collisions.[6,7] In the case of a homogeneous (or slightly inhomogeneous) magnetic field, $\mathbf{j}_{st}$ can be written in the form:

$$j_\alpha = e^2 e'^2 \int d^3 p'\, d^3\kappa \int_{-\infty}^{0} d\tau \exp[i\boldsymbol{\kappa} \int_0^\tau \mathbf{u}_{\tau'}\, d\tau'] \times \frac{\kappa_\alpha \boldsymbol{\kappa}}{\kappa^4}\left(\frac{\partial f}{\partial \mathbf{p}_\tau} f' - f \frac{\partial f'}{\partial \mathbf{p}'_\tau}\right), \tag{I.1.7}$$

where $\mathbf{p}_\tau$, $\mathbf{p}'_\tau$ are the particle momenta in the magnetic field as a function of time with an initial condition of $\mathbf{p}_0 = \mathbf{p}$, $\mathbf{p}'_e = \mathbf{p}'$. Integration over $\kappa$ is contained within the limits of $\rho_{\max}^{-1}$ and $\rho_{\min}^{-1}$.

It is convenient to break up $\mathbf{p}'_\tau$ for electrons into components which are longitudinal and transverse to the magnetic field:

$$\mathbf{p}'_\tau = \mathbf{p}'_H + \mathbf{p}'_\perp(\tau).$$

The phase of the exponent in (I.1.7) is

$$\sigma = \boldsymbol{\kappa}(\mathbf{v} - \mathbf{v}'_H)\tau + \boldsymbol{\kappa}_\perp \int_0^\tau \mathbf{v}'_\perp(\tau')d\tau' \equiv \boldsymbol{\kappa}\mathbf{u}_H + \boldsymbol{\kappa}_\perp \mathbf{r}'_\perp(\tau). \tag{I.1.8}$$

In order of magnitude, $|\mathbf{r}'_\perp(\tau)| \lesssim \rho_L$. In the range $\kappa\rho_L \gg 1$ the small values of $\tau$ are substantial ($\omega_L \tau \ll 1$); in this case $\sigma = \boldsymbol{\kappa}\mathbf{u}\tau$. This means that the contribution of this range of impact parameters ($\kappa = \rho^{-1}$) has the form (I.1.5) where $L = L_1 = \ln(\rho_L/\rho_{\min})$, $\mathbf{u} = \mathbf{v} - \mathbf{v}'$. In the range $\kappa\rho_L < 1$ the second term in the expression (I.1.8) can be ignored, which corresponds to the substitution $\mathbf{u} \to \mathbf{u}_H = \mathbf{v} - \mathbf{v}'_H$. Thus, for this range, $\mathbf{j}_{st}$ can also be written in the form (I.1.5), where

$$L = L_2 = \ln\left(\frac{\rho_{\max}}{\rho_L}\right), \qquad \mathbf{u} = \mathbf{v} - \mathbf{v}'_H, \qquad \rho_{\max} \gg \rho_L. \tag{I.1.9}$$

This structure of $\mathbf{j}_{st}$ has a simple physical meaning: in the range $\kappa\rho_L \ll 1$ the collisions occur adiabatically in relation to the Larmor rotation of electrons and the degrees of freedom which are transverse to $H$ do not contribute to the momentum and energy exchange. (More accurately speaking, the condition of adiabaticity is $\kappa u_H \ll \omega_L$; if $u_H \gg v_T$, then $\rho_L$ in (I.1.9) must be replaced by $\rho_1 = u/\omega_L$)†).

For future use, it is convenient to write (I.1.5) in the form

$$j_\alpha^{st} = \langle \Delta p_\alpha \rangle^e f - \frac{1}{2}\frac{\partial}{\partial p_\beta}\langle \Delta p_\alpha\, \Delta p_\beta \rangle^e f \tag{I.1.10}$$

where

$$\langle \Delta p_\alpha \rangle^e = F_\alpha^e + \frac{1}{2}\frac{\partial}{\partial p_\beta}\langle \Delta p_\alpha \Delta p_\beta \rangle^e \tag{I.1.11}$$

$$\mathbf{F}^e = \frac{\partial}{\partial \mathbf{v}} \int d^3 p' \frac{u_\beta}{u}\frac{\partial f'}{\partial p'_\beta} \cdot 2\pi e^2 e' 2L \tag{I.1.12}$$

$$\langle \Delta p_\alpha \Delta p_\beta \rangle^e \equiv d_{\alpha\beta}^e = 4\pi e^2 e'^2 L \int d^3 p' f \frac{u^2 \delta_{\alpha\beta} - u_\alpha u_\beta}{u^3}. \tag{I.1.13}$$

In the physical sense, $\mathbf{F}^e$ is the mean momentum transferred to the particle by the medium in a unit of time, if no account is taken of scattering (to which corresponds $M \to \infty$). As a result, the force $\mathbf{F}$ is referred to as the dynamic frictional force.[8,9] The tensor of diffusion $d_{\alpha\beta}$, on the contrary, corresponds to the interaction with infinitely heavy moving Coulomb centres. The true mean force is equal to $\langle \Delta \mathbf{p} \rangle$. As can be seen from (I.1.12), the force $\mathbf{F}^e$ has a potential in velocity space:

$$\mathbf{F}^e = -\frac{\partial}{\partial \mathbf{v}} U(\mathbf{v}, \mathbf{r}), \tag{I.1.14}$$

where

$$U(\mathbf{v}, \mathbf{r}) = \frac{2\pi e^2 e'^2}{m} L \int d^3 p' \left(\frac{\partial}{\partial \mathbf{v}'}\frac{\mathbf{u}}{u}\right) = \frac{4\pi e^2 e'^2}{m} L \varphi_v. \tag{I.1.15}$$

For a current $\mathbf{j}_1$, when $\mathbf{u} = \mathbf{v} - \mathbf{v}'$, $\varphi_v$ as a function of the velocity $\mathbf{v}$ has the form of a potential of

† Below in our work we shall not deal in detail with the effects connected with magnetic field and take $H = 0$. For careful study of the magnetic field effect on the cooling kinetics further investigations will be required.

attraction, created by the distributed Coulomb sources:

$$\varphi_v = -\int \frac{d^3p'}{u} f', \qquad \Delta_v \varphi_v = 4\pi m^3 f'. \quad \text{(I.1.16)}$$

This analogy, thanks to its familiarity, can conveniently be used for practical calculations and evaluations.

For a current $\mathbf{j}_2$, when $\mathbf{u} = \mathbf{v} - \mathbf{v}'_H$, $\varphi_v$ has no Coulomb analogue:

$$\begin{aligned} \varphi_v &= \frac{1}{2}\int d^2p' f'\left(\frac{\partial}{\partial \mathbf{v}'_H}\frac{\mathbf{u}_H}{u_H}\right) \\ &= -\frac{1}{2}v_\perp^2 \int \frac{d^3p'}{u_H^3} f' \\ &= -\frac{\mathbf{v}_\perp}{2}\frac{\partial}{\partial \mathbf{v}_\perp}\varphi'_H, \end{aligned} \quad \text{(I.1.17)}$$

where $\varphi_H$ is the potential (I.1.16) created by a $\delta$-type distribution in a direction transverse to the magnetic field

$$f'(\mathbf{p}'_H, \mathbf{r}) = \delta(\mathbf{p}'_\perp) \int d^2p'_\perp f'(\mathbf{p}', \mathbf{r}).$$

Let us now obtain $\mathbf{j}_0^{st}$ for collisions with atoms of residual gas. Its influence should be in lateral scattering of protons on nuclei and deceleration (entrainment in the accompanying system) on electrons. As the relative velocities of protons and atoms may be relativistic, it is necessary to use the expression $\mathbf{j}_{st}$ obtained in the relativistic case by Belyaev and Budker:[4]

$$j_\alpha = 2\pi e^4 \sum_a Z_a^2 L_a \int d^2p' S_{\alpha\beta}\left(\frac{\partial f}{\partial p_\beta} f_a - f\frac{\partial f_a}{\partial p'_\beta}\right),$$

where $Z_a$ is the charge of type "a" particles, $L_a$ is the corresponding Coulomb logarithm, $S_{\alpha\beta}$ is the collision matrix, for which a cumbersome expression covering the general case is given in Ref. 4. In our case, we can obtain a simple expression for $S_{\alpha\beta}$ if we take advantage of the small velocity of the protons $v$ in relation to the velocity of the gas $v'$ and eliminate in $S_{\alpha\beta}$ the members of the order $v/v'$, $v/c$ and higher,

$$S_{\alpha\beta} = \frac{v'^2\delta_{\alpha\beta} - v'_\alpha v'_\beta}{v'^3} + O\left(\frac{v}{v'}, \frac{v}{c}\right).$$

If we take, for the particles of residual gas, the distribution function in the form $f_\alpha = n_a\delta(\mathbf{p}' + \gamma m_a \mathbf{v}_0)$, where $\mathbf{v}_0$ is the mean proton velocity in the laboratory system, we will obtain

$$\langle \Delta p_\| \rangle_0 = F_{0\|} = -4\pi e^4 \sum_a \frac{L_a Z_a^2 n_a}{\gamma m_a v_0^2} \quad \text{(I.1.18)}$$

$$\langle (\Delta p_r)^2 \rangle_0 = \langle (\Delta p_z)^2 \rangle_0 = 4\pi e^4 \sum_a \frac{L_a Z_a^2 n_a}{v_0}$$

$$\langle (\Delta p_\|)^2 \rangle_0 = \langle \Delta p_\alpha \Delta p_\beta \rangle_{0\,\alpha\neq\beta} = 0$$
$$(\mathbf{p}_\perp = \{p_r, p_z\}) \quad \text{(I.1.19)}$$

as must be the case.

It is sufficient in the expression $\langle \Delta p_\| \rangle_0$ to take into account only the electron component, in view of the electron's small mass. The scattering, on the contrary, occurs mainly on the nuclei.

## 2 *Scattering on Coherent Fluctuations of the Space Charge*

Coulomb scattering may not be the only mechanism of diffusion (or "heating") of the proton beam in an electron current. Apart from the basic stationary part, it is also necessary to take into account in the space-charge field an irregular "random" part which is associated with collective fluctuations in density and velocity, caused by sources of an "external" origin (oscillations in the controlling voltage, cathode scintillation, etc.).

