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CLUSTER DECOMPOSITION OF VENEZIANO AMPLITUDES *)

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A B S T R A C T

A simple general rule is given for writing down the factorized form of an N point Veneziano amplitude when it is decomposed into several clusters separated by high energies. It is hoped that while the explicit form of the cluster vertices may be useful phenomenologically, the manner in which they are obtained may also be instructive when considering the factorization of N point functions in general. Note that factorization along trajectories as considered here differs in meaning from that of residues at resonance poles as embodied in the operator formalism.

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*) Dedicated to the memory of a friend, L. Michejda, who died 7 August 1971.

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1. - CLUSTER DECOMPOSITION

Consider first a single term in the N point Veneziano amplitude [1] defined by a certain cyclic ordering of the N external lines. We wish to make a cluster decomposition of this amplitude in the manner illustrated in Fig. 1.

Such a term is given as a function of those Mandelstam variables corresponding to the various partitions of the diagram without changing the ordering of the external lines. However, the number of these variables is in general bigger than the number of independent dynamical variables required to describe the N particle process in physical four-dimensional space. They thus satisfy certain algebraic relations which define a sub-manifold in the space spanned by the original set of Mandelstam variables. For reasons which will be apparent, it is convenient for us first to treat these variables as independent, working as if we were in a physical space of unlimited dimensions. Only when the need arises shall we impose the four-dimensionality of space and restrict the amplitude to the physical manifold.

In a physical space of unlimited dimentions then, the cluster decomposition of Fig. 1 is defined as the limit of the amplitude when :

(i) $|\alpha_P| \to \infty$ for all α_P dual to the Reggeized lines α_{AB}, α_{BC}, etc ;

(ii) all other variables are held fixed.

The directions in which the various α_P in (i) approach infinity will depend on the physical region we are considering, or in other words, on which of the N lines we choose as incoming and which as outgoing.

Now the integral representations with which one usually defines the Veneziano amplitudes are convergent only in the region where all $\mathrm{Re}\,\alpha_P < 0$. In taking the limit of Fig. 1, therefore, it is convenient to consider first the (unphysical) limit $\alpha_P \to -\infty$ for all P dual to α_{AB}, α_{BC},... The physical limits for various physical regions where some $\alpha_P \to +\infty + i\varepsilon$ will be obtained by analytic continuation.

It turns out that a very simple rule exists which allows one to write down explicitly the limits of the amplitude in general when $\alpha_P \to -\infty$ corresponding to Fig. 1. Before explaining the rule in full generality, we shall first illustrate it with a few examples.

A. - Six-point function, decomposed as in Fig. 2

Starting with the representation

$$
\begin{aligned}
B_6 = \iiint dx_1\, dx_2\, dy\; & x_1^{-\alpha_{12}-1}\; x_2^{-\alpha_{56}-1}\; (1-x_1)^{-\alpha_{23}-1} \\
& (1-x_2)^{-\alpha_{45}-1}\; y^{-\alpha_{13}-1}\; (1-y)^{-\alpha_{34}-1}\; (1-x_1 y)^{-\alpha_{24}+\alpha_{23}+\alpha_{34}} \\
& (1-x_2 y)^{-\alpha_{35}+\alpha_{34}+\alpha_{45}}\; (1-x_1 x_2 y)^{-\alpha_{61}-\alpha_{34}+\alpha_{24}+\alpha_{35}}
\end{aligned}
\tag{1}
$$

we wish to take the limit $\alpha_{34}, \alpha_{61}, \alpha_{24}, \alpha_{35} \to -\infty$ with all other variables held fixed. This is most conveniently done by changing the variable of integration y to z [2] :

$$
y = -z/\alpha_{34} \tag{2}
$$

On taking the limit, some factors in (1) become exponentials, e.g. :

$$
\lim_{\alpha_{34}\to-\infty} (1-x_1 y)^{-\alpha_{24}+\alpha_{23}+\alpha_{34}} = \exp\left\{ \frac{x_1 z}{\alpha_{34}} (-\alpha_{24}+\alpha_{34}) \right\} \tag{3}
$$

One of the integrals can then be done explicitly yielding :

$$
\begin{aligned}
B_6 \longrightarrow \Gamma(-\alpha_{13}) \int dx_1\; & x_1^{-\alpha_{12}-1} (1-x_1)^{-\alpha_{23}-1} \int dx_2\; x_2^{-\alpha_{56}-1} (1-x_2)^{-\alpha_{45}-1} \\
& \times \left[-\alpha_{34}(1-x_1)(1-x_2) - \alpha_{61} x_1 x_2 - \alpha_{24} x_1 (1-x_2) - \alpha_{35} x_2 (1-x_1) \right]^{\alpha_{13}}
\end{aligned}
\tag{4}
$$

We note first that (4) contains two factors which are identical to the two four-point vertices A and B of the Reggeon α_{13} were on shell. Indeed, if we put $\alpha_{13} = 0$, the last factor in (4) disappears and one just has a product of two B_4, as one should. It seems thus that the continuation of the vertices to off-mass-shell Reggeon is accomplished simply by the last factor, raised to the appropriate power of the (complex) Reggeon spin. Moreover, the factor

$$
\langle A B \rangle = \left[-\alpha_{34}(1-x_1)(1-x_2) - \alpha_{61} x_1 x_2 - \alpha_{24} x_1 (1-x_2) - \alpha_{35} x_2 (1-x_1) \right] \tag{5}
$$

which links A and B, has a simple structure ; it is minus the sum of all variables α_P which are dual to the Reggeized line $\alpha_{AB} = \alpha_{13}$, each α_P multiplied by its own dual conjugates in both halves of the diagram.

