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SIGNATURE, FACTORIZATION AND UNITARITY IN MULTI-REGGE THEORY : THE FIVE-POINT FUNCTION

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A B S T R A C T

We develop the multi-Regge theory of the five-point function from the Sommerfeld-Watson approach and emphasize the use that can be made of t-channel unitarity. We include signature systematically so that signature phase factors come out naturally and discontinuities can be taken in a straightforward way. We give a general method for defining Froissart-Gribov continuations. Using the unitarity relations they satisfy we prove factorization of the individual helicity amplitudes contributing to Regge pole residues. This is sufficient to prove factorization of the full amplitude and its discontinuities in multi-Regge limits as well as the factorization of discontinuities in helicity pole limits. We define Reggeon scattering amplitudes which are continuations of t-channel helicity amplitudes to complex helicity. We show that the t-channel unitarity relation for these amplitudes contains an extra term, besides the usual phase space integral, which vanishes when the trajectory function goes through integer points. We give a generalized crossing relation which relates s- and t-channel Reggeon amplitudes.

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1. INTRODUCTION

Multi-Regge asymptotic behaviour of multi-particle amplitudes is now so familiar that it may be thought that there can be little basic theory left to develop. However, until recently the only systematic model-independant development of the formalism has been based on the use of group-theoretic SO(2,1) expansions[1-5]. Although this approach defines the limits in which Regge behaviour will appear and the general form it will take, it does not go very far beyond this. Analyticity and unitarity can only be exploited to a limited extent. This is a particularly serious problem when inclusive reactions are considered since the discontinuity in a missing mass variable of a multi-particle amplitude must be taken to obtain a multi-Regge representation of the inclusive cross-section. Since multi-particle angular momentum theory may well be the most direct way to calculate asymptotic behaviour even in a complete theory it is clearly desirable to understand the implications of analyticity and unitarity.

Also it seems inevitable that the constraints imposed by analyticity and unitarity on a multi-Regge representation of a multi-particle amplitude will play a significant role in the future analysis of both inclusive and exclusive experimental data on multi-particle processes. In fact the lack of analytic structure in the multi-Regge representation given by the group theory approach is often used as justification for the use of the dual resonance model for multi-Regge phenomenology in situations where the duality properties of the model are not necessarily exploited[6,7].

A major purpose of the present paper is to show that the Sommerfeld-Watson[8-11] (S-W) approach to multi-Regge theory provides an alternative to the group-theoretic approach. The S-W approach has the obvious virtue that it is a straightforward generalization of the two-body Regge theory with which most people are familiar. It also has many other advantages over the group-theoretic approach and we can list some of the most important as follows:

i) t-channel unitarity (the t-channel being the resonance channel for a Regge pole) can be taken into account completely in the sense that the full multi-particle equations can be continued in the complex angular momentum plane[12] and all the anticipated constraints of unitarity in this channel can be derived. We can

a) prove factorization of Regge pole residues

b) show that Regge poles must be accompanied by cuts for which discontinuity formulae can be derived[13-15]. The discontinuities being given in terms of the Froissart-Gribov amplitudes that appear in the Sommerfeld-Watson transform.

c) define Reggeon scattering amplitudes as continuations to complex helicity of t-channel helicity amplitudes, as well as deriving the unitarity equations they satisfy.

We should perhaps also mention that t-channel unitarity requires that the same Regge trajectories which appear in two-body amplitudes also appear in multi-particle amplitudes and therefore (using the Mueller theorem[16]) in inclusive cross-sections. This point is sometimes questioned.

ii) We can give a complete description of asymptotic behaviour in all Regge and helicity-pole limits[8,17,18]. This includes

a) the helicity structure of vertex functions. This goes beyond their factorization properties and is a consequence of the analytic structure of amplitudes. The Steinmann relations[19] are vital for constructing multi-particle Froissart-Gribov continuations and as a result a structure for vertex functions consistent with these relations, emerges naturally from the Sommerfeld-Watson representation.

b) the phases of asymptotic contributions, that is signature factors

c) we emphasize that the structure of vertex functions in helicity-pole as well as Regge pole limits comes out straightforwardly.

iii) The analytic structure of the amplitude is built in and we can

a) take discontinuities easily so that the Mueller discontinuity formula[16] can be used to study the Regge behaviour of inclusive cross-sections

b) analytically continue to particle poles on a Regge trajectory in a well-defined way. The helicity structure of the Sommerfeld-Watson transform is particularly important here.

We shall not give a complete discussion of all the above points in this paper. In particular we shall say very little about Regge cuts since we have already discussed them extensively in previous papers[14,15]. Also we restrict attention almost entirely to the five-point function. This means that our discussion of helicity-pole limits, the factorization of discontinuities and Reggeon amplitudes, does not have the immediate physical relevance to inclusive cross-sections that is obtained by discussing higher amplitudes. Our reason for considering only the five-point function is that it is the simplest multi-particle amplitude. By presenting together the various aspects of multi-Regge theory for this simple case we hope to give an understanding of how more complicated amplitudes can be treated.

In fact many of the problems we consider are perhaps over-simplified by considering the five-point function. We give a systematic treatment of signature phase factors by relating boundary-values onto cuts in invariant variables to boundary-values for the angular variables of the Sommerfeld-Watson transform. This also enables discontinuities to be taken in a straightforward way. However, the five-point function has the special simplification that the gram-determinant constraints do not appear as they do for the higher multi-particle amplitudes. These constraints make any discussion of boundary-values onto cuts considerably more complicated[10,20,21]. Nevertheless, we anticipate that when the Steinmann relations[19] are used in conjunction with the Bergman-Oka-Weil theorem[9,22] to break the amplitude up into components with only certain combinations of simultaneous singularities, a simple discussion of boundary-values and signature factors will still be possible for the higher amplitudes. We expect that the breaking up of amplitudes in this way will be important for dealing with the problem of multiple helicity sums[9-11], which again only arises when considering amplitudes with at least six external particles.

Factorization is also particularly simple in the five-point function in that no problems arise with phase factors[20,21]. Again we anticipate that the factorization of generalized "helicity amplitudes" that we show follows from the factorization of four-particle amplitudes, will be the basic building block for proving the factorization of higher amplitudes and their discontinuities. The factorization of these helicity amplitudes (by helicity amplitudes we mean amplitudes where all helicity labels are linked to some angular momentum, which may be that of a Reggeon) is derived from the unitarity relations these amplitudes satisfy. Since the asymptotic behaviour in all Regge or helicity-pole limits is expressible in terms of what we call helicity amplitudes we can immediately show what factorization holds in any such limit. We hope that the generalization of our results to higher amplitudes will lead to the rules for the factorization of discontinuities recently given by Weis[23)] as well as providing a check on the non-factorization of the eight-particle amplitude in a helicity-pole limit recently suggested by Moen and Zakrzewski[24)].

The study of Reggeon scattering amplitudes now has the motivation that these amplitudes can be directly measured in inclusive experiments[25,26)]. An important point is that these amplitudes are defined through helicity-pole limits of inclusive cross-sections (or the corresponding multi-particle amplitude) which pick out a definite "t-channel" helicity for the Reggeon. We regard this as the correct definition of a Reggeon amplitude and distinguish it from that defined in terms of a Regge limit of a multi-particle amplitude. The amplitudes we consider reduce to physical t-channel helicity amplitudes when the trajectory goes through a physical integer point. They also satisfy generalized unitarity relations. The t-channel discontinuity of a Reggeon amplitude can be expressed as a normal unitarity phase-space integral plus an extra term which is an integral over real unphysical values of the momentum transfer variables. This extra term can be regarded as resulting from the "structure" of a Reggeon.

We show that we can also define "s-channel" Reggeon amplitudes which will satisfy a unitarity relation in the s-channel. The s- and t-channel Reggeon amplitudes are related through a generalization of the usual crossing relation for helicity amplitudes. This crossing relation can be expressed either in the form of a complex helicity integral or as an infinite sum over helicities. We find that there are some subtleties in the way the s- and t-channel pole structures of a Reggeon amplitude are related through the crossing transformation. That t-channel Reggeon amplitudes do not satisfy simple s-channel unitarity relations has important implications for the Reggeon amplitudes measured in inclusive experiments. We discuss why this is also important in comparing the s-channel derivation of cut discontinuities by Abarbanel[27,28)] with the t-channel derivation[13,15)].

We should note the overlap of part of the work presented in Sections 2 and 4 with the papers by Weis[10] and by Abarbanel and Schwimmer[11]. Weis has given representations of multi-particle amplitudes in terms of invariant variables which are based on Sommerfeld-Watson representations. The advantage of using representations in terms of invariants is that kinematic singularity problems do not appear and the representations can be compared with models in a straightforward way. In particular the connection with the dual resonance model is particularly striking and perhaps indicates the advantage of using the Sommerfeld-Watson representation instead of the dual resonance model for phenomenological analyses that are really testing only the Regge and analytic (and not the dual) properties of an amplitude. We prefer to use the full S-W representation because this involves partial-wave amplitudes whose unitarity and factorization properties we know and for which we have Regge cut discontinuity formulae. We can also explicity see the factorization into physical helicity amplitudes at particle poles. One topic which we discuss only briefly but which is extensively discussed by Weis[10] is nonsense wrong signature fixed-poles.

In section 2 we go over the development of the Sommerfeld-Watson transform given in Ref. 9 with the difference that we discuss signature more completely. We give a complete discussion of the representation of the two Reggeon/particle vertex that emerges from the Sommerfeld-Watson representation, both as a helicity integral and in the form of an asymptotic series in the Toller angle.

In section 3 we give a general method of defining Froissart-Gribov continuations which should be applicable to amplitudes with the general Landau singularity structure suggested by perturbation theory. This section is perhaps rather technical and certainly is not essential for understanding the other sections of the paper in which we assume the existence of Froissart-Gribov continuations. The method we give starts from a general integral representation of the full amplitude in terms of angular variables, which is derived from the Bergman-Oka-Weil theorem. Since this representation breaks the amplitude up into terms with only right- or left-hand cuts in the various invariant variables the concept of a signatured multi-particle amplitude[10] can be introduced using this representation and it is clear that this analytic definition of signature coincides with the group-theoretic definition[3]. As in Ref. 9 we are not able to rigorously define Froissart-Gribov amplitudes simultaneously satisfying the necessary Carlson condition in two angular momenta. However, we discuss why we expect that the asymptotic behaviour of the amplitude can be discussed using the full S-W transform. In section 4, we show that the asymptotic behaviour of the amplitude in all Regge and helicity-pole limits can then be expressed in terms of Froissart-Gribov amplitudes evaluated at points that can be rigorously reached by Carlson continuation. Therefore, the exact unitarity relations for these amplitudes can be used to prove factorization of the full amplitude and its discontinuities in the various asymptotic limits.

Section 5 is devoted to Reggeon amplitudes and contains the derivation of the unitarity equation for these amplitudes as well as the crossing relation for s- and t-channel amplitudes.

2. SIGNATURE IN THE SOMMERFELD-WATSON TRANSFORM

One of the obvious advantages that the S-W transform has over the group-theoretic approach to two-body Regge theory is the simple way in which the phase of a Regge pole contribution is simply given by the signature factor. This emerges naturally from the S-W transform because of the direct way that the S-W transform reflects the cut-plane analyticity of the amplitude. Since the transform is obtained from the t-channel partial-wave expansion the integral representation for the amplitude initially holds in a region where the amplitude is analytic and real (for t below threshold). Provided that the representation converges it can be continued away from this region and in particular can be used to study Regge asymptotic limits which are taken in regions close to singularities of the amplitude where, of course, the amplitude is no longer real. Since the Froissart-Gribov amplitudes are real, it follows that (provided the representation converges) the phase of the amplitude must be given by the phase of the Legendre functions in the representation. To determine this it is only necessary to determine how the limit has been taken with respect to the cuts of the Legendre functions. That is boundary-values on to the cuts of the full amplitude in the invariant variables have to be re-interpreted as boundary-values in the angular variables in which the cuts of the Legendre functions appear. This process is straightforward and familiar for four-particle amplitudes. However, we want to generalize it to five-particle amplitudes. We know that in this case the phase of Regge pole contributions is not quite so straightforward[29]. Nevertheless, we shall show that a careful treatment of signature in the S-W transform will give us the phase of the amplitude in a simple way.

To obtain the S-W representation of the five-point function we must introduce the "group theoretic" or angular variables which are the analogue of the t-channel centre-of-mass scattering angle for the four-point function. We introduce the variables corresponding to the tree diagram[2,9] of Fig. 2.1. Before discussing the physical significance of the angular variables we give the expressions which relate them to the invariant variables. We use a slightly different notation to that used in Ref. 9. We write $t_1 = Q_1^2$, $t_2 = Q_2^2$ and denote all other invariants by $(P_1 - P_5)^2 = S_{15}$, etc., so that

$$S_{15} = \frac{1}{2}\left(3m^2 - t_1 + t_2\right) + \left(\frac{t_1 - 4m^2}{4t_1}\right)^{\frac{1}{2}} \lambda^{\frac{1}{2}}(t_1, t_2, m^2)\, z_1 \tag{2.1}$$

with S_{25} given by $z_1 \to -z_1$, S_{35} by $t_1 \leftrightarrow t_2$, $z_1 \to z_2$ and S_{45} by $t_1 \leftrightarrow t_2$, $z_1 \to -z_2$. Also

$$\begin{aligned} S_{13} = 2m^2 &- \tfrac{1}{2}(t_1 t_2)^{\frac{1}{2}} \cosh\zeta - \tfrac{1}{2}\left(t_2(t_1 - 4m^2)\right)^{\frac{1}{2}} \sinh\zeta\, z_1 \\ &+ \tfrac{1}{2}\left(t_1(t_2 - 4m^2)\right)^{\frac{1}{2}} \sinh\zeta\, z_2 + \left((t_1 - 4m^2)(t_2 - 4m^2)\right)^{\frac{1}{2}} \times \\ &\times \left(\cosh\zeta\, z_1 z_2 - \cos\omega\, (1 - z_1^2)^{\frac{1}{2}} (1 - z_2^2)^{\frac{1}{2}}\right) \end{aligned} \tag{2.2}$$

where $\sinh\zeta = \lambda^{\frac{1}{2}}(t_1, t_2, m^2)/2(t_1 t_2)^{\frac{1}{2}}$. S_{23}, S_{14}, S_{24} are given by similar expressions to (2.2). The definition of z_1, z_2 and $\cos\omega$ through (2.1) and (2.2) is complicated. However, the physical significance of these variables is simple, at least in the physical regions where t_1 and t_2 are energy variables. In the physical region for the process $1 + 2 \to 3 + 4 + 5$, t_1 is the total energy and t_2 is a sub-energy variable We can then write $z_1 = \cos\theta_1$ where θ_1 is the centre-of-mass scattering angle for the process $P_1 + P_2 \to P_5 + Q_2$ and $z_2 = \cos\theta_2$ where θ_2 is the centre-of-mass scattering angle for the process $Q_1 - P_5 \to P_3 + P_4$. ω is the Toller angle between the two scattering planes. Similarly z_1 and z_2 are the cosines of scattering angles in the physical region for the process where t_2 is the total energy and t_1 is a sub-energy variable. We shall refer to channels where t_1 and t_2 are positive as direct channels.

In channels where t_1 and t_2 are both momentum transfer variables and are negative (for example the channel where $1 + 4 \to 2 + 3 + 5$) z_1, z_2 and $\cos\omega$ continue directly into the Bali, Chew and Pignotti variables[1]. That is z_1 and z_2 become hyberbolic cosines of boost variables, while ω remains an angular variable. In a six-particle amplitude where particle 5 can effectively have negative mass it is possible to go to physical regions where ω becomes a boost variable and asymptotic limits in which $\cos\omega \to \infty$ can be considered. (These are helicity-pole limits, of course). Note that real values of the invariants only corresponds to real values of z_1, z_2 and $\cos\omega$, when either $t_1, t_2 > 4m^2$ or $t_1, t_2 < 0$ because of the various square root branch-points in (2.1) and (2.2).

The partial-wave expansion of A_5 in the physical regions where t_1 and t_2 are positive is ($u = e^{i\omega}$)

$$A_5 = \sum_{\ell_1,\ell_2=0}^{\infty} \sum_{|n|\leq \ell_1,\ell_2} u^n P_{\ell_1}^{-|n|}(z_1) P_{\ell_2}^{-|n|}(z_2)\, a_{\ell_1\ell_2 n}(t_1,t_2) \tag{2.3}$$

We have discussed the problems of the convergence of this expansion in the presence of physical region singularities in Ref. 9. The simplest way to remove this problem and to make all our discussions of signature, etc., straightforward is to assume that we can analytically continue the amplitude as a function of the internal mass[9] to a point where there are no physical region singularities. (The problem only arises for the contribution of a small number of sub-energy threshold branch points which we do not expect to be of significance in determining the asymptotic behaviour of the amplitude so that we shall not discuss it in detail here.) We then have from (2.1) and (2.2) that for t_1 and t_2 above threshold we can keep away from cuts in S_{15}, S_{25}, S_{35} and S_{45} by keeping $|z_1|, |z_2| \leq 1$. We can then reach the cuts in S_{14}, S_{13}, S_{23}, S_{24} by taking $\cos\omega$ large. It then follows that the boundary-value for $\cos\omega$ is simply determined by the boundary-values onto the cuts in S_{14}, S_{13}, S_{23}, S_{24}. As a result we have the simple prescription that the take "physical" (i.e. + $i\varepsilon$) limits with respect to these cuts (for z_1 and z_2 real) we take $\cos\omega \to \cos\omega \pm i\varepsilon$ for $\cos\omega \to \pm\infty$. Having reached the cuts in S_{14}, etc., we can then approach the cuts in S_{15}, S_{25}, S_{35}, S_{45} simply by taking $|z_1|$, $|z_2| \to \infty$ so that physical limits with respect to these cuts correspond to $z_{1,2} \to z_{1,2} \pm i\varepsilon$ as $z_{1,2} \to \pm\infty$.

For t_1, $t_2 < 0$ the Froissart-Gribov coefficients which appear in the S-W transform will be real analytic and so the above boundary-values for the angular variables should be sufficient to determine the phase of Regge pole contributions to the asymptotic behaviour. As we shall show, this is essentially the case.

