Archives

To appear in Physics Report   Ref.TH.1448-CERN

FROM SOFT TO REAL PIONS IN NUCLEAR PHYSICS

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ABSTRACT

This review treats of the applications of soft-pion theorems to nuclei. After a general introduction of the soft-pion techniques, the partially conserved axial vector current hypothesis is discussed and we emphasize its particular nature in the nuclear case.

The rest of the article treats of the mass extrapolation problem. The relation between the physical and the soft-pion amplitude is derived for several processes, and its physical interpretation is discussed.

Ref.TH.1448-CERN
4 January 1972

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I. INTRODUCTION

In recent years, a considerable amount of effort has been devoted to understanding the low-energy properties of the pion-nucleon interaction. This has culminated in the recent rapid development of the current algebra and its associated low-energy theorems, and many striking predictions have been made with remarkable quantitative agreement with experiments. By now this is a well-established fact in particle physics and has been reviewed in numerous summer schools and monographs[1]. Any further effort toward the exposition of such a subject would be considered superfluous.

There remains, however, a highly non-trivial question which deserves a lot more attention, and that is to ask whether or not the same theoretical tools can be equally well applied to the interaction of pions with nuclei, and whether by studying pion-nuclear interaction one could not gain some new insight into the fundamental pion-nucleon dynamics which are perhaps unattainable from a free nucleon, and also into some properties of nuclei hitherto unprobed by other means. We may emphasize that this is of course a question that one always faces when one does non-classical nuclear physics. It would be simply disappointing if the low-energy theorems and their corrections worked as well for a nucleus as for a nucleon without bringing in any new information. It is this new information that we would like to explore.

The purpose of our review is to treat these new features in a unified way in various pion-nuclear processes. We believe that the moment is ripe for such a review, as there is an increasing effort among nuclear physicists to learn something about the interaction of nuclei with pions and other mesons.

Let us now introduce the concept of the low-energy theorems. These theorems aim at a description of processes involving one particle (or several particles) -- photon, pion, etc. -- of small kinetic energy. They give an exact description in the limit when the particle becomes soft, i.e. in the limit where <u>all the components</u> of its four-momentum vanish, $k_\mu \to 0$. The amplitude in that limit is called a soft (photon or pion) amplitude.

The idea of considering the soft limit has first arisen for photons where it comes naturally as the long wavelength limit. In that limit, which is physically realizable since the photon is massless, one obtains very simple results. The scattering of soft photons becomes a universal expression in the sense that it does not depend on any detailed structure of the target. This scattering is described by the well-known Thomson formula

$$\frac{d\sigma}{d\Omega} = \frac{1}{2}(1 + \cos^2\theta)\left(\frac{e^2}{4\pi m}\right)^2 \tag{I.1}$$

where e is the charge and m the mass of the target. This universality is easily understandable. It just reflects the fact that the infinite wavelength of the photon makes it impossible to explore the shape and structure of the target. This point is aptly described by the following statement[2]: "If you want to explore the shape of your hand you pour sand over it, rather than marbles or apples".

There exist other soft-photon theorems. Consider a bremsstrahlung process where a soft photon is emitted. It may be emitted by an external line, or by an internal line as shown in Fig. 1. Consider the external emission. In the soft-photon limit the photon does not transfer momentum. The intermediate particle is then on the mass shell and hence its propagator vanishes so that this graph is divergent. The singularity is of the order k^{-1} (where k is the photon momentum). Thus in this limit these external graphs constitute the dominant part of the amplitude. On the other hand, the electromagnetic current is conserved, and this imposes a gauge condition on the amplitude. This gauge condition makes possible[3] the determination of the amplitude from the external emission graphs, not only to the leading order k^{-1} but to the next order $(k)^0$. These properties are the essential content of the soft-photon theorems.