B. - Eight-point function, decomposed as in Fig. 3

We may start from the twisted multiperipheral representation (Fig. 4) of B_8 which may be found, e.g., in Ref. 3). Then using the same technique as in A, one obtains the following expression for the limit :

$$B_8 \to \int d\varphi_A \, d\varphi_B \, d\varphi_C \, \langle AB\rangle^{\alpha_{AB}} \langle BC\rangle^{\alpha_{BC}} \, U(\alpha_{AB}, \alpha_{BC}; \varkappa) \tag{6}$$

where

$$\int d\varphi_{A(B,C)} \quad = \text{vertex } A(B,C) \text{ when the connected Reggeon lines are on-shell, i.e., when } \alpha_{AB} = \alpha_{BC} = 0. \tag{7}$$

$$U(\alpha_1, \alpha_2; \varkappa) = \int dz_1 \, dz_2 \; z_1^{-\alpha_1 - 1} z_2^{-\alpha_2 - 1} \exp\left[-z_1 - z_2 - z_1 z_2/\varkappa\right] \tag{8}$$

$$\varkappa = \frac{\langle AB\rangle\langle BC\rangle}{\langle ABC\rangle} \tag{9}$$

In (9), we have generalized the notation of (5) in example A, so that $< ABC >$ is the sum of all Mandelstam variables linking A to C each multiplied by its dual conjugates in A, B and C. Explicitly for the decomposition of Fig. 3, in the representation of Fig. 4 :

$$\langle ABC\rangle = -\alpha_{23}(1-x_1)(1-y_1)z - \alpha_{13}\, x_1(1-y_1)z - \alpha_{24}(1-x_1)y_1 z - \alpha_{14}\, x_1 y_1 z \tag{10}$$

The definitions of $< AB >$ and $< BC >$ are similar.

One notes that again the same factors as in (4) and (5) continue the Reggeons off-mass-shell. In addition, there is a two-Reggeon vertex function which describes in a sense the relative polarization of the two Reggeons in terms of a Toller-type variable $\varkappa$. This vertex U replaces the simple Γ function in (4).

Comparing the form of $\mathcal{U}$ given in (8) with the integral representation of $\Gamma(-\alpha_{123})$ which occurs in (4), namely :

$$\Gamma(-\alpha) = \int_0^\infty dz\, z^{-\alpha-1} \exp\{-z\} \tag{11}$$

it is not difficult to guess the manner in which the formulas (4) and (6) are to be generalized. For the general N point function decomposed in an arbitrary way as shown in Fig. 1, we have

$$B_N \longrightarrow \int d\varphi_A\, d\varphi_B \cdots d\varphi_Z\, \langle AB\rangle^{\alpha_{AB}} \langle BC\rangle^{\alpha_{BC}} \cdots \langle YZ\rangle^{\alpha_{YZ}}\, \mathcal{U} \tag{12}$$

where

$$\begin{aligned} \mathcal{U} = \int_0^\infty & dz_{AB}\, dz_{BC} \cdots dz_{YZ}\, z_{AB}^{-\alpha_{AB}-1} z_{BC}^{-\alpha_{BC}-1} \cdots z_{YZ}^{-\alpha_{YZ}-1} \\ & \times \exp\Big\{ -z_{AB} - z_{BC} - \cdots - z_{YZ} - \frac{\langle ABC\rangle}{\langle AB\rangle\langle BC\rangle} z_{AB} z_{BC} - \cdots \\ & - \frac{\langle XYZ\rangle}{\langle XY\rangle\langle YZ\rangle} z_{XY} z_{YZ} - \cdots - \frac{\langle AB\cdots YZ\rangle}{\langle AB\rangle\cdots\langle YZ\rangle} z_{AB}\cdots z_{YZ} \Big\} \end{aligned} \tag{13}$$

The symbol $< J...P >$ denotes as before minus the sum of all Mandelstam variables linking J to P, each multiplied by their dual conjugates in all clusters J,...,P.

That the formulas (12) and (13) are indeed valid in general can be seen most readily from the original definition of the beta functions [1] :

$$B_N = \int \prod_{P'} du_{P'} \left(\frac{1}{J}\right) \prod_{P} u_P^{-\alpha_P - 1} \tag{14}$$

where P runs over all partitions of the N point diagram, P' runs over some independent set, and J is a volume element factor whose exact form is irrelevant for our present discussion.

Consider first the simple case of a decomposition into two culsters A and B, as in Fig. 5a. We can assume without loss of generality that the independent set P' contains the line AB which is Reggeized. The decomposition is defined by the limit $\alpha_Q \to -\infty$ for all partitions Q dual to AB. In this limit, the integral (14) is dominated by the region $u_{AB} \to 0$ and it is convenient as in examples A and B to examine this by a simple change of variables. Let a be an arbitrary line in cluster A and b another in cluster B. Define z by :

$$u_{AB} = - z / \alpha_{ab} \tag{15}$$

as $\alpha_{ab} \to -\infty$, $u_{AB} \to 0$. The factor J in (14) being a product of u variables factorizes as usual. In the remaining product in the integrand, we can distinguish three types of terms :

(i) $u_P = u_{AB}$; this just gives by (15) a factor $[-\alpha_{ab}]^{\alpha_{AB}+1} \quad z^{-\alpha_{AB}-1}$;

(ii) $u_P = u_R$, where R is not dual to AB ; these factors are not affected by the limit $u_{AB} \to 0$ and make up exactly those terms represented by the symbols $d\varphi_A$ and $d\varphi_B$ in the notation of examples A and B ;

(iii) $u_P = u_Q$, where Q is dual to AB ; these terms will exponentiate when $\alpha_{ab} \to -\infty$; using the duality condition

$$u_Q = 1 - \prod_{\bar{Q}} u_{\bar{Q}} \tag{16}$$

one can write

$$u_Q^{-\alpha_Q - 1} = \left[1 + \frac{z}{\alpha_{ab}} \prod_{\bar{Q}'} u_{\bar{Q}'}\right]^{-\alpha_Q - 1} \tag{17}$$

where $\bar{Q}' \neq AB$ is dual to Q. Now when $u_{AB} \to 0$ all u variables dual to $AB \to 1$. Hence

$$u_Q^{-\alpha_Q - 1} \longrightarrow \exp\left\{- z \frac{\alpha_Q}{\alpha_{ab}} \prod_{\bar{Q}''} u_{\bar{Q}''}\right\} \tag{18}$$

where now $\bar{Q}''$ runs over only those partitions dual to Q which are neither equal nor dual to AB.