We shall first make a formal derivation of the expression of the corresponding collision term, departing in (I.1.4) from the stationary part of the potential $\varphi_c = \varphi_c(\mathbf{r}) + \varphi_c(\mathbf{r}, t)$ and the Coulomb collisions. In accordance with the theory of disturbances, we shall expand $f(\mathbf{p}, \mathbf{r}, t)$ into a series by degrees of

$$f = f^{(0)} + f^{(1)} + f^{(2)} + \cdots. \quad \text{(I.2.1)}$$

Since the coherent fluctuation times may not be small in comparison with the period of movement in the external field, it is convenient to change immediately in the kinetic equation to the $C_i$ variables:

$$\frac{\partial}{\partial t} f + [\tilde{V}; f] = 0.$$

If we insert here the expansion (I.2.1), we will obtain:

$$\frac{\partial}{\partial t} f^{(1)} + [\tilde{V}; f_0] = 0; \qquad \frac{\partial}{\partial t} f^{(2)} + [V; f^{(1)}] = 0.$$

The average speed of variation of $f(C, t)$ is

$$\left\langle \frac{\partial}{\partial t} f \right\rangle = \left\langle \frac{\partial}{\partial t} f^{(2)} \right\rangle = \left\langle \left[ \tilde{V}; \left[ \int_{-\infty}^{t} \tilde{V}(C, t')\mathrm{d}t'; f \right]\right]\right\rangle, \quad (\mathrm{I}.2.2)$$

since by definition

$$\left\langle \frac{\partial}{\partial t} f^{(1)} \right\rangle = [\langle \tilde{V} \rangle; f] = 0.$$

We shall introduce the designation

$$\tilde{S} = \int_{-\infty}^{t} \tilde{V}(C, t')\mathrm{d}t'$$

and transform (I.2.2), using the characteristics of the Poisson brackets:

$$\left\langle \frac{\partial}{\partial t} f \right\rangle = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\langle [\tilde{S}; [\tilde{S}; f]]\rangle + [U_c; f], \quad (\mathrm{I}.2.3)$$

where $U_c = \frac{1}{2}[\tilde{V}; \tilde{S}]$ is the correlative potential.[10]

The last term in (I.2.3) is the usual Poisson bracket and does not contribute to diffusion: the potential $U_c$ can be considered as a minor correction to the regular part of $e\varphi_c$. If we use the notation of the Poisson brackets in the form of a divergence of the current vector in the phase space we will obtain the diffusion equation

$$\frac{\partial}{\partial t} f - \frac{1}{2}\frac{\partial}{\partial C_i}\left\langle \frac{\mathrm{d}}{\mathrm{d}t}(\tilde{\Delta} C_i \tilde{\Delta} C_\kappa)\right\rangle \frac{\partial}{\partial C_\kappa} f = 0, \quad (\mathrm{I}.2.4)$$

where

$$\tilde{\Delta} C_i = [\tilde{S}; C_i].$$

If, as usual, we change over from time averaging to probability averaging, the scattering tensor in (I.2.4) can be written in the form

$$\tilde{D}_{i\kappa} = \langle [\tilde{V}; C_i][\tilde{S}; C_\kappa] + [\tilde{V}; C_\kappa][\tilde{S}; C_i]\rangle.$$

$\tilde{D}_{i\kappa}$ can be expressed explicitly by fluctuations in the electrical field:

$$[\tilde{V}; C_i] = e\frac{\partial C_i}{\partial \mathbf{p}}\tilde{\Delta}\mathbf{E}; \qquad [\tilde{S}; C_i] = e\int_{-\infty}^{t} \frac{\partial C_i}{\partial \mathbf{p}_{t'}}\tilde{\Delta}\mathbf{E}(\mathbf{r}_{t'}, t')\mathrm{d}t'$$

$$\tilde{D}_{i\kappa} = e^2 \int_{-\infty}^{t} \langle \tilde{\Delta} E_\alpha(\mathbf{r}, t)\tilde{\Delta} E_p(\mathbf{r}_{t'}, t')\rangle \times \left(\frac{\partial C_i}{\partial p_\alpha}\frac{\partial C_\kappa}{\partial p_{\alpha t'}} + \frac{\partial C_\kappa}{\partial p_\alpha}\frac{\partial C_i}{\partial p_{\beta t'}}\right). \quad (\mathrm{I}.2.5)$$

The mean $\langle\cdots\rangle$ in (I.2.5) is, by definition, a correlation function of fluctuations in the field $\mathscr{K}_{\alpha\beta}(\mathbf{r}|\mathbf{r}', t/t')$. Provided there is a spatial and temporal homogeneity, $\mathscr{K}_{\alpha\beta}$ is a function of the difference of the arguments and may be expressed by the spectral density of the fluctuations:[9,11]

$$\mathscr{K}'_{\alpha\beta} = \sum_{\boldsymbol{\kappa}, \omega} (E_\alpha E_\beta^*)_{\boldsymbol{\kappa}, \omega} \exp[i\boldsymbol{\kappa}(\mathbf{r} - \mathbf{r}') - i\omega(t - t')]. \quad (\mathrm{I}.2.6)$$

In this case

$$\langle \Delta E_\alpha(\mathbf{r}, t)\Delta E_\beta(\mathbf{r}, t)\rangle = \sum_{\boldsymbol{\kappa}, \omega} (E_\alpha E_\beta^*)_{\boldsymbol{\kappa}, \omega}. \quad (\mathrm{I}.2.7)$$

In the spatially inhomogeneous case, $\mathscr{K}_{\alpha\beta}$ is not a function only of the difference $\mathbf{r} - \mathbf{r}'$. The form of notation of (I.2.6) and (I.2.7) can be retained if the "local" spectral density is introduced:

$$(E_\alpha E_\beta^*)^{\mathbf{r}}_{\boldsymbol{\kappa}, \omega} \cdot \delta(\omega - \omega') = \int \mathrm{d}^3\kappa' \langle E_\alpha(\boldsymbol{\kappa}', \omega')E_\beta^*(\boldsymbol{\kappa}, \omega)\rangle e^{i(\boldsymbol{\kappa}' - \boldsymbol{\kappa})\mathbf{r}} \quad (\mathrm{I}.2.8)$$

(the correlation of Fourier harmonics is now not proportional to $\delta(\boldsymbol{\kappa} - \boldsymbol{\kappa}')$). The diffusion tensor in the space of integrals $C_i$ can thus be written in the form:

$$\tilde{D}_{i\kappa} = e^2 \sum_{\boldsymbol{\kappa}, \omega} (E_\alpha E_\beta^*)^{\mathbf{r}}_{\boldsymbol{\kappa}, \omega} \int_{-\infty}^{0} \mathrm{d}\tau \times \exp[i\boldsymbol{\kappa}(\mathbf{r} - \mathbf{r}_\tau) + i\omega\tau]A^{\alpha\beta}_{i\kappa}, \quad (\mathrm{I}.2.9)$$

where $A^{\alpha\beta}_{i\kappa}$ is the expression in curved brackets in (I.2.5).

As we know,[9] the integral of paired collisions (I.1.10) can also be obtained in the spirit of the general method for composing the Fokker–Plank equation, taking into account the frictional force calculated from the undisturbed movement of the charge, and scattering on thermodynamic fluctuations of the field of the medium. In the case of a homogeneous plasma the expressions $\mathbf{F}$ and $\mathrm{d}_{\alpha\beta}$, which take into account dynamic polarization of the medium by interacting particle, have the form

$$F_\alpha = 2e^2 e'^2 \sum_{\kappa, \omega} \frac{\kappa_\alpha \kappa_\beta}{\kappa^4 |\varepsilon_{\|}|^2} \int \mathrm{d}^3 p' \frac{\partial f'}{\partial p'_\beta} \times \delta(\omega - \boldsymbol{\kappa}\mathbf{v}')\delta(\omega - \boldsymbol{\kappa}\mathbf{v}) \quad (\mathrm{I}.2.10)$$

$$\mathrm{d}_{\alpha\beta} = \pi e^2 \sum_{\boldsymbol{\kappa}, \omega} (E_\alpha E_\beta^*)^T_{\boldsymbol{\kappa}, \omega}\delta(\omega - \boldsymbol{\kappa}\mathbf{v}), \quad (\mathrm{I}.2.11)$$

where

$$(E_\alpha E_\beta^*)^T_{\boldsymbol{\kappa},\omega} = \frac{2\kappa_\alpha\kappa_\beta}{\pi\kappa^4|\varepsilon_{\|}|^2}\int \mathrm{d}^3p' f'\delta(\omega - \boldsymbol{\kappa}\mathbf{v}'), \tag{I.2.12}$$

and $\varepsilon_{\|}(\boldsymbol{\kappa}, \omega)$ is the electrical permeability.[6,9,12] The factor $|\varepsilon_{\|}|^{-2}$ takes into account Debye screening ($|\varepsilon_{\|}| \sim 1 + (\kappa\mathrm{d})^{-2}$). (I.2.10) and (I.2.11) coincide, with logarithmic accuracy, with (I.1.12) and (I.1.13).

It can be seen from a comparison of (I.2.9) and (I.2.11) that (I.2.9) may be a general expression of the tensor of diffusion through the total spectral density, including both (I.2.12) and an "epithermal" part.

In the case of a thermodynamically stable plasma, when $\langle E_\alpha E_\beta^*\rangle_{\boldsymbol{\kappa},\omega} = (E_\alpha E_\beta^*)^T_{\boldsymbol{\kappa},\omega}$, taking into account the absorption of the energy of the fluctuation field leads to thermal equilibrium of test particles (protons) with the medium: $T_p = T_e$. But if waves with random phases (coherent fluctuations) are now induced in the plasma, the spectral density may considerably exceed (I.2.12). Physically, this means that the energy of the waves is comparable with or much greater than the Debye energy of the Coulomb interaction of charges of the plasma:

$$\langle(\tilde{\Delta}\mathbf{E})^2\rangle \gtrsim \frac{T_e}{\mathrm{d}^3}, \tag{I.2.13}$$

i.e. it exceeds the themodynamic equilibrium value[9,13] (at the same time, if the representation concerning the waves is to be applicable, it is necessary that $\langle(\tilde{\Delta}\mathbf{E})^2\rangle \ll nT$, i.e. when the waves are taken into account, there must be no variation in the "gross" characteristics of the medium). The diffusion rate increases accordingly. The rate of the Coulomb losses

$$-\left\langle\frac{\mathrm{d}W}{\mathrm{d}t}\right\rangle = -\mathbf{F}\cdot\mathbf{v}$$

practically does not change, since the action of the particle on the medium is due principally to paired collisions, on which the presence of "weak" coherent noise does not have any significant influence. In this way, the coherent fluctuations raise the final temperature of the proton beam, without affecting the damping rate. However, if cooling is to take place at all, it is essential that the maximum value of Coulomb losses exceeds the diffusion rate:

$$|\mathbf{F}\mathbf{v}|_{v-v_T} \gg \frac{\langle(\Delta\mathbf{p})^2\rangle}{2M}.$$

Let us now discuss certain special features of an interaction with coherent noises which are related to the specific problem. In an isolated volume of plasma, the spectral density of the waves is concentrated in the range $\kappa \ll \mathrm{d}^{-1}$, where a development of instabilities is possible, while in the range $\kappa \gtrsim \mathrm{d}^{-1}$ the waves are quickly attenuated. Consequently, the coherent part of the spectrum $(E_\alpha E_\beta^*)_{\boldsymbol{\kappa},\omega}$ is sharply separated from the thermodynamic part (interaction with "plasmons").[13] In our case, when the time-of-flight of the proton through the beam may be comparable with the period of Langmuir oscillations $\mathrm{d}/v_T$, and the beam and "waves" in it are constantly being renewed, the spectral density may be substantially different from (I.2.12) also in the range $\kappa \gtrsim \mathrm{d}^{-1}$, since there is not enough time for the waves to be attenuated. For the same reason, possible plasma instabilities cannot play a substantial part either, since their times are normally $\tau \gg \mathrm{d}/v_T$. The fluctuations caused by an "external" source may only be deformed substantially when spreading occurs in the electron current. Another special feature is due to the oscillatory character of proton motion near the equilibrium orbit: slow variations in current and density in the beam with frequencies of $\omega \ll \omega_i$ ($\omega_i$ are the partial frequencies of proton oscillations) are adiabatic in relation to the oscillation of protons and do not lead to an increase in amplitude. For a given spectral density, this is automatically achieved by the structure of the tensor diffusion (I.2.9).

