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UNITARITY BOUNDS ON ELASTIC ABSORPTIVE PARTS IN TERMS OF THE DIFFRACTION PEAK WIDTH

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A B S T R A C T

Upper and lower bounds on the absorptive contribution to the elastic differential cross-sections are given in terms of the forward slope and the elastic and total cross-sections. They are compared with recent experimental data.

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A long time ago, McDowell and Martin [1] derived a lower bound on the forward logarithmic slope b of the absorptive contribution to the elastic differential cross-section $d\sigma^A/dt$ of two spinless particles,

$$b(s) \equiv \frac{d}{dt} \mathrm{Log} \frac{d\sigma^A}{dt}(s,t)\Big|_{t=0} \quad , \tag{1}$$

in terms of the total cross-section σ_{tot} and the absorptive contribution to the elastic cross-section $\sigma^A_{e\ell}$. This bound, which follows from unitarity alone, is valid at all energies and reads :

$$b(s) > \frac{\sigma^2_{tot}(s)}{18\pi\, \sigma^A_{e\ell}(s)} - \frac{2}{9k^2} \quad . \tag{2}$$

Later on, this result has been extended by Roy and Singh [2], who obtained a sharp unitarity upper bound on the absorptive amplitude $A(s,t)$ at negative t, and hence on

$$\frac{d\sigma^A}{dt}(s,t) = \frac{\pi}{sk^2} A^2(s,t) \quad , \tag{3}$$

in terms of the same quantities σ_{tot} and $\sigma^A_{e\ell}$. Remarkably enough, this bound is almost saturated by the high energy experimental data in the near forward region of the diffraction peak, under the assumption of purely absorptive, spin-independent, scattering. On the other hand, this bound being everywhere positive does not allow us to predict the existence of physical zero(s) of $A(s,t)$ [and hence of dip(s) in the elastic cross-section] on the basis of unitarity. Further, no non-trivial lower bound on $d\sigma^A/dt$ exists, given only the values of σ_{tot} and $\sigma^A_{e\ell}$. In this letter, we show that by adding the experimental information the bound (2) on $b(s)$ is close to saturation ; we get both a non-trivial lower bound on $d\sigma^A/dt(s,t)$ as well as information on the existence (and location) of zeros of $A(s,t)$. This is suggested by the fact that the exact saturation of the McDowell-Martin bound completely determines the amplitude $A(s,t)$, which sisplays infinitely many physical zeros in the asymptotic limit. A Actually, inserting the known [1] partial wave distribution $a_\ell(s)$ which saturates inequality (2) into the expansion :

$$A(s,t) = \frac{\sqrt{s}}{k} \sum_{\ell=0}^{\infty} (2\ell+1)\, a_\ell(s)\, P_\ell\left(1 + \frac{t}{2k^2}\right) , \qquad (4)$$

one easily obtains for $s \to \infty$:

$$A(s,t) \simeq \frac{3s}{16\pi}\, \sigma_{tot}(s)\, \frac{J_2\left(4\sqrt{\frac{T}{3}}\right)}{T} , \qquad (5)$$

where

$$T = -t\, \frac{1}{16\pi}\, \frac{\sigma_{tot}^2(s)}{\sigma_{e\ell}^A(s)} . \qquad (6)$$

One can then expect that, when b gets sufficiently close to its minimal allowed value, the qualitative picture exhibited by any unitary amplitude should not be too different from that given by Eq. (5). This in turn should imply the existence of unitarity upper and lower bounds tending to a common limit given by the right-hand side of Eq. (5) when b reaches its lower bound. We may also expect that improving the value of b would improve the upper bound of Ref. 2) in such a way that it could become interesting at higher values of t.