Collecting all these factors, then, one has [4]

$$B_N \longrightarrow - \frac{1}{\alpha_{ab}} \int d\varphi_A \, d\varphi_B \int_0^\infty dz \left(-\frac{z}{\alpha_{ab}}\right)^{-\alpha_{AB}-1} \exp\left\{\frac{\langle AB \rangle z}{\alpha_{ab}}\right\} \tag{19}$$

which by a simple change of variable

$$z' = - \frac{\langle AB \rangle}{\alpha_{ab}} z \tag{20}$$

reduces to the general form (12) as required.

The arguments given above can readily be generalized to a decomposition into more than two clusters. For each of the Reggeized lines, one makes a change of variables as in (15) :

$$u_{AB} = -\frac{z_{AB}}{\alpha_{ab}} , \quad u_{BC} = -\frac{z_{BC}}{\alpha_{bc}} \quad \dots \tag{21}$$

The only new point in the argument is that the product in the integrand of (14) now contains terms which are dual to several of the Reggeized lines. These in the limit will give exponents involving several z variables corresponding to the last terms in (13). For example, let S be dual to BC and CD ; then as α_{bc}, $\alpha_{cd} \to -\infty$, we have

$$u_S^{-\alpha_S - 1} \longrightarrow \exp\left\{ \frac{\alpha_S}{\alpha_{bc}\alpha_{cd}} z_{BC}\, z_{CD} \prod_{\bar{S}''} u_{\bar{S}''} \right\} \tag{22}$$

where $\bar{S}''$ runs over all partitions dual to S which are neither equal nor dual to either BC or CD. The formulas (12) and (13) are otherwise obvious.

Finally, we note that the integral expression (12) is not factorized in the usual sense, since the symbols $< J\ldots P >$ involve the integration variables in all the clusters J,...,P. Factorization will only result when we restrict the dimension of our physical space, as we shall explain in the next section.

2. - FACTORIZATION

In Fig. 1, the Reggeon α_{AB} carries spin. Thus the orientation of the vertex A in space will in general affect the orientation of the vertex B. Similarly, the orientation of B will affect the orientation of C, and so on, so that in a space of unrestricted dimensions, the information of the A orientation will propagate down the chain and there is no factorization. In the physical space of four dimensions, however, the information will not propagate. Here, it is obvious that once we fix the orientation of B relative to A, and C relative to B, we fix also the orientation of C relative to A. We thus expect that when we restrict the amplitude to the physical manifold by imposing the four-dimensionality of space, the amplitude will properly factorize. In this section, we shall demonstrate how this comes about for the integral (12) obtained above.

Consider an N point diagram divided into two parts separated by a Reggeon line, as shown in Fig. 5a. Let i_A, j_A and i_B, j_B be any two pairs of lines in the parts A and B respectively [5]. In this section, we shall demonstrate how this following relation is valid asymptotically between the Mandelstam variables :

$$\frac{s_{i_A i_B}\ s_{j_A j_B}}{s_{i_A j_B} \cdot s_{j_A i_B}} = 1 \tag{23}$$

where we have used the notation :

$$s_{ij} = (p_i + p_{i+1} + \cdots + p_j)^2 \tag{24}$$

The relation (23) is particularly easy to remember in terms of the dual diagram Fig. 5b, where the left-hand side takes the form of a cross ratio.

The asymptotic relation (23) can be established in various ways. We shall prove it here by evaluating the Gram determinant. Let $p_1, p_2, \ldots, p_6$ be any six vectors in space satisfying the condition :

$$p_1 + p_2 + \cdots + p_6 = 0 \tag{25}$$

A necessary and sufficient condition that these vectors be contained in a space of four dimensions is that

$$\Delta \equiv \det(p_i \cdot p_j) = 0 \tag{26}$$

where $i,j = 1,\ldots,5$. Note that although we have written (26) in a form asymmetric between p_i and p_6, the condition is in fact symmetric because of (25). Moreover, because of (25) again, only nine of the (p_i, p_j) are linearly independent. We may therefore choose to express them all in terms of the set of nine Mandelstam variables obtained by partitions of the six-point diagram with $p_1, \ldots, p_6$ as external lines arranged in cyclic order, namely : s_{12}, s_{23}, s_{34}, s_{45}, s_{56}, s_{61}, s_{13}, s_{24}, s_{35} defined as in (24). In four-dimensional space, then, these Mandelstam variables satisfy a condition which can be derived from (26). In particular, for the six-point function, this condition $\Delta = 0$ defines our physical manifold.

Consider now the asymptotic limit defined by Fig. 2 of example A. A straightforward evaluation of Δ to leading non-vanishing order yields the following expression :

$$\Delta \rightarrow \frac{1}{16}\, s_{13}\,(s_{34}\, s_{61} - s_{24}\, s_{35})^2 \tag{27}$$

The condition (26) then implies :

$$\frac{s_{34} \cdot s_{61}}{s_{24} \cdot s_{35}} = 1 \tag{28}$$

which is a special case of (23). Further, if in the preceding arguments, one replaces $p_1, \ldots, p_6$ by the six vectors represented by the sides of the hexagon (solid lines) in Fig. 5b, one obtains the asymptotic condition (23) in general.

It is noteworthy that the relation (23) remains valid even when some of the particles in the clusters A and B of Fig. 5 are boosted to form new clusters, as shown in Fig. 6 in a four-cluster case. Moreover, the derivation of (23) given above is entirely algebraic and analytic. The asymptotic condition is thus valid not merely in some physical regions of the multi-particle amplitude, but in all (complex) directions on the physical manifold. Thus, in particular, it holds also in the unphysical region with all $\alpha < 0$ for which the integral representation of B_N is convergent.

With the relation (23), we shall now prove that the symbol $< AB >$ introduced in Section 1 factorizes into two parts depending only on variables respectively in clusters A and B. Choose two arbitrary lines, a in A and b in B. Introduce the notation :

$$\langle A] = \frac{1}{\alpha_{ab}} \sum_{i_A} \alpha_{i_A b} \prod_{P_{i_A}} u_{P_{i_A}} \tag{29}$$

$$[B\rangle = \frac{1}{\alpha_{ab}} \sum_{i_B} \alpha_{a i_B} \prod_{P_{i_B}} u_{P_{i_B}} \tag{30}$$

where the sum in (29) runs over all i_A, while the product is taken over all u_P in A which are dual conjugate to $\alpha_{i_A b}$. Diagrammatically, the terms in the sum of (29) are represented by the dotted lines in Fig. 7a ; the conjugate variables $u_{P_{i_A}}$ which multiply a certain $\alpha_{i_A b}$ correspond to all lines in A which cross the line representing $\alpha_{i_A b}$. The definition of $[B>$ is similar.