## 3 *Averaging over Phases*

Let us now write, in $C_i$ variables, a full kinetic equation which takes into account collisions with electrons of the beam, residual gas and scattering on coherent noise,

$$\frac{\partial}{\partial t}f + \frac{\partial}{\partial C_i}\left(Q_i f - \frac{1}{2}D_{i\kappa}\frac{\partial}{\partial C_\kappa}f\right) = 0, \tag{I.3.1}$$

where

$$Q_i = \frac{\partial C_i}{\partial p}(F^e + F^0)$$

and

$$D_{i\kappa} = \frac{\partial C_i}{\partial p_\alpha}\frac{\partial C_\kappa}{\partial p_\beta}(\mathrm{d}^e_{\alpha\beta} + \mathrm{d}^0_{\alpha\beta}) + \tilde{D}_{i\kappa}. \tag{I.3.2}$$

As canonical integrals of motion we shall take three pairs of conjugate variables, action $I_i$ and phase $\varphi_i$, through which radial, axial and longitudinal deviations of the coordinates and momenta from the equilibrium phase trajectory are expressed as follows:[14]

$$r = r_b + r_c; \qquad p_r = \frac{\partial r}{\partial \theta}\frac{p}{R}$$
$$r_b = \frac{R}{p}\sqrt{\frac{I_r}{2}}\, f_r(\theta)\exp[i\nu_r\theta + \varphi_r] + C.C.;$$
$$r_c = \frac{\gamma R\psi(\theta)p_{\parallel}}{p}$$
$$z = \frac{R}{p}\sqrt{\frac{I_z}{2}}\, f_z(\theta)\exp[i\nu_z\theta + \varphi_z] + C.C.; \tag{I.3.3}$$
$$p_z = \frac{p}{R}\frac{\partial z}{\partial \theta}$$
$$\vartheta = \theta - \theta_s = \vartheta_c + \vartheta_b:$$
$$\vartheta_b = \frac{(\psi\, \partial r_b/\partial\theta - r_b\, \mathrm{d}\psi/\mathrm{d}\theta)}{R}$$
$$\left|\frac{\mathrm{d}\vartheta_c}{\mathrm{d}\theta}\right| \sim \left|\frac{p_{\parallel}}{p}\right| \ll 1.$$

Here, $\theta_s = \omega_s t$, $p = \gamma\beta MC$, $2\pi R$ are respectively, the azimuth, momentum and length of the orbit of the equilibrium particle, $f_r$ and $f_z$ are Floquet functions, $\psi$ is the forced solution of the equation

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}\theta^2} + [1 - n(\theta)]\psi = \frac{R}{R(\theta)},$$

and $p_{\parallel}$ is the longitudinal momentum in the accompanying system. In the absence of an rf field $p_{\parallel} = I_c$, $\mathrm{d}\vartheta_c/\mathrm{d}\theta \sim p_{\parallel}/p = \text{const}$, but for the bunched beam

$$p_{\parallel} = -\sqrt{2I_c}\sin(\nu_c\theta_s + \varphi_c), \qquad \frac{\partial I_c}{\partial p_{\parallel}} = p_{\parallel}. \tag{I.3.4}$$

Together with the coordinates and momenta of particles, the coefficients of the kinetic equation (I.3.1) are periodic functions of "fast" phases $\psi = \omega_i t + \varphi_i$. When considered as functions of $I_i$ and $\varphi_i$, they oscillate with time, in which case the oscillations generally cannot be considered small. It is, however, possible to replace (I.3.1) by a much simpler equation with coefficients which are not time-dependent, if the variation of $f(C, t)$ over a time of the order $\omega_i^{-1}$ is small. For this we shall apply to (I.3.1) the averaging method, and write the equation in the following form, for shortness:

$$\frac{\partial}{\partial t} f + \hat{L}(t) f = 0.$$

The operator $\hat{L}(t)$, as a function with a line-spectrum, can be represented in the form

$$\hat{L}(t) = \bar{\hat{L}} + \tilde{\hat{L}}(t),$$

where $\bar{\hat{L}}$ represents the mean value of $\hat{L}$ over the period $T_0 \sim \omega_i^{-1}$ and $\tilde{\hat{L}}$ is its oscillating part. (By the period $T_0$ we should understand a sufficiently large period of time such that

$$T_0^{-1}\left|\int_t^{t+T} \tilde{\hat{L}}\, \mathrm{d}t'\right| \ll \bar{\hat{L}}.$$

In this sense, the operator $\tilde{\hat{L}}$ can be treated as a periodic function of time, with a mean value equal to zero.) The averaged equation, which gives the correct change in the function $f$ over the period $T_0$ with an accuracy up to terms of the second order, is

$$\frac{\partial}{\partial t} f + \bar{\hat{L}} f + \frac{1}{2}\overline{[\hat{L}\hat{M}]} f = 0, \tag{I.3.5}$$

where $[\hat{L}\hat{M}]$ is the commutator of the operators $\hat{L}$ and $\hat{M}$, where $\hat{M}$ is given by

$$\hat{M} = \int_{t_0}^{t_0+t} \hat{L}\, \mathrm{d}t'.$$

If the third term in (I.3.5) represents a minor correction ($\hat{M} \ll 1$), taking it into account cannot change the nature of the solution even over great lengths of time, since the solution of the kinetic equation with time either tends to a stable stationary state or diverges. It is sufficient to use the equation

$$\frac{\partial}{\partial t} f = -\bar{\hat{L}} f = -\frac{\partial}{\partial C_i}\left(\bar{Q}_i f - \frac{1}{2}\bar{D}_{i\kappa}\frac{\partial}{\partial C_\kappa} f\right). \tag{I.3.6}$$

The condition $\hat{M} \ll 1$ means that the relaxation time $\tau \gg T_0$ since $\hat{M} \sim T_0\hat{L}$, and $\hat{L} \sim \tau^{-1}$, in a physical sense. An averaging of the coefficients of the kinetic equation for the time "on the trajectory" with this condition provides the basis for so-called anti-diffusion approximation.[10]

In practice, averaging over time can almost always be replaced by averaging over phases $\psi_i$. The operator $\hat{L}$ as a function of phases can be represented by a Fourier series

$$\hat{L}(I, \psi) = \sum_{\{m\}} \hat{L}_{\{m\}}(I) \exp\left[ i \sum_i m_i \psi_i \right].$$

The frequencies of variation with time in the harmonics are

$$\omega_{\{m\}} = \sum_i m_i \omega_i .$$

Apart from the "zero" harmonic, $m_i = 0$, which represents simply the mean value for the phase $\hat{L}$, when averaging for time is effected, nonzero contributions to $L$ may also come, generally speaking, from harmonics with frequencies of

$$\omega_{\{m\}} \lesssim \tau^{-1}. \qquad \text{(I.3.7)}$$

Formally speaking, it is always possible to select a combination of $\{m\}$ such that $\omega_{\{m\}}$ can be made as small as desired. However, the condition (I.3.7) can in the general case be justified only for quite high $m$, if consideration is made of the fact that $\omega_i \tau \gg 1$, and the frequencies $\omega_i$, as a rule, do not form, among themselves, rational relations of a low order. Consequently, the values of such harmonics will be small enough to be ignored. In reality, it is necessary to consider also the nonlinearity of particle oscillations, which leads to the dependence of frequencies on amplitudes. In spite of the relatively small variation of frequency with amplitude:

$$|\Delta\omega_i| = \left| \frac{\Delta I_\kappa \, \partial\omega_i}{\partial I_\kappa} \right| \ll \omega_i,$$

this dependence leads to violation of the "resonance" condition (I.3.7) when, under the effect of collisions, $I$ receives an increment $\Delta I$ such that

$$|\Delta\omega_{\{m\}}| > \tau^{-1}.$$

This circumstance is particularly important when estimating the part played by resonating harmonics of a low order, which may be comparable in size with the zeroth harmonic. An exception to this may be those cases when the resonance condition is maintained by self-phasing, which occurs as a result of an "external" disturbance, for example, the electron-beam field. In the case of nonlinear resonances, this phenomenon may be ignored if the phase volume inside the separatrix of the resonance is relatively small.

Thus, provided that $\tau \gg T_0$, it may almost always be considered that the kinetic coefficients $\bar{Q}_i$ and $\bar{D}_{i\kappa}$ do not depend on the phases $\varphi_i$. This makes it possible to change over to a much simpler equation for the function of three variables $I_i$, after having integrated equation (I.3.6) for phases:

$$\frac{\partial}{\partial t} f + \frac{\partial}{\partial I_i} \left\{ \bar{Q}_i f - \frac{1}{2} \bar{D}_{i\kappa} \frac{\partial}{\partial C_\kappa} f \right\} = 0. \qquad \text{(I.3.8)}$$

The sign ($-$) now indicates averaging for phases of transverse and longitudinal oscillations, and for the azimuth on the equilibrium proton orbit.

The equation (I.3.8) can be conveniently written in the standard form of Fokker–Planck[15,16]

$$\frac{\partial}{\partial t} f + \frac{\partial}{\partial I_i} \left\{ \langle \Delta I_i \rangle f - \frac{1}{2} \frac{\partial}{\partial I_\kappa} \langle \Delta I_i \Delta I_\kappa \rangle f \right\} = 0, \qquad \text{(I.3.9)}$$

where

$$\langle \Delta I_i \rangle = \bar{Q}_i + \frac{1}{2} \frac{\partial}{\partial I_\kappa} \langle \Delta I_i \Delta I_\kappa \rangle \equiv \langle \Delta I_i \rangle_d + \langle \Delta I_i \rangle_{f\ell} \qquad \text{(I.3.10)}$$

$$\langle \Delta I_i \Delta I_\kappa \rangle = \bar{D}_{i\kappa}. \qquad \text{(I.3.11)}$$

In the physical sense $\langle \Delta I_i \rangle_d = \bar{Q}_i$ gives the speed $\dot{I}$ on account of dissipative processes ($\bar{Q}_i$ is the "power" of the frictional force) and $\langle \Delta I_i \rangle_{f\ell}$ describes the average increment of $I_i$ in a unit of time on account of absorption of the energy of the fluctuation field of the "medium." In the oscillation condition, when $I_i$ defines the energy of the oscillator, the values of $\langle \Delta I_i \rangle$ characterize, for a group of particles with close values of $I_i$, the direction of the kinetic process as a whole. In the case of infinity of motion, when $I_i \sim p_i$, the moments of (I.3.11) also are important in this sense.

The coefficients $\partial I_i / \partial p_\alpha$, which are necessary for composing (I.3.10–I.3.11), can be found directly from the expressions (I.3.3–I.3.4) by making use of the characteristic of canonical transformation of $(\partial I_i / \partial p_\alpha) = (\partial q_\alpha / \partial \psi_i)$. Let us, furthermore put forward the usual expressions of $\langle \Delta I_i \rangle$ in terms of the moments $\langle \Delta p_\alpha \rangle$ and $\langle \Delta p_\alpha \Delta p_\beta \rangle$ (these are, naturally, identical to (I.3.10)). Taking (I.3.3) into account we obtain

$$\langle \Delta I_r \rangle = \overline{\frac{\langle \Delta p_r \rangle \, \partial I_r}{\partial p_r}} + \overline{\frac{\langle \Delta p_\parallel \rangle \, \partial I_r}{\partial p_\parallel}} + \frac{1}{2} \overline{\frac{\langle (\Delta p_r)^2 \rangle \, \partial^2 I_r}{\partial p_r^2}} + \frac{1}{2} \overline{\frac{\langle (\Delta p_\parallel)^2 \rangle \, \partial^2 I_r}{\partial p_\parallel^2}} + \overline{\frac{\langle \Delta p_r \Delta p_\parallel \rangle \, \partial^2 I_r}{\partial p_r \, \partial p_\parallel}}, \qquad \text{(I.3.12)}$$

$$\langle \Delta I_z \rangle = \overline{\langle \Delta p_z \rangle \frac{\partial I_z}{\partial p_z}} + \frac{1}{2} \overline{\langle (\Delta p_z)^2 \rangle \frac{\partial^2 I_z}{\partial p_z^2}}. \quad \text{(I.3.13)}$$

$$\langle \Delta I_c \rangle = \overline{\langle \Delta p_\parallel \rangle \frac{\partial I_c}{\partial p_\parallel}} + \frac{1}{2} \overline{\langle (\Delta p_\parallel)^2 \rangle \frac{\partial^2 I_c}{\partial p_\parallel^2}}. \quad \text{(I.3.14)}$$

The amplitude of the radial betatron oscillations $a_r \sim \sqrt{I_r}$, as can be seen from (I.3.12), changes under the effects of impacts not only in a radial, but also in an azimuthal direction, since the position of equilibrium, determined by the total energy (momentum) of the proton, changes, in this case, by successive jumps (in contradistinction to the adiabatically slow oscillations in synchrotron motion).