Is it possible to obtain similar results for other projectiles, for example for a pion or a kaon? There are certain similarities between these particles and the photon. They are the quanta of the nuclear (or strong) field, as the photon is the quantum of the electromagnetic field. Their masses are small on the hadronic scale as the photon is massless; but otherwise they are very different. Nevertheless, the remarkable fact is that in the infinite wavelength limit one finds very similar results[4-9], i.e. the same universality for the meson-nucleon scattering and analogously for the emission of the mesons.

But what kind of wavelength are we considering? Is it sufficient to take the ordinary wavelength as being infinite, i.e. a pion at rest? Certainly not, because a pion at rest has still a finite Compton wavelength that enables it to explore the features of the target. What we should do then is to immerse the target in an infinite wavelength so that there is no variation at all on the target size. To do so we should have the pion not only at rest $\underset{\sim}{q} = 0$, but also massless $q_0 = 0$. Clearly it is not the physical pion, which has a mass of 140 MeV. And the fact that the soft amplitudes which are calculated do not apply to the physical pion but to a pion off the mass-shell raises a problem which is not present in the photon case.

There is another reason why the soft-pion theorems are not mere repetitions of the results obtained for photons. In general, if one takes a process involving a photon, the difficulty in calculating the amplitude arises from the large number of graphs that may contribute. The soft-photon limit greatly simplifies the problem for small k, since the dominant contributions arise only from the external emission graphs, for which the coupling of a photon with a nucleon or with a pion is well-known. For pions, however, we do not know the basic interaction. The π-nucleon coupling, for instance, may be taken to be pseudoscalar or pseudovector, even if one chooses the simplest ones. There is no way of testing these couplings by a perturbation expansion. This is why the soft-pion theorems require more than just the softness assumption; they need another hypothesis in order to fix the coupling. This hypothesis is that of the partial conservation of the axial current (PCAC). Furthermore, when more than one soft pion is involved (as in the scattering problem), or when the emission takes place in the presence of an electromagnetic perturbation (as in the photoproduction), certain commutation relations are also needed.

The subjects which will be treated consistently by means of the low-energy theorems are the elastic pion-nuclear scattering, the photoproduction of pions and its inverse reaction, and the pion production in the nucleon-nucleon or nucleon-nucleus collision. In all processes, the pion will be at rest or have small kinetic energy. Throughout the discussion, we shall rely heavily on the techniques already available for pion-nucleon processes.

We will now say a few words about the plan of this review and the conventions and notations that we shall adopt.

The remainder of this section is devoted to a discussion on the PCAC which, in the soft-pion theorem, plays a role equivalent to the gauge invariance of the electromagnetic current, essential to the soft-photon theorem. We empasize the delicate nature of the PCAC when applied to nuclei. A few examples of the soft-pion theorems are given in Section II, where their successes and limitations are discussed together with a general remark on the mass-extrapolation procedure. In succeeding sections, each of the processes under consideration will be analysed in detail, with comparisons to experiments whenever they are feasible.

Our conventions and notations are given in various places, but we summarize them here for completeness. Although we will be concerned with the asymptotic states, the "in" and "out" symbols will not be explicitly shown. In relativistic formulas, the metric will be that of Drell and Bjorken[10]. The states, however, will be normalized covariantly, $\langle p|p'\rangle = (2\pi)^3\, 2p_0\, \delta(\underset{\sim}{p}-\underset{\sim}{p}')$ for both fermions and bosons. The normalization for non-relativistic states will be the conventional one, $\langle p|p'\rangle = \delta(\underset{\sim}{p}-\underset{\sim}{p}')$.

P C A C

Let us now discuss the fundamental ingredient of the soft-pion theorem, the hypothesis of the PCAC. One should recall that the hadronic current of the weak interactions has a vector part V_μ and an axial part A_μ, which have different properties under space reflection. Here we are concerned only with the strangeness-conserving part of the axial current.