We now claim the following :

$$\langle AB \rangle = -\alpha_{ab} \langle A] . [B\rangle \tag{31}$$

This can be seen from Fig. 7b. In the product (31), there is a term with $\alpha_{i_A b} \cdot \alpha_{a i_B} / \alpha_{ab}$ multiplied by the appropriate u variables. However, one sees immediately from Fig. 7b that any line in A which crosses $\alpha_{i_A b}$ and any line in B which crosses $\alpha_{a i_B}$ must also cross $\alpha_{i_A i_B}$ and vice versa. The u variables multiplying $\alpha_{i_A b} \cdot \alpha_{a i_B} / \alpha_{ab}$ in the product $\alpha_{ab} < A] \cdot [B>$ are thus the same as those multiplying $\alpha_{i_A i_B}$ in $< AB >$. Moreover, by (23), we have indeed

$$\alpha_{i_A i_B} = \alpha_{i_A b} \cdot \alpha_{a i_B} / \alpha_{ab} \tag{32}$$

which therefore proves (31).

We note here two points. First, the vectors a and b appear only as frames of reference to describe the orientations of the clusters in space and have no special significance. It can easily be seen using (23) that $\langle A]$ as defined by (29) only depends on the choice of a and not on b

$$\langle A] = \frac{1}{\alpha_{ab}} \sum_{i_A} \alpha_{i_A b} \prod_{P_{i_A}} u_{P_{i_A}} = \frac{1}{\alpha_{ac}} \sum_{i_A} \alpha_{i_A c} \prod_{P_{i_A}} u_{P_{i_A}} \tag{33}$$

where c need not even belong to cluster B. Second, the relation (31) is valid for any two adjacent clusters which may occur anywhere along the chain of Fig. 1. Thus we have in general

$$\langle R S \rangle = - \alpha_{rs} \langle R] . [S\rangle \tag{34}$$

The factorization theorem (34) is readily extended to symbols linking more than two clusters. For example, consider the decomposition represented by the dual diagram of Fig. 8. It is clear that all those and only those lines in B which cross from one side of the diagram to the other are dual to variables linking A to C. Hence the expression $\langle ABC \rangle$ has a common factor, namely

$$[B] = \text{product of all } u \text{ variables corresponding to lines in } B \text{ traversing the dual diagram} \tag{35}$$

Applying again the arguments in the preceding paragraphs, one easily sees then that the following holds,

$$\langle ABC \rangle = - \alpha_{ac} \langle A] . [B] . [C\rangle \tag{36}$$

The generalization to more than three clusters is obvious

$$\langle J K \ldots P \rangle = - \alpha_{jp} \langle J] . [K] . \ldots [P\rangle \tag{37}$$

We turn now to the factorization of the integrals (12). For the decomposition into two clusters, one sees immediately that the amplitude factorizes :

$$\Gamma(-\alpha_{AB}) \int d\varphi_A \, d\varphi_B \, \langle AB \rangle^{\alpha_{AB}}$$
$$= \int d\varphi_A \, \langle A]^{\alpha_{AB}} \cdot \left\{ \Gamma(-\alpha_{AB}) [-\alpha_{ab}]^{\alpha_{AB}} \right\} \cdot \int d\varphi_B \, [B\rangle^{\alpha_{AB}} \tag{38}$$

For an end-cluster attached to only one Reggeon line, we have thus a vertex function :

$$V_E(A) = \int d\varphi_A \, \langle A]^{\alpha_{AB}} \tag{39}$$

and for the Reggeon line, we have a factor :

$$D(AB) = \Gamma(-\alpha_{AB}) [-\alpha_{ab}]^{\alpha_{AB}} \tag{40}$$

Next, from the factorization into three clusters [see (8) and (9) for definitions] :

$$\int d\varphi_A \, d\varphi_B \, d\varphi_C \, \langle AB \rangle^{\alpha_{AB}} \langle BC \rangle^{\alpha_{BC}} \; \mathcal{U}(\alpha_{AB}, \alpha_{BC}; \varkappa)$$
$$= V_E(A) \, D(AB) \, V_I(B) \, D(BC) \, V_E(C) \tag{41}$$

one defines also the vertex function for an internal cluster attached to two Reggeon lines, namely

$$V_I(B) = \left[\Gamma(-\alpha_{AB}) \Gamma(-\alpha_{BC}) \right]^{-1} \int d\varphi_B \, [B\rangle^{\alpha_{AB}} \langle B]^{\alpha_{BC}}$$
$$\cdot \; \mathcal{U}\left(\alpha_{AB}, \alpha_{BC}; \varkappa = -\frac{[B\rangle\langle B]}{[B]} \frac{\alpha_{ab} \cdot \alpha_{bc}}{\alpha_{ac}}\right) \tag{42}$$

Together with (39) and (40), this then allows one to write down the factorized form of (12) decomposed into any number of clusters. The proof that (12) in general does factorize in such a manner is reduced by (37) to the proof of factorization in the multi-Regge limit which may be found, e.g., in Ref. 6) and need not be repeated here.

It should be noted, however, that our procedure of factorization applies so far only to the unphysical region where all $\alpha < 0$ for which (12) is valid. The continuation of the factorized form to the various physical regions is in general not trivial and will be dealt with in the next section.

3. - CONTINUATION TO PHYSICAL REGIONS

The factorized form of the amplitude obtained in the preceding section was based on two conditions :

(i) the asymptotic limit defined by Fig. 1,

(ii) the four-dimensionality of space as embodied in Eq. (23).

Therefore, whether one can continue it from the unphysical region where it is defined to a certain physical region depends on whether there exists a path connecting the two regions which lies entirely in the asymptotic limit of the physical manifold as defined by (i) and (ii). A simple example will demonstrate that this is in general not possible.