## PART II THE KINETICS OF ELECTRON COOLING

Using Eq. (I.3.9), we shall now investigate the basic characteristics of the kinetic process associated with deviations from thermodynamic equilibrium in an electron current. Here the most important factors seem to be the dependence of attenuation rate and equilibrium distribution on the shape of the electron velocity distribution and spatial inhomogeneity of the beam of electrons, and heating of the proton beam by coherent "noise" in the electron current.

### 1 *The "Monochromatic" Instability*

Let us first establish the general nature of the dependence of the attenuation rate and stabilized mean amplitudes on the velocity distribution of electrons.

For greater clarity, we shall disregard scattering on coherent noise and residual gas, and consider the electron current as being spatially homogeneous. We shall also ignore, for simplicity, the azimuthal inhomogeneity of proton focusing ($|f_r| = \text{const}, |f_z| = \text{const}$).

In these conditions, the expressions of the kinetic coefficients (I.3.12)–(I.3.14) take the form:

$$\langle \Delta I_z \rangle = \overline{p_z F_t^e} + \frac{1}{2} \overline{\mathrm{d}_{zz}^e}, \quad \text{(II.1.1)}$$

$$\langle \Delta I_r \rangle = \overline{p_r F_r^e} + \frac{1}{2} \overline{\mathrm{d}_{rr}^e} + \overline{\frac{\mathrm{d}_\parallel^e p^2}{2 v_c^2}}, \quad \text{(II.1.2)}$$

$$\langle \Delta I_c \rangle = \overline{p_\parallel F_\parallel^e} + \frac{1}{2} \overline{\mathrm{d}_\parallel^e}. \quad \text{(II.1.3)}$$

As has been pointed out (I.1.14)–(I.1.16), the frictional force $\mathbf{F}^e(\mathbf{v})$ is the analogue of the field of attraction created by a distribution of Coulomb sources $f'(\mathbf{v})$. From this we can immediately obtain the behaviour of $\mathbf{F}^e(\mathbf{v})$ when the distribution in the accompanying system is close to the Maxwellian $f_T(\mathbf{v})$:

$$\mathbf{F}^e \simeq \frac{-g}{m} \begin{cases} \dfrac{\mathbf{v}}{v_T^3} & v < v_T \\ \dfrac{\mathbf{v}}{v^3} & v > v_T, \end{cases} \quad \text{(II.1.4)}$$

where $g = 4\pi e^4 nL$.

$\mathrm{d}_{\alpha\alpha}$ can also be easily evaluated:

$$\mathrm{d}_{\alpha\alpha} \simeq g \begin{cases} \dfrac{1}{v_T} & v < v_T \\ \dfrac{1}{v} & v > v_T. \end{cases} \quad \text{(II.1.5)}$$

The stabilized value of $v_i^2$ is found from the condition

$$v_i^2 \simeq \left(\frac{m}{M}\right) v_T^2,$$

as must be the case.

Let us now examine the case where the Maxwellian distribution is "shifted" in the accompanying system by $\langle \mathbf{v}' \rangle = \mathbf{\Delta}$: $f' = f_T(\mathbf{v}' - \mathbf{\Delta})$ (the error in the mean velocity of the electrons). If $\Delta < v_T$, then (II.1.4) and (II.1.5) remain correct, since the mean value of the frictional force $\langle \mathbf{F} \rangle = g\mathbf{\Delta}/mv_T^3$ does not contribute to $\bar{Q}_i$ (the shift $\Delta < v_T$ does not alter the characteristic of the friction which determines the decrement of the attenuation).

A quite different situation occurs if $\Delta \gg v_T$. Let the error $\mathbf{\Delta}$ be directed along the normal degree of freedom 1. Let us find the mean power of friction on this degree for small oscillations:

$$\bar{Q}_1 \simeq -g \frac{M}{m} \left\langle \frac{v_1(v_1 - \Delta)}{|v_1 - \Delta|^3} \right\rangle \simeq 2g \frac{M}{m} \frac{v_1^2}{\Delta^3} > 0 \quad \text{(II.1.6)}$$

Thus, $\bar{Q}_1$ reverses sign, and the oscillations start to increase for this degree. The "transverse" degrees of freedom remain stable:

$$(\bar{Q}_\alpha)_1 \sim -g \frac{M}{2m} \frac{v_\alpha^2}{\Delta^3}, \qquad \langle \Delta I_i \rangle_{f\ell} \sim \frac{g}{\Delta}. \quad \text{(II.1.7)}$$

The reason for the appearance of the instability in the direction $\mathbf{\Delta}$ (here it is an important fact,

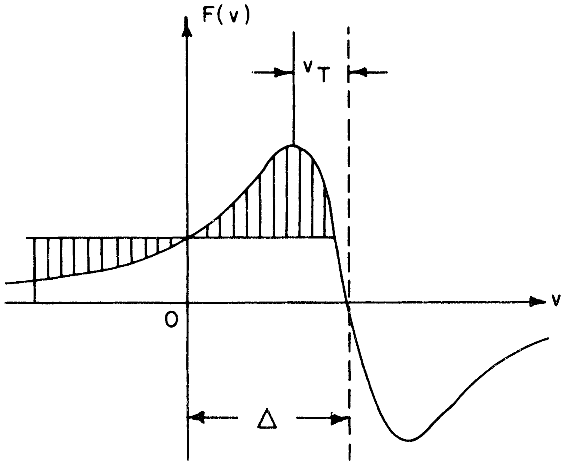


FIGURE 1 Graph of the frictional force, corresponding to the distribution $f'(\mathbf{v}) \sim \exp[-(\mathbf{v}' - \mathbf{\Delta})^2/v_T'^2]$. The power $\bar{Q}_1 = \overline{F_1 v_1}$ is positive for all amplitudes $v_1^0 < \Delta$.

however, that $\mathbf{\Delta}$ is directed along the normal degree of freedom) is that when there is a large shift in the mean velocity $\Delta > v_T$, small oscillations enter into the region where the characteristic friction is negative, and a build-up of oscillation energy occurs (Figure 1). In spite of the evaluation (II.1.6) being based upon the condition $\Delta \gg v_T$, it is clear that for the occurrence of the instability of small oscillations, it is sufficient that the sign of the characteristic of friction changes when the shift takes place. For degrees of freedom which are lateral to $\mathbf{\Delta}$, the appearance of an error is equivalent to a rise in the temperature of the electron beam in the relation $(\Delta/v_T)^2$, whereas the characteristic of friction remains positive.

In the condition $\Delta \gg v_T$, if small oscillations ($v < \Delta$) are considered, the nonmonochromaticity of the electron beam may be ignored. The instability which occurs then has a simple interpretation: there is a pendulum around which flows a "wind," the frictional force having a negative characteristic and being small in relation to the elastic force. The oscillations then occur around an almost unchanged position of equilibrium ($X = 0$), but become unstable: the energy build-up over the half-period of movement "with the wind" exceeds the losses during movement "against it."

Let us note that from (II.1.6) and (II.1.7) it follows that the sum of oscillation decrements is zero if they are defined as $\tau_i^{-1} = -\bar{Q}_i/I_i$:

$$\tau_1^{-1} + \tau_2^{-1} + \tau_3^{-1} \simeq \left(\frac{g}{Mm\Delta^3}\right)(-2 + 1 + 1) = 0. \tag{II.1.8}$$

This approximate result is a specific instance of the general theorem established in Section 4.

Let us now evaluate the stabilized mean amplitudes. As can be seen from Figure 1 the buildup continues, in any case, until the amplitude value of the velocity $a$ approaches $\Delta$:

$$\frac{\Delta - a}{v_T} \sim \frac{v_T^2}{\Delta^2} \ll 1;$$

(as shown below, $u_\perp \sim v_T$).

As the amplitude continues to grow, the phase section for which $a \sin \psi_i > \Delta$ starts to give a contribution to power in which the frictional force has a different sign, and, when averaged, compensates the section for which $a \sin \psi_1 < \Delta$. In the range $|a - \Delta| \lesssim v_T$, power varies within the limits $\sim \pm |Q|_{\max}$, turning to zero at a certain point $a_s > \Delta$, $a_s - \Delta \sim v_T$.

One can evaluate $|Q|_{\max}$ in order of magnitude:

$$|Q|_{\max} \sim \frac{a_s |F|_{\max} M \delta\psi}{2\pi} = g\left(\frac{M}{m}\right)\frac{\Delta}{v_T^2}\left(\frac{v_T}{\Delta}\right)^{1/2} = g\frac{M}{mv_T}\sqrt{\frac{\Delta}{v_T}},$$

where $\delta\psi \sim (v_T/\Delta)^{1/2}$ is the time spent by a particle in the region $|v - \Delta| \sim v_T$. This value is confirmed by model calculations given in the appendix.

Thus, in the region $|a - a_s| \lesssim v_T$ oscillation amplitudes damp to $a_s$ with decrement

$$\delta \sim \frac{|Q|_{\max}}{M^2 v_T \Delta} \sim \frac{g}{mMv_T^2(v_T\Delta)^{1/2}}.$$

The diffusion rate of electrons is of the order of magnitude

$$\left(M^2 \frac{da^2}{dt}\right)_{f\ell} \sim \left(\frac{v}{a}\right)^2\left(\frac{dp^2}{dt}\right)_{f\ell} \sim \frac{g\delta\psi}{2\pi v_T} \sim \frac{g}{(v_T\Delta)^{1/2}}$$

and for the equilibrium spread of amplitudes we obtain

$$\langle(a - a_s)^2\rangle = \frac{1}{\delta}\left(\frac{da^2}{dt}\right)_{f\ell} \sim \left(\frac{m}{M}\right)v_T^2.$$

Thus, when detuning takes place, the oscillator equilibrium distribution is concentrated close to the amplitude $a_s \simeq \Delta$ with the same absolute spread of amplitudes as that at thermodynamic equilibrium. Since the relative spread is small,

$$\frac{\langle(a - a_s)^2\rangle}{a_s^2} \sim \frac{m}{M}\frac{v_T^2}{\Delta^2}.$$

This feature distinguishes dissipative "heating" from thermal heating. Energy is transferred to the oscillator, not from the thermal motion of the electron current, but from its orderly motion. Consequently, the amplitude distribution is also concentrated in a narrow range near the mean value.

For oscillations in a direction transverse to Δ,

$$(\bar{Q}_\alpha)_\perp \simeq -g\frac{M}{m}\frac{a_\alpha^2}{v_T^2(v_T\Delta)^{1/2}}.$$

and

$$(\Delta I_\alpha)_{f\ell} \simeq \left(\frac{g}{v_T\Delta}\right)^{1/2}$$

whence

$$\frac{\langle a_\alpha^2\rangle}{v_T^2} \sim \frac{m}{M}.$$

For the lateral degrees of freedom, practically the same amplitudes are established as in the case of thermodynamic equilibrium.