Consider the β-decay of the neutron. The weak Hamiltonian may be written as

$$H_W = \frac{G \cos\theta_c}{\sqrt{2}} \quad J_\mu \ell^\mu$$

where G is the fundamental weak-interaction coupling constant ($G = 1.0 \times 10^{-5}/m_p{}^2$, where m_p is the proton mass), θ_c is the Cabibbo angle, J_μ and ℓ_μ are the hadronic and weak currents respectively. Now the nucleons are subject to the strong interactions, as, for example, pions can be exchanged between the weak and the strong vertices. The hadronic matrix element of the axial current is then modified over the value it would have for a point-like Dirac particle, and from the Lorentz invariance properties one writes (for β^- decay)

$$\langle p(p_2) | A_\mu^+ | n(p_1) \rangle = \bar{u}_p(p_2) \left[g_A(q^2) \gamma_\mu \gamma_5 + h_A(q^2) q_\mu \gamma_5 \right] \tau^+ u_n(p_1) \tag{I.2}$$

with $q = p_2 - p_1$. Here p_1 and p_2 stand for four-momenta of the respective nucleons, g_A and h_A are the axial and pseudoscalar form factors respectively, and the suffix + is associated with an isospin-raising operator such that $\tau^+|n\rangle = |p\rangle$. (For more general definitions, see Appendix A.) For neutron β-decay, $q^2 \approx 0$ and $g_A(0) \approx 1.23$. The fact that g_A is not far from unity is significant. To realize this, consider the similar expansion for the vector current

$$\langle p(p_2) | V_\mu^+ | n(p_1) \rangle = \bar{u}_p(p_2) \left[g_V(q^2) \gamma_\mu + i\, g_M(q^2) \sigma_{\mu\lambda} q^\lambda \right] \tau^+ u_n(p_1) \tag{I.3}$$

In general one would expect that the vector form factor is also renormalized by the strong interactions. But the conservation of the vector current implies that there is no renormalization effect and that $g_V = 1$, i.e., the same as the μ decay.

The value found for g_A close to one had first suggested that maybe the axial current was also conserved. But it was quickly realized that if this were the case, the pion would live for ever, or nearly so, since the main decay mode of the charged pions $\pi \to \mu + \nu_\mu$ would be suppressed. This can be seen in the following way. The vector current cannot contribute to this decay because we cannot build a pseudovector with the only vector available, the pion momentum. Thus the matrix element of this process is that of the axial current between the pion state and the vacuum,

$$\langle 0 | A_\mu^{\pm} | \pi^{\mp} \rangle = i f_\pi q_\mu \tag{I.4}$$

where q_μ is the pion momentum, f_π the charged pion decay constant ($= 0.94\ m_\pi$), and suffix on the pion stands for charge states (see Appendix A). Take now the matrix element of the divergence of the axial current $D = \partial_\mu A^\mu$ between the same states. It amounts to multiplying the previous matrix element (I.4) by $-i\ q_\mu$; thus

$$\langle 0 | D^{\pm} | \pi^{\mp} \rangle = f_\pi q^2 = f_\pi m_\pi^2 \tag{I.5}$$

The conservation of the axial current would imply $\partial_\mu A^\mu = 0$ and therefore from the relation (I.5) one sees that the decay matrix element (I.4) would have to vanish unless $m_\pi = 0$. Thus the smallness of the renormalization effect in the axial coupling constant led instead to the idea of a <u>partial conservation</u> of the axial current, a somewhat misleading terminology which is explained in the following.

Let us consider the matrix element of the operator $\partial_\mu A^\mu \equiv D$ between two hadronic states A and B. This matrix element is made of all the objects that can be exchanged between D and the system A, B. The quantity D is a pseudoscalar quantity. It has the quantum numbers of the pion. Hence among the objects that can be exchanged, the pion is the lightest of these objects. This is illustrated in Fig. 2. The matrix element is then singular when the exchanged pion is on the mass-shell, i.e. when the momentum transfer is the pion mass $t = (p_B - p_A)^2 = m_\pi^2$. When A and B