Consider the six-point function decomposed as in Fig. 2. We wish to continue the factorized amplitude to the physical region R, say, where 2 and 5 are incoming lines, while all others are outgoing. One easily sees then that in R ; $s_{34}, s_{61} > 0$, while $s_{24}, s_{35} < 0$. Now, the amplitude has cuts for $s_{34}, s_{62} > 0$, and R is defined such that both s_{34} and s_{61} have to remain above the cut. In order to reach R therefore from the unphysical region where $s_{34}, s_{61} < 0$, one has to rotate both s_{34} and s_{61} along an infinite arc in the clockwise direction. This one has to do continuously under the constraint :

$$\frac{s_{34} \cdot s_{61}}{s_{24} \cdot s_{35}} = 1 \tag{43}$$

which is clearly impossible without bringing either s_{24} or s_{34} across their cuts into their unphysical sheets.

It is seen that the situation met with in the preceding example is quite general. Only in exceptional cases can the factorized form of the amplitude be continued from the unphysical region (all $\alpha < 0$) to a physical region, or from one physical region to another. This is true for any N particle amplitude, not merely for the Veneziano model.

To obtain the factorized form of the amplitude in physical regions, therefore, one has to give up either (i) or (ii), namely, either (i) continue the amplitude through the finite region of the physical manifold, or (ii) continue it outside the physical manifold. We choose here the second alternative. The asymptotic form (12) will first be continued treating all α's as independent to the desired physical region. Only then shall we impose the four-dimensionality condition to factorize the amplitude.

Consider first a two-cluster decomposition as in Fig. 5. In a certain physical region, say, some of the variables α dual to α_{AB} are positive [7]. Treating all α's as independent, the amplitude can be continued just by relating all such α's along an infinite arc in the clockwise direction. Depending on the integration variables, $< AB >$ will then sometimes be negative with phase $e^{-i\pi}$. Hence,

$$B(AB) = \Gamma(-\alpha_{AB}) \int d\varphi_A \, d\varphi_B \, |\langle AB\rangle|^{\alpha_{AB}} \times \left[\Theta(\langle AB\rangle) + e^{-i\pi\alpha_{AB}} \, \Theta(-\langle AB\rangle)\right] \tag{44}$$

where

$$\Theta(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases}$$

If now we express $\langle AB \rangle$ in terms of $\langle A]$ and $[B\rangle$ we find that (44) does not factorize in general. This is understandable, since (44) has a mixture of positive and negative signature exchanges. Even though the amplitudes for both signatures factorize, a mixture of them does not if the vertex functions for positive and negative signatures are different.

In order to factorize the full amplitude, we form the signaturized combinations, $B(\tau)$ ($\tau = \pm$), defined as follows [8] :

$$B(\tau) = B(AB) + \tau \, B(A \times B) \tag{45}$$

where $B(A\times B)$ is obtained from $B(AB)$ by twisting the Reggeized line, which means that the ordering of the lines in cluster B is reversed. Under the reversal of the ordering in B, the only quantity which changes in (44) is $\langle AB \rangle$ ($d\varphi_B$ being a Veneziano integrand, is invariant under reflection). Hence,

$$B(A\times B) = \Gamma(-\alpha_{AB}) \int d\varphi_A \, d\varphi_B \, |\langle A\times B\rangle|^{\alpha_{AB}} \times \left[\Theta(\langle A\times B\rangle) + e^{-i\pi\alpha_{AB}} \, \Theta(-\langle A\times B\rangle)\right] \tag{46}$$

where $\langle A\times B \rangle$ is defined as $\langle AB \rangle$ but with the ordering of the lines in B reversed.

Consider now an arbitrary term in the sum of $\langle AB \rangle$: $-\alpha_{i_A i_B}$ multiplied by its dual conjugate variables in A and B. There is a corresponding term in $\langle A\times B \rangle$ that involves the same conjugate variables, multiplied by the variable

$$-\bar{\alpha}_{i_A i_B} = -\alpha\left[\left(p_{i_A} + \cdots + p_{N_A} + p_{i_{B+1}} + \cdots + p_{N_A+N_B}\right)^2\right] \tag{47}$$

(see Fig. 9 for notation). By momentum conservation

$$\bar{\alpha}_{i_A i_B} = \alpha\left[\left(p_{i_A} + \cdots + p_{N_A} - p_1 - p_2 - \cdots - p_{i_B}\right)^2\right] = -\alpha_{i_A i_B} \tag{48}$$

to leading order. Hence we find that

$$\langle A \times B \rangle = - \langle AB \rangle \tag{49}$$

As before, the phase of $< A \times B >$ is $e^{-i\pi}$ whenever it is negative. Thus

$$B(A \times B) = \Gamma(-\alpha_{AB}) \int d\varphi_A \, d\varphi_B \, |\langle AB \rangle|^{\alpha_{AB}} \times \left[e^{-i\pi\alpha_{AB}} \, \Theta(\langle AB \rangle) + \Theta(-\langle AB \rangle) \right] \tag{50}$$

The signaturized amplitude (45) now becomes

$$B(\tau) = \Gamma(-\alpha_{AB}) (1 + \tau \, e^{-i\pi\alpha_{AB}}) \times \int d\varphi_A \, d\varphi_B \, |\langle AB \rangle|^{\alpha_{AB}} \left[\Theta(\langle AB \rangle) + \tau \, \Theta(-\langle AB \rangle) \right] \tag{51}$$

Using (34) and

$$\Theta(x_1 x_2) + \tau \, \Theta(-x_1 x_2) = \left[\Theta(x_1) + \tau \, \Theta(-x_1) \right] \left[\Theta(x_2) + \tau \, \Theta(-x_2) \right] \tag{52}$$

we have then the factorized form :

$$B(\tau) = V_E(A, \tau) \, D(AB, \tau) \, V_E(B, \tau) \tag{53}$$

where

$$V_E(A, \tau) = \int d\varphi_A \, |\langle A]|^{\alpha_{AB}} \left[\Theta(\langle A]) + \tau \, \Theta(-\langle A]) \right] \tag{54}$$

$$D(AB, \tau) = \Gamma(-\alpha_{AB}) \, \tau \, (1 + \tau \, e^{-i\pi\alpha_{AB}}) \, |\alpha_{ab}|^{\alpha_{AB}} \left[\mathrm{sign}\,(\alpha_{ab}) \right]^{(1-\tau)/2} \tag{55}$$

which define respectively the one-Reggeon vertex and the Reggeon "propagator" for signaturized Reggeons.