Let us now consider the case in which **Δ** has projections of the same order of magnitude on two or three normal degrees of freedom. For small oscillations,

$$\bar{Q}_\alpha \sim -g\frac{M}{m}\left\langle\frac{v_\alpha(v_\alpha-\Delta_\alpha)}{|\bar{\mathbf{v}}-\mathbf{\Delta}|^3}\right\rangle$$

$$\simeq -g\frac{M}{m}\frac{\Delta^2-3\Delta_\alpha^2}{\Delta^5}\overline{v_\alpha^2} \qquad \text{(II.1.11)}$$

The appearance of an error in the other degrees of freedom may thus compensate instability in the degree of freedom considered, since it is equivalent in its action to the increase in the temperature of the electron gas, as was pointed out above. The characteristic of force remains positive if $3\Delta_\alpha^2 < \Delta^2$, in spite of the fact that $\Delta_\alpha^2 > v_T^2$. At the same time, the sum of the decrements, as can be seen from (II.1.11), remains equal to zero. In reality, as will be shown later, when there is a large error $\Delta^2 \gg v_T^2$, it is difficult to avoid the instability (although theoretically this is possible) but if **Δ** exceeds $v_T$ in an insignificant manner (but in such a way that the small oscillations lie in the region where the characteristic is negative, if **Δ** is directed along the normal degree of freedom), then all of the oscillations will be attenuated on condition that $\Delta_1 \simeq \Delta_2 \simeq \Delta_3$. This anisotropy of attenuation in the direction **Δ** is explained by the existence of discrete directions of normal oscillations (nondegenerate three-dimensional oscillator).

The occurrence of instability in the case of a distribution of the form $\sim\exp[-(\mathbf{v}'-\mathbf{\Delta})^2/v_T^2]$ is a specific characteristic of oscillatory motion. If the motion is infinite (absence of self-phasing in synchrotron motion), instability does not occur, but there is an entrainment of a proton beam by electrons.

Although we have considered here the case in which the mean velocity of the electron current in the accompanying system is different from zero, this is not necessary for the occurrence of instability. For example, if we take $f'(\mathbf{v}')$ in the form of two extended Maxwellian distributions $\exp[-(\mathbf{v}'\pm\mathbf{\Delta})^2/v_T^2]$, then $\langle\mathbf{v}'\rangle = 0$, but the characteristic of friction, when $v = 0$, in the direction of the shift will be negative if $\Delta > v_T$, which leads to instability. In general, for the occurrence of instability it is essential that the energy of orderly motion in the electron current exceeds the thermal energy,† i.e. the distribution must be qualitatively different from the Maxwellian. An important characteristic here is that the stabilized value of the energy of the oscillator is $M/m$ times greater in order of magnitude than the energy of "orderly motion of the electron," since there is an equalization of speeds, but not of temperatures: $a_{st}^2 \simeq \Delta^2$.

Belyaev and Budker[4] also pointed out the case of spherical distribution

$$f'(v) \sim v_0^{-1}\delta(v^2 - v_0^2).$$

In this case, the frictional force and momentum transfer are equal to zero‡ if $v < v_0$ (field of a charged sphere) and heating of the proton beam occurs. This case clearly demonstrates the characteristics of the Coulomb interaction, although it appears, in practice, to be exceptional.

Let us also evaluate the attenuation rate in a case which is important in practice, when the error **Δ** oscillates with time. The oscillations may, for example, be due to oscillations in the accelerating voltage. Let $w(\mathbf{\Delta})$ be the distribution of probability of error:

$$\int w(\mathbf{\Delta})\, d^3\Delta = 1.$$

The mean probable value of the frictional force may

† This condition justifies, in fact, the designation of instability.

‡ More correctly, are decreased $L$ times as compared with the case for the Maxwellian distribution.

be written in the form:

$$\langle \mathbf{F} \rangle = -\frac{g}{m} \int \frac{\mathbf{v} - \mathbf{\Delta} - \mathbf{v}'}{|\mathbf{v} - \mathbf{\Delta} - \mathbf{v}'|^3} f'(v') w(\Delta) \, d^3v' \, d^3\Delta, \tag{II.1.13}$$

where $f'(\mathbf{v}')$ is the velocity distribution of the electrons with regard to the mean velocity $\langle \mathbf{v}' \rangle = \mathbf{\Delta}$, which is close in form to the Maxwellian: $\langle v'^2 \rangle \sim v_T^2$. The case $\langle \Delta^2 \rangle \gg v_T^2$ is of interest to us. If the oscillations occur in three dimensions then, obviously, for all degrees of freedom this is equivalent to an increase in thermal spread of the electrons up to a value of $\langle \Delta^2 \rangle$. (It is assumed that the distribution $w(\Delta)$ is bellshaped.) In the case of one-dimensional oscillations directed along a normal degree. for small oscillations of protons

$$\bar{Q}_1 \simeq -g \frac{M}{m} \frac{\bar{v}_1^2}{\langle \Delta^2 \rangle^{3/2}}.$$

The characteristic of friction thus remains positive although the effective temperature of the electron beam increases just as in the case of three-dimensional oscillations. For the remaining degrees,

$$(\bar{Q}_\alpha)_\perp \sim -g \frac{M}{m} \frac{\bar{v}_\alpha^2}{v_T^2 \langle \Delta^2 \rangle^{1/2}}.$$

This result is explained simply: in view of the sharp dependence of the "lateral" frictional force on the error $\sim \Delta^{-3}$, the basic contribution to the power is given by the section for which $\Delta \sim v_T$; the fraction of the time "passed" by the error on this section is equal to $v_T / \langle \Delta^2 \rangle^{1/2}$.

Let us note that for $\delta$-type oscillations ($w(\mathbf{\Delta}) = (\Delta_0/2)\delta(\Delta_\perp)\delta(\Delta_T^2 - \Delta_0^2)$), the integral in (II.1.13) gives the previous result (II.1.6), i.e. an instability, as must be the case.

Thus, the appearance of a variable error with a distribution $w(\mathbf{\Delta})$ in the case $\langle \Delta^2 \rangle \gg v_T^2$, is equivalent to the establishment of a stationary distribution of electrons $\omega(\mathbf{v}')$ (in the case of one-dimensional or two-dimensional oscillations the electron spread which is lateral to them remains equal to $v_T$.) Bearing this analogy in mind, we may extend the above qualitative criterion of instability (or warming up) also to the case of non-stationary velocity distribution of electrons

$$\overline{E^2} > \overline{(\Delta E)^2} = \overline{(E - \bar{E})^2}, \qquad \left(E = \frac{mv'^2}{2}\right),$$

where $(-)$ designates the averaging for the "instantaneous" distribution and for time. This condition is necessary but not sufficient. A strict necessary and sufficient condition is the formal requirement of negativity (equality to zero) of the characteristic of friction in the direction of a normal oscillation.

## 2 *Effects of Spatial Inhomogeneity*

Let us now examine the effect of a spatial inhomogeneity of the distribution of electrons $f'(\mathbf{p}', \mathbf{r})$ on the damping rate of small amplitudes. We will define spatial inhomogeneity by the gradients of the average velocity, temperature and density in the electron current, assuming that in the absence of gradients the proton motion is attenuated (in the moving system). For this, it is sufficient to represent

$$f' = f_T(\mathbf{v}' - \mathbf{\Delta}(\mathbf{r}))n(\mathbf{r}), \tag{II.2.1}$$

where $f_T$ is a distribution of the Maxwellian type with a temperature $T(\mathbf{r})$, and the error $\mathbf{\Delta}(\mathbf{r})$ does not exceed in order of magnitude the velocity $v_T$:

$$|\mathbf{\Delta}| < v_T. \tag{II.2.2}$$

In this case, the frictional force for low proton velocities $v < v_T$ has the form (see II.1.4)

$$\mathbf{F}(\mathbf{v}, \mathbf{r}) \simeq -\frac{g}{m} \frac{\mathbf{v} - \mathbf{\Delta}}{v_T^3}. \tag{II.2.3}$$

If we assume focusing to be homogeneous, we obtain, with the aid of expressions (I.1.13), (I.3.2) and (I.3.5):

$$\bar{Q}_r \sim +M(\overline{v_r F_r} + \omega_s \overline{r_b F_{\parallel}}) \simeq -\frac{v_r^2 n}{T^{3/2}} - \omega_s r_b^2 \overline{\frac{\partial}{\partial r} \frac{n\Delta}{T^{3/2}}}$$

$$\bar{Q}_r \sim -\frac{v_z^2 n}{T^{3/2}}; \tag{II.2.4}$$

where

$$r = r_c + r_b, \qquad r_c = \frac{v_{\parallel}}{\omega_s v_r^2}$$

$$r_b = a_r \cos \psi_r, \qquad v_r = -a_r v_r \omega_s \sin \psi_r. \tag{II.2.5}$$

In the longitudinal direction, it is sufficient to obtain a force $F_{\parallel}$ averaged for betatron oscillations:

$$F_{\parallel} \sim -\frac{v_{\parallel} - \Delta^0}{T^{3/2}} n + r_c \frac{\partial}{\partial r}\left(\frac{n\Delta}{T^{3/2}}\right), \tag{II.2.6}$$

where $\Delta^0 = \Delta|_{z=0}$ (in working conditions without an rf field it is necessary to assume $\Delta^0 = 0$, since the equilibrium velocity is determined from the condition $F_{\parallel} = 0$).

Taking (II.2.5) into account, the expressions (II.2.4) and (II.2.6) take the form

$$\bar{Q}_r \sim -\overline{v_r^2}\left[\overline{\left(\frac{n}{T^{3/2}}\right)} + \frac{1}{v_r^2\omega_s}\overline{\frac{\partial}{\partial r}\left(\frac{n\Delta}{T^{3/2}}\right)}\right] \quad \text{(II.2.7)}$$

$$F_{\parallel} \sim -v_{\parallel}\left[\overline{\left(\frac{n}{T^{3/2}}\right)} - \frac{1}{v_r^2\omega_s}\overline{\frac{\partial}{\partial r}\left(\frac{n\Delta}{T^{3/2}}\right)}\right]. \quad \text{(II.2.8)}$$

In this way, the axial oscillations are always damped if the condition (II.2.2) is fulfilled, whereas in (II.2.4) and (II.2.6), terms appear which are proportional to the gradient $F_{\parallel}$ in a radial direction on the equilibrium orbit. These terms are due to the coupling of radial and longitudinal motion, or, as is said, to the fact that the equilibrium orbit is closed, and produce decrements of damping which are identical in value but of different sign. From a comparison of (II.2.7) and (II.2.8), the condition of stability can be obtained

$$\left|\frac{\partial}{\partial r}\overline{\left(\frac{nr_\Delta}{T^{3/2}}\right)}\right| < \overline{\left(\frac{n}{T^{3/2}}\right)}, \qquad r_\Delta \equiv \frac{\Delta}{v_r^2\omega_s}. \quad \text{(II.2.9)}$$

Although the influence of spatial inhomogeneity disappears if $\Delta \equiv 0$, the condition $|\Delta| > v_T$, is not at all essential for the occurrence of a "gradient" instability, as in the case of "monochromatic" instability which was examined above. Let, for example $\partial/\partial r(n/T^{3/2}) = 0$. Then, it follows from (II.2.9) that instability is possible on condition that

$$\left|\frac{\partial\Delta}{\partial r}\right| > v_r^2\omega_s = \frac{dv_{\parallel}(r)}{dr}, \quad \text{(II.2.10)}$$

where $v_{\parallel}(r)$ is the azimuthal velocity as a function of the radial deviation on the trajectory of the proton. If $v_{\parallel} < v_T$, then (II.2.10) and (II.2.2) may be compatible, since intrinsically (II.2.10) denotes that $|\Delta| > |v_{\parallel}|$. In the case of $\Delta = \text{const}$, the condition of instability is

$$|\Delta| > \left|\overline{\left(\frac{n}{T^{3/2}}\right)}\Big/\frac{\partial}{\partial r}\left(\frac{n}{T^{3/2}}\right)\right|\frac{\mathrm{d}v_{\parallel}}{\mathrm{d}r} \equiv b_r\frac{\mathrm{d}v_{\parallel}}{\mathrm{d}r}, \quad \text{(II.2.11)}$$

where $b_r$ is the dimension of inhomogeneity if the relative variation $\delta\ln(n/T^{2/3}) \sim 1$. If the last two conditions are united, a general qualitative criterion of gradient instability can be formulated: on the dimension of radial inhomogeneity, the mean-radius value of the error $\Delta$ must exceed the variation of $v_{\parallel}(r)$. If there is no rf field, instability may arise only if there is a mean velocity gradient; for the bunched beam the gradients of density and temperature also contribute to the decrements if there is an error in velocity on the equilibrium orbit. It would appear in practice that the velocity gradient is the most dangerous.