In view of these considerations, we have looked for the best possible upper and lower bounds on $A(s,t)$ for $t < 0$, given the values of σ_{tot}, $\sigma_{e\ell}^A$ and b. Here, we present our results, and compare them with a few experimental data in order to stress their usefulness. The details of the derivation, together with some related questions, will be discussed elsewhere. Two simplifying assumptions have been made.

i) Spin effects are neglected [preliminary results [3] seem to indicate that spin effects can spoil the scalar bounds only weakly, as they do for the McDowell-Martin bound (2) *)].

ii) The asymptotic approximation is used. This means that the energy is high enough to make all terms of relative order $O(1/s)$ negligible. In particular, it is assumed that $s\sigma_{tot}^2/\sigma_{e\ell} \gg 1$, so that one

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*) They have been shown to reduce the right-hand side of Eq. (2) by a factor of less than 1% [4].

is allowed to replace the Legendre polynomials by their Bessel approximation $J_o(2\ell\sqrt{-t/s})$ and the sum over ℓ by an integral in Eq. (4). This approximation, although very good in the energy range considered, is by no means essential. It has the virtue of producing bounds on $A(s,t)/A(s,o)$ which no longer depend explicitly on s, but only on two dimensionless parameters, namely the rescaled transfer variable T defined by Eq. (6) and the R parameter, defined as

$$R = 18\pi \frac{\sigma_{el}^{A}(s)}{\sigma_{tot}^{2}(s)} b(s) \tag{7}$$

[the numerical factor in the right-hand side is adjusted so that the bound (2) simply reads $R \geq 1$].

Then, according to Eqs. (1), (3), (4) and the optical theorem, the problem to be solved can be stated as follows.

For each $t < 0$, find the supremum and the infimum of

$$A(s,t) = 2 \int_0^\infty d\ell \; 2\ell \; a_\ell(s) \, J_o\left(2\ell\sqrt{\tfrac{-t}{s}}\right) , \tag{8}$$

given the quantities

$$\sigma_{tot} = \frac{16\pi}{s} \int_0^\infty d\ell \; 2\ell \; a_\ell(s) , \tag{9}$$

$$\sigma_{el}^{A} = \frac{16\pi}{s} \int_0^\infty d\ell \; 2\ell \; a_\ell^2(s) , \tag{10}$$

$$\sigma_{tot} \cdot b = \frac{64\pi}{s^2} \int_0^\infty d\ell \; 2\ell \; a_\ell(s) \frac{\ell^2}{2} , \tag{11}$$

and the positivity constraints

$$a_\ell(s) \geq 0 \quad \text{for all } \ell\text{'s} \tag{12}$$

[we have omitted the remaining unitarity constraints, $a_\ell(s) \leq 1$, as they turn out to be automatically satisfied by the solutions, given high energy data on σ_{tot}, $\sigma_{e\ell}^{A}$ and b].

The solution is obtained in a standard way. First, the variational method with Lagrange multipliers is applied heuristically to derive partial wave distributions $a_\ell(s)$. Then, the corresponding extremal amplitudes are shown to be true bounds by a "direct subtraction" proof [5]. For the upper bound, it is found that there is an interval $0 \leq T \leq T_c$ (with T_c depending on R) where the variational equations have no solutions. In this case, the best upper bound actually coincides with the b independent bound of Ref. 2), and corresponds to a supremum of the functional (8) under the constraints (9)-(12) which is not attained but only approached arbitrarily well. In the other cases, the computation of the Lagrange multipliers is tedious and has to be done partly numerically, so that the resulting bounds $g_{U,L}(T,R)$ on $A(s,t)/A(s,o)$:

$$g_L(T,R) \leq \frac{A(s,t)}{A(s,o)} \leq g_U(T,R) \qquad (T \geq 0, R \geq 1) \tag{13}$$

cannot be expressed in a closed form. That these bounds actually depend only on T and R is obvious from Eqs. (8)-(12), after an appropriate rescaling of ℓ and $a_\ell(s)$.