The S-W transform of the sum over n in (2.3) is straightforward[8,9]

$$A_5 = \sum_n a_n u^n \tag{2.4}$$

$$= \frac{-1}{2i} \sum_{\tau_3} \int \frac{dn}{\sin\pi n} \left[a_>^{\tau_3}(n) + a_<^{\tau_3}(n)\right]\left[(-u)^n + \tau_3 (u)^n\right] \tag{2.5}$$

where the $\gtrless$ labels refer to continuations made from $n \gtrless 0$, and τ_3 is a signature in n. The next step is to note[9] that

$$\hat{a}_{\gtrless}^{\tau_3}(n) = (1-z_1^2)^{\mp\frac{n}{2}}(1-z_2^2)^{\mp\frac{n}{2}}\; a_{\gtrless}^{\tau_3}(n)$$

are regular at $z_1, z_2 = \pm 1$ and have only "dynamic" (as opposed to kinematic) singularities in z_1 and z_2. Therefore, if we make an expansion of $\hat{a}_>^{\tau_3}$ and $\hat{a}_<^{\tau_3}$ in terms of Jacobi polynomials we can anticipate that inserting the correct signature factors in the S-W transform of this expansion will correctly reflect the phase of $\hat{a}_>^{\tau_3}$ and $\hat{a}_<^{\tau_3}$ at large z_1 and z_2. For $|z_1|$, $|z_2| < |$ it is clear that we can absorb $(1 - z_1^2)^{n/2}$ $(1 - z_2^2)^{n/2}$ into $(\pm u)^n$ and that since it is $(1 - z_1^2)^{\frac{1}{2}}$ $(1 - z_2^2)^{\frac{1}{2}}$ u which for large u is linearly related to the invariants, it is this quantity which should be given $\pm i\varepsilon$ boundary values in the physical limit. (We shall see later that only $\hat{a}_{>}^{\tau_3}(n)$ contributes to $u \to \infty$, while only $\hat{a}_{<}^{\tau_3}(n)$ contributes to $u \to 0$.)

Therefore, we rewrite the helicity transform (2.5) as

$$A_5 = \frac{-1}{2i}\sum_{\tau_3}\int \frac{dn}{\sin\pi n}\Big\{\hat{a}_{>}^{\tau_3}(n)\Big[\big(-(1-z_1^2)^{\frac{1}{2}}(1-z_2^2)^{\frac{1}{2}}u\big)^n + \tau_3\big((1-z_1^2)^{\frac{1}{2}}(1-z_2^2)^{\frac{1}{2}}u\big)^n\Big]$$

$$+\ \hat{a}_{<}^{\tau_3}(n)\Big[\big(-(1-z_1^2)^{-\frac{1}{2}}(1-z_2^2)^{-\frac{1}{2}}u\big)^n + \tau_3\big((1-z_1^2)^{-\frac{1}{2}}(1-z_2^2)^{-\frac{1}{2}}u\big)^n\Big]\Big\} \qquad (2.6)$$

Now we write

$$\hat{a}_{>}^{\tau_3}(n) = \sum_{\ell_1 - n = 0}^{\infty}\ \sum_{\ell_2 - n = 0}^{\infty} P_{(\ell_1-n)}^{(n,n)}(z_1)\, P_{(\ell_2-n)}^{(n,n)}(z_2)\, a_{\ell_1\ell_2}^{\tau_3}(n) \qquad (2.7)$$

where $P_{(\ell-n)}^{(n,n)}(z)$ is a Jacobi polynomial of order $(\ell - n)$. [Note that since $P_{(\ell-n)}^{(n,n)}(z)\left(\frac{1-z^2}{2}\right)^{n/2} = P_\ell^{-n}(z)$, (2.7) can be inserted into (2.4) to regain (2.3).] If we now assume that we can SW transform (2.7) we obtain

$$\hat{a}_{>}^{\tau_3}(n) = -\frac{1}{4}\sum_{\tilde{\tau}_1,\tilde{\tau}_2}\int \frac{d\ell_1\, d\ell_2}{\sin\pi(\ell_1-n)\sin\pi(\ell_2-n)}\Big[P_{(\ell_1-n)}^{(n,n)}(-z_1) + \tilde{\tau}_1 P_{(\ell_1-n)}^{(n,n)}(z_1)\Big]$$

$$\times \Big[P_{(\ell_2-n)}^{(n,n)}(-z_2) + \tilde{\tau}_2 P_{(\ell_2-n)}^{(n,n)}(z_2)\Big]\ a_{>}^{\tilde{\tau}}(\ell_1,\ell_2,n) \qquad (2.8)$$

where $\tilde{\tau}_1$ and $\tilde{\tau}_2$ are signatures w.r.t. $(\ell_1 - n)$ and $(\ell_2 - n)$ respectively. If (2.8) together with the analogous expression for $\hat{a}_{<}^{\tau_3}(n)$ is inserted into (2.6) then the full SW transform of A_5 is obtained. Since the physical limit as $|z_{1,2}| \to \infty$ is $z_{1,2} \to z_{1,2} \pm i\varepsilon$ as $z_{1,2} \to \pm\infty$ it is clear that if $a_{>}^{\tau}(\ell_1,\ell_2,n)$ is real, then the phases of asymptotic contributions to A_5 are determined.

To explicitly determine the asymptotic behaviour from the SW transform it is, of course, necessary to use a generalization of the Mandelstam-

Sommerfeld-Watson transform. For completeness we give the necessary equations[30], that is

$$\frac{P_\ell^{-n}(z)}{\sin\pi(\ell-n)} = \frac{e^{-in\pi}\,\Gamma(\ell-n+1)}{\pi\cos\pi\ell\;\Gamma(\ell+n+1)}\left[Q_\ell^{n}(z) - Q_{-\ell-1}^{n}(z)\right] \tag{2.9}$$

$\left(P_\ell^{-n}(z) = \left(\frac{1-z^2}{2}\right)^{\frac{n}{2}} P_{(\ell-n)}^{(n,n)}(z)\right)$ together with

$$e^{-in\pi} Q_{-\ell-1}^{n}(z) \underset{|z|\to\infty}{\sim} \frac{2^{\ell}\pi^{\frac{1}{2}}\,\Gamma(n-\ell)}{\Gamma(-\ell+\frac{1}{2})}\, z^{\ell} \tag{2.10}$$

If (2.9) is inserted into (2.8) the term involving $Q_\ell^{+n}(z)$ will not contribute to the Regge asymptotic behaviour since if a generalized Mandelstam symmetry[31] holds at half-integer ℓ the appropriate ℓ-contour can be moved arbitarily far to the right (giving a continually decreasing asymptotic behaviour) without encounting Regge poles or cuts. Therefore, to obtain the Regge asymptotic behaviour from (2.8) we can replace $P_\ell^{-n}(z)/\sin\pi(\ell-n)$ by

$$\frac{2^{\ell}\,\Gamma(n-\ell)\,\Gamma(\ell-n+1)}{\pi^{\frac{1}{2}}\cos\pi\ell\;\Gamma(-\ell+\frac{1}{2})\,\Gamma(\ell+n+1)}\; z^{\ell} \tag{2.11}$$

Since the coefficient of z^ℓ in (2.11) is real analytic we can absorb all the factors apart from $\Gamma(n-\ell)$ to define a new partial-wave coefficient $b^{\tau}_{>}(\ell_1,\ell_2,n)$ and so obtain from (2.6) and (2.8)

$$A_5 \underset{z_1,z_2\to\infty}{\sim} \Big\{ \sum_{\tau} \int_{C_>} dn\, \frac{\left[(-z_1z_2u)^{n} + \tau_3\,(z_1z_2u)^{n}\right]}{\sin\pi n} \int d\ell_1 d\ell_2\, b^{\tau}_{>}(\ell_1,\ell_2,n)$$

$$\times\,\Gamma(n-\ell_1)\,\Gamma(n-\ell_2)\left[(-z_1)^{\ell_1-n} + \tilde{\tau}_1(z_1)^{\ell_1-n}\right]\left[(-z_2)^{\ell_2-n} + \tilde{\tau}_2(z_2)^{\ell_2-n}\right]\Big\} \tag{2.12}$$

$$+\left\{ > \to < \right\}$$

Since $\tilde{\tau}_1$ and $\tilde{\tau}_2$ are signatures in $(\ell_1 - n)$ and $(\ell_2 - n)$ we have that $\tilde{\tau}_1 = \tau_1\tau_3$ and $\tilde{\tau}_2 = \tau_2\tau_3$ where τ_1 and τ_2 are signatures in ℓ_1 and ℓ_2, respectively. Therefore, the contribution to (2.12) of Regge poles α_1 and α_2 with signatures τ_1 and τ_2 is

$$A_5 \underset{z_1,z_2 \Rightarrow \infty}{\sim} \int_{C_>} dn \frac{\left[(-z_1 z_2 u)^n + \tau_3 (z_1 z_2 u)^n\right]}{\sin \pi n} \; b_>^{\tau}(\alpha_1, \alpha_2, n)\, \Gamma(n-\alpha_1)\Gamma(n-\alpha_2)$$
$$\times \left[(-z_1)^{\alpha_1 - n} + \tau_3\tau_1 (z_1)^{\alpha_1 - n}\right]\left[(-z_2)^{\alpha_2 - n} + \tau_3\tau_2 (z_2)^{\alpha_2 - n}\right] \quad (2.13)$$
$$+ \{ > \rightarrow < \}$$

We should note that the complicated singularity structure of five-particle amplitudes makes the whole problem of the region of convergence of the SW transform much more difficult than for the four-particle amplitude. We have partly discussed this problem in Ref. 9 and we shall comment briefly on it at various points in this paper. The presence of complex singularities in the amplitude means that the representation may not converge at all near the high-energy physical regions where we want to take asymptotic Regge limits. If this is the case then there are reasons we can give for ingnoring it as a potential problem. The simplest way out would be to say that we are only considering the contributions of normal threshold and pole singularities (to the integral representation of the next section) which are real, since we believe these are all that contribute to the multi-Regge asymptotic behaviour. Since this may not be true we can instead say that we extract the Regge asymptotic behaviour in some region where the representation converges and continue this to the relevant physical region. To justify this, we can appeal to the corresponding SO(2.1) expansion. In general the convergence of this expansion is complementary to that of the SW transform[2]. However, the net result will be the same as ignoring the divergence of the S-W transform. As we are emphasizing in this paper the S-W expansion contains the most information about the amplitude and so it is obviously simplest to use this expansion directly.

We could obviously obtain a general expression for the two Reggeon/particle vertex contributing to the asymptotic behaviour (2.13) by simply extracting $|z_1|^{\alpha_1}|z_2|^{\alpha_2}$ and dividing through by the two-particle/Reggeon vertices $\beta_{\alpha_1}(t_1)$ and $\beta_{\alpha_2}(t_2)$. However, we shall obtain a more familiar form for this vertex if we close the contour $C_>$ to the left in the n-plane to obtain an asymptotic expansion as $u \to \infty$. There will be contributions from poles in $\Gamma(n - \alpha_1)$ and $\Gamma(n - \alpha_2)$ and also potentially from poles at negative integers arising from the $\sin \pi n$ factor. However, because we are deriving an asymptotic expansion for $u \to \infty$ we must also pull the $C_<$ contour in $\{> \to <\}$ to the left. [Since the spinless five-particle amplitude we are considering is a function of $\cos \omega = 2(u + 1/u)$ the amplitude is symmetric under $u \leftrightarrow 1/u$ and we could obviously obtain the same asymptotic expansion by considering $u \to 0$. This would simply involve moving both $C_>$ and $C_<$ to the right instead of the to the left.]

The singularities in the left-half n-plane that might contribute are the poles at integer n arising from $\sin \omega n$ together with nonsense wrong signature poles of $b_<^{\tilde{\tau}}(j_1,j_2,n)$ at $j_1 - n = -1, -2, \ldots$ $j_2 - n = -1, -2, \ldots$ However, the signature factors in (2.13) will prevent the nonsense poles from contributing. The contributions from integer n will simply cancel those from the > integral if

$$\Gamma(N-\alpha_1)\,\Gamma(N-\alpha_2)\, b_<^{\tilde{\tau}}(\alpha_1,\alpha_2,N) = \Gamma(-N-\alpha_1)\,\Gamma(-N-\alpha_2)\, b_>^{\tilde{\tau}}(\alpha_1,\alpha_2,N) \tag{2.14}$$

for N a positive integer. This property is reminiscent of the Mandelstam symmetry in ℓ referred to earlier. We shall discuss why we expect this property to hold in the next section. If we assume that it does then we only need consider the contributions of the poles of $\Gamma(n - \alpha_1)$ and $\Gamma(n - \alpha_2)$ to the asymptotic expansion of (2.13) as $u \to \infty$. It is important to note that all the contributions from integer n are cancelled precisely because the contour $C_<$ is not quite the mirror reflection of the $C_>$ contour in that the pole at $n = 0$ lies to the right of both contours.

At $(\alpha_1 - n) = 0$ we must have $\tilde{\tau}_1 = \tau_3\tau_1 = +1$ and so $\tau_3 = \tau_1$ and the pole at $(\alpha_1 - n) = 0$ gives

$$\frac{b^{(\tau_1,\tau_2,\tau_1)}(\alpha_1,\alpha_2,\alpha_1)\,\Gamma(\alpha_1-\alpha_2)}{\sin \pi \alpha_1}\left[(-z_1z_2u)^{\alpha_1} + \tau_1 (z_1z_2u)^{\alpha_1}\right]\left[(-z_2)^{\alpha_2-\alpha_1} + \tau_1\tau_2(z_2)^{\alpha_2-\alpha_1}\right] \tag{2.15}$$

$$= |z_1|^{\alpha_1} |z_2|^{\alpha_2} |u|^{\alpha_1} \frac{b^{\tau} \Gamma(\alpha_1-\alpha_2)}{\sin \pi\alpha_1} \left[e^{-i\pi\alpha_1} + \tau_1\right]\left[e^{-i\pi(\alpha_2-\alpha_1)} + \tau_1\tau_2\right] \tag{2.16}$$

The contributions of the other poles of $(n - \alpha_1)$ and $(n - \alpha_2)$ can be extracted in the same way and it can be checked that we can write

$$A_5 \underset{z_1,z_2\to\infty}{\sim} |z_1|^{\alpha_1}|z_2|^{\alpha_2} \; V^{\tau_1\tau_2}(t_1,t_2,u)\,\beta_1(t_1)\beta_2(t_2) \tag{2.17}$$

where

$$V^{\tau_1\tau_2} \underset{|u|\to\infty}{\sim} |u|^{\alpha_1} V_1^{\tau_1\tau_2} + |u|^{\alpha_2} V_2^{\tau_1\tau_2} \tag{2.18}$$

and (assuming that for large u the asymptotic series converges and we can neglect the background integrals in n)

$$\beta_1\beta_2 V_1^{\tau_1\tau_2} = \left[e^{-i\pi\alpha_1} + \tau_1\right]\left[e^{i\pi(\alpha_2-\alpha_1)} + \tau_1\tau_2\right]\left[\sum_{N=0}^{\infty} \frac{\Gamma(\alpha_1-\alpha_2-N)\, b^{\tau}(\alpha_1,\alpha_2,\alpha_1-N)\,|u|^{-N}}{\sin\pi(\alpha_1-N)\;(N-1)!}\right] \tag{2.19}$$

with $V_2^{\tau_1\tau_2}$ given by the same expression with 1 and 2 interchanged.

Since $\eta = S_{23}/S_{15}S_{45} \underset{|u|\to\infty}{\sim} u$ it is clear that (2.19) is equivalent to the familiar expressions for $V_1^{\tau_1\tau_2}$ and $V_2^{\tau_1\tau_2}$ as power series in η [18,29,32]. (2.18) together with (2.19), provides a convenient way of expressing $V^{\tau_1\tau_2}$ for several reasons. Firstly it is clear that for large u the asymptotic behaviour (2.13) can be expressed entirely in terms of the Froissart-Gribov amplitudes $a^{\tau}_{\gtrless}(\alpha_1,\alpha_2,\alpha_1\mp N,t_1,t_2)$, $a^{\tau}_{\gtrless}(\alpha_1,\alpha_2,\alpha_2\mp M,t_1,t_2)$ N,M = 0,1,2, ... and it is these "helicity amplitudes" which we shall be able to define uniquely in terms of the Carlson condition in the next section. As a result we can prove factorization of these amplitudes from unitarity directly and so factorize out $\beta_1\beta_2$ from the right-hand side of (2.19). It also follows from (2.19) that $V_1^{\tau_1\tau_2}$ has poles at α_1 = integer (α_2 not equal to an integer) while $V_2^{\tau_1\tau_2}$ has poles at α_2 = integer (α_1 not equal to an integer). Therefore $V_1^{\tau_1\tau_2}$ contributes to the particle amplitudes obtained by taking α_1 to an integer and $V_2^{\tau_1\tau_2}$ does not. This structure has proved most important in the recent discussion of Pomeron decoupling theorems[33,37] because of its relation to the singularity structure of A_5 as we discuss further in section 4.

It is also interesting to compare the nonsense decoupling (which takes place at integer α_1 and α_2) using both (2.13) and (2.19). This decoupling follows naturally from the (2.13) because of the singularity structure in the n-plane shown in Fig. 2.2. It is clear from this figure that at integer α_1 a finite number of the poles of $\Gamma(\alpha_1 - n)$ pinch with the poles at n = integer from $\sin \pi n$ to give a pole with a finite number of helicity amplitudes in the residue. The > integral gives the helicity amplitudes from $n = 0 \ldots \alpha_1$, while the < integral gives the amplitudes from $n = -\alpha_1 \ldots -1$. This decoupling, that is the reduction of the vertex to a polynomial in $\cos \omega$ in the residue of the pole at α_1 = integer, is not evident from (2.19) as it stands. Since (2.19) must give the helicity amplitudes with both positive and negative n the pole at $\alpha_1 = M$ which appears to occur in all terms in the sum over N because of the $\sin \pi (\alpha_1 - N)$ factor must only occur for $N \leq 2M$. In fact $b^{\underset{\sim}{\tau}}(\alpha_1,\alpha_2,\alpha_1-N)$ will have the necessary nonsense zeros essentially because of the factor $\Gamma(\ell+n+1)$ in the denominator of (2.11).