Let us evaluate the maximum amplitudes achieved with gradient instability. For this, without assuming the smallness of $|\mathbf{v} - \mathbf{\Delta}|/v_T$, we shall take the force $\mathbf{F}$ in the form

$$\mathbf{F} \sim -\frac{\mathbf{v} - \mathbf{\Delta}}{[(v - \Delta)^2 + v_T^2]^{3/2}}n: \qquad \mathbf{\Delta} = \{0, \Delta_{\parallel}, 0\}.$$

If the radial betatron oscillations are unstable, we may consider that $v_{\parallel}^2 \ll v_T^2$, since the longitudinal motion will be damped (let $\Delta_{\parallel}^0 < v_T$). Then,

$$\bar{Q}_r \sim -\langle(v_r^2 + \omega_s r_0\Delta_{\parallel})(\Delta_{\parallel}^2 + v_r^2 + v_T^2)^{-3/2}\rangle,$$

$$\Delta_{\parallel} \simeq \Delta_{\parallel}^0 + r_0\frac{\partial\Delta_{\parallel}}{\partial r}$$

Let us take, for concreteness,

$$\frac{\partial v_T}{\partial r} = 0, \qquad \frac{\partial\Delta_{\parallel}}{\partial r} = \frac{\Delta'}{r_0},$$

where $r_0$ is the radial dimension of the beam. Bearing in mind the conditions (II.2.10) and (II.2.11), we may conclude that the stabilized amplitude $a_r > r_0$ is determined in the general case by the equality

$$\overline{v_r^2} \simeq \langle\Delta_{\parallel}^2\rangle.$$

In this case it is not essential that $\Delta_{\parallel}^2 < v_T^2$. In the case of instability of the synchrotron motion

$$F_{\parallel} \sim -(v_{\parallel} - \Delta_{\parallel})[(v_{\parallel} - \Delta_{\parallel})^2 + v_T^2]^{3/2},$$

$$\Delta_{\parallel} = \Delta_{\parallel}^0 + \frac{v_{\parallel}}{v_c^2\omega_s}\frac{\partial\Delta_{\parallel}}{\partial r}$$

If the motion is infinite, the "anti-damping" ceases when the radial deviation exceeds the dimension of the beam:

$$|v_{\parallel}| \sim r_0 v_r^2\omega_s.$$

For the bunched beam, the oscillations build up indefinitely. This is obvious if $\Delta_{\parallel}^0 = 0$ and the condition (II.2.10) is fulfilled. If, however $\partial\Delta_{\parallel}/\partial r = 0$, the condition (II.1.11) is then fulfilled for all amplitudes since the energy buildup occurs only at "small" velocities $|v_{\parallel}| < |\Delta_{\parallel}|$, when the particle trajectory passes through the beam.

3 *The Critical Level of Coherent Fluctuations and other Questions*

Let us evaluate the permissible level of fluctuations of the space charge of an electron beam, starting from the condition that the diffusion growth of the amplitude does not exceed the maximum power of Coulomb losses:

$$\langle \Delta I_i \rangle_{f\ell} < |\bar{Q}_i|_{\max},$$

where

$$\langle \Delta I_i \rangle_{f\ell} = \tfrac{1}{2}\langle (\Delta p_i)^2 \rangle, \; \bar{Q}_i = p_i F_i$$

Let us consider scattering on the fluctuations of an electric field (in the moving system), the correlation time of which is $\tau_c < \omega_i^{-1}$, and the spatial dimension (wavelength) is $\kappa^{-1} > v\tau_0$, where $v$ is the velocity of the protons and $\tau_0 = l/\beta c$ is the time taken by the proton to pass through the section of the orbit occupied by the electron beam. "Collisions" then occur instantaneously in relation to the periodic motion of the protons and the scattering cross section does not depend on the momentum. In a single passage, a proton gathers the momentum

$$\Delta \mathbf{p} = e\Delta \mathbf{E} \cdot \tau_0$$

whence

$$\langle (\Delta p_i)^2 \rangle = \frac{e^2[(\Delta E_i)^2]\tau_0^2}{T_0},$$

where $[(\Delta E_i)^2]$ denotes the statistical mean.

As the frictional force in the region $v > v_T$, decreases as $v^{-2}$, the maximum $|\bar{Q}_i|$ is reached when $v \sim v_T$ and in order of magnitude is equal to

$$|\bar{Q}|_{\max} \simeq \left(\frac{4\pi e^4 nLM\tau_0}{mv_T T_0}\right). \quad \text{(II.3.1)}$$

For a relative fluctuation, we obtain

$$\left(\frac{[(\Delta \mathbf{E})^2]}{E^2}\right)^{1/2}_{cr} \sim \left(\frac{\gamma\theta_e nVm}{LM}\right)^{-1/2},$$

where $L$ is the Coulomb logarithm, $\theta_e$ the angular spread of electrons in the laboratory system, $\gamma$ the relativistic factor, and $n$ and $V$ are the density and volume of the electron beam. Let us take, for example. the numerical values

$$L \simeq 20, \quad \gamma = 2, \quad \theta_e = 3 \times 10^{-3},$$
$$n = 10^8 \text{ cm}^{-3}, \quad V = 3 \cdot 10^3 \text{ cm}^3.$$

We then have $(\Delta E/E)_{cr} \sim 5 \times 10^{-3}$.

The type of fluctuation under consideration is apparently the most dangerous as far as the size of the scattering cross section is concerned.

Let us also make here an evaluation of the critical value of the density of the residual gas $n$,[1] scattering on which can be included in the over-all "fluctuation background," (I.1.18):

$$(n_0^{cr})_d \sim \left(\frac{LMn\tau_0}{L_z mZ^2\gamma^3\theta_e}\right),$$

where $Z$ is the charge of the nucleus, $L_z$ the corresponding Coulomb logarithm and the densities refer to the laboratory system.

For the condition without an rf field, the question may arise as to whether deceleration on the electrons of the residual gas will not prevail over the "entrainment" of the protons by the electron beam. With the aid of the expression for the force of friction on the residual gas (I.1.18), it is simple to obtain the relation

$$\left[\frac{(n_0^{cr})_{fz}}{(n_0^{cr})_d}\right] \sim \left(\frac{zmL_z}{\theta_e L_e M}\right),$$

where $L_e$ is the Coulomb logarithm for scattering on the electrons of the gas. In practice, this relation does not differ markedly from unity.

Finally, let us also make an evaluation of the lifetime of the protons with regard to the process of recombination (formation of atoms) in the electron beam. The relationship of $\tau_{rec}$ to the relaxation time $\tau$ is equal to:

$$\frac{\tau_{rec}}{\tau} \sim \frac{m}{M} L \frac{\sigma_R}{\sigma_{rec}} \sim \frac{m}{M} L \times \left\{\alpha(\gamma\beta\theta)^2 \ln\left[1 + \left(\frac{\alpha}{\gamma\beta\theta}\right)^3\right]\right\}^{-1}, \quad \text{(II.3.2)}$$

where $\sigma_R$ is the Rutherford cross section for wide-angle scattering:

$$\sigma_R \simeq \frac{4r_e^2}{(\gamma\beta\theta)^4},$$

$\sigma_{rec}$ is the total recombination cross section for all levels of the hydrogen atom, $\theta$ is the relative angular spread of particles in the laboratory system, and $\alpha$ is the fine-structure constant. Equation (II.3.2) is an interpolation formula[17] and gives the correct behaviour for small velocities $\gamma\beta\theta/\alpha \ll 1$ and large ones $1 < (\gamma\beta\theta/\alpha)^2 \lesssim 10$. For example, for $\theta = 3 \times 10^{-3}$, $\gamma = 2$,

$$\frac{\tau_{rec}}{\tau} \sim 3 \cdot 10^4$$

## 4 *The Kinetics of Small Amplitudes*

We shall now investigate the solution of the kinetic equation (I.3.9), assuming the condition that the total diffusion rate is much less than the critical level at which attenuation does not occur at all. Let us first consider the region of low velocities $v < v_T$, assuming also that the spatial inhomogeneity for the corresponding range of amplitudes can be characterized with a high degree of accuracy by the gradient $f(\mathbf{p}, \mathbf{r})$. As the characteristic scale of variation of the friction force $\mathbf{F}(\mathbf{v}, \mathbf{r})$ is of the order of $v_T$, the latter can be expanded in the series

$$F_\alpha(\mathbf{v}, \mathbf{r}) = (F_\alpha)_0 + \left(\frac{\partial F_\alpha}{\partial \mathbf{v}}\right)_0 \mathbf{v} + \left(\frac{\partial F_\alpha}{\partial \mathbf{r}_\perp}\right) \mathbf{r}_\perp .$$

We shall make a similar assumption for the quadratic fluctuations

$$\langle \Delta p_\alpha \Delta p_\beta \rangle = \langle \Delta p_\alpha \Delta p_\beta \rangle_0 + \cdots .$$

If we confine ourselves in these expansions to terms of the lowest order that make a nonvanishing contribution when $Q_i$ and $D_{i\kappa}$ are averaged over phases, we obtain after averaging

$$\bar{Q}_i = 2\lambda_i I_i \tag{II.4.1}$$

$$\bar{D}_{i\kappa} = 2\mu_i I_i \equiv D_i,\ i = \kappa; \qquad \bar{D}_{i\kappa} = 0,\ i \neq \kappa, \tag{II.4.2}$$

where (see I.3.3)

$$\lambda_r = \frac{1}{2M}\left[ -\overline{\left(\frac{\partial F_r}{\partial \vartheta_r}\right)_0} + \overline{\frac{\psi}{\omega_s}\left(\frac{\partial F_\parallel}{\partial r}\right)_0} + \gamma \overline{\frac{d\psi}{d\theta}\left(\frac{\partial F_\parallel}{\partial \vartheta_r}\right)_0} \right] \tag{II.4.3}$$

$$\lambda_\parallel = -\frac{1}{2M}\left[ \overline{\left(\frac{\partial F_\parallel}{\partial \vartheta_\parallel}\right)} + \overline{\frac{\psi}{\omega_s}\left(\frac{\partial F_\parallel}{\partial r}\right)} + \gamma \overline{\frac{d\psi}{d\theta}\left(\frac{\partial F_\parallel}{\partial \vartheta_r}\right)} \right] \tag{II.4.4}$$

$$\lambda_z = -\frac{1}{2M}\overline{\left(\frac{\partial F_z}{\partial v_z}\right)}$$

$$\mu_z = \frac{1}{2}\overline{|f_r|^2 \langle (\Delta p_r)^2 \rangle} + \frac{\gamma^2}{2}\overline{|\zeta|^2 \langle (\Delta p_\parallel)^2 \rangle} + \overline{\gamma Im(f_r^* \zeta)\langle \Delta p_r \Delta p_\parallel \rangle_0} \tag{II.4.5}$$

$$\mu_\parallel = \tfrac{1}{2}\overline{\langle (\Delta p_\parallel)^2 \rangle}, \qquad \mu_z = \tfrac{1}{2}\overline{|f_z|^2 \langle (\Delta p_z)^2 \rangle}.$$

Here

$$\zeta = v_r f_r \psi - i\left(\psi \frac{df_r}{d\theta} - f_r \frac{d\psi}{d\theta}\right).$$

The physical meaning of the coefficients $\mu_i$ is clear from the relation $\langle \Delta I_i \rangle_{f\ell} = \frac{1}{2}\partial D_{i\kappa}/\partial I_\kappa$, whence $\langle \Delta I_i \rangle_{f\ell} = \mu_i$. In the condition without an rf field, it is necessary to take $D_\parallel = 2\mu_\parallel$, since $I_\parallel = p_\parallel$.