For the needs of application, we have found approximate formulae for $g_{U,L}(T,R)$, the error of which does not exceed 0.001 when $0 \leq T \leq 6$ and $1 \leq R \leq 1.222$:

$$g_U(T,R) \cong \frac{3J_2\left(4\sqrt{\frac{T}{3}}\right)}{2T} + \frac{T^3}{100}\sqrt{R-1}\Big[\left(1.848 - 5.127\sqrt{R-1} + 7.434(R-1)\right) - \left(0.5435 - 1.907\sqrt{R-1} + 2.760(R-1)\right)T + \left(0.4364 - 1.773\sqrt{R-1} + 2.564(R-1)\right)\frac{T^2}{10}\Big] \tag{14}$$

$$g_L(T,R) \cong \frac{3J_2\left(4\sqrt{\frac{RT}{3}}\right)}{2RT} - \frac{T^2}{100}\sqrt{R-1}\Big[\left(3.884 - 4.36\sqrt{R-1} + 8.78(R-1)\right) - \left(0.813 - 1.474\sqrt{R-1} + 2.768(R-1)\right)T + \left(0.528 - 1.306\sqrt{R-1} + 2.289(R-1)\right)\frac{T^2}{10}\Big] \tag{15}$$

Squaring $g_{U,L}(T,R)$ provides us with the desired bounds on the differential cross-section. Notice that $g_U^2(T,R)$ $(g_L^2(T,R))$ gives a lower (upper) bound wherever $g_U(T,R)$ $(g_L(T,R))$ turns out to be negative :

$$\underset{U,L}{\mathrm{Min}}\ g_{U,L}^2(T,R) \leq \frac{d\sigma^A}{dt}(s,t)\Big/\frac{d\sigma^A}{dt}(s,0) \leq \underset{U,L}{\mathrm{Max}}\ g_{U,L}^2(T,R) \, . \tag{16}$$

One can show that these optimal bounds remain true bounds when $\sigma^A_{e\ell}(s)$ is replaced everywhere (in T and R) by $\sigma_{e\ell}(s)$. It is worth mentioning that the variable T defined by Eq. (6) is nothing but the natural variable occurring in the "weak scaling" property of Cornille and Martin [6)]. Whenever such a scaling takes place (which happens if $s\,\sigma^2_{tot}/\sigma_{e\ell} \to \infty$ and R remains bounded when $s \to \infty$), the limit function $f(T)$ of Ref. 6) has to obey the bounds of Eq. (6).

Coming back to the question of zeros of $A(s,t)$, we find a partial answer in the fact that $g_U(T,R)$, as a function of T, exhibits changes of sign when $1 \leq R \leq 1.028$. This entails the existence of physical zeros, if R is in this range. Also some information on their location is obtained. If, e.g., $R = 1.028$, the first zeros of $g_U(T,1.028)$ and $g_L(T,1.028)$ occur at $T = 7.4$ and 4.2, respectively, which means that $A(s,t)$ must vanish at least once in the interval $(4.2)16\pi\,\sigma_{e\ell}/\sigma^2_{tot} \leq$ $\leq -t \leq (7.4)16\pi\,\sigma_{e\ell}/\sigma^2_{tot}$. This result already has some practical interest. Consider for example pp scattering at $p_{lab} \gtrsim 100$ GeV/c : from the absence of zero below $T = 7.4$ (the t value of the dip in the differential cross-section corresponds to $T \simeq 15$), we immediately infer that $R > 1.028$. It must be stressed, however, that the condition $R < 1.028$, although sufficient for enforcing at least one physical zero, is by no means optimal. This is due to the fact that the upper bound $g_U(T,R)$ <u>cannot</u> be saturated simultaneously for all T's. The true maximal value of R for which unitarity requires a physical zero must lie somewhere between 1.028 and 1.125 [$A(s,t) \propto \exp(-\alpha t)$ meets the positivity condition and gives $R = 1.125$]. Its determination demands the solution of a separate problem.

More interesting are the bounds themselves. On Fig. 1, they are plotted against T for values of R ranging between 1 and 1.222, together with some experimental data at various energies, taken from Refs. 7). Of course, the upper and lower bounds coincide for $R = 1$ [and are obtained by squaring Eq. (5)]. The points a, b, c correspond to T values where the (negative) lower bound on the amplitude begins to exceed the upper bound in magnitude, so that the upper bound on the cross-section is given by $g_L^2(T,R)$ for higher values of T [as explained before Eq. (6)]. Still, the "extrapolated" upper bounds $g_U^2(T,R)$ (faint lines) retain a sensible meaning, because they continue to represent true upper bounds if one knows that no zero has occurred below the T value considered. One observes that : i) the lower bounds are good only in the low transfer region $(0 < T \lesssim 2)$; ii) for reasonable values of R, the upper bounds are quite good in an extended range $(0 < T \lesssim 6)$, and improve considerably the R independent bound of Ref. 2) ; they can be used to put lower bounds on R. As a rough illustration, consider pp scattering at $p_{lab} = 100$ GeV/c : comparing the data points between $T = 3$ and 5 with the UB curves, and assuming that the real part effects are negligible, one deduces that R is certainly larger than 1.13, whereas the purely experimental value is $R = 1.09 \pm 0.05$.

For the sake of comparison with experimental data, the use of the variable T is not very convenient, because the horizontal error bars (due mainly to the uncertainty on $\sigma_{e\ell}$) are "correlated" with the value of R (which also depends on $\sigma_{e\ell}$). We have made a more detailed analysis of the recent (and preliminary) results of Ankenbrandt et al. [8] on pp elastic scattering at $p_{lab} = 200$ GeV/c by coming back to the t variable (Figs. 2 and 3). The two relevant parameters are $b\,[(\mathrm{GeV/c})^{-2}]$ and $\sigma_{e\ell}/\sigma_{tot}^2\,[\mathrm{mb}^{-1}]$. Extrapolating the data points of Ref. 8) down to $t = 0$ gives $9.7 < b < 10.5$, whereas [9] $0.0042 < \sigma_{e\ell}/\sigma_{tot}^2 < 0.0048$. The corresponding lower bounds are plotted in Fig. 2 [the range of $\sigma_{e\ell}/\sigma_{tot}^2$ has been reduced to (0.00468, 0.0048), as $R \geq 1$ implies $\sigma_{e\ell}/\sigma_{tot}^2 > 0.00468$ for $b = 9.7$]. It appears that the lower bound is not violated by the experimental points in the region $0.07 \lesssim t \lesssim 0.13\ [(\mathrm{GeV/c})^2]$ only if $b > 10.3$, whatever value of $\sigma_{e\ell}/\sigma_{tot}^2$ is chosen within its full experimental range. Further information is gained by looking at the upper bounds plotted in Fig. 3 (where we have confined the two parameters within their new allowed intervals). Demanding that the bounds be not violated in the region $0.3 \lesssim t \lesssim 0.5$ leads to the conclusion that $b > 10.5$ and $\sigma_{e\ell}/\sigma_{tot}^2 > 0.0048$. Of course, these results are reliable only to the extent that the real part (and spin) effects for the t values considered are small as compared to the level of accuracy we are working with (~5% on the cross-sections). At $p_{lab} = 200$ GeV/c, this is a rather reasonable assumption, since the real part at $t = 0$ has been found to be practically vanishing (≲3% on the amplitude [10]). Moreover, as far as lower bounds are concerned, notice that the only assumption of a vanishing forward real part suffices to make them exact bounds on $d\sigma^A/dt(s,t)$ (because the forward slope b is entirely due to the absorptive part in that case).

Then $d\sigma/dt|_{exp} - d\sigma^A/dt|_{LB}$ provides us with an interesting upper bound for the real part effects in non-forward directions. Let us finally point out a general feature which is apparent on Figs. 2 and 3 : for fixed, and not too high, t values $(t \lesssim 0.2)$, the lower bound is rather insensitive to the variations of the parameter $\sigma_{e\ell}/\sigma^2_{tot}$, whereas the upper bound is rather insensitive to the variations of b.

We believe that our bounds are of use in the analysis of high energy elastic scattering data. They offer a means of narrowing the range of the less accurately known quantities, like b, $\sigma_{e\ell}$, and real part contribution. Our discussion of the preliminary results of Ankenbrandt et al. just intended to be a possible paradigm for such a use.

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REFERENCES

1) S.W. McDowell and A. Martin - Phys.Rev. 135B, 960 (1964).

2) S.M. Roy and V. Singh - Phys.Rev.Letters 24, 28 (1970) ; Phys.Rev. 1D, 2638 (1970).

3) G. Mennessier - Private communication.

4) G. Auberson, A. Martin and G. Mennessier - in preparation.

5) See, e.g. : A. Martin - Phys.Rev. 129, 1432 (1963).

6) H. Cornille and A. Martin - Nuclear Phys. B115, 163 (1976).

7) C.W. Akerlof et al. - University of Michigan Preprint UM-HE 76-6 (1976).;
G. Giacomelli - Physics Reports 23C, No 2 (1976) ;
E. Bracci et al. - CERN/HERA 75-2 compilation (1975).

8) C. Ankenbrandt et al. - Communication to the XVIII International Conference on High Energy Physics, Tbilisi (1976).

9) E. Bracci et al. - CERN/HERA 73-1 compilation (1973).

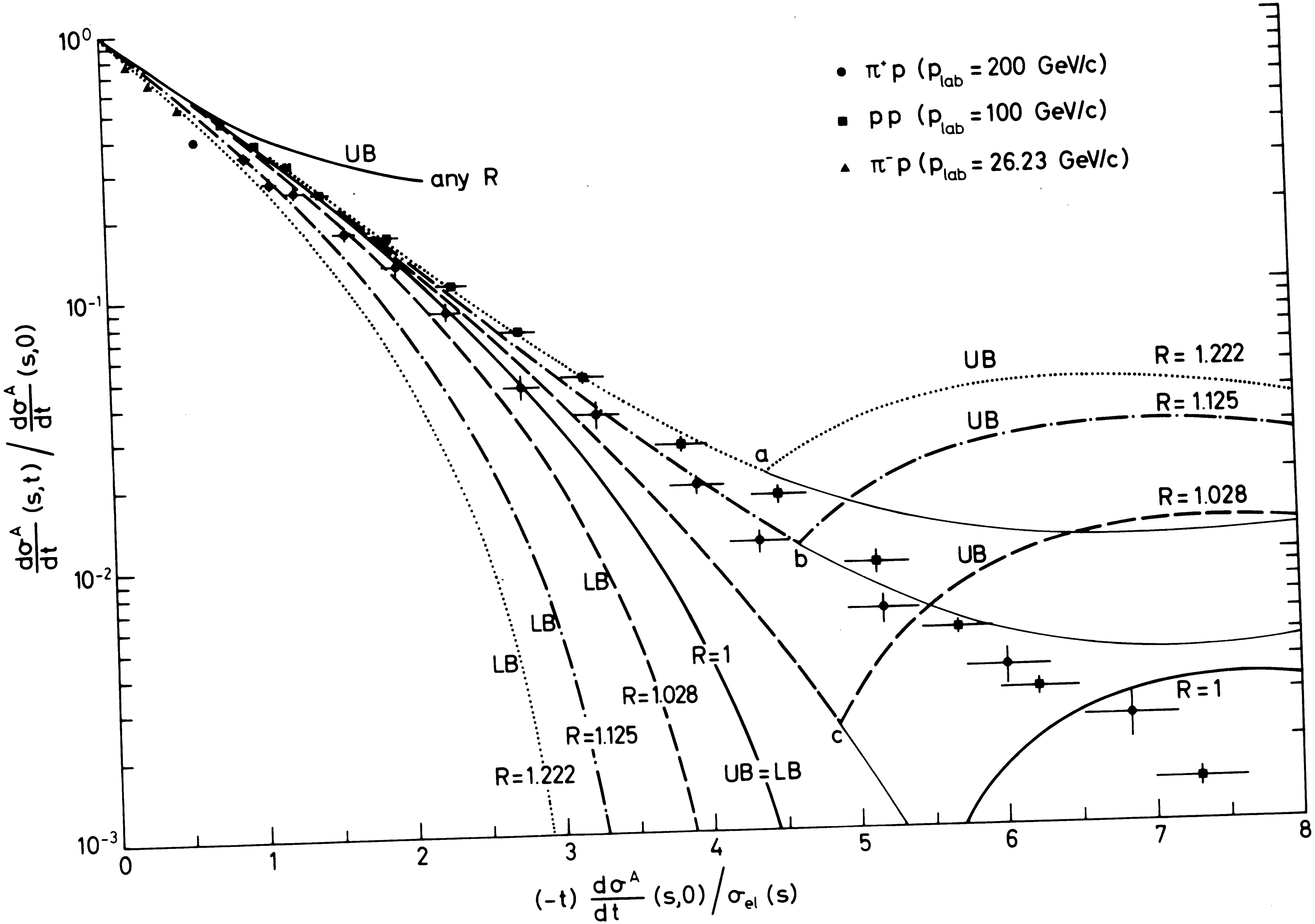
10) C. Ankenbrandt et al. - FNAL pub. Fermilab-Conf-75/61 exp. (1975).

FIGURE CAPTIONS

Figure 1 Upper (UB) and lower (LB) bounds on $(d\sigma^A/dt(s,t))/(d\sigma^A/dt(s,o))$ as functions of $T = -t(d\sigma^A/dt(s,o))/\sigma_{e\ell}(s)$ for various values of R. The faint lines are extrapolations of the upper bound in the region where the (negative) lower bound on A(s,t) exceeds the upper bound in magnitude. The line "any R" is the R independent upper bound of Ref. 2). Experimental data are taken from Refs. 7).

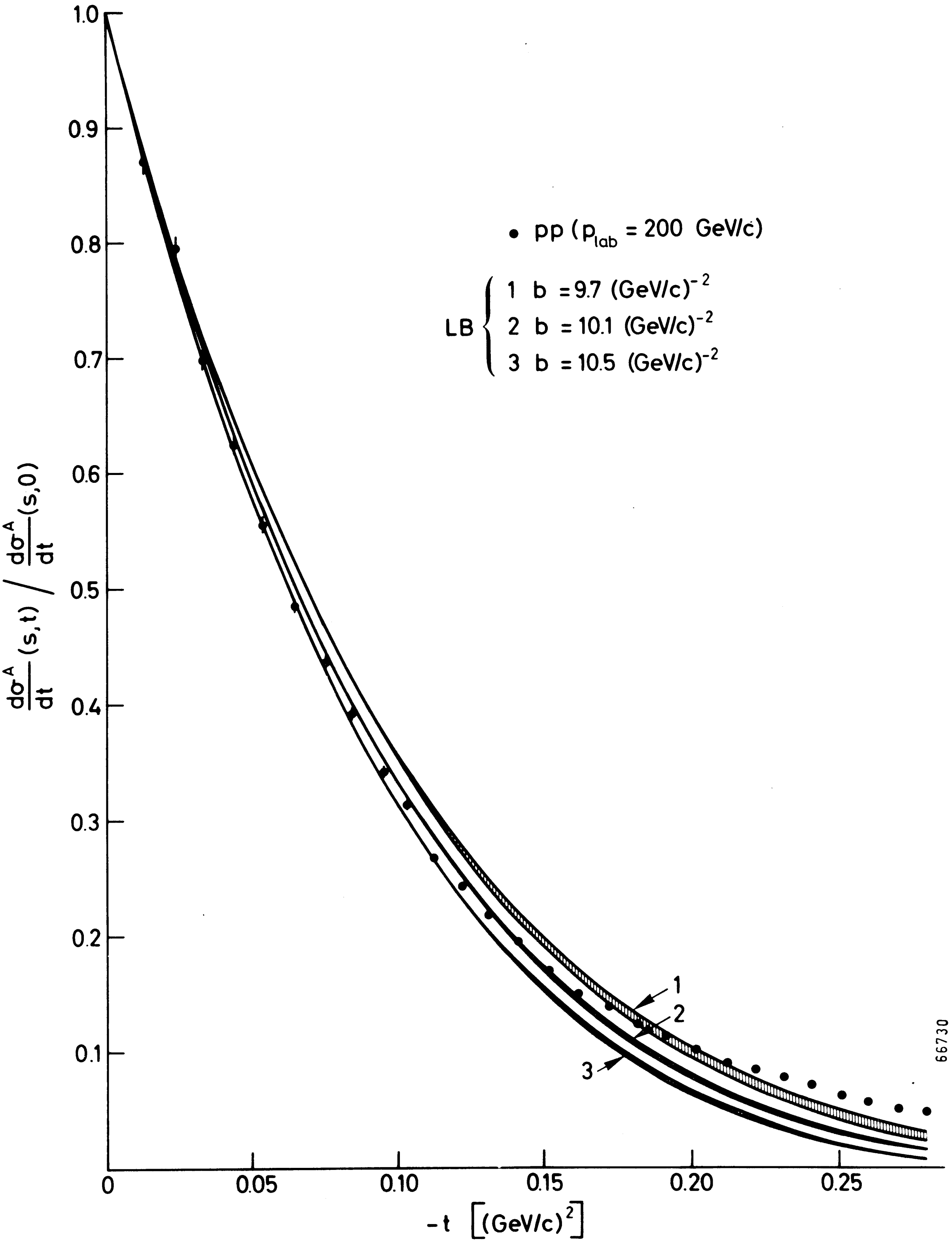
Figure 2 Lower bounds on $(d\sigma^A/dt(s,t))/(d\sigma^A/dt(s,o))$ in the small t region as compared to the (preliminary) experimental data of Ankenbrandt et al. [8]. Each shaded strip corresponds to a given value of b (9.7, 10.1 and 10.5 $(GeV/c)^{-2}$) and a common range for $\sigma_{e\ell}/\sigma^2_{tot}$: 0.00468 - 0.00480 mb^{-1}. At fixed t, the bounds are decreasing functions of the latter parameter.

Figure 3 Upper bounds on $(d\sigma^A/dt(s,t))/(d\sigma^A/dt(s,o))$ for two values of b (10.3 and 10.5 $(GeV/c)^{-2}$) and two values of $\sigma_{e\ell}/\sigma^2_{tot}$ (0.00468 and 0.00480 mb^{-1}), as compared to the experimental data of Ref. 8).



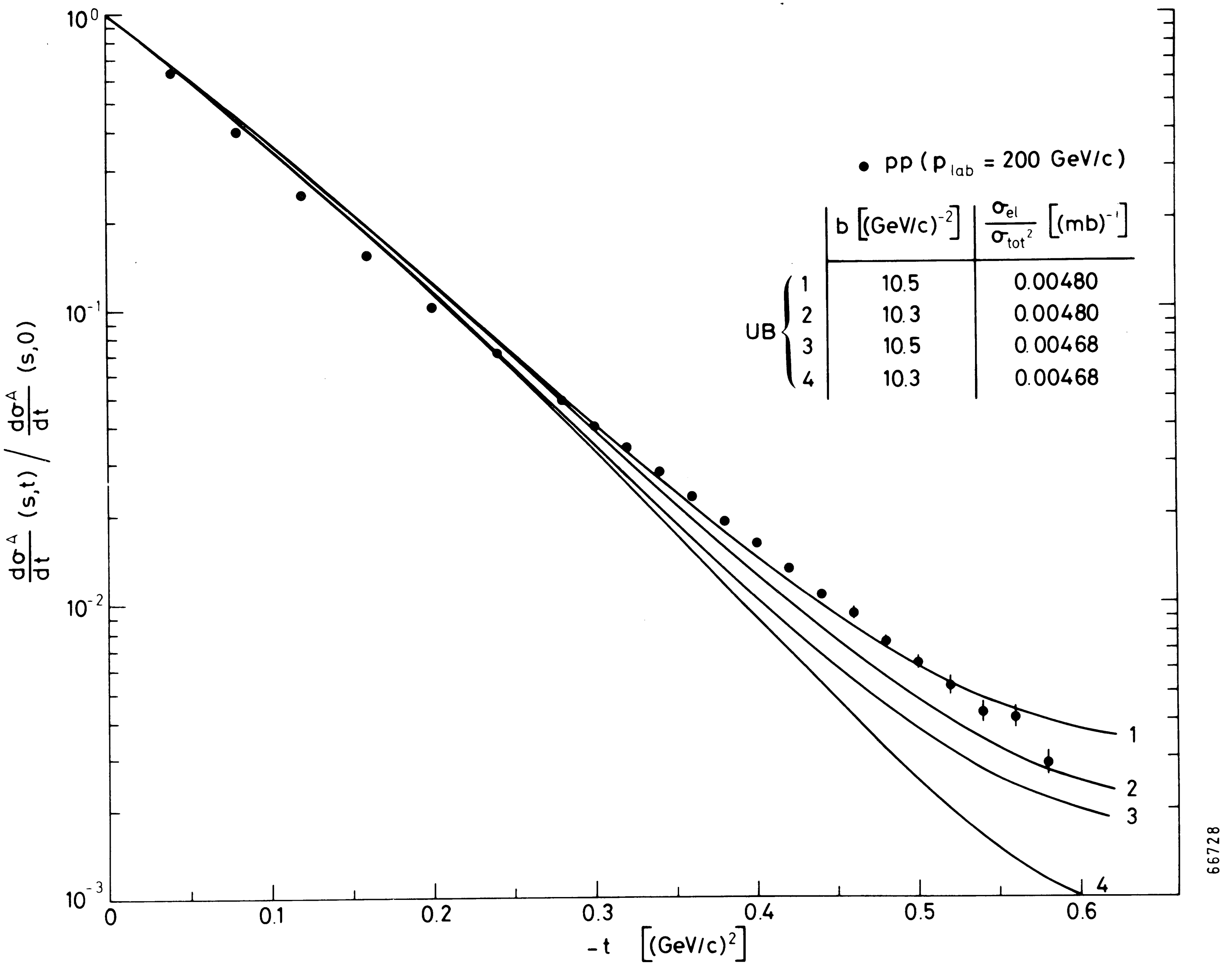
$\pi^+ p$ (p_{lab} = 200 GeV/c)
pp (p_{lab} = 100 GeV/c)
$\pi^- p$ (p_{lab} = 26.23 GeV/c)
UB
any R
UB
R = 1.222
UB
R = 1.125
R = 1.028
UB
R = 1
a
b
c
LB
LB
LB
R=1
R=1.028
R=1.125
R=1.222
UB = LB
10^0
10^{-1}
10^{-2}
10^{-3}
0
1
2
3
4
5
6
7
8
$\frac{d\sigma^A}{dt}(s,t) \Big/ \frac{d\sigma^A}{dt}(s,0)$
$(-t)\,\frac{d\sigma^A}{dt}(s,0) \Big/ \sigma_{el}(s)$
66729

Fig.1



1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0.05
0.10
0.15
0.20
0.25
-t [(GeV/c)^2]
dσ^A/dt (s,t) / dσ^A/dt (s,0)
● pp (p_lab = 200 GeV/c)
LB
1 b = 9.7 (GeV/c)^-2
2 b = 10.1 (GeV/c)^-2
3 b = 10.5 (GeV/c)^-2
1
2
3
66730

Fig. 2



• pp (p lab = 200 GeV/c)
b [(GeV/c)^-2]
σel / σtot^2 [(mb)^-1]
UB
1 10.5 0.00480
2 10.3 0.00480
3 10.5 0.00468
4 10.3 0.00468
dσ^A/dt (s,t) / dσ^A/dt (s,0)
10^0
10^-1
10^-2
10^-3
0
0.1
0.2
0.3
0.4
0.5
0.6
-t [(GeV/c)^2]
1
2
3
4
66728

Fig. 3