3. FROISSART-GRIBOV CONTINUATION OF THE FIVE-PARTICLE AMPLITUDE

In this section we give a method for defining Froissart-Gribov amplitudes which is potentially applicable to amplitudes with the most general singularity structure of perturbation theory. The method is a generalization of that given in Ref. 9 and essentially parallels the usual method for defining the Froissart-Gribov continuation of the four-particle amplitude. We first write an integral representation for the amplitude in terms of the angular variables z_1, z_2 and u. From (2.2) it is clear that for fixed t_1, t_2, A_5 will have kinematic singularities at $z_1, z_2 = \pm 1$. However, continuation around any of these branch-points simply results in

$$A_5(\cos\omega) \rightarrow A_5(-\cos\omega) \tag{3.1}$$

and so if we first define amplitudes $A_5^{\pm}$ which have definite signature with respect to cos ω by

$$A_5^{\pm} = A_5(\cos\omega) \pm A_5(-\cos\omega) \tag{3.2}$$

and then write

$$\hat{A}_5^{+} = A_5^{+} \quad , \quad \hat{A}_5^{-} = \cos\omega\,(1-z_1^2)^{\frac{1}{2}}(1-z_2^2)^{\frac{1}{2}} A_5^{-} \tag{3.3}$$

$\hat{A}_5^{\pm}$ will be non-singular at z_1, $z_2 = \pm 1$. We can therefore obtain an integral representation for these amplitudes by applying the Bergman-Oka-Weil (BOW) theorem[9,22] to them regarded as functions of z_1, z_2, u with t_1 and t_2 fixed. This theorem is a generalization of Cauchy's theorem for a function of one complex variable. We can write the resulting representation in the form

$$\hat{A}_5^{\pm} = \sum_{i<j<k} \int du' dz_1' dz_2' \; \underset{\sim}{q}^{i} \cdot (\underset{\sim}{q}^{j} \times \underset{\sim}{q}^{k}) \; \hat{\rho}_{ijk}^{\pm}(u', z_1', z_2', t_1, t_2) \tag{3.4}$$

where

$$\underset{\sim}{q}^{\nu} = \left(q_u^{\nu} \, , \, q_{z_1}^{\nu} \, , \, q_{z_2}^{\nu} \right) \qquad \nu = i, j, k \tag{3.5}$$

The q functions are a generalization of the usual dispersion relation denominator. They are functions of z_1, z_1', z_2, z_2', u and u$'$ and satisfy

$$(u'-u)\, q_u^{\nu} + (z_1'-z_1)\, q_{z_1}^{\nu} + (z_2'-z_2)\, q_{z_2}^{\nu} = 1 \qquad \nu = i, j, k \tag{3.6}$$

They must also have the property that as functions of u, z_1, z_2 they are analytic in the domain of analyticity of $\hat{A}_5^{\pm}$ when z_1', z_2' and u$'$ lie on any combination of cuts and poles of $\hat{A}_5^{\pm}$. The sums over i,j,k in (3.4) are over the intersections of the cuts and poles of $\hat{A}_5^{\pm}$ (taken three at a time). $\hat{\rho}_{ijk}^{\pm}$ is a generalized spectral function and can be expressed in terms of $A_5^{\pm}$ evaluated in all possible combinations of boundary-values onto the cuts involved.

The specific construction of the q-functions is not really essential for the following method and the existence of such functions is not in doubt. Nevertheless, it is interesting to discuss their construction. First we consider fixed invariant singularities with their associated cuts. For invariants depending linearly on z_1 or z_2 as in (2.1) the answer is obvious

$$q^i_{z_1,z_2} = \frac{1}{(z'_{1,2} - z_{1,2})} \quad , \quad q^i_{z_2,z_1} = q^i_u = 0 \tag{3.7}$$

For the invariants related to the group-variables as in (2.2) we can use essentially the functions given in Ref. 9, with an extra dependance on $\cos \omega$ introduced. We construct these functions as follows. If the invariant, S_{13} say, is given by $S_{13} = F(z_1, z_2, u)$ then F will have square-root branch-points at $z_1, z_2 = \pm 1$. However,

$$\hat{P}(z_1,z_2,u',z'_1,z'_2,u') = [F(z_1,z_2,u) - F(z'_1,z'_2,u')][F(z_1,z_2,-u) - F(z'_1,z'_2,u')] \tag{3.8}$$

will be a polynomial in z_1, z_2, $\cos \omega$ and if we multiply through by u^2 we will obtain a function P, polynomial in z_1, z_2, u which vanishes at $z_1 = z'_1$, $z_2 = z'_2$, $u = u'$. Therefore, we can obviously write

$$P = R_{z_1}(z'_1 - z_1) + R_{z_2}(z'_2 - z_2) + R_u(u' - u) \tag{3.9}$$

and so define

$$q_{z_1} = \frac{R_{z_1}}{P} \quad , \quad q_{z_2} = \frac{R_{z_2}}{P} \quad , \quad q_u = \frac{R_u}{P} \tag{3.10}$$

Note that the singularities of q_{z_1}, etc., occur only when $S_{13}(z_1, z_2, u) = S'_{13} \equiv F(z'_1, z'_2, u')$ or $S_{13}(z_1, z_2, -u) = S'_{13}$ which are outside the domain of analyticity of $\hat{A}^{\pm}$ provided that S'_{13} is on the cut (or pole) in the S_{13}-plane. Therefore the functions q_{z_1}, etc. defined in this way satisfy the conditions of the BOW theorem.

For a general Landau singularity the equation of the singularity takes the form $Q(\underset{\sim}{s}) = 0$ where Q is some polynomial in the invariants $\underset{\sim}{s}$. If we could ignore the "positive α" condition on the singularity of the amplitude[38] we could simply proceed in the same way as above treating Q as we did F. Unfortunately, the positive α condition means that the amplitude is not singular on the whole of the curve $Q(\underset{\sim}{s}) = 0$ and so we cannot use q-functions that are. Only certain branches of $Q(\underset{\sim}{s}) = 0$ are singular. It seems that this makes it unavoidable to introduce q-functions which are not meromorphic as functions of z_1, z_2, u. This is not a serious problem but it means that the expressions for the Froissart-Gribov continuations will be slightly more complicated as we discuss briefly below. In general it seems safe to assume that the polynomial form of the Landau surfaces will make the construction of the q-functions fairly straightforward in practize.

One point that we should note in the application of the BOW theorem is that although there are two branches in the u-plane for each invariant singularity (because $z \cos \omega = u + \frac{1}{u}$), the resulting cuts should be regarded as one. That is intersections of the two branches should not be included in (3.4). This is because the intersection of such intersections with another cut will give only a two-dimensional set and not a three-dimensional set as is the case for proper intersections[22].

We can now define partial-wave projections

$$a^{+}_{\ell_1 \ell_2 n} = \frac{1}{8\pi i \, \Lambda(\ell_1, \ell_2, |n|)} \int_{|u|=1} du \int_{-1}^{+1} dz_1 dz_2 \, P_{\ell_1}^{-|n|}(z_1) P_{\ell_2}^{-|n|}(z_2) \, u^{-n-1} A^{+} \tag{3.11}$$

where

$$\Lambda(\ell_1, \ell_2, n) = \frac{\Gamma(\ell_1 - n + 1)\,\Gamma(\ell_2 - n + 1)}{\Gamma(\ell_1 + n + 1)\,\Gamma(\ell_2 + n + 1)} \tag{3.12}$$

and inserting (3.4) into (3.11) and interchanging orders of integration, we obtain for $n \geq 0$

$$a^{+}_{\ell_1\ell_2 n} = \frac{1}{8\pi i\Lambda} \sum_{i,j,k} \int du' dz_1' dz_2' \, \rho^{+ijk}(u_1', z_1', z_2') \int dz_1 dz_2 du$$

$$\times \, \underline{q}^{i} \cdot (\underline{q}^{j} \times \underline{q}^{k}) \; P^{-n}_{\ell_1}(z_1) \, P^{-n}_{\ell_2}(z_2) \, u^{-n-1} \tag{3.13}$$

We now move the u-contour out to ∞ keeping $-1 \leq z_1, z_2 \leq 1$ and consider the behaviour of (3.13) as $n \to \infty$ with $(\ell_1 - n)$, $(\ell_2 - n)$ fixed at integer values. $P^{-n}_{\ell}(z)$ is given by $(1 - z^2/2)^{n/2} \times$ polynomial in z[30] and we can consider the asymptotic behaviour of the polynomial term by term. The coefficients are ratios of Γ-functions of the form

$$\frac{\Gamma(M-N)\,\Gamma(2n+N+M+1)}{\Gamma(-N)\,\Gamma(2n+N+1)\,\Gamma(1+n+M)} \qquad (N = \ell - n) \tag{3.14}$$

$$\underset{|n|\to\infty}{\sim} n^{-n} \tag{3.15}$$

whereas

$$\Lambda \underset{|n|\to\infty}{\sim} (2n)^{-2n} \tag{3.16}$$

Therefore, if we define "group-theoretic" SO(3) partial-wave coefficients by

$$\tilde{a}^{+}_{\ell_1\ell_2 n} = \Lambda^{\frac{1}{2}} \, a^{+}_{\ell_1\ell_2 n} \tag{3.17}$$

then the behaviour as $|n| \to \infty$ of the continuation of $\tilde{a}^{+}_{\ell_1\ell_2 n}$ defined by (3.13) is determined by the factors $(1 - z_1^2)^{+n/2}$, $(1 - z_2^2)^{+n/2}$ [in $P^{-n}_{\ell_1}(z_1)$ and $P^{-n}_{\ell_2}(z_2)$] and by u^{-n-1}. Since $|z_1|, |z_2| < 1$ and $|u| > 1$ it follows that if we introduce a signature factor for singularities in the left-half u-plane then we can define $\hat{a}^{+}_{>\ell_1-n,\ell_2-n}(n)$ which satisfies the Carlson condition as $|n| \to \infty$ in the right-half n-plane with $(\ell_1 - n)$ and $(\ell_2 - n)$ fixed. In an analogous way we can define continuations $\hat{a}^{-}_{>\ell_1-n,\ell_2-n}(n)$. Also by moving the u-contour into $u = 0$ we can define continuations $\tilde{a}^{\pm}_{<\ell_1-n,\ell_2-n}(n)$ from $n < 0$ which satisfy the Carlson condition in the left-half n-plane.

Note that we have proved a slightly different result here from that in Ref. 9. There we proved directly the existence of $a^{\tau_3}_{>}(n)$ satisfying the Carlson condition, and this is what is needed for the helicity transform (2.5). We could obviously have derived this result here by considering only the projection with respect to u in (3.11) instead of performing the z_1 and z_2 projections as well. In fact the result also follows from inserting (3.15) and (3.16) into (2.7) (assuming that the sums over ℓ_1 and ℓ_2 do not cause any problems). However, we have chosen to construct the functions $\tilde{a}^{\pm}_{\gtrless\ell_1-n,\ell_2-n}(n)$ partly because we have in mind the direct continuation of multi-particle unitarity equations to complex helicity and angular momentum. We use this to prove factorization in the next section and we also used this extensively in our treatment of Regge cuts in Refs. 14 and 15. The unitarity equations considered are always completely partial-wave projected so that both the angular momentum and the helicity labels are diagonalized. To continue equations involving $\tilde{a}_{j\ell n}$ to complex helicity and angular momentum it is necessary to first go to complex n using the continuations $\tilde{a}^{\pm}_{\gtrless\ell_1-n,\ell_2-n}(n)$ which we have now shown, satisfy the Carlson condition. Points with $(\ell_1 - n)$ or $(\ell_2 - n)$ complex can then be reached using the Carlson condition on the amplitudes as functions of $(\ell_1 - n)$ or $(\ell_2 - n)$ with n kept fixed.

To go to complex $(\ell_1 - n)$ or $(\ell_2 - n)$ is now very similar to the method used in Ref. 9. First we note that poles in u of $\underset{\sim}{q}^i$, etc., will occur at positions u_0 where

$$(1-z_1^2)^{\frac{1}{2}}(1-z_2^2)^{\frac{1}{2}}\, u_0 - \gamma_0(z_1,z_2) = 0 \tag{3.18}$$

and γ_0 is regular at z_1, $z_2 = \pm 1$. Therefore, if we perform the u-integration in (9) by pushing the contour out to ∞ and picking up pole contributions the factor u^{-n-1} will give a factor $(-z_1^2 + 1)^{-n/2}\,(1 - z_2^2)^{-n/2}$ in the pole residues. (For general Landau surfaces there will be cuts in u but this factor can still be expected to emerge from the integral over u of the discontinuities across such cuts.) Now, as usual, we can put

$$(1-z_1^2)^{\frac{n}{2}}(1-z_2^2)^{\frac{n}{2}} P_{\ell_1}^{-n}(z_1)\, P_{\ell_2}^{-n}(z_2) \rightarrow -\frac{\Lambda(\ell_1,\ell_2,n)}{\pi^2}(z_1^2-1)^{\frac{n}{2}}(z_2^2-1)^{\frac{n}{2}}\, Q_{\ell_1}^{n}(z_1)\, Q_{\ell_2}^{n}(z_2)$$

with the integrals over z_1 and z_2 replaced by contour integrals.

We have intentionally not restricted t_1 and t_2 to the physical region for a specific scattering process since we shall want to consider both high-energy behaviour and direct-channel unitarity in the following. As we discussed in the previous section we shall assume that for all our purposes an analytic continuation in the internal mass[9,12] will be sufficient to ensure that there are no "physical region" (i.e. $|z_1|$, $|z_2| \leq 1$) singularities and so we can equally well go to complex $(\ell_1 - n)$ or $(\ell_2 - n)$.

To define a continuation to complex $(\ell_1 - n)$ we move the z_2-contour out to infinity first, followed by the z_1-contour.

In contrast to the single variable situation we do not now simply get contributions only from the singularities of the q-functions in (3.4). There will also be singularities arising from the u-integration and, when we move the z_1-contour, from the z_2-integration. This effect was discussed in Ref. 9 and as we discussed there the main problem arises from singularities of the q-functions depending on z_1 and z_2 [for example the singularities which occur for fixed values of S_{13} which will be given by (2.2)]. We can enumerate the main contributions as follows. We shall not go into full detail in considering them since this would essentially duplicate the analysis of Ref. 9.

3.1 Three separate singularities of the q functions

These contributions take the form (if the q functions are meromorphic)

$$\int dz_1' dz_2' du\; \rho^{+ijk}(u',z_1',z_2')\, Q_{\ell_1}^{n}(z_1')\, Q_{\ell_2}^{n}(z_2')\, (u')^{-n-1} \tag{3.19}$$

In fact we only get contributions of this sort from intersections of three singularities if one of them occurs only for fixed-values of z_1. A simple way to see this is to plot the movement of the z_2 contour in three dimensions as in Fig. 3.1, using z_2 for two of the dimensions and making the third dimension the direction in the z_1-plane in which the contour is moved out to infinity. The initial contour is equivalent to the small loops around the individual singularities shown in Fig. 3.1. In moving z_1 out to infinity it is clear that because of the equivalence of the two contours shown in Fig. 3.2, there will be no trapping of the contours except by a fixed z_1 singularity which would appear as an "impassable" plane in this picture.

The boundary of the z_1'-integration in (3.19) is determined therefore by the poles and normal threshold in the invariants linearly related to z_1. Since these always lie in either the right or the left-half z_1-plane it is straightforward to define signatured Froissart-Gribov continuations to complex $(\ell_1 - n)$ from (3.19). It is interesting to note that contributions of the above sort would be equivalent to those given by intersections of "L_1 and L_3 singularities" in the method of Ref. 9.

3.2 From the intersection of two singularities with those of $Q^n_{\ell_2}(z_2)$ at $z_2 = \pm 1$

The position of these intersections in the z_1-plane is independant of u since putting $z_2 = \pm 1$ in (2.2) removes u from the equation. Therefore the results of Ref. 9 can be used directly to show that for normal thresholds and poles these intersections occur entirely in either the right or left-half z_1-plane. The form these contributions take is more complicated than (3.19). If $z_1^{\pm}(u', z_1', z_2')$ are the points of intersection of the (intersection of the) two singularities with $z_2 = \pm 1$ then there will be cuts in the z_1-plane attached to branch-points at $z_1 = z_1^{\pm}$. The contributions to the z_1-integration from these cuts takes the form

$$\sum_k \frac{i}{2\pi} \int dz_1' dz_2' du' \, \rho^{+ijk}(u', z_1', z_2') \int_{z_1^{\pm}}^{\pm\infty} dz_1 \, (z_1^2 - 1)^{\frac{n}{2}} Q^n_{\ell_1}(z_1)$$
$$\times \operatorname*{disc}_{\tilde{z}_2 = \pm 1} \left[Q^n_{\ell_2}(\tilde{z}_2) (\tilde{z}_2^2 - 1)^{\frac{n}{2}} (\tilde{u})^{-n-1} R \right] \qquad (3.20)$$

$\tilde{z}_2$, $\tilde{u}$ and the residue function R will all be functions of z_1, u_1', z_1', z_2'. It is straightforward to now introduce signature factors in z_1. These contributions would arise from the "L_3" singularities of Ref. 9.

3.3 From the two u-plane branches of one singularity intersecting with another singularity

These contributions arise because the residue of a q_u function taken at one u-plane branch will have the point of intersection with the other branch (u = 1) as a singularity. This gives an extra branch-point at $\hat{z}_2(z_1, z_1', z_2', u')$ which will contribute to the z_2-integration if we take the contribution to the z_1-integration from another singularity of the q-functions. As in 3.1 this other singularity must be a fixed z_1-singularity. Therefore, contributions of this sort can be written in the form

$$\sum_k' \frac{i}{2\pi} \int dz_1' dz_2' du' \rho^{+ijk}(u', z_1', z_2') Q_{\ell_1}^n(z_1') (z_1'^2 - 1)^{\frac{n}{2}} \int_{\hat{z}_2}^{\infty} dz_2 Q_{\ell_2}^n(z_2) (z_2^2-1)^{\frac{n}{2}} \operatorname*{disc}_{\hat{z}_2} \left[(\hat{u})^{-n-1} \hat{R} \right] \tag{3.21}$$

where $\hat{u}$ and $\hat{R}$ are functions of u', z_1', z_2', z_2. The boundary of the z_1'-integration is again determined by normal thresholds and poles occuring for fixed values of z_1' and it is straightforward to introduce a signature factor in z_1' to define signatured continuations to complex $(\ell_1 - n)$. These contributions correspond to intersections of the "L_1 and L_2 singularities" of Ref. 9 since they can be considered as arising from pinches of the u-contour at u = 1.