The values of $\lambda_i$ represent damping decrements of the phase-space volume for normal degrees of freedom. Equation (II.4.4) coincides with the usual determination of the decrement and for the coasting beam

$$\langle \Delta p_\parallel \rangle = -2\lambda_\parallel p_\parallel .$$

The energy spread, however, is attenuated in this case at twice as fast a rate as in the oscillation condition

$$\frac{d}{dt} p_\parallel^2 = -4\lambda_\parallel p_\parallel^2 .$$

This difference has a simple physical meaning: the specific heat of the oscillators is twice that of the free particles.

The decrements of radial and longitudinal motion include terms which are proportional to the derivative of $F_\parallel$ with respect to the velocity and coordinate in the radial direction. The influence of the terms $\sim \partial F_\parallel/\partial r$ has been discussed in detail above. The appearance of the terms $\sim \partial F_\parallel/\partial v_r$ is associated with the modulation of the velocity $v_r$ by synchrotron motion, in the case of azimuthal inhomogeneity of focusing and curvature of the equilibrium orbit. Qualitatively, their role may be important, provided the error $\Delta$ has nonzero components $\Delta_r \sim \Delta_\parallel \gtrsim v_T$.

As can be seen from the expressions for the decrements, the sum $\lambda_r + \lambda_\parallel$ does not depend on the coupling of radial and longitudinal motion. We also know of a similar result for decrements of radiation damping in the theory of accelerators.[14] For the full sum, a remarkable relation occurs†

$$\lambda_r + \lambda_\parallel + \lambda_z = -\frac{1}{2M}\operatorname{div}_v F = \frac{8\pi^2 e^4 L}{mM}\bar{f}_0, \tag{II.4.6}$$

† This relation, all considered, on the whole is valid with an accuracy up to terms due to Coulomb-logarithm dependence on the relative particle velocity.

(I.1.14)–(I.1.6) where $\overline{f_0}$ is the mean value of the distribution function of the electrons $f(\mathbf{v}, \mathbf{r})$ on the equilibrium trajectory of protons.

Thus the sum of decrements does not depend on the "orientation" of the anisotropic velocity distribution and is determined only by its size. This conclusion is general irrespective of the value and sign of the value and sign of the individual decrements and the shape of $f(\mathbf{v}, \mathbf{r})$.

For an isotropic spatially homogeneous distribution of electrons

$$\lambda_r = \lambda_{\|} = \lambda_z = \lambda = \frac{8\pi^2 e^4 L}{3mM}\overline{f_0} \simeq \frac{8\pi^2 e^4 Ln}{3mMv_T^3}, \tag{II.4.7}$$

which corresponds to the usual formula for the relaxation time of the plasma, when the velocities of the ions $v$ are less than those of the electrons $v_T$.[1–3]

From (II.4.6) in particular, it follows that all $\lambda_i$ can remain positive even if the error $\Delta \gg v_T$. In this case, however, their value becomes quite small, since it is proportional to the "tail" of the distribution. Thus, for the Maxwellian distribution $\sum \lambda_i \sim \exp(-\Delta^2/v_T^2)$. This conclusion agrees with the results of the approximate investigation performed in Section 1.

Let us also note that in conditions of spatial homogeneity, the values of the decrements cannot, exceed the value (II.4.7) in order of magnitude. But in the case of a strong spatial inhomogeneity, as follows from the results of Section 2, the value $|\lambda_r - \lambda_{\|}|$ can become much greater than $\lambda$. Naturally, the frictional power cannot in any circumstances exceed the maximum value (II.3.1).

Let us construct a solution of the kinetic equation

$$\frac{\partial}{\partial t} f - \sum_i \frac{\partial}{\partial I_i}\left(2\lambda_i I_i f + \mu_i I_i \frac{\partial}{\partial I_i} f\right) = 0, \tag{II.4.8}$$

assuming the condition that all the $\lambda_i > 0$. By the method of separation of variables $f = \prod_i f_i(I_i)e^{-\chi_i t}$, we obtain the equation (for brevity we shall leave out the index $i$)

$$\chi f + \frac{d}{dI}\left(2\lambda I f + \mu I \frac{df}{dI}\right) = 0,$$

which, by means of the substitutions $I = (\mu/2\lambda)x$, $f = ye^{-x}$ is brought to the equation for Laguerre polynomials:[18]

$$xy'' + (1 - x)y' + \alpha y = 0;$$

$$\alpha = \frac{\chi}{2\lambda} = n = 0, 1, 2, \ldots.$$

The normal solution of the equation (II.4.8) is

$$f_{\{n\}} = \prod_i L_n(x_i)\exp(-x_i - 2n_i\lambda_i t).$$

The general solution can be expressed by the fundamental solution or by the Green's function

$$G(I\,|\,I', t) = \prod_i g_i(x_i\,|\,x_i', t),$$

where

$$g_i = e^{-x}\sum_{n=0}^{\infty} L_n(x_i)L_n(x_i')e^{-2n\lambda_i t}$$

$$= \frac{\exp\left(-\dfrac{x_i - x_i' e^{-2\lambda i t}}{1 - e^{-2\lambda i t}}\right)}{1 - e^{-2\lambda i t}} I_0\left(\frac{\sqrt{x_i x_i'}}{sh(\lambda_i t)}\right), \tag{II.4.9}$$

where $I_0$ is the Bessel function of imaginary argument.

The general solution is then

$$f(I, t) = \int d^3I' G(I\,|\,I', t) f(I', 0).$$

A direct check will confirm that

$$g(x\,|\,x', 0) = \delta(x - x'), \qquad g(x\,|\,x', \infty) = e^{-x}$$

$$\int g(x\,|\,x', t)\,dx' = 1.$$

The equilibrium distribution and evolution of the mean amplitudes can also be obtained directly from (II.4.8):

$$f_{st} = \prod_i \exp\left(-\frac{I_i}{I_{is}}\right), \qquad I_{is} = \frac{\mu_i}{2\lambda_i}$$

$$\frac{d}{dt}\langle I_i\rangle = -2\lambda_i I_i + \mu_i,$$

as must be the case.

In the condition without an rf field, the solution of the equation for $f(p_{\|})$,

$$\chi_{\|} f + \frac{d}{dp_{\|}}\left(2\lambda_{\|} p_{\|} f + \mu_{\|}\frac{df}{dp_{\|}}\right) = 0;$$

$$p_{\|} = (\mu_{\|}/\lambda_{\|})x$$

is $f_n = e^{-x^2}H_n(x)$,[18] where $H_n$ is a Hermite polynomial. The fundamental solution[19] is

$$g(x\,|\,x', t) = e^{-x^2}\sum_n H_n(x)H_n(x')e^{-2\lambda n t}$$

$$= \frac{1}{2\sqrt{\pi}}\left[\sqrt{\mathrm{cth}(\lambda t)}\exp\left[-\frac{(x - x'e^{-2\lambda t})^2}{1 - e^{-4\lambda t}}\right] + \sqrt{\mathrm{th}(\lambda t)}\exp\left(-\frac{(x + x')e^{-2\lambda t}}{1 - e^{-4\lambda t}}\right)\right]. \tag{II.4.10}$$

The equilibrium solution is $e^{-x^2}$, and the evolution of $\langle p_\parallel^2 \rangle$ is determined by the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle p_\parallel^2 \rangle = -4\lambda \langle p_\parallel^2 \rangle + 2\mu_\parallel .$$

Knowledge of the fundamental solution makes it possible, if necessary, to obtain directly the evolution from the initial distribution to a state of equilibrium. Let us take, for example,

$$f(I, 0) = I_0^{-1} \exp\left( -\frac{I}{I_0} \right).$$

If we integrate $f(I, 0)$ with the Green function (II.4.9), we obtain[19]

$$f(I, t) = I^{-1}(t)\exp\left( -\frac{I}{I(t)} \right),$$

$$I(t) = I_s(1 - e^{-2\lambda t}) + I_0 e^{-2\lambda t},$$

i.e., the shape of the distribution is retained, and only $\langle I \rangle$ is changed. A similar result is obtained by means of (II.4.10) in the case of the initial distribution

$$f(p_\parallel, 0) = (\sqrt{2\pi} p_0)^{-1} \exp\left( -\frac{p_\parallel^2}{2p_0^2} \right):$$

$$f(p_\parallel, t) = (\sqrt{2\pi} p(t)^{-1} \exp\left( -\frac{p_\parallel^2}{2p^2(t)} \right),$$

$$p^2(t) = \frac{\mu_\parallel}{\lambda_\parallel}(1 - e^{-4\lambda\|t}) + p_0^2 e^{-4\lambda\|t}.$$

## 5 *Evaluation of a Solution in the Region of Large Amplitudes*

By the term "large amplitudes" we shall understand the general case, when the kinetic coefficients cannot be linearized in the variables $I_i$. This may be due basically to the nonlinear behaviour of the frictional force at velocities $v > v_T$ or to a strong spatial inhomogeneity of the electron beam (for example, when the amplitudes of the oscillations exceed the lateral dimension of the beam).

We shall first examine the nature of the kinetic process in conditions of spatial homogeneity, ignoring the azimuthal inhomogeneity of focusing. In the range $v^2 > v_T^2$, the coefficients $Q_i$ decrease as $v^{-1}$ (or faster, $\sim v_i^2/v^3$, if $v_i^2 \ll v^2$). In the presence of a fluctuating background (we shall assume the diffusion speed on this to be constant $\langle \Delta I_i \rangle_{f\ell} = \text{const} = \mu_i$) which considerably exceeds the thermodynamic level of fluctuations in the electron beam, for sufficiently large amplitudes

$$|\bar{Q}_i| < \mu_i,$$

and the particles are not captured in the damping conditions. Let us evaluate the region of captured amplitudes in the condition $|\bar{Q}|_{\max} \gg \mu$. For one-dimensional oscillations,

$$|\bar{Q}_i| \simeq 4\pi L e^4 n \overline{(v_0 |\sin\psi|)^{-1}}$$
$$\simeq \frac{2}{\pi}\frac{v_T}{v_0} |\bar{Q}|_{\max} \cdot \ln\left(\frac{v_0}{v_T}\right)$$

where $v_0$ is the amplitude of the velocity,

$$|\bar{Q}|_{\max} \simeq \frac{4\pi e^4 L n M}{m v_T} \frac{\tau_0}{T_0}.$$

Thus,

$$\left(\frac{v_0^{cr}}{v_T}\right) \simeq \frac{2}{\pi}\frac{|\bar{Q}|_{\max}}{\mu} \ln\left(\frac{|\bar{Q}|_{\max}}{\mu}\right).$$

For two- or three-dimensional oscillations the expression in the integral when averaging for phases has no singularities. Consequently

$$\left(\frac{v_0^{cr}}{v_T}\right) \simeq \left(\frac{|\bar{Q}|_{\max}}{\mu}\right).$$

Accordingly, the time taken to pass through the region of large amplitudes in the damping conditions $v_T < v_0 < v_0^{cr}$ is equal to

$$\tau_{(1)} = \frac{\tau}{\ln(v_{in}/v_T)}$$

in the one-dimensional case; also

$$\tau_{(2)} = \left(\frac{\chi}{\ln(1 + \sqrt{2})}\right)\tau \qquad \frac{2}{\pi} < \chi < 1$$

for the two-dimensional case, if $v_{in}^1 = v_{in}^2$, where

$$\tau = \left(\frac{m M v_{in}^3 T_0}{24 n L e^4}\right)$$

and $v_{in} \gg v_T$ is the initial amplitude of the velocity along one degree of freedom. In the three-dimensional case $\tau_{(3)}$ also differs from $\tau$ by a numerical factor which is close to unity.