A similar procedure can be applied to a three-cluster decomposition to derive the two-Reggeon vertex for signaturized Reggeons. For this purpose it is convenient to write the vertex function $\mathcal{V}$ defined in (8) in terms of the confluent hypergeometric function $\phi(a,b;x)$ as [9]

$$\mathcal{U}(\alpha_1,\alpha_2;x) = x^{-\alpha_1}\,\Gamma(-\alpha_1)\,\Gamma(\alpha_1-\alpha_2)\,\Phi(-\alpha_1, 1+\alpha_2-\alpha_1; x)$$
$$+ x^{-\alpha_2}\,\Gamma(-\alpha_2)\,\Gamma(\alpha_2-\alpha_1)\,\Phi(-\alpha_2, 1+\alpha_1-\alpha_2; x) \tag{56}$$

Since $\phi(a,b;x)$ is analytic for finite x, substitution of (56) into the integral (6) will exhibit the singularity structure in simple factors such as $< AB >^{\alpha}$, etc. Again, forming as before the combinations of amplitudes with definite signature exchanges, an analogous calculation yields a factorized form for the amplitude decomposed into three clusters :

$$B(\tau_{AB}, \tau_{BC}) = V_E(A,\tau_{AB})\,D(AB,\tau_{AB})\,V_I(B,\tau_{AB},\tau_{BC})$$
$$D(BC,\tau_{BC})\,V_E(C,\tau_{BC}) \tag{57}$$

where the internal vertex function V_I takes the following form :

$$V_I(B,\tau_{AB},\tau_{BC})$$
$$= \frac{1+\tau_{AB}e^{-i\pi\alpha_{AB}}+\tau_{BC}e^{-i\pi\alpha_{BC}}+\tau_{AB}\tau_{BC}e^{-i\pi(\alpha_{BC}-\alpha_{AB})}}{[1+\tau_{AB}e^{-i\pi\alpha_{AB}}][1+\tau_{BC}e^{-i\pi\alpha_{BC}}]}\cdot\frac{\Gamma(\alpha_{AB}-\alpha_{BC})}{\Gamma(-\alpha_{BC})}$$
$$\times \left|\frac{\alpha_{ab}\alpha_{bc}}{\alpha_{ac}}\right|^{-\alpha_{AB}} \int d\varphi_B\, |\langle B]|^{\alpha_{BC}-\alpha_{AB}}\,[B\rangle^{\alpha_{AB}}\left[\tau_{AB}\Theta(\langle B])+\tau_{BC}\Theta(-\langle B])\right]$$
$$\times\,\Phi\left(-\alpha_{AB}, 1+\alpha_{BC}-\alpha_{AB}; -\frac{[B\rangle\langle B]}{[B]}\,\frac{\alpha_{ab}\alpha_{bc}}{\alpha_{ac}}\right)$$
$$+ \left\{A \longleftrightarrow C\,;\ \langle B] \longleftrightarrow [B\rangle\right\} \tag{58}$$

With the vertices (54) and (58), and the Reggeon propagators (55), one can now write down the factorized form of a general N point Veneziano amplitude decomposed into any number of clusters in any physical region. Our program is thereby completed.

ACKNOWLEDGEMENTS

Two of us (P.H. and P.V.R.) are grateful for the kind hospitality extended to them at the Theoretical Study Division of CERN. P.H. thanks the University of Helsinki for a grant.

FOOTNOTES AND REFERENCES

1) For a review and references, see, e.g. :
Chan Hong-Mo - Proc.Roy.Soc.London A318, 379 (1970).

2) K. Bardakçi and H. Ruegg - Phys.Letters 28B, 342 (1968).

3) B. Hasselacher, C.S. Hsue and D.K. Sinclair - Stony Brook Preprint (1971), to be published in Phys.Rev.

4) The expression (19) in a different form has already been obtained previously by :
D.K. Campbell, D.I. Olive and W.J. Zakrzewski - Nuclear Phys. B14, 319 (1969).
The present authors are grateful to D. Olive for reminding them of this paper.

5) Factorization as considered here should be distinguished from the factorization at resonance poles as embodied in the operator formalism. In the latter case, the residue at a resonance pole is expressed as a finite sum of factorized terms, the number of which depend on the position of the pole. It has no meaning along the trajectory except at integers, and is independent of the dimension of physical space. In the present context, however, factorization means that the asymptotic form of the amplitude is expressed as a simple product of terms, in such a way that the vertex function describing a cluster along the chain is independent of the internal structure of its neighbours. It has a meaning also for non-integral values of the trajectory function. For this factorization to hold, the dimension of physical space is crucial. In four dimensions, one sees that two clusters moving at high relative energy appear as discs to each other because of Lorentz contraction. To describe their relative orientation, one needs fix only one vector in each cluster. The angle between the transverse components of these vectors (namely the Toller angle) is sufficient to determine their orientation. The situation will not be so simple in higher dimensions.

6) K. Bardakçi and H. Ruegg - Phys.Rev. 181, 1884 (1969).

7) We are of course going to continue to positive values only those variables dual to α_{AB}. The variables α that are completely contained in one cluster are always going to be kept negative, in order for the integral representation to converge.

8) Our approach is analogous to that of Weis in the special case of the multiperipheral limit.
J.H. Weis - M.I.T. Preprint (1971), to be published in Phys.Rev.

9) A. Białas and S. Pokorski - Nuclear Phys. B10, 399 (1969).

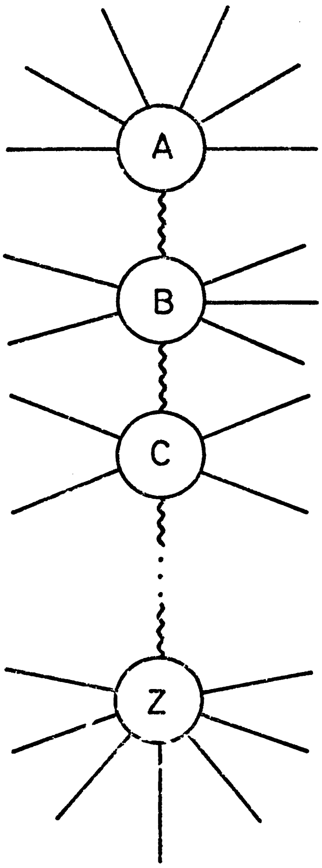

FIG.1

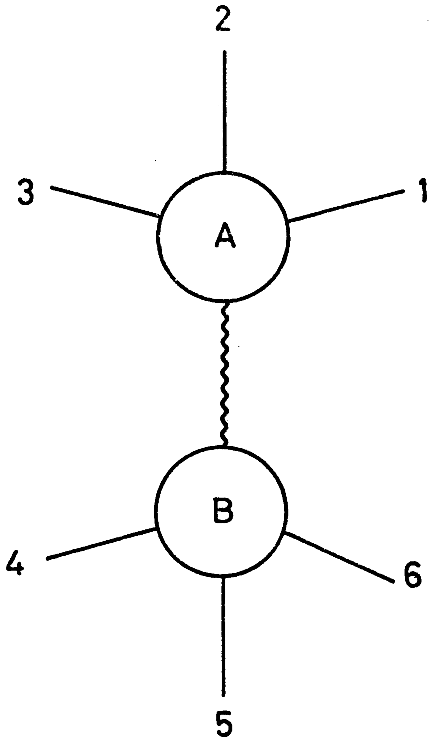

FIG.2

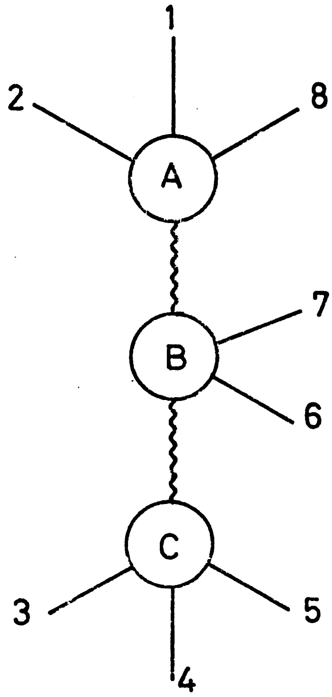

FIG.3

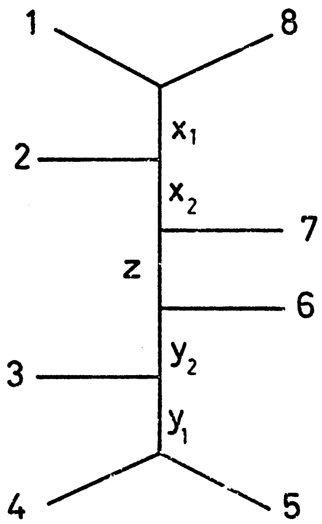

FIG.4

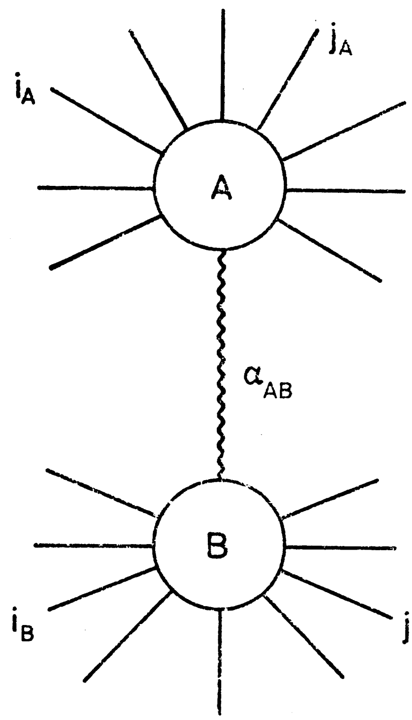

FIG. 5 (a)

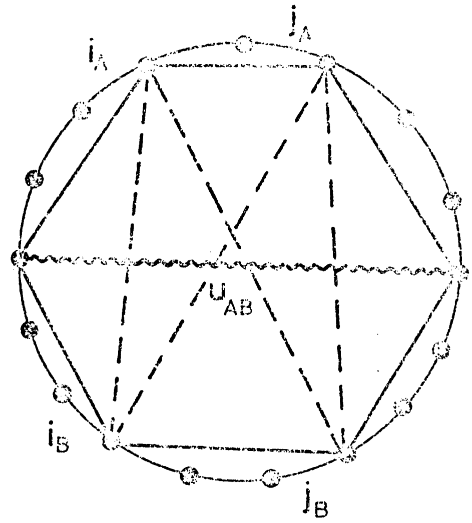

FIG. 5 (b)

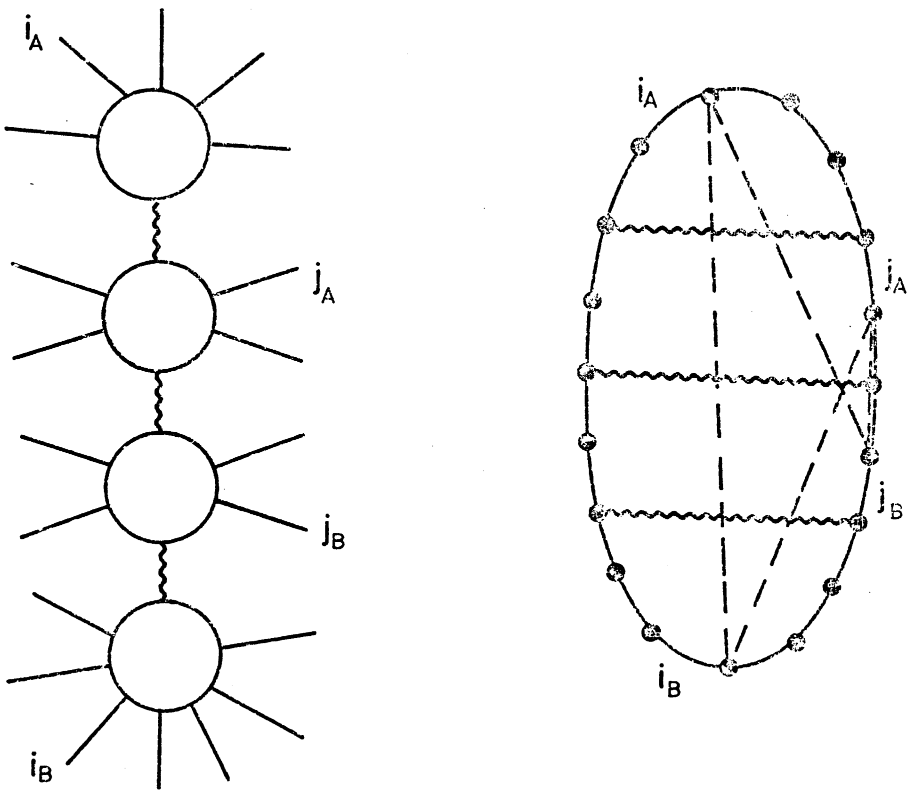

FIG. 6

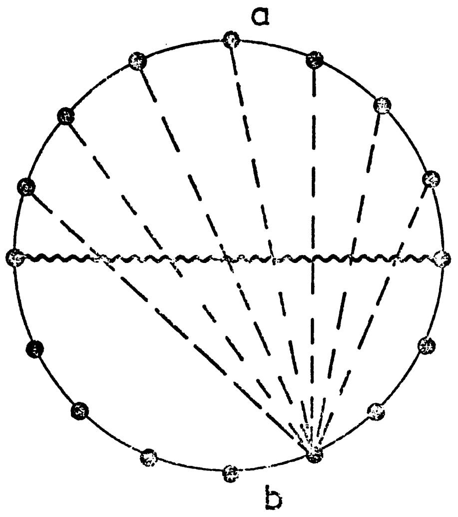

FIG. 7(a)

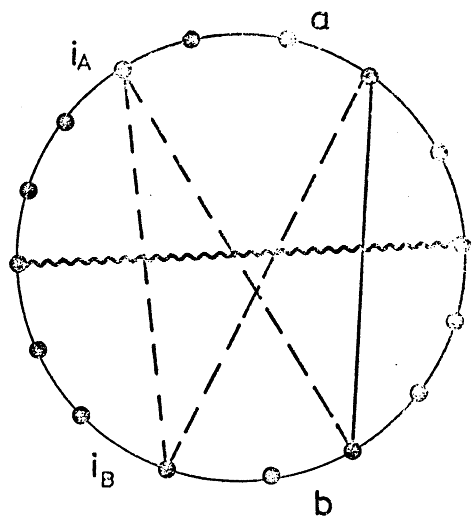

FIG. 7(b)

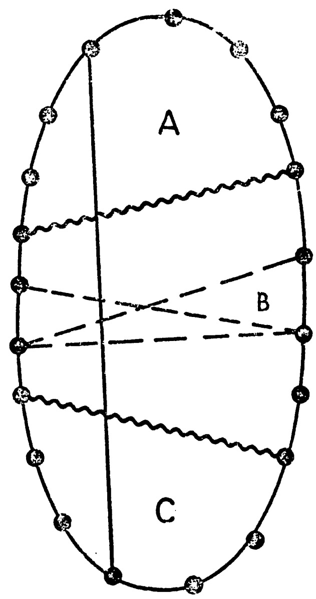

FIG. 8

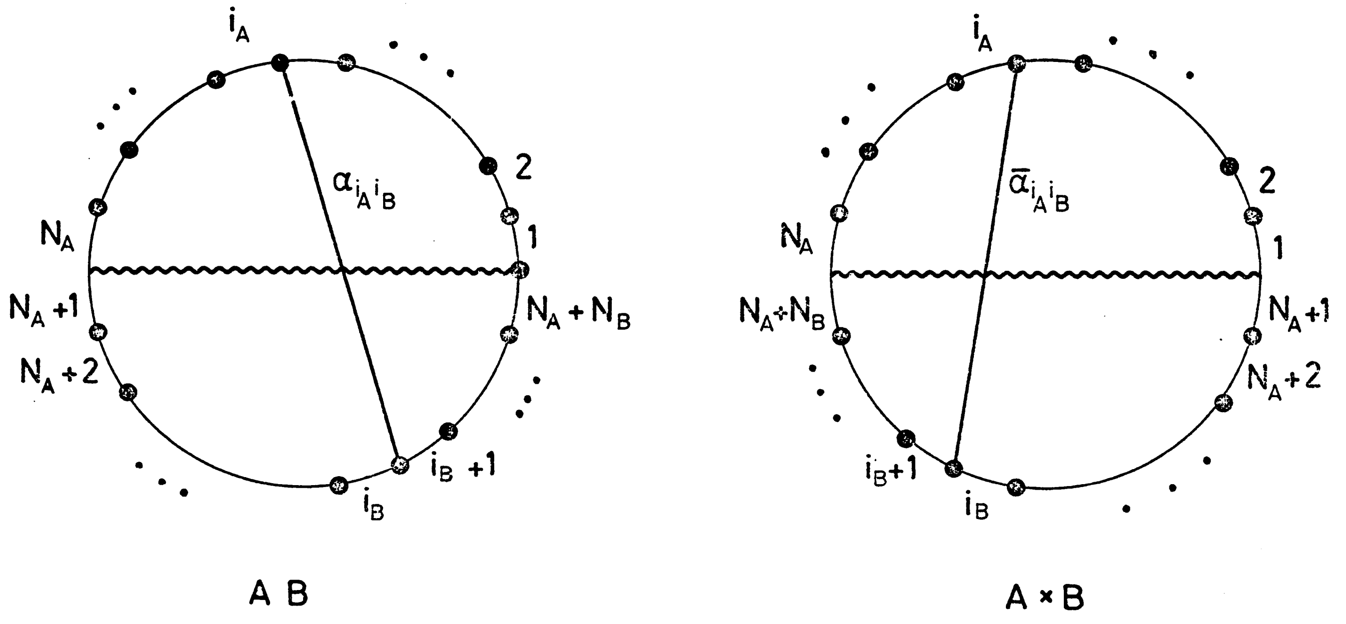
i_A
$\alpha_{i_A i_B}$
2
1
N_A
$N_A + N_B$
$N_A + 1$
$N_A + 2$
$i_B + 1$
i_B
A B
i_A
$\bar{\alpha}_{i_A i_B}$
2
1
N_A
$N_A + N_B$
$N_A + 1$
$N_A + 2$
$i_B + 1$
i_B
A × B

FIG. 9