3.4 From intersections of the two u-plane branches of one singularity with $z_2 = \pm 1$

These contributions arise from the $\hat{z}_2$ branch-points of 3.3 trapping the z_2-contour with the branch-points of $Q_{\ell_2}^n(z_2)$ at $z_2 = \pm 1$. This gives extra branch-points $\hat{z}_1^{\pm}$ in the z_1-plane and these contributions can be written in the form

$$\sum_{j,k} \frac{-1}{4\pi^2} \int dz_1' dz_2' du' \rho^{+ijk}(z_1', z_2', u) \int_{\hat{z}_1^{\pm}}^{\pm\infty} dz_1 (z_1^2-1)^{\frac{n}{2}} Q_{\ell_1}^n(z_1) \times \operatorname*{disc}_{\hat{z}_1^{\pm}} \int_{\hat{z}_2}^{\infty} dz_2 Q_{\ell_2}^n(z_2)(z_2^2-1)^{\frac{n}{2}} \operatorname*{disc}_{\hat{z}_2} \left[(\hat{u})^{-n-1} \hat{R} \right] \tag{3.22}$$

Again the results of Ref. 9 give that $\hat{z}_1^{\pm}$ occur always in the same half z_1-plane for fixed-invariant poles or cuts and so signatured continuations to complex $(\ell_1 - n)$ can again be defined. These last contributions arise from the L_2 singularities of Ref. 9.

Although we have identified the above contributions to the Froissart-Gribov continuations with those obtained using the method of Ref. 9 the explicit forms we have given are slightly different. This is simply because we have started with an integral representation of the full amplitude and introduced corresponding q-functions. In Ref. 9 we introduced an integral representation and analogous q-functions for $\hat{a}_{\gtrless}^{\tau_3}(n, t, t_2, z_1, z_2)$.

We have not paid much attention to general Landau singularities in the above but we can anticipate that they will lie in signatured positions since in general it seems that for large internal mass they are either contained in or they approach real regions in the invariant variables which are bounded by normal thresholds. In this case they will contribute to (3.4) through spectral regions which lie close to the normal threshold regions. The virtue of the above method of defining Froissart-Gribov continuations is that it is then obviously applicable to such contributions. The fact that the q functions are not meromorphic will simply give more complicated expressions for the Froissart-Gribov continuations For example, the simplest of the above expressions (3.19) would have the form

$$\int dz_1' dz_2' du' \rho^{+ijk}(u', z_1', z_2') \int_{z_1'}^{\infty} dz_1 \int_{z_2'}^{\infty} dz_2 \int_{u'}^{\infty} du \; Q_{\ell_1}^{n}(z_1) Q_{\ell_2}^{n}(z_2)$$
$$\times u^{-n-1} \, \mathrm{Disc}\left[\underset{\sim}{q}^i \cdot (\underset{\sim}{q}^j \times \underset{\sim}{q}^k) \right] \tag{3.23}$$

where Disc refers to the relevant multiple discontinuity of $\{\underset{\sim}{q}^i \cdot (\underset{\sim}{q}^j \times \underset{\sim}{q}^k)\}$. We should perhaps note that tree diagram contributions to the full amplitude (this would include dual model tree amplitudes) will have at most only two simultaneous singularities in z_1, z_2, u and so will not immediately satisfy a representation of the form of (3.4). In fact, it is straightforward to introduce subtractions into (3.4) and the net result is that tree diagrams give only contributions of the form 3.3 and 3.4 in the Froissart-Gribov continuations.

For the contributions 3.3 and 3.4, it is relatively easy to discuss the generation of Regge poles in the continuations and the relationship with the Steinmann relations. It is clear that simultaneous Regge poles in ℓ_1 and ℓ_2 can occur in 3.3, since there are two independent possibilities for the divergence of the integral. The z_1'-integration can obviously diverge. Since $\hat{z}_2$ and $\hat{u}$ are determined by the same singularity of the q-functions they do not lead to independent divergences of the integral depending on ℓ_2 and n separately but instead give one extra possible divergence on top of the z_1'-divergence.

For the contributions 3.4, $\hat{z}_1$, $\hat{z}_2$ and $\hat{u}$ are all related to the same singularity and so must give only one possibility for divergence of the integral. Without examining this problem in detail it seems that this must be, when the values of z_1', z_2', u' are such that the invariant associated with the singularity is large.

Therefore, the double Regge poles in the continuation to complex $(\ell_1 - n)$ originate only from that part of the amplitude with fixed z_1 singularities. Similarly the double Regge poles in the continuation to complex $(\ell_2 - n)$ originate only from that part of the amplitude with fixed z_2 singularities. Fixed z_1 and z_2 singularities correspond to normal thresholds and (poles) in different channels which cannot occur simultaneously according to the Steinmann relations. (This is at the level where we consider only the poles or normal thresholds.) It follows then that the double Regge poles in the continuations to complex $(\ell_1 - n)$ and $(\ell_2 - n)$ originate from different parts of the amplitude and we shall discuss the relation of this to the "asymptotic Steinmann relations" in the next section.

In the ladder graph model of Regge poles the important singularities are the normal thresholds and we can again expect the relevent parts of the Froissart-Gribov continuations to be of the form 3.3 and 3.4.

Contributions of the form 3.1 must involve fixed z_1-singularities but not those of the form 3.2. To establish a general connection between the normal thresholds in z_1 and the double Regge poles in the continuation to complex (ℓ_1 - n) it would only be necessary to show that the contributions 3.2 do not have double Regge poles. We would like to make this connection because of the results obtained from the Sommerfeld-Watson transform which we discuss in the next section. However, it seems difficult to do this in general.

We have now shown that the continuation of partial-wave amplitudes to complex (ℓ_1 - n) or (ℓ_2 - n) can be performed for amplitudes with a quite general singularity structure. As we have emphasized in previous papers these continuations are sufficient for a complete analysis of the multi-particle unitarity equations in both the t_1 and t_2-channels. For this analysis it is desirable that these continuations can be defined for the whole amplitude and that no singularities are neglected and no approximations made. It is in this context that we regard the potential of the above method for dealing with general Landau singularities as most important.

To apply the S-W transform as we did in the previous section we need to define continuations to both complex (ℓ_1 - n) and (ℓ_2 - n). This does not appear to be possible because of the contributions 3.2 and 3.4 as we discussed in detail in Ref. 9. The problem is caused by the singularities in the invariants S_{13}, S_{23}, S_{14}, S_{24} which are non-linear singularities in z_1 and z_2. The S-W transformation nevertheless gives results (for vertex functions for example) which are in agreement with models such as the dual resonance model and the Feynman graph ladder model. We can understand this as follows.

As we have said the familiar models give only contributions of the form of 3.3 and 3.4 to the Froissart-Gribov continuations. The analysis of Ref. 9 shows that if only intersections of singularities associated with non-overlapping channels are considered then the integration over z_2 in (3.21) will lie entirely in a single half-plane. Therefore, if signature factors in z_2 are introduced in (3.21) we will immediately obtain a continuation of this expression to complex (ℓ_2 - n) which satisfies the Carlson condition. Note that the Steinmann relations in the simple form of no singularities in overlapping channels are vital for this result. (Strictly we showed in Ref. 9 that the z_2-integration in (3.21) only lay in a single half-plane in the cross-channel region t_1, t_2 < 0).

We cannot define a continuation of 3.4 to complex $(\ell_2 - n)$ because (3.22) involves

$$\underset{z_2=\pm 1}{\mathrm{disc}}\; Q^n_{\ell_2}(z_2) \sim P^n_{\ell_2}(z_2)$$

However, we can argue that for the part of the amplitude which contributes to (3.21), that is the part with fixed z_1-singularities, the Froissart-Gribov continuation obtained from (3.21) will give the full asymptotic behaviour. Using the approach of Ref. 9 we can rewrite (3.21) in the form

$$-\frac{1}{4\pi^2}\int dz_1' dz_2' \; \psi_{>}^{+ij} \, (z_1'^2-1)^{\frac{n}{2}} \, Q^n_{\ell_1}(z_1') \, (z_2'^2-1)^{\frac{n}{2}} \, Q^n_{\ell_2}(z_2') \tag{3.24}$$

where $\psi_{>}^{ij}$ is the spectral function appearing in the double integral representation of $\hat{a}_{>}^{+}(n, z_1, z_2, t_1, t_2)$. Essentially this form results from interchanging the $\hat{z}_2$ and z_2', u' integrations in (3.21). The integration is now over the intersections of the cuts of $a_{>}^{+}$ which is the region $z_1', z_2' > 1$ apart from a finite region near $z_1' = z_2' = 1$. Since

$$Q^n_\ell(z) \underset{|z|\to\infty}{\sim} z^{-\ell-1} \tag{3.25}$$

and $(z_1'^2 - 1)^{n/2}$ $(z_2'^2 - 1)^{n/2}$ $\psi_{>}^{+ij}$ can be expressed in terms of boundary-values of $a_{>}^{+}$ it is clear that to leading order in z_1' and z_2' (3.24) is essentially the Mellin transform of $a_{>}^{+}$ and so must give the correct residues for the leading asymptotic behaviour of $a_{>}^{+}$.

In the same way we can argue that the correct asymptotic behaviour for that part of the amplitude having fixed z_2-singularities can be obtained by taking the contribution 3.3 to the continuation to complex $(\ell_2 - n)$ defined as above. So for an amplitude with only normal thresholds and poles, the integral representation (3.4), will break the amplitude up into parts having only fixed z_1 or z_2 singularities (or neither) because of the Steinmann relations and Froissart-Gribov continuations giving the full asymptotic behaviour can be separately defined for the parts having fixed z_1 and z_2 singularities. The parts having neither fixed z_1 or fixed z_2 singularities give only contributions of the form

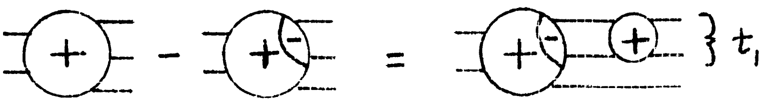
3.4 at integer $(\ell_1 - n)$ and $(\ell_2 - n)$ so that it seems safe to assume that (as is found in models) they do not contribute to double Regge asymptotic behaviour. For the contributions of more complicated singularities we must assume that contributions of the form 3.1 and 3.3 give the double Regge asymptotic behaviour since only for these will we be able to define continuations to both complex $(\ell_1 - n)$ and $(\ell_2 - n)$.

The above discussion gives a reasonable if not rigorous justification for the use of the full Sommerfeld-Watson transform to discuss asymptotic behaviour. We shall find that a consistent picture emerges in the next section in that as in the double Regge limit discussed in the previous section, the asymptotic behaviour in all limits of physical interest can be expressed in terms of the continuations at integer $(\ell_1 - n)$ or $(\ell_2 - n)$, which we can study rigorously. This is provided that the position of Regge singularities depends only on ℓ_1 and ℓ_2 and not n. This has been argued for by Weis[10)] on the basis of the Steinmann relations and by Abarbanel and Schwimmer[11)] on the basis of Lorentz invariance.

The result that only integer values of $(\ell_1 - n)$ or $(\ell_2 - n)$ contributed to the two Reggeon/particle vertex in the last section depended on the neglect of contributions from integer n which required the symmetry property (2.14). It is not difficult to see that this property follows if $a_{>}^{\tau_3}(N, z_1, z_2, t_1, t_2) = a_{<}^{\tau_3}(N, z_1, z_2, t_1, t_2)$ for N an integer. If we define $a_{\gtrless}^{\tau_3}$ simply by distorting the u-contour in the Fourier projection of A_5 as in Ref. 9. Then the integration contours are as shown in Fig. 3.3. For non-integer n the contours differ by an integral over the cut from u^{-n-1} (plus contributions from zero and infinity). For $n = -N$ the contours are equivalent and so the symmetry will hold-provided that the contributions from zero and infinity can be neglected. For Regge pole and cut contributions we can argue that this will be true for sufficiently negative t_1 and t_2 and the result can be continued from this region. An analogous argument to this can also be given for the Mandelstam symmetry at half-integer ℓ referred to in the last section.

4. FACTORIZATION AND DISCONTINUITIES IN REGGE AND HELICITY POLE LIMITS

The amplitudes $a^{\tau}_{\gtrless}(\ell_1,\ell_2,n,t_1,t_2)$ satisfy unitarity equations in both the t_1 and t_2 channels. The full multi-particle equations are important for deriving Regge cut discontinuity formulae, but to prove the factorization of Regge pole residues it is only necessary to use the two-particle equations in both the t_1 and t_2 channels. In a theory with only one kind of particle the discontinuity across the two-particle threshold in the t_1-channel can be written in the usual S-matrix notation[38,39] as


(4.1)

The partial-wave projection of this equation is[14])

$$a_{\ell_1\ell_2 n}(t_1,t_2) - a^{-}_{\ell_1\ell_2 n}(t_1,t_2) = \rho\, a^{-}_{\ell_1\ell_2 n}(t_1,t_2)\, a_{\ell_1}(t_1) \qquad (4.2)$$

where $a^{-}_{\ell_1\ell_2 n}$ is the partial-wave projection of , $a_{\ell_1}(t_1)$ is the partial-wave projection of the four-particle amplitude =⊕=, and ρ is a phase-space factor. (4.2) can be directly continued to complex n and either complex $(\ell_1 - n)$ or $(\ell_2 - n)$, and so can be continued to Regge poles in ℓ_1 and ℓ_2. However, (4.1) holds initially for $t_1 > 4m^2$ and a Regge pole at $\ell_1 = \alpha_1$ will have the branch point at $t_1 = 4m^2$ as a singularity of the trajectory function. As a result $a^{\tau}_{\gtrless}(\ell_1,\ell_2,n,t_1,t_2)$ will be singular at $\ell_1 = \alpha_1(t_1)$ while $a^{\tau -}_{\gtrless}(\ell_1,\ell_2,n,t_1,t_2)$ will be singular at $\ell_1 = \alpha_1^*(t_1)$ and so the pole at $\ell_1 = \alpha_1$ will occur only in $a^{\tau_1}(\ell_1,t_1)$ in the continuation of the right-hand side of (4.2) to complex ℓ_1. So going to the pole at $\ell_1 = \alpha_1$ we will obtain

$$R^{\tau}_{>}(\alpha_1,\ell_2,n,t_1,t_2) = \rho\, a^{\tau -}_{>}(\alpha_1,\ell_2,n,t_1,t_2)\, \beta^{L}_{\alpha_1}(t_1)\, \beta^{R}_{\alpha_1}(t_1) \qquad (4.3)$$

if $R^{\tau}_{\tilde{>}}$ is the residue of the pole at $\ell_1 = \alpha_1$ in $a^{\tau}_{\tilde{>}}$ and $\beta^L_{\alpha_1} = \beta^R_{\alpha_1}$ is the two-particle/Reggeon vertex appearing in $a^{\tau_1}(\ell_1, t_1)$. We therefore have the "factorization"

$$R^{\tilde{\tau}}_{>\alpha_1} = A^{\tilde{\tau}}_{>\alpha_1} \beta^R_{\alpha_1} \tag{4.4}$$

where

$$A^{\tilde{\tau}}_{>\alpha_1} = \rho\, a^{\tilde{\tau}-}_{>} \beta^L_{\alpha_1} \tag{4.5}$$

Of course, in a theory with only one kind of particle, factorization is meaningless at this level and can be achieved simply by definition. However, we now consider a theory involving several different kinds of particles, some of which have spin and so will give rise to several different helicity amplitudes for some processes. We consider the discontinuity of a particular amplitude across the lowest two-particle threshold in the t_1-channel. For simplicity we first assume that these two particles are spinless so that the discontinuity is still given by an equation of the form of (4.1) which will be diagonalized by partial-wave projections as in (4.2). If the external particles have spin then on the right-hand side of (4.2) a_{ℓ_1} will carry the helicity labels of the particles in the t_1-channel, while $\bar{a}_{\ell_1 \ell_2 n}$ will carry the labels of the remaining external particles.

The equivalents of (4.3) to (4.5) can now be used to prove factorization of the amplitudes. The β vertices appearing in (4.4) and (4.5) will no longer be identical. The vertex $\beta^R_{\alpha_1}$ in (4.4) will give the coupling of the Reggeon α_1 to the two external particles in the t_1-channel, while that ($\beta^L_{\alpha_1}$) in (4.4) will give the coupling to the two "internal" particles whose threshold discontinuity is being considered. The $a^{\tilde{\tau}-}_{>}$ appearing in (4.5) will depend only on the two internal particles and the remaining external particles. It then follows that (4.4) and (4.5) define a genuine factorization of the five-particle amplitudes in the theory given the factorization of the four-particle amplitudes[40].

In fact it is not necessary to consider the lowest threshold in the theory or to assume that the particles involved are spinless. The same argument will go through using the total discontinuity across any set of two-particle thresholds, since this will be a sum of terms of the form of (4.1). The definition of $A^{\tau}_{>\alpha_1}$ would then involve a sum of terms of the form of (4.5).

Having proved factorization of $a^{\tau}_{>}$ at $\ell_1 = \alpha_1$ we can now use unitarity in the t_2-channel in exactly the same way to prove factorization at $\ell_2 = \alpha_2$. We therefore have the fundamental result that the Froissart-Gribov continuations defined with either $(\ell_1 - n)$ or $(\ell_2 - n)$ complex factorize in the form

$$(\ell_1-\alpha_1)(\ell_2-\alpha_2)\; a^{\tau}_{>}\left(\ell_1,\ell_2,\left\{\begin{matrix}\ell_1-N\\ \ell_2-N\end{matrix}\right\},t_1,t_2\right) \underset{\substack{\ell_1\to\alpha_1\\ \ell_2\to\alpha_2}}{\sim} \beta^{\tau_1}_{\alpha_1}(t_1)\; g^{\tau_1\tau_2}_{\alpha_1\alpha_2}\left(\left\{\begin{matrix}\alpha_1-N\\ \alpha_2-N\end{matrix}\right\},t_1,t_2\right)\beta^{\tau_2}_{\alpha_2}(t_2) \tag{4.6}$$

This result will enable us to give a complete discussion of the factorization of the full amplitude and the discontinuity in a sub-channel in all high-energy Regge and helicity-pole limits.

First consider the single Regge limit $|z_1| \to \infty$ with z_2, u, t_1, t_2 fixed. Using (2.5), (2.7) and transforming only the $(\ell_1 - n)$ sum we can write (temporarily ignoring signature and $\gtrless$ labels)

$$A_5 = -\frac{1}{4}\int d\ell_1 \int \frac{dn\; u^n\, P^{-n}_{\ell_1}(z_1)}{\sin\pi n\,\sin\pi(\ell_1-n)} \sum_{\ell_2-n=0}^{\infty} P^{-n}_{\ell_2}(z_2)\; a(\ell_1,\ell_2,n) \tag{4.7}$$

and using (2.9) to (2.11) we obtain

$$A_5 \underset{|z_1|\to\infty}{\sim} |z_1|^{\alpha_1} \int \frac{dn\; u^n\,\Gamma(n-\alpha_1)}{\sin\pi n} \sum_{\ell_2-n=0}^{\infty} P^{-n}_{\ell_2}(z_2)\,\tilde{b}(\alpha_1,\ell_2,n) \tag{4.8}$$

where $\tilde{b}$ is simply related to $a(\ell_1,\ell_2,n)$ and so the factorization of $a(\ell_1,\ell_2,n)$ with $\ell_2 - n$ = integer leads directly to

$$A_5 \underset{|z_1|\to\infty}{\sim} |z_1|^{\alpha_1}\,\beta_{\alpha_1}(t_1)\; B_{\alpha_1}(t_1,t_2,z_2,u) \tag{4.9}$$

where

$$B_{\alpha_1} = \int \frac{dn\, u^n\, \Gamma(n-\alpha_1)}{\sin\pi n} \sum_{\ell_2 - n = 0}^{\infty} P_{\ell_2}^{-n}(z_2)\, \tilde{g}(\alpha_1, \ell_2, n) \tag{4.10}$$

B_{α_1} is sometimes called a Reggeon-particle scattering amplitude[1]. If we pull back the n contour in (4.10) in the left-half n-plane there will be poles at $n = \alpha_1 - N$ coming from the $\Gamma(n - \alpha_1)$. However, there will also be singularities arising from the Regge singularities of $\tilde{g}$ in ℓ_2. In fact there will a sequence of singularities at $n = \alpha_2 - N$ for each Regge singularity at $\ell_2 = \alpha_2$ because of the sum over $(\ell_2 - n)$. Therefore, we can write

$$B_{\alpha_1} = \sum_N \frac{u^{\alpha_1 - N}}{(N-1)!\,\sin\pi(\alpha_1 - N)} A_{\alpha_1 - N}(t_2, z_2) + \ell_2 \text{ singularities} \tag{4.11}$$

Clearly the poles at α_1 = integer of B_{α_1} come only from the sum over N in (4.11) and it is the amplitudes $A_{\alpha_1 - N}$ which give the physical helicity amplitudes at α_1 = integer. We shall show in the next section that the $A_{\alpha_1 - N}$ satisfy generalized unitarity equations and so it is these quantities which we shall define as Reggeon scattering amplitudes.

We have already shown in (2.19) that the asymptotic behaviour in the double Regge limit $|z_1|$, $|z_2| \to \infty$ with u, t_1, t_2 fixed can be expressed in terms of $a_{\gtrless}^{\tau}$ evaluated at $(\ell_1 \mp n)$ or $(\ell_2 \mp n)$ = integer and so factorization in this limit immediately follows.

We can consider some further "helicity-pole" limits[8,17,18] by allowing $|u| \to \infty$. These limits are not physical for the five-particle amplitude but they are analogous to limits of higher amplitudes that give physical limits of inclusive cross-sections (once the Mueller theorem is used) and so it is of interest to consider them in the five-point function. First we consider the "pure helicity-pole" limit $|u| \to \infty$ with z_1, z_2, t_1, t_2 fixed. It follows from (2.5) that the asymptotic behaviour is governed by poles in the n-plane of $a_{>}^{\tau_3}(n)$. From (2.8) we see that Regge poles at $\ell_1 = \alpha_1$ and $\ell_2 = \alpha_2$ will give poles in the n-plane at $n = \alpha_1$ and α_2 as a result of the pinching of the ℓ_1 and ℓ_2 contours by the Regge poles and the poles at $\ell_1 - n = 0$, $\ell_2 - n = 0$ coming from $\sin\pi(\ell_1 - n)$ and $\sin\pi(\ell_2 - n)$. Further since $P^{(n,n)}_{\ell-n}(z)$ is a constant for $\ell - n = 0$, we can write

$$A_s \underset{|u|\to\infty}{\sim} \frac{|(1-z_1^2)^{\frac{1}{2}}u|^{\alpha_1} A_{\alpha_1}(t_2,z_2)\,\beta_{\alpha_1}}{\sin\pi\alpha_1} + \frac{|(1-z_2^2)^{\frac{1}{2}}u|^{\alpha_2} A_{\alpha_2}(t_1,z_1)\,\beta_{\alpha_2}}{\sin\pi\alpha_2} \tag{4.12}$$

and so we see explicitly the factorization of the residues of the two "helicity poles" at $n = \alpha_1$ and α_2. However, the vertices involve only the single "helicity amplitudes" A_{α_1} and A_{α_2} and not a sum over such amplitudes as appeared in the single Regge limit. Also one helicity trajectory depends on t_1 while the other depends on t_2, and so the resulting factorizations are with respect to t_1 and t_2 respectively.

Next we consider the mixed "Regge/helicity-pole" limit in which $|u|, |z_2| \to \infty$ with z_1, t_1, t_2 fixed. Since

$$A_{\alpha_1} = \frac{1}{2i}\int \frac{d\ell_2}{\sin\pi(\ell_2-\alpha_1)}\, P^{-\alpha_1}_{\ell_2}(z_2)\, \tilde{g}(\alpha_1,\ell_2,\alpha_1) \tag{4.13}$$

a Regge pole at $\ell_2 = \alpha_2$ gives

$$A_{\alpha_1} \underset{|z_2|\to\infty}{\sim} |z_2|^{\alpha_2}\, \Gamma(\alpha_1-\alpha_2)\; g(\alpha_1,\alpha_2,\alpha_1)\,\beta_{\alpha_2} \tag{4.14}$$

and so we obtain

$$A_s \underset{\substack{|z_2|\to\infty\\|u|\to\infty}}{\sim} \frac{|(1-z_1^2)^{\frac{1}{2}}u|^{\alpha_1}|z_2|^{\alpha_2}\Gamma(\alpha_1-\alpha_2)\beta_{\alpha_1} g(\alpha_1,\alpha_2,\alpha_1)\beta_{\alpha_2}}{\sin\pi\alpha_1} + \frac{|u|^{\alpha_2}|z_2|^{\alpha_2}A_{\alpha_2}\beta_{\alpha_2}}{\sin\pi\alpha_2} \tag{4.15}$$

So in this limit we obtain a sum of two terms. The second term gives effectively single Regge behaviour and involves only a single factorization of the vertex function. The first term gives double Regge behaviour and the vertex involved factorizes with respect to both Reggeons. Notice now that only a single helicity amplitude $g(\alpha_1,\alpha_2,\alpha_1)$ appears as the "two Reggeon particle vertex" instead of the sum over helicity amplitudes which appears in (2.18) and (2.19). We could also have obtained (4.14) by directly simultaneously pulling back the ℓ_2 and n contours in the full Sommerfeld-Watson transform instead of first taking $|u| \to \infty$ (pulling back the n-contour) and then $|z_2| \to \infty$ (pulling back the ℓ_1-contour).

Finally we consider the "double Regge/helicity-pole" limit $|z_1|$, $|z_2|$, $|u| \to \infty$ with t_1, t_2 fixed. This can be obtained by letting $|u| \to \infty$ in (2.17) as we have done in (2.18), so that

$$A_5 \sim |z_1|^{\alpha_1} |z_2|^{\alpha_2} |u|^{\alpha_1} \beta_{\alpha_1} g(\alpha_1,\alpha_2,\alpha_1) \beta_{\alpha_2} + |z_1|^{\alpha_1} |z_2|^{\alpha_2} |u|^{\alpha_2} \beta_{\alpha_1} g(\alpha_1,\alpha_2,\alpha_2) \beta_{\alpha_2} \tag{4.16}$$

The only difference between (2.18) and (4.16) is the reduction of V_1 and V_2 to single helicity amplitudes by the limit $|u| \to \infty$.

We now consider the problem of taking discontinuities across the cuts in the various invariant variables. We can do this in a straightforward way because of our discussion in section 2 of the relation between $i\varepsilon$ boundary values for the invariants and boundary values for the angular variables. In the physical region where t_1 and t_2 are negative, S_{23} is the total energy and S_{15}, S_{45}, S_{14} are large and positive the discontinuities we shall be interested in will be those in S_{15} and S_{45}. We know from the Steinmann relations that the amplitude cannot have simultaneous discontinuities in both S_{15} and S_{45}. It follows from (2.1) and (2.2) that we can evaluate $A(S_{15} + i\varepsilon) - A(S_{15} - i\varepsilon)$ (S_{23}, S_{14}, S_{45} all having $+ i\varepsilon$ prescriptions) by evaluating $A(z_1 + i\varepsilon) - A(z_1 - i\varepsilon)$ with z_2 and $(1 - z_1^2)^{\frac{1}{2}} (1 - z_2^2)^{\frac{1}{2}} \cos\omega$ being given $+ i\varepsilon$ prescriptions. Similarly the S_{45} discontinuity can be obtained from $A(z_2 + i\varepsilon) - A(z_2 - i\varepsilon)$.

From (2.6) and (2.8) it follows that $A(z_2 + i\varepsilon) - A(z_2 - i\varepsilon)$ can be evaluated by eliminating $P^{(n,n)}_{\ell_2-n}(+z_2)$ since this will have no right-hand cut in z_2, and replacing $P^{(n,n)}_{\ell_2-n}(-z_2)$ by $P^{(n,n)}_{\ell_2-n}(-z_2 - i\varepsilon) - P^{(n,n)}_{\ell_2-n}(-z_2 + i\varepsilon)$. In this way we obtain a general Sommerfeld-Watson representation of $\underset{S_{45}}{\text{disc}}\, A_5$ in the form

$$\underset{s_{45}}{\text{disc}}\, A_5 = \frac{1}{8\pi i} \sum_{\substack{\tau_3 \\ \tilde{\tau}_1, \tilde{\tau}_2}} \left\{ \int \frac{dn \left[a_>^{\tau_3}(n) \right]}{\sin \pi n} \left[\left(-(1-z_1^2)^{\frac{1}{2}}(1-z_2^2)^{\frac{1}{2}} u \right)^n + \tau_3 \left((1-z_1^2)^{\frac{1}{2}}(1-z_2^2)^{\frac{1}{2}} u \right)^n \right] \right.$$

$$\times \int dl_1 dl_2 \left[P_{(l_1-n)}^{(n,n)}(-z_1) + \tilde{\tau}_1 P_{(l_2-n)}^{(n,n)}(z_1) \right] \left[P_{(l_2-n)}^{(n,n)}(-z_2 - i\varepsilon) \right. \qquad (4.17)$$

$$\left. \left. - P_{(l_2-n)}^{(n,n)}(-z_2 + i\varepsilon) \right] a_>^{\tilde{\tau}}(l_1, l_2, n) \right\} + \{ > \to < \}$$

and so this can be used directly to study the asymptotic limits of $\underset{s_{45}}{\text{disc}}\, A_5$.

First consider the pure helicity-pole limit $|u| \to \infty$. The helicity-pole at $n = \alpha_1$ will arise in (4.17), as before, but the pole at $n = \alpha_2$ would be given by the pinching of the pole at $\ell_2 = n$ with that at $\ell_2 = \alpha_2$. However, the residue of the pole at $\ell_2 = R_2$ is now $P_0^{(n,n)}(-z_2 - i\varepsilon) - P_0^{(n,n)}(-z_2 + i\varepsilon) = 0$ since $P_0^{(n,n)}(z)$ is a constant. Therefore, the second term in (4.12) does not contribute to $\underset{s_{45}}{\text{disc}}\, A_5$. The contribution of the pole at $n = \alpha_1$ takes the form

$$\underset{s_{45}}{\text{disc}}\, A_5 \underset{|u| \to \infty}{\sim} \left| (1-z_1^2)^{\frac{1}{2}}(1-z_2^2)^{\frac{1}{2}} u \right|^{\alpha_1} \left[e^{-i\pi\alpha_1} + \tau_1 \right] \frac{1}{2i} \sum_{\tau_2} \int \frac{d\ell_2}{\sin \pi(\ell_2 - \alpha_1)} \qquad (4.18)$$

$$\times \tilde{g}^{\tau}(\alpha_1, \ell_2, \alpha_1) \left[P_{(\ell_2-\alpha_1)}^{(\alpha_1,\alpha_1)}(-z_2 - i\varepsilon) - P_{(\ell_2-\alpha_1)}^{(\alpha_1,\alpha_1)}(-z_2 + i\varepsilon) \right]$$

Comparing this result with (4.13) we see that this can be written in the form

$$\underset{s_{45}}{\text{disc}}\, A_5 \underset{|u| \to \infty}{\sim} \left| (1-z_1^2)^{\frac{1}{2}} u \right|^{\alpha_1} \left[e^{-i\pi\alpha_1} + \tau_1 \right] \left| (1-z_2^2)^{\frac{\alpha_1}{2}} \right| \underset{s_{45}}{\text{disc}} \left[\frac{A_{\alpha_1}(t_2, s_{45})}{(1-z_2^2)^{\frac{\alpha_1}{2}}} \right] \qquad (4.19)$$

So the asymptotic limit of disc A_5 in the helicity-pole limit involves the discontinuity of the Reggeon scattering amplitude A_{α_1}. (The kinematic singularity factor $(1 - z_2^2)^{\alpha/2}$ having been extracted before the discontinuity is taken.) If we now consider the Regge/helicity pole limit of (4.15) we obtain from (4.18)

$$\underset{S_{45}}{\text{disc}\, A_5} \underset{\substack{|z_2|\to\infty \\ |u|\to\infty}}{\sim} \frac{|(1-z_1^2)^{\frac{1}{2}} u|^{\alpha_1} \left[e^{-i\pi\alpha_1} + \tau_1\right] |z_2|^{\alpha_2} \sin\pi(\alpha_2-\alpha_1)}{\sin\pi\alpha_1} \tag{4.20}$$

$$\times\, \Gamma(\alpha_1-\alpha_2)\, \beta_{\alpha_1}\, g(\alpha_1,\alpha_2,\alpha_1)\, \beta_{\alpha_2}$$

The $\sin\pi\,(\alpha_2 - \alpha_1)$ factor arises from the presence of $P^{(\alpha_1,\alpha_1)}_{\alpha_2-\alpha_1}(-z_2 - i\varepsilon) - P^{(\alpha_1,\alpha_1)}_{\alpha_2-\alpha_1}(-z_2 + i\varepsilon)$ and kills the spurious poles at $\alpha_1 - \alpha_2$ = integer which cannot appear in the discontinuity. (The P's are, of course, converted to Q's in the asymptotic limit.) So in this limit we have the important result that taking the S_{45} discontinuity eliminates the term with single Regge behaviour and leaves only a double Regge term with a factorizing vertex. This Regge/helicity pole limit can therefore be considered as a double Regge limit in the same sense that the analogous Regge/helicity pole limit of the "M^2 discontinuity of the six-particle amplitude which gives the "triple Regge" behaviour of the inclusive cross-section, is considered as a triple Regge limit.

The appearance of the "absorptive part" of a Reggeon amplitude in the helicity-pole limit of the S_{45} discontinuity as in (4.19) is also analogous to the appearance of the absorptive part of a Reggeon-particle scattering amplitude in the inclusive cross-section.

As in (4.18) and (4.20) the helicity amplitudes with $(\ell_2 - n)$ = integer will not contribute to the double Regge limit of the S_{45}-discontinuity. Therefore, it follows from (2.13) to (2.19) that $V_2^{\tau_1\tau_2}$ will not contribute to the S_{45} discontinuity and similarly $V_1^{\tau_1\tau_2}$ will not contribute to the S_{15} discontinuity.

In fact, the simplest way to see this last result is to use (2.18) to rewrite (2.17) as[18-20]

$$A_5 \sim S_{23}^{\alpha_1} S_{45}^{\alpha_2-\alpha_1} \beta_1\beta_2 V_1^{\tau_1\tau_2} + S_{23}^{\alpha_2} S_{15}^{\alpha_1-\alpha_2} V_2^{\tau_1\tau_2} \tag{4.21}$$

Assuming that the asymptotic power behaviour in a particular invariant is built up by finite singularities in that variable, we see that the "asymptotic Steinmann relations" are obviously satisfied by (4.21). The S_{45} discontinuity now obviously involves only $V_1^{\tau_1\tau_2}$.

So contributions to the double Regge asymptotic behaviour with $(\ell_1 - n)$ and $(\ell_2 - n)$ an integer arise respectively from the parts of the amplitude with singularities in S_{45} and S_{15} which are necessarily distinct because of the Steinmann relations. This relation between the double Regge poles in the amplitude with $(\ell_1 - n)$ and $(\ell_2 - n)$ an integer and the singularities in S_{15} and S_{45} is just that which we discussed using the Froissart-Gribov formula in section 3.

This relation when combined with the pole structure in α_1 and α_2 of (2.19) has, as we remarked in section 2, been of great importance in deriving Pomeron decoupling theorems from sub-channel discontinuity formulae[33,36].

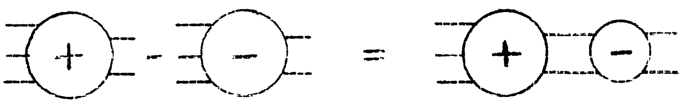
Finally, we remark that the above discussion of the five-point function should generalize in a straightforward way to higher multi-particle amplitudes. By considering the two-particle unitarity equations in the channels of all the Reggeons involved in an asymptotic limit it should be possible to prove factorization of the individual helicity amplitudes contributing to the vertex functions involved. Provided a simple boundary-value prescription (in terms of the angular variables) can be given for each contribution to the multiple integral representation of the amplitude obtained by using the BOW theorem, it should be straightforward to take discontinuities and discuss their factorization. We hope to return to this problem in a future paper.

5. UNITARITY EQUATIONS FOR REGGEON AMPLITUDES

In the previous section we have defined Reggeon scattering amplitudes $A_{\alpha_1(t_1)-N}(t_2, z_2)$ which we can represent pictorially by Fig. 5.1. We defined these amplitudes in terms of a helicity-pole limit of the five-particle amplitude rather than a Regge pole limit. This has the advantage that we can derive unitarity equations for these amplitudes simply by continuing t-channel unitarity equations to complex helicity using the Carlson condition. We shall consider only the two-particle intermediate state here, but we anticipate that the method can be extended to many particle states.

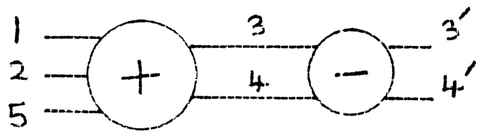
From (4.12) it is clear that we can define A_{α_1} from the residue of the helicity-pole at $n = \alpha_1$ in $a_{>}^{\tau_3}(n)$ defined as in section 2. As we noted in section 4, the pole arises from the pinching of poles at $\ell_1 = \alpha_1$ and $\ell_1 = n$ in (2.8). Since $(\ell_1 - n) = 0$ gives $\tilde{\tau}_1 = 1 = \tau_3\tau_1$ it follows that if the Regge pole α_1 has signature τ_1 then it will appear in the helicity continuation with $\tau_3 = \tau_1$. In the same way it can easily be checked which helicity continuation $A_{\alpha_1 - N}$ will appear in. We could also define A_{α_1} in terms of a pole at $n = -\alpha_1$ in $a_{<}^{\tau_3}(n)$. This is because of the symmetry $-a_{<0}^{\tau_3}(-n) = a_{>}^{\tau_3}(n)$. In a more general theory we would define Reggeon amplitudes $A_{\alpha_1}^{\gtrless}$ which would separately give the positive and negative helicity amplitudes at integer points.

We start with the extended unitarity equation[39] in the t_2-channel



(5.1)

The problem is, of course, to derive the continuation to complex helicity of the integral I on the right-hand side. We label the particles as follows



We can introduce variables $\cos\theta_1$, $\cos\theta_2$, $\cos\omega$ for $\equiv\!\oplus\!=$ $(=A_5)$ as in section 2 but we shall also need the analogous variables for the whole integral I regarded as a five-point function, and we shall need to relate the two sets. The simplest way to do this is to introduce little group transformations (that is rotations). We write

$$u_1 = u_x(\theta_1)\, u_z(\nu_1) \quad , \quad u_2 = u_z(\nu_2)\, u_x(\theta_2) \tag{5.2}$$

where $u_x(\theta_1)$ is a rotation through θ_1 about the x-axis, etc., and $\omega = \nu_1 + \nu_2$. u_1 transforms from the centre-of-mass frame of P_1 and P_2 to the centre-of-mass frame of Q_1 and P_5, while u_2 transforms from the centre-of-mass frame of P_5 and Q_2 to that of P_3 and P_4. In the same way we can introduce transformations u_1 and u_3 to describe the integral I. The four-particle amplitude $=\ominus=$ $(=A_4^-)$ will then be a function of the transformation $u_2^{-1}\, u_3$ which transforms from the centre-of-mass frame of P_3 and P_4 to that of P_3' and P_4'. We can therefore write the integral I in the form[14]

$$I(u_1, u_3) = \int du_2 \; A_5(u_1, u_2) \; A_4^-(u_2^{-1} u_3) \tag{5.3}$$

which, introducing the helicity expansion of A_5 we can write as

$$I(\theta_1, \nu_1+\nu_3, \theta_3) = \int_{-1}^{+1} d(\cos\theta_2) \int_0^{2\pi} d\nu_2 \sum_n a_n(\theta_1, \theta_2)\, e^{in(\nu_1+\nu_2)} A_4^- . \tag{5.4}$$

$$= \sum_n e^{in(\nu_1+\nu_3)} \int d(\cos\theta_2)\, d\nu_2\, a_n(\theta_1, \theta_2)\, e^{in(\nu_2-\nu_3)} A_4^- \tag{5.5}$$

and so we can write the helicity projection of I in the form

$$I_n = \int d(\cos\theta_2) \int_0^{2\pi} d\omega_2\, a_n(\theta_1, \theta_2)\, e^{-in\omega_2}\, A_4^-(\theta_2, \omega_2, \theta_3) \tag{5.6}$$

where $\omega_2 = \nu_3 - \nu_2$.

Since

$$u_2^{-1} u_3 = u_x(-\theta_1)\, u_z(-\nu_2)\, u_z(\nu_3)\, u_x(\theta_3) \tag{5.7}$$

$$= u_x(-\theta_1)\, u_z(\nu_3 - \nu_2)\, u_x(\theta_3) \tag{5.8}$$

it is clear that A_4^- will be a function of ω only and will have no separate dependence on ν_2 or ν_3. It can easily be checked that $t = (P_3 - P_3')^2$ and $u = (P_3 - P_4')^2$ are given by expressions of the form of (2.2) with $t_1 = t_2$, $\omega = \omega_2$, $Q_2 = Q_3$, and ch $\zeta = 1$ so that they depend linearly on cos ω_2. We have, of course, suppressed the dependence on t_1 and t_2 in (5.3) to (5.6).

To define signatured continuations of I_n to complex n which satisfy the Carlson condition we must obviously introduce the continuations $a_{\gtrless}^{\tau_3}(n)$ of a_n. However, the presence of the factor $e^{-in\omega_2}$ in (5.6) means that we must also distort the ω_2-contour in analogy with the definition of Froissart-Gribov continuations to complex helicity[8,9]. Since t and u depend linearly on cos ω_2, A_4^- will have cuts only on the real-axis in the cos ω_2-plane. The contour for the ω_2-integration appears as the unit. circle in the v_2-plane, where $v_2 = e^{i\omega_2}$ and the right and left-hand cuts of A_4^- appear as in Fig. 5.2. To define continuations $I_>^{\tau_3}(n)$ we distort the contour to enclose the right and left-hand cuts of A_4^- in $|v_2| > |$ as shown in Fig. 5.3. (To define continuations $I_<^{\tau_3}(n)$ we distort the contour into $|v_2| < |$.) We can therefore write

$$I_>^{\tau_3}(n) = \int d(\cos\theta_2)\, a_>^{\tau_3}(n) \Big[\int_{C_R} dv_2\, (v_2)^{-n-1} A_4^- + \tau_3 \int_{C_L} dv_2\, (-v_2)^{-n-1} A_4^- \Big] \qquad (5.9)$$

which clearly satisfies the Carlson condition in Re n > 0 provided that $a_>^{\tau_3}(n)$ satisfies this condition. The continuation of the helicity-projection of (5.1) to complex helicity is therefore

$$\underset{t_2 = 4m^2}{\mathrm{disc}}\; a_>^{\tau_3}(n) = I_>^{\tau_3}(n) \qquad (5.10)$$

with $I_>^{\tau_3}(n)$ defined by (5.9). If we now continue this equation to a pole at $n = \alpha_1(t_1)$, we obtain

$$\underset{t_2 = 4m^2}{\mathrm{disc}}\; A_{\alpha_1(t_1)}(t_2, \cos\theta_2) = \int d(\cos\theta_2)\, A_{\alpha_1(t_1)}(t_2, \cos\theta_2)$$
$$\times \Big[\int_{C_R} dv_2\, (v_2)^{-\alpha_1-1} A_4^- - \tau_1 \int_{C_L} dv_2\, (-v_2)^{-\alpha_1-1} A_4^- \Big] \qquad (5.10)$$

This then is the basic unitarity equation for the Reggeon amplitude A_{α_1}. An analogous equation can be derived for A_{α_1-N} by going to the helicity pole at $n = \alpha_1 - N$.

The form (5.10) does not display the reduction of the equation to the usual unitarity equation at integer points in a very transparent way. We can express it in a different form by unwrapping C_R from the right-hand cut of A_4^- as shown in Fig. 5.4. Thus C_R is equivalent to an integral over the unit circle plus a left-hand cut integral. The integral over the unit circle now gives a normal phase-space integral

$$u = \int_{-1}^{+1} d(\cos\theta_2) \int_0^{2\pi} d\omega_2 \, (e^{i\omega_2})^{-\alpha_1} A_{\alpha_1} A_4^- \tag{5.11}$$

As we have shown in Fig. 5.4 the factor $(\nu_2)^{-\alpha_1-1}$ in (5.10) introduces an extra left-hand cut in the ν_2-plane. Therefore (5.10) can be rewritten in the form

$$\underset{t_2=4m^2}{\mathrm{disc}} A_{\alpha_1(t_1)}(t_2, \cos\theta_2) = u + \hat{u} \tag{5.12}$$

where

$$\hat{u} = \int d(\cos\theta_2) A_{\alpha_1}\Big[-\int_{-\infty}^{-1} d\nu_2 \sin\pi\alpha_1 (-\nu_2)^{-\alpha_1-1} A_4^- \\ - \int_{C_L} d\nu_2 \left[(\nu_2)^{-\alpha_1-1} + \tau_1 (-\nu_2)^{-\alpha_1-1}\right] A_4^- \tag{5.13}$$

Since $\hat{u}$ clearly vanishes at right signature integer points we see that only the usual phase-space integral u remains in (5.12). Alternatively $\hat{u}$ gives the correction to the equation for non-integer α_1.

The unsymmetric form for $\hat{u}$ in (5.13) occurs because we chose to unways just C_R in (5.10). We can obtain a symmetric form if we assume that the right and left-hand cuts of A_4^- separately have good asymptotic behaviour. We can then use a dispersion relation for A_4^- to write $A_4^- = A_R^- + A_L^-$ where A_R^- and A_L^- have only right-hand and only left-hand cuts respectively. We can then replace A_4^- by A_R^- and A_L^- in the associated integrals over C_R and C_L in (5.10). Then unwrapping both C_R and C_L as in Fig. 5.4 leads to

$$\underset{t_2=4m^2}{\text{disc}} A_{\alpha_1} = U_1 + \hat{U}_1 \tag{5.14}$$

where now

$$U_1 = \int_{-1}^{+1} d(\cos\theta_2) \int_0^{2\pi} d\omega_2\, A_{\alpha_1} \left[(e^{i\omega_2})^{-\alpha_1} A_R^- + \tau_1 (-e^{i\omega_2})^{-\alpha_1} A_L^- \right] \tag{5.15}$$

and

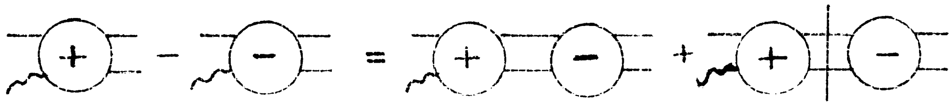
$$\hat{U}_1 = \sin\pi\alpha_1 \int_{-1}^{+1} d(\cos\theta_2)\, A_{\alpha_1} \left[-\int_{-\infty}^{-1} d v_2\, v_2^{-\alpha_1-1} A_R^- + \int_1^{\infty} d v_2 (-v_2)^{-\alpha_1-1} A_L^- \right] \tag{5.16}$$

Now $\hat{U}_1$ vanishes at each integer α_1 and U_1 reduces to an integral over A_4^- only at right signature points.

The v_2 integrations in (5.13) and (5.16) are over unphysical regions for A_4^-. However, since the momentum transfer variables t and u for A_4^- are linearly related to cos ω_2 and v_2 real implies cos ω_2 real we can easily locate these regions. $v_2 < -1$ corresponds to t negative and outside the physical region while $v_2 > 1$ corresponds to u negative and unphysical. Therefore (5.15) and (5.16) together give that the phase-space integration is extended to $t = -\infty$ for the t-cut in A_4^- and to $u = -\infty$ for the u-cut.

There is an important point which we have so far neglected, which is the question of the convergence of the v_2-integrations extending to $\pm\infty$. This is clearly controlled by the Regge behaviour of A_4^-. If the α_1 trajectory is also present in the t_2-channel then we must have $t_1 > t_2$ to obtain convergence of the v_2-integrations. In general, we can expect to have to take $t_1 \gtrsim t_2$ to obtain convergence. This is why we called (5.1) an extended unitarity equation since we were anticipating using the equation outside of the physical region where it initially applies. To extend (5.12) or (5.14) to negative t_1 say, it would be necessary to define $\hat{U}$ and $\hat{U}_1$ by analytic continuation from the region where (5.13) and (5.16) hold.

In general then we can represent the Reggeon unitarity equation pictorially as



(5.17)

where the first term on the right-hand side represents a conventional phase-space integral while the second term represents an integration over unphysical phase-space (when the integral converges) and vanishes at integer α_1. This result can be expected to generalize to a general multi-particle state and to multi-Reggeon amplitudes also. We can expect that Reggeon unitarity relations will in general contain extra terms besides the contributions from the usual intermediate state integrations which will vanish only when all Reggeon trajectories are at physical integer values. If we take the second term in (5.17) to be $\hat{U}_1$ then we can see that this extra term arises from the factor $\nu_2^{-\alpha_1-1}$ which we can say represents the "structure" of the Reggeon. The ν_2 or $\cos\omega_2$ dependence associated with a Reggeon α_1 arises from the dependence of the five-point function on variables like S_{23} (see Fig. 2.1). The dependence on $\cos\omega_2$ is built up by singularities in these variables which arise from intermediate states which can be seen only by "cutting through the Reggeon". This is illustrated in Fig. 5.5 using a Feynman ladder graph as a model of a Reggeon. In this sense then extra terms arise in Reggeon unitarity equations as a result of the structure of a Reggeon being more complicated than that of a particle or resonance.

A further important point is that the amplitudes A_{α_1-N} that we have shown satisfy unitarity-like equations in the t_2-channel are "t_2-channel helicity amplitudes". They give t_2-channel helicity amplitudes at α_1 = integer. We can call these amplitudes "t-channel amplitudes" to make contact with the usual convention in defining s and t-channel helicity amplitudes. We arrived at the amplitudes "$A^t_{\alpha_1-N}$" because we started from a t_2-channel partial-wave analysis of A_5. To define amplitudes "$A^s_{\alpha_1-N}$" which would give s-channel helicity amplitudes at integer points we should have to start with a partial-wave analysis corresponding to the tree diagram of Fig. 5.6 where $S_{45} = (P_4 + P_5)^2$ would be the relevant "s". Repeating the procedure used to define $A^t_{\alpha_1-N}$ would then lead to amplitudes which we can denote by $A^s_{\alpha_1-N}$. The question then arises whether the crossing relation which relates these amplitudes at integer α_1 has any analogue for complex α_1. We anticipate that such a crossing relation does exist. We shall not attempt to prove such a relation here but simply outline how we anticipate the relation can be derived.

We rewrite the partial-wave expansion (2.3) in the form

$$A_5 = \sum_{\ell_1, n} a^t_{\ell, n} \, D^{\ell_1}_{on}(u_1) \tag{5.18}$$

where the t on $a^t_{\ell,n}$ indicates that although we have displayed the partial-wave analysis in the t_1-channel only, we have used t_2 implicitly in defining n. If we introduce the crossing transformation c which will be a function of t_1, t_2 and S_{45} then we can write

$$A_5 = \sum_{\ell_1, n'} \left(\sum_n a^t_{\ell, n} \, D^{\ell_1}_{n'n}(c) \right) D^{\ell_1}_{on'}(u_1') \tag{5.19}$$

where $u_1' = c^{-1} u_1$. c is in fact a rotation which connects the frame in which Q_1 lies along the t ($\equiv$ time)-axis and P_5 lies in the z-t plane, with the frame in which Q_1 still lies along the t-axis but P_3 lies in the z-t plane. If we compare (5.19) with the partial-wave analysis using S_{45} to define n' we have

$$a^s_{\ell, n'} = \sum_n a^t_{\ell, n} \, D^{\ell_1}_{n'n}(c) \tag{5.20}$$

which is the usual crossing relation for crossing t channel helicity amplitudes into s-channel amplitudes. To obtain the generalization of this for complex ℓ_1 we need to Sommerfeld-Watson transform the sums over ℓ_1, n and n' in (5.19). We can ignore signature problems for simplicity but we must take account of the need to distinguish $n \gtrless n'$. If we consider n, $n' \geq 0$ in (5.19) then we can rewrite the triple sum as

$$\sum_{n=0}^{\infty} \sum_{n' \geq n}^{\infty} \sum_{\ell_1 \geq n'}^{\infty} + \sum_{n'=0}^{\infty} \sum_{n > n'}^{\infty} \sum_{\ell \geq n}^{\infty} \tag{5.21}$$

which we Sommerfeld-Watson transform in the form

$$\int\frac{dn}{\sin\pi n}\int\frac{dn'}{\sin\pi(n'-n)}\int\frac{d\ell_1}{\sin\pi(\ell_1-n')} + \int\frac{dn'}{\sin\pi n'}\int\frac{dn}{\sin\pi(n-n')}\int\frac{d\ell_1}{\sin\pi(\ell_1-n)} \tag{5.22}$$

To extract $A^s_{\alpha_1}$ from the transformation of (5.19) that we obtain, we need to perform the n'-integration last in both terms of (5.22) and extract the residue of the pole at $n' = \alpha_1$ which results from the presence of a pole at $\ell_1 = \alpha_1$ in $a^t(\ell_1, n)$. The pole at $\ell_1 = \alpha_1$ will directly give a pole at $n' = \alpha_1$ in the first term in (5.22) because of the factor $\sin\pi(\ell_1 - n')$ in the ℓ_1-integration. The second term can only give a pole at $n' = \alpha_1$ if a pole is produced first at $n = \alpha_1$ which can then pinch the n-contour with the pole at $n = n'$ coming from $\sin\pi(n - n')$ to give a pole at $n' = \alpha_1$. However, we have divided the sums in (5.21) so that the second term in (5.22) only needs to produce the part of the sum with $n > n'$. Therefore, the n-contour in this term will actually lie to the right of the point $n = n'$ in the n-plane and as a result cannot pinch with the pole at $n = \alpha_1$, since the contour also lies to the right of this pole.

It follows then that only the first term in (5.22) will contribute to the pole in the n'-plane at $n' = \alpha_1$ and so extracting the residue of this pole we obtain a crossing relation in the form

$$A^s_{\alpha_1} = \frac{\sin\pi\alpha_1}{2i}\int dn\,\frac{D^{\alpha_1}_{\alpha_1,n}(c)\,a^t(\alpha_1,n)}{\sin\pi n\,\sin\pi(\alpha_1-n)} + \{n\to -n\} \tag{5.23}$$

The contribution to (5.23) from $n < 0$ will be of the same form as the contribution from $n > 0$ but with $n \to -n$. The reduction of (5.23) to the finite-dimensional crossing-relation at α_1 = integer follows in an analogous way to the nonsense decoupling of vertex functions described in section 2. The two sets of poles from $\sin\pi n$ and $\sin\pi(\alpha_1 - n)$ pinch the n-contour to give a pole with a finite sum over helicities in its residue, which is extracted by the $\sin\pi\alpha_1$ factor.

We can obtain an alternative form for the crossing relation by pulling back the n-contour in the left-half n-plane (again as we did for the two Reggeon/particle vertex). We then obtain

$$A^{s}_{\alpha_1} = \sum_{N=0}^{\infty} D^{\alpha_1}_{\alpha_1,\alpha_1-N}(c)\, A^{t}_{\alpha_1-N} \tag{5.24}$$

A similar expression can also be obtained for $A^{s}_{\alpha_1-N}$ and so we see that the finite dimensional relation for helicity amplitudes at integer α_1 generalizes to an infinite dimensional relation for non-integer α_1. In this context it is perhaps interesting to note that the matrices $D^{\alpha_1}_{n,n'}$ give a special irreducible (non-unitary) representation of SO(3) for $\alpha_1 - n$, $\alpha_1 - n'$ = integers[41]. We can therefore say that in a sense a Reggeon corresponds to a representation of SO(3) of this kind and that Reggeon scattering amplitudes describe the scattering of this object. However, the sum over N in (5.24) is effectively an asymptotic expansion in the associated azimuthal angle and so presumably (5.24) will only converge for large values of this variable.

Since we have s and t-channel unitarity equations for Reggeon amplitudes it is possible to discuss the analytic structure of the amplitudes as functions of s and t. In particular, we can look for the maximally analytic solution of the equations as is done for particle amplitudes[38]. Although the extra term in the Reggeon equations makes this more complicated, it should not lead to important differences between the singularity structure of Reggeon and particle amplitudes. The generation of singularities in the unphysical region integrations will presumably be similar to that in the physical phase-space integration. If the infinite dimensional crossing matrix does not introduce extra significant singularities then it seems reasonable to assume that a Landau singularity structure similar to that of particle amplitudes will emerge. That is Reggeon amplitudes will have Mandelstam cut-plane analyticity[42]. This is obviously important for the application of Finite Energy Sum Rules to these amplitudes in inclusive experiments[25,26].

There is however, at least one important distinction between the analytic properties of Reggeon amplitudes and particle amplitudes. The Sommerfeld-Watson representation of A_{α_2} given by (4.13) shows that a

Regge pole at $\ell_1 = \alpha_1$ will lead to poles in A_{α_2} at $\alpha_1 - \alpha_2$ = integer but not to poles at α_1 = integer. This results from the occurrence of $\sin \pi\ (\ell_1 - \alpha_2)$ in the denominator of (4.13) rather than the usual $\sin \pi\ell$. Therefore, the particle pole (of Reggeised particles) in the t-channel of a Reggeon amplitude move with the trajectory of the Reggeon. At first sight it would appear that this is inconsistent with any equation of the form of (5.17). Since $=\ominus=$ certainly has particle poles at the usual positions, it would seem that these poles must directly carry over into at least one of $\tilde{\tau}\oplus=$ and $\tilde{\tau}\ominus=$.

The vital point here is that the residue of a particle pole in $=\ominus=$ is a polynomial in the momentum transfer-variables. This means that this residue will have neither right or left-hand cuts to contribute to (5.10). As a result a particle pole in A_4^- will be cancelled by zeros of the integrations in (5.10). The convergence of the integrals is, of course, essential and this is ensured by the factors $(\nu_2)^{-\alpha_1-1}$, $(-\nu_2)^{-\alpha_1-1}$ for α_1 sufficiently large.

Since the particle poles in the t-channel of a t-channel Reggeon amplitude move with the trajectory function it might be thought that the s-channel poles (of a t-channel amplitude) also move. This would appear to be suggested by the crossing relation (5.24). However, it can easily be seen that the s-channel poles cannot move by considering the asymptotic behaviour of the five-point function. In particular, we can consider the asymptotic behaviour of the contribution of a pole in the S_{45}-channel, (as shown in Fig. 5.7) to the helicity-pole limit of section 4. This limit will be equivalent to the usual Regge limit of the four-point function shown in Fig. 5.7, that is $S_{23} \to \infty$ with t_{12} fixed. As a result the asymptotic behaviour as $u \to \infty$ of this contribution will certainly be of the form u^{α_1}. Therefore, using the definition of $A^t_{\alpha_1}$ given by (4.12) it is clear that Fig. 5.7 will contribute to $A^t_{\alpha_1}$ and that it will give a particle pole in the S_{45}-channel at the normal position.

In fact, if we use the crossing relation in the form (5.23), we can show that the s-channel poles do not move. The Sommerfeld-Watson representation of $a^t(\alpha_1,n)$ will be

$$a^t(\alpha_1,n) = \frac{1}{2i}\int \frac{d\ell_2 \; P_{\ell_2}^{-n}(z_2)}{\sin\pi(\ell_2-n)}\, \tilde{g}(\alpha_1,\ell_2,n,t_1,t_2) \qquad (5.25)$$

and so a pole $\ell_2 = \alpha_2(t_2)$ will give a pole in $a^t(\alpha_1,n)$ at $n = \alpha_2(t_2)$. However, this pole will lie on the left-hand side of the n-contour of (5.23) in the n-plane. As a result it can only give a pole in $A^s_{\alpha_1}$ by pinching with poles on the right-hand side of the contour. The wrong-signature nonsense poles of $a^t(\alpha_1,n)$ at $n = \alpha_1 + 1$, $\alpha_1 + 2$, .. etc. will in fact be cancelled by signature factors. This means that the pole at $n = \alpha_2(t_2)$ can only pinch with the poles at integer n coming from $\sin\pi n$. Therefore, a Regge pole in the t_2-channel gives only normal particle poles in $A^s_{\alpha_1}$. Or equivalently $A^t_{\alpha_1}$ will have only the normal particle poles in the s-channel. Although, each term in (5.24) has poles in the t_2 channel at $\alpha_2(t_2) - \alpha_1(t_1) = 0$, the whole sum does not.

Despite the fact that the crossing transformation actually simplifies the pole structure of Reggeon amplitudes it seems unlikely that this is the case for the multi-particle thresholds. It seems therefore that a t-channel Reggeon amplitude will not satisfy any simple unitarity relation in the s-channel. The s-channel amplitudes will, of course, satisfy an analogous relation to (5.17) but the crossing transformation will not commute with the integration involved in this equation.

By considering some of the simplest Feynman graphs, Gribov and Migdal[43] have suggested that the particle-Reggeon scattering amplitude (defined through a helicity-pole limit of the six-point function as indicated by Fig. 5.8) will satisfy a simple "s-channel" unitarity relation. ("s-channel" is equivalent to the M^2-channel in Fig. 5.8.) The particle-Reggeon amplitude has to give a physical t-channel helicity amplitude at integer points on the trajectories and these physical amplitudes do not satisfy simple unitarity relations in the s-channel. Therefore, a unitarity relation in the form given by Gribov and Migdal cannot be exact. We can again expect that only s-channel Reggeon-particle scattering amplitudes will satisfy a simple s-channel unitarity relation and even this will contain extra terms besides the usual phase-space integration.

These extra terms will arise from azimuthal angle factors of the form "$(e^{i\omega})^{\alpha}$" associated with a Reggeon amplitude. In the Feynman graph approach of Gribov-Migdal these factors correspond to the extra "Sudakov parameter" dependence in the vertices of the Feynman graphs used to define the Reggeon-particle scattering amplitude. Gribov and Migdal argue that in general these extra factors will not affect the form of the unitarity relation. Consistency with our results would seem to require that this cannot be the case and that complications must arise from the Sudakov parameter dependence of, at least, some Feynman graphs.

As Gribov and Migdal point out, they do not have a rigorous definition of the Reggeon-particle scattering amplitude (as we do) and so their definition of this amplitude in terms of Feynman graphs may not coincide with ours. However, if the Gribov treatment of the asymptotic behaviour of Feynman graphs is right, then it would seem that the definition of the Reggeon-particle scattering amplitude through the asymptotic behaviour of the six-point function would be sufficient to ensure the coincidence of the two definitions.

The precise definition of what Gribov and Migdal call the amplitude for a particle and a Reggeon to scatter into many particles (Fig. 5.9) is not so clear. This is important since, as we now show, the application of the crossing transformation to the unitarity relation (5.10) does give a relation which can, in a sense, be approximated by simple phase-space integrals. Applying the transformation in the form (5.23) to (5.9) (written in an unsignatured form since we are still ignoring signature for simplicity) we obtain the s-channel relation for $A^t_{\alpha_1}$ in the form

$$A^t_{\alpha_1} = \frac{\sin\pi\alpha_1}{2i}\int \frac{dn\, D^{\alpha_1}_{\alpha_1 n}(c)}{\sin\pi n \sin\pi(\alpha_1 - n)}\left[\int d(\cos\theta_2)\, a_>(n)\int d\nu_2\, \nu_2^{-n-1} A_4^-\right] \tag{5.26}$$

$$+ \{n \to -n,\ > \to <\}$$

and interchanging the n-integration with the cos θ_2 and ν_2 integrations, we obtain

$$A^t_{\alpha_1} = \int d(\cos\theta_2)\int d\nu_2\, A_4^- \frac{1}{2i}\int\frac{dn \sin\pi\alpha_1}{\sin\pi n \sin\pi(\alpha_1 - n)}$$

$$\times\ D^{\alpha_1}_{\alpha_1, n}(c)\ a_>(n)\ (\nu_2)^{-n-1} + \{n \to -n\,;\ > \to <\} \tag{5.27}$$

Now instead of moving the n-contour to the left as usual, we can consider moving it to the right. The only poles which will contribute will be those at integer n. The cut in the v_2-plane coming from the $(v_2)^{-n-1}$ factor will not be present in these contributions and so the v_2-contour can be closed to give the usual unit circle phase-space integration. Therefore, if we could evaluate the n integral entirely in terms of these poles we would have an expression for the s-channel discontinuity of $A^t_{\alpha_1}$ involving only the usual phase-space integration. However, to similarly evaluate the $\{n \to -n, > \to <\}$ term in (5.27) it would be necessary to move the analogous n-contour to the left. We can only expect to be able to ignore the background n-integral in such manipulations if we consistently move the n-contour in the same way in both the > and < terms. Since only in this case are we deriving an asymptotic expansion in the associated azimuthal angle.

In general, then we expect to be able to write (5.27) in the form

$$A^t_{\alpha_1} = \frac{1}{2i}\int d(\cos\theta_1)\int d\omega_2 \sum_N D^{\alpha_1}_{\alpha_1,N}(c)\; a_>(N)\left(e^{i\omega_2}\right)^{-N-1} A_4^- \tag{5.28}$$

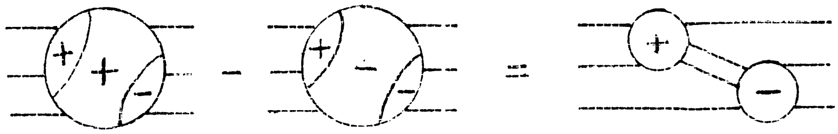
$$+ \text{background integral} + \{n \leftrightarrow -n\,,\ > \to <\}$$

Even if it were possible to neglect the background integral in this expression, it is clear that

$$\sum_N D^{\alpha_1}_{\alpha_1,N}(c)\; a_>(N)\; \left(e^{i\omega_2}\right)^{-N-1}$$

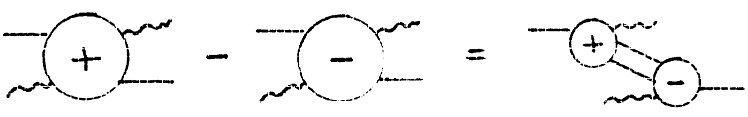
is not a simple Reggeon amplitude. Firstly the crossing transformation involved is that for the external particles and secondly the sum over integer helicities does not produce anything easily identifiable.

It seems then that part of the difficulty in reconciling our results with the argument of Gribov and Migdal may lay in a rigorous definition of the Reggeon/many particle amplitudes introduced by them. The need to pay careful attention to the helicities involved in defining Reggeon amplitudes is emphasized also by considering the implications of the Mueller discontinuity formula for Reggeon amplitude unitarity equations. Consider the contribution of the two-particle state to the Mueller formula



(5.29)

If we take the helicity-pole limit of this equation and assume that the limit commutes with the phase-space integration on the right, we obtain



(5.30)

This appears to be an s-channel unitarity relation for a t-channel Reggeon amplitude of just the form that we have been arguing against. However, we should first note that the limit certainly may not commute with the phase-space integration. Secondly, the amplitudes involved on the right-hand side of (5.30) will not be the amplitudes A^t_α we have been considering, but rather the amplitudes B_α defined in section 4. This is because the limit of the five-particle amplitudes involved is a physical region limit and as a result the relevant Toller angle variable ($\cos \omega$) cannot be taken large to pick out a particular helicity of the Reggeon. Therefore, if a relation of the form of (5.30) can be obtained from the Mueller formula, it does not correspond in any simple way to the relations we have derived from t-channel unitarity. As we have shown both our unitarity and crossing relations continue to integer points on the Regge trajectory in a well-defined way and give satisfactory results. It is difficult to see how this can also be the case for a relation of the form of (5.30). If (5.30) does not hold the difference between the two sides could in principle be detected by the inclusive measurement of the left-hand side and the exclusive measurement of the right-hand side.

Finally, we comment briefly on the Reggeon amplitudes that appear in Regge cut discontinuity formulae. First we consider the Reggeon-particle cut whose discontinuity[14] in the four-particle amplitude is illustrated pictorially in Fig. 5.10. We consider the ℓ_2-projection of A_{α_1}, that is

$$A_{\alpha_1}(\ell_2, t_2) \equiv \lim_{\ell_1 \to \alpha_1} \frac{a(\ell_1, \ell_2, \ell_1)}{\beta_{\alpha_1}(\ell - \alpha_1)} \tag{5.31}$$

[Note that the ℓ_2-plane unitarity condition for this amplitude is obtained by direct continuation to complex ℓ_2 of the t_2-channel analogue of (4.2). It contains no extra terms. This does not lead to any inconsistency because the equation cannot be obtained by direct partial-wave projection of either (5.12) or (5.14).] $A_{\alpha_1}(\ell_2, t_2)$, as we have defined it, would have a nonsense zero at $\ell_2 = \alpha_1 - 1$ if $A_{\alpha_1}(\cos\theta_2, t_2)$ had no third double spectral function. Therefore, $A_{\alpha_1}(\alpha_1 - 1, t_2)$ is determined only by this spectral function, and it is this amplitude which appears in the discontinuity formula for the Reggeon-particle cut. [If $\tilde{A}_{\alpha_1}(\alpha_1 - 1, t)$ is defined using the group-theoretic coefficiencs of section 3 then $\tilde{A}_{\alpha_1}$ will have an inverse square root branch-point at $\ell_2 = \alpha - 1$ and $A_{\alpha_1}(\alpha_1 - 1, t_2)$ with be the "residue" of this branch-point.] A Froissart-Gribov formula for $A_{\alpha}(\ell_2, t_2)$ can be obtained by writing a dispersion relation for

$$A_{\alpha_1}(z_2, t_2) \,/\, (z_2^2 - 1)^{\alpha_1}$$

as a function of z_2. The result is then a simple generalization of the usual Froissart-Gribov formula

$$\begin{aligned} A_{\alpha_1}^{\tau_2}(\ell_2, t_2) = \frac{1}{2\pi} \int_{C_R} dz_2\, Q_{\ell_2}^{\alpha_1}(z_2)\, A_{\alpha_1}(z_2, t_2) \\ + \frac{\tau_1 \tau_2}{2\pi} \int_{C_L} dz_2\, Q_{\ell_2}^{\alpha_1}(-z_2)\, A_{\alpha_1}(z_2, t_2) \end{aligned} \tag{5.32}$$

The integrals C_R and C_L being round the right and left-hand cuts of $(-z_2^2 + 1)^{-\alpha_1} A_{\alpha_1}$ respectively. Since $Q_{\ell_2}^{\alpha_1}(z_2)$ reduces to a constant at the nonsense wrong-signature point $\ell_2 = \alpha_1 - 1$, the integrals over C_R and C_L are equal (neglecting the contour at infinity) and so

$$A_{\alpha_1}(\alpha_1 - 1, t_2) \propto \int_{C_R} dz_2 \, A_{\alpha_1}(z_2, t_2) \tag{5.33}$$

So the "fixed-pole residue" which appears in the discontinuity of the Reggeon particle cut, can be expressed as an integral over the discontinuity of A_{α_1}. This discontinuity is precisely what appears in the asymptotic limit of $\underset{S_{45}}{\text{disc}} A_5$ as we showed in (4.19). Of course, (5.33) is only true when the integral converges.

For the two Reggeon cut, the discontinuity in the four-particle amplitude (Fig. 5.11), involves the Reggeon-particle amplitude $A_{\alpha_1\alpha_2}(j,t)$ evaluated at the "fixed-pole" point $j = \alpha_1 + \alpha_2 - 1$. This can also be written as an integral of the discontinuity of the Reggeon-particle scattering amplitude, which as we have already said can be measured in the inclusive cross-section. We should note that this integral will not converge near the branch-point and so the one-particle contribution to the inclusive cross-section cannot be used[44] to obtain the Gribov-Migdal lower bound[43] for the cut contribution to the four-particle amplitude. It is important to note that it is the t-channel derivation of Regge cut discontinuities which gives that the Reggeon particle amplitude measured in the inclusive cross-section is directly related to that appearing in the two-Reggeon discontinuity.

The amplitude, which Abarbanel[27] and also Chew and Ellis[28,45] have suggested should be measured for the two-Reggeon cut discontinuity formula is potentially different from this. The s-channel treatment of the two-Reggeon cut given by Abarbanel associates the cut directly with the presence of Reggeon exchange in production amplitudes. As a result the amplitude which appears in Abarbanel's formula is obtained by measuring all multi-Reggeon amplitudes of the form shown in Fig. 5.9 and performing the unitarity sum and integration over states shown in Fig. 5.12. This will certainly give a contribution to the inclusive cross-section in the relevant helicity-pole limit. However, this can only be identified with the complete "absorptive part" of the Reggeon-particle amplitude if (5.30) and its generalization to multi-particle states holds.

If the only unitarity equations that hold for Reggeon amplitudes are the ones analogous to those we have given, then this will not be the case. Since Fig. 5.12 represents an "s-channel" phase-space integral both the crossing relation and the extra terms in our equations would be involved in relating this to the absorptive part of the t-channel Reggeon/particle amplitude. It seems that the structure of a Reggeon is playing an important role in preventing the comparison of Abarbanel's formula for cuts with that obtained from the t-channel. This bears on the problem of the sign of the cut and a further discussion of this point will be given in a forthcoming paper[46] in the context of the Pomeron problem.

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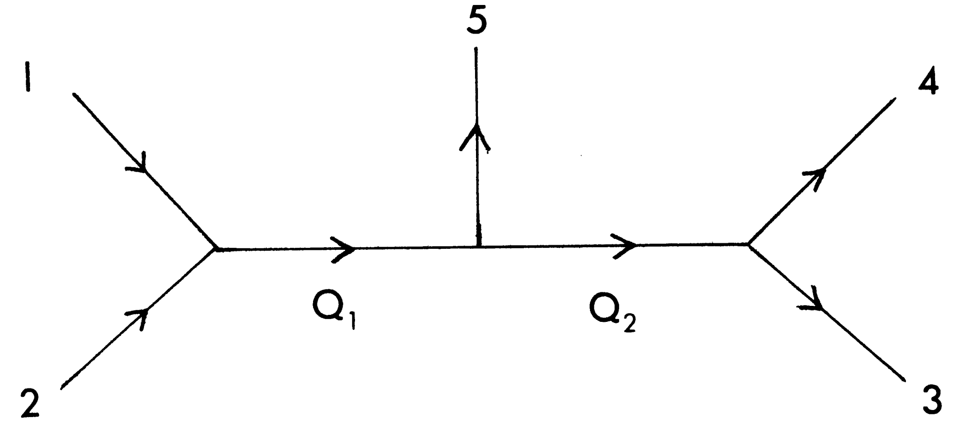
Figure captions



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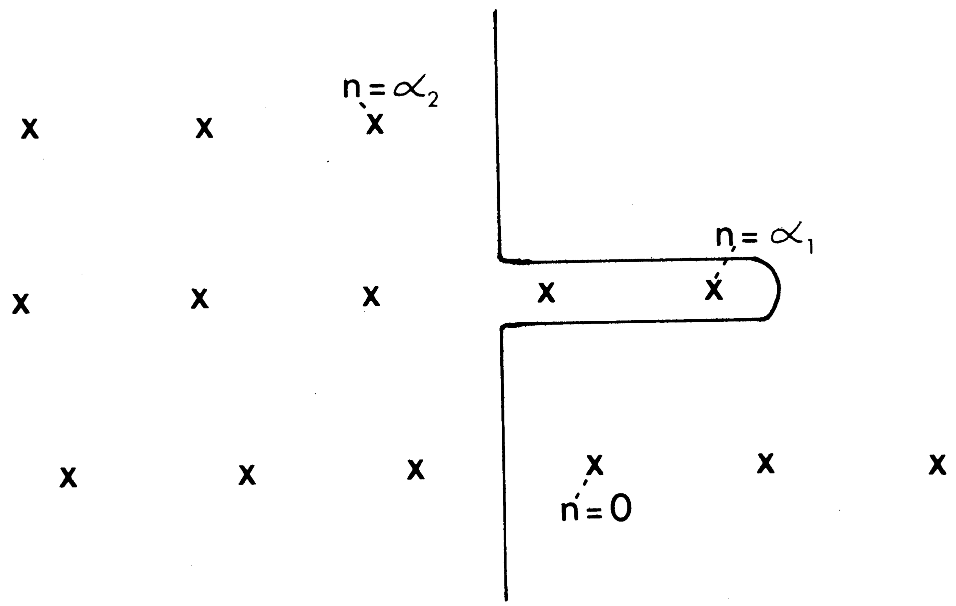

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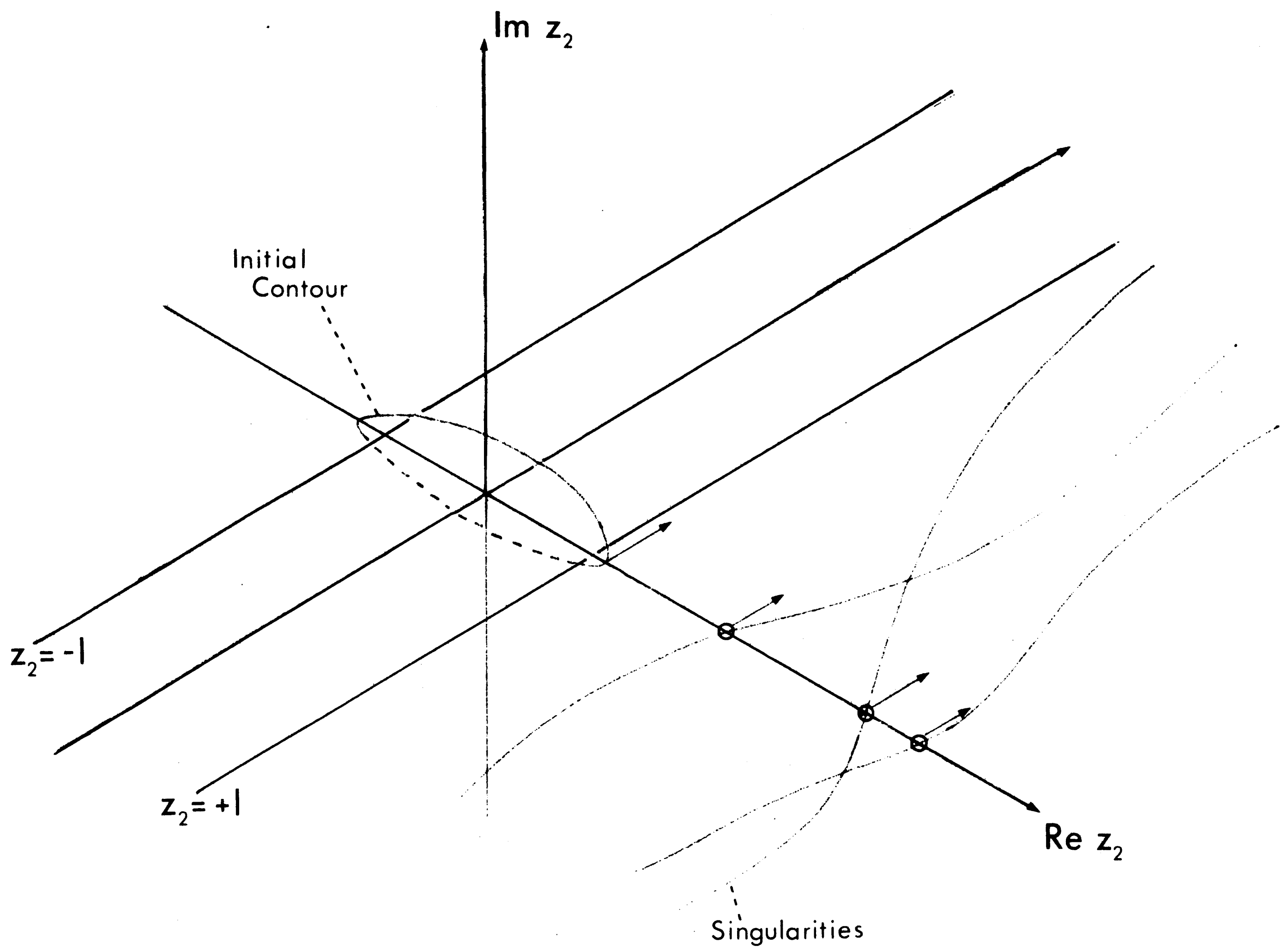

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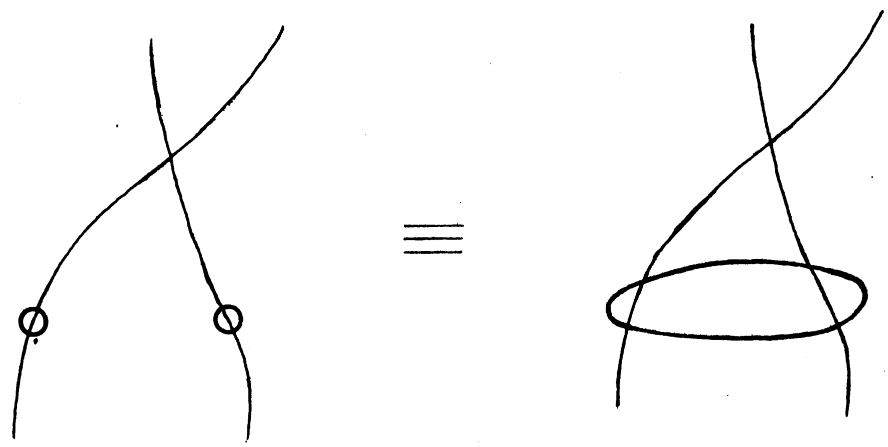

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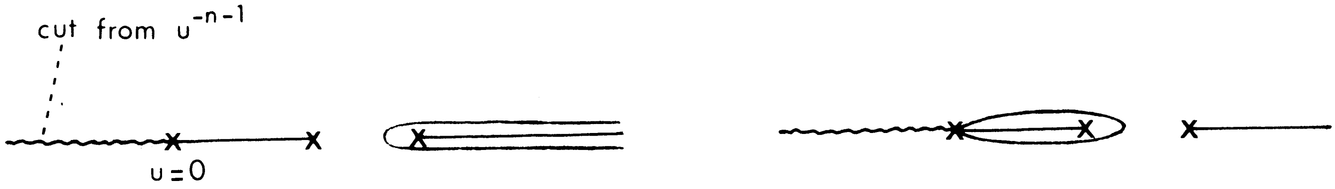

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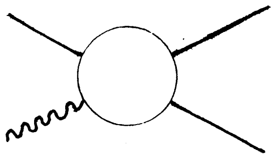

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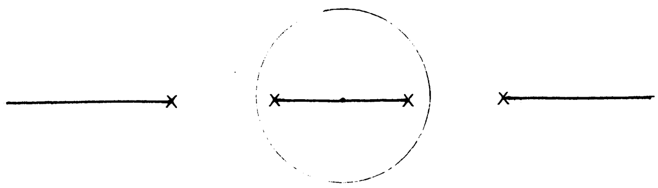
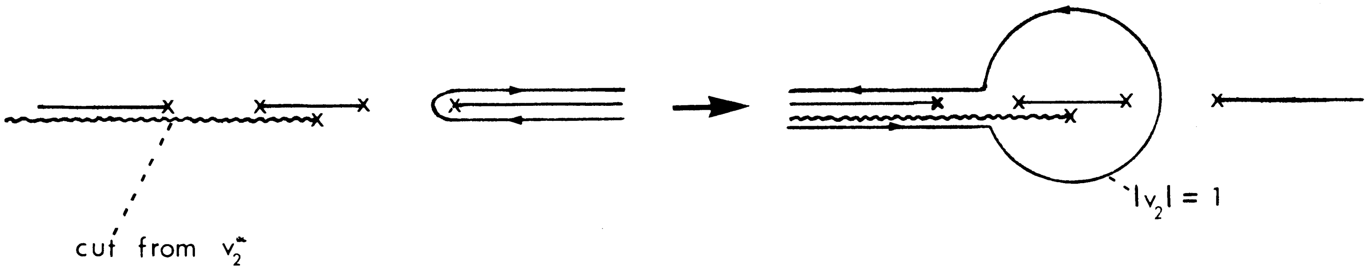


Fig. 5.2

Fig. 5.3



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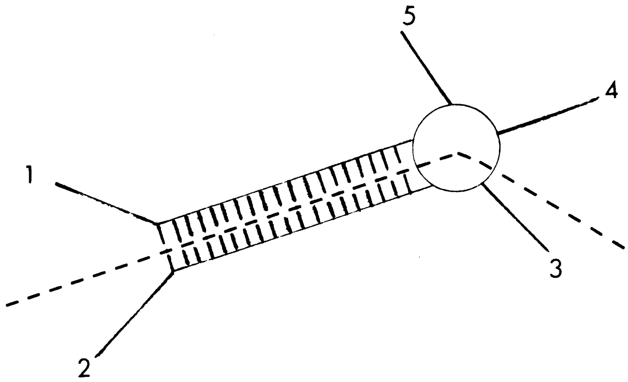

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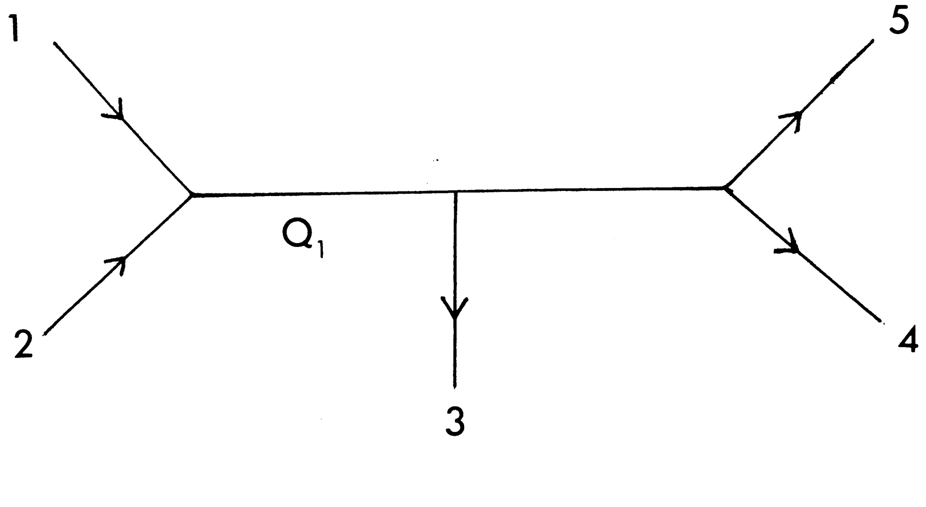

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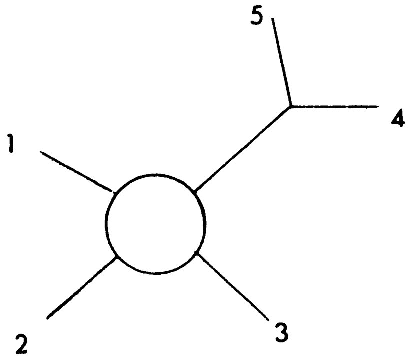

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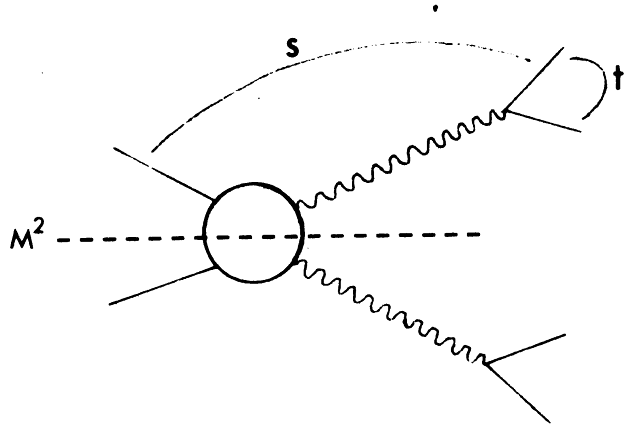

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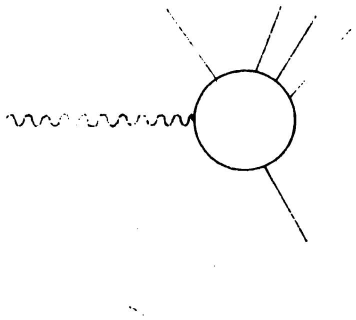

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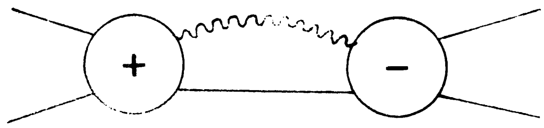

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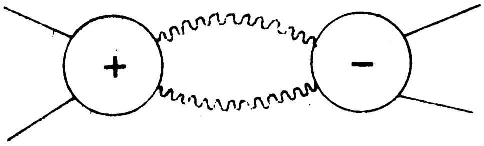

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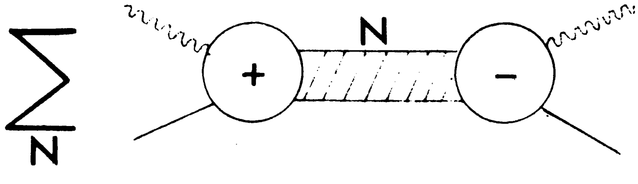

Fig. 5.12