The over-all picture of the movement of the amplitudes, which is described approximately by the equations

$$\frac{\mathrm{d}I_i}{\mathrm{d}t} = \bar{Q}_i + \mu_i \equiv \langle \Delta I_i \rangle,$$

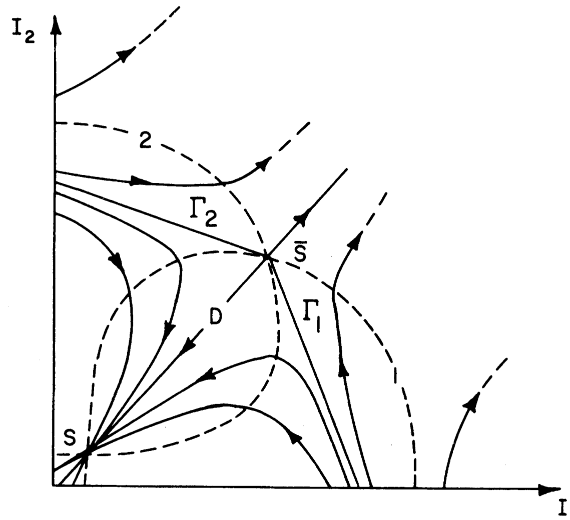


FIGURE 2 Trajectory of movement of amplitudes

$$\dot{I}_1 = \bar{Q}_1 + \langle \Delta I_1 \rangle_{fl}, \qquad \dot{I}_2 = \bar{Q}_2 + \langle \Delta I_2 \rangle_{fl}.$$

is fairly complex, but accessible to a qualitative analysis. The nature of the process is illustrated by Fig. 2, showing the trajectories of two-dimensional movement

$$\left(\frac{dI_s}{\langle \Delta I_s \rangle}\right) = \left(\frac{dI_2}{\langle \Delta I_2 \rangle}\right)$$

when $I_3$ is constant (or equal to zero). The dotted curves 1 and 2 correspond to the equations $\langle \Delta I_1 \rangle = 0$ and $\langle \Delta I_2 \rangle = 0$. A simultaneous damping of the amplitudes occurs only in the region of $D$ which is bounded by these curves. The curves $\Gamma_1$ and $\Gamma_2$ bound the region of captured amplitudes. As can be seen from the diagram, when there is a strong excitation of one degree in the region of capture the other degree at first "warms up," after which the trajectory passes into the region $D$ where both amplitudes are damped. The points $S$ and $\bar{S}$ correspond to the stable and unstable positions of equilibrium. When the third degree of freedom is "switched on," the figure can be considered as a projection of the three-dimensional picture on to a plane. The region $D$ is transformed into a "cocoon," and the over-all nature of the movement is unchanged.

Strictly speaking, a stationary distribution does not exist, since the region of captured amplitudes is limited. It is, however, possible to talk of a quasi-stationary distribution and life of the particles in the region of capture (or in the region of permissible amplitudes $I < I_{per}$), if $|\bar{Q}|_{\max} \gg \mu$. The "equilibrium" distribution is found from the equation

$$\sum_d \frac{\partial}{\partial I_i}\left[\bar{Q}_i f - \mu_i I_i \frac{\partial}{\partial I_i} f\right] = 0.$$

The solution can be found in the general form, if $\mu_1 = \mu_2 = \mu_3 = \mu$, and if advantage is taken of the property of the frictional force $\mathbf{F} = -(\partial/\partial v)U$ (I.1.14). As $v_i = (\sqrt{2I_i/M})\sin\psi_i$, then in the conditions considered here,

$$\bar{Q}_i = -\overline{\frac{p_i\,\partial U}{\partial v_i}} = -2MI_i \frac{\partial}{\partial I_i}\bar{U}.$$

If we assume $j_i = 0$, we obtain the equations

$$2M\frac{\partial U}{\partial I_i} f + \mu \frac{\partial f}{\partial I_i} = 0,$$

which have a joint solution

$$f = C\exp\left(-\frac{2M\bar{U}}{\mu}\right). \qquad \text{(II.5.1)}$$

In accordance with what has been said above, this solution cannot be normalized, since $U \to$ const at infinity. Its use has a meaning if, in the interval $0 \leq I \leq I_{per}$, the large majority of the particles are concentrated in the region $I \ll I_{per}$. The index of the exponential can be written in the form

$$-\frac{2M}{\mu}U = 2\frac{|\bar{Q}|_{\max}}{\mu}\left\langle\frac{v_T}{u}\right\rangle,$$

where $u = |v - v'|$, and $\langle\cdots\rangle$ denotes averaging over the distribution of the electrons. If the distribution is close to Maxwellian, the solution in the region $v < v_T$ is

$$f \sim \exp\left[-\frac{2\lambda}{\mu}(I_1 + I_2 + I_3)\right],$$

where $\lambda$ coincides with the expression (II.4.7). The "normalized" solution when $|\bar{Q}|_{\max} \gg \mu$ has the form

$$f = \left(\frac{2\lambda}{\mu}\right)^3 \exp\left[2\frac{|\bar{Q}|_{\max}}{\mu}\left\langle\frac{v_T}{u} - \frac{v_T}{v^1}\right\rangle\right]. \qquad \text{(II.5.2)}$$

The solution (II.5.2) can in fact be used for evaluating the "tail" of the distribution even when the $\mu_i$ are sharply different in value, if we simply assume for degrees of freedom with small $\mu_i$ that $v_i = 0$.

Let us note that from (II.5.1) it follows that, in accordance with the evaluation in Section 1, when an error $|\Delta| > v_T$ is present, the distribution near $I = I_{st}$ is Gaussian:

$$f \sim \exp\left[-(I - I_{st})^2\left(\frac{\partial \bar{Q}}{\partial I}\right)_{st}\Big/2I_{st}\right], \qquad (\bar{Q}(I_{st}) = 0).$$

It is easy to evaluate that $\langle(I - I_{st})^2\rangle \ll I_{st}^2$, i.e., the distribution in the case of "monochromatic" instability is concentrated in the proximity of $I_{st}$.

Finally, let us discuss the dependence of the damping rate on the transverse dimensions of the electron beam. Let the beam be situated symmetrically in relation to the equilibrium orbit of the protons. In the case of excitation of two-dimensional betatron oscillations, a decrease in the transverse dimensions $b_r$, $b_z$ always proves to be advantageous, since the power, here, "builds up" independently of the dimensions, at velocities $|v_i| \sim v_{i0}$, and the product of the density $n$ and the fraction of the phases, when the particles are situated in the beam, does not, in any case, decrease. Consequently, at a fixed current, the integral time of attentuation decreases with reduction of the dimensions.

For one-dimensional oscillations $v_0 \gg v_T$, when the dimension is reduced in the direction of the oscillations from a value $b = a$ (oscillation amplitude) to a certain $b < a$, the power decreases in the ratio $\ln(v_0/v_T)$, since at small velocities $|v| \sim v_T$ which give the basic contribution for $b \gtrsim a$, the particle is outside the beam

$$(v = -v_0 \sin \psi, \qquad r = a \cos \psi).$$

The situation is different when synchrotron motion is excited. For the coasting beam, the effect of bypassing the beam is manifest when synchrotron deflection of the radius exceeds $b_r$. This effect leads to a sharp decrease in the power $\bar{Q}_c$ in the condition of oscillations too since $r_c \sim v_{\|}$. It can easily be evaluated that for amplitudes of $v_0 \lesssim v_T$, the power decreases in the relation $\eta \sim \overline{r_c^2}/b_r^2$ when $b_r$ is reduced from the value $b_r^2 = \overline{r_c^2}$. If,

$$1 < \frac{v_0^2}{v_T^2} < \frac{\overline{r_c^2}}{b_r^2}$$

then

$$\eta = \frac{v_0^2}{v_T^2}\frac{b_r^2}{\overline{r_c^2}} > 1.$$

The radial dimension of the beam of electrons must thus be kept at a level of $b_r^2 \sim \overline{r_c^2}$.

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# Appendix

Let us find a mean (for an oscillation period) frictional power in the presence of "the proton and electron mean velocities detuning." The frictional-force dependence on the velocity is taken in the form

$$\mathbf{F}(\mathbf{v}) = -\frac{g}{m}\frac{(\mathbf{v}-\mathbf{\Delta})}{[(\mathbf{v}-\mathbf{\Delta})^2+v_T^2]^{3/2}}.$$

Let us consider a situation when detuning $\mathbf{\Delta}$ is directed along one of three normal (and space-orthogonal degrees of freedom). Then, for investigation of an equilibrium state of a proton beam, one can take the proton velocity transverse to $\mathbf{\Delta}$ to be small in comparison with $v_T$:

$$v_\perp \ll v_T,$$

and for degrees of freedom in the direction $\mathbf{\Delta}$

$$\frac{1}{2}\left\langle\frac{d}{dt}Ma^2\right\rangle = -\frac{g}{\pi m}\int_{-a}^{a}\frac{v-\Delta}{[(v-\Delta)^2+v_T^2]^{3/2}} \times \frac{vdv}{\sqrt{a^2-v^2}} \equiv -\frac{g}{\pi m}J. \quad \text{(A-1)}$$

We are interested in the case $|a-\Delta| \ll \Delta$. Bearing in mind this condition, one can substitute for the integral in (A.1) an approximate one:

$$J \simeq -\sqrt{\frac{\Delta}{2}}\int_{-\infty}^{a-\Delta}\frac{xdx}{(x^2+v_T^2)^{3/2}\sqrt{a-\Delta-x}}.$$

Introduce a designation $\xi = (a-\Delta)/v_T$ and rewrite $J$ in the form

$$J = -\sqrt{\frac{\Delta}{2v_T^3}}\cdot\int_{-\infty}^{\xi}\frac{xdx}{(x^2+1)^{3/2}\sqrt{\xi-x}}.$$

For the final evaluation of the integral it is convenient to transform it to the form

$$J = \sqrt{\frac{\Delta}{2v_T^3}}\left[\int_{\xi}^{\infty}\frac{xdx}{(x^2+1)^{3/2}\sqrt{\xi+x}} = -2\int_0^{\xi}\frac{x^2dx}{(x^2+1)^{3/2}\sqrt{\xi^2-x^2}\,x(\sqrt{\xi+x}+\sqrt{\xi-x})}\right. \quad \text{(A.2)}$$

It is clear that at $\xi = 0$, $J \sim 1$, while at $\xi \gg 1$

$$J \sim \sqrt{\frac{\Delta}{2v_T^3}}\left(\frac{1}{\sqrt{\xi}}\int_{\xi}^{\infty}\frac{dx}{x^2} - \frac{2}{\sqrt{\xi}}\int_1^{\xi}\frac{dx}{\xi x}\right) \sim -\sqrt{\frac{\Delta}{(a-\Delta)^3}}\ln\xi < 0.$$

At the same time, maximum values of each integral in (A.2) lie in the region $\xi \sim 1$ and in order of magnitude are equal to 1. That means, that the power $\bar{Q}$ in this region varies within the range $\sim \pm(gM/m)\sqrt{\Delta/v_T^3}$ and its derivative at the point $\bar{Q}(\xi) = 0$ is equal to

$$\frac{d\bar{Q}}{da} \sim -\frac{|\bar{Q}|_{\max}}{v_T} \sim \frac{gM}{mv_T^2}\sqrt{\frac{\Delta}{v_T}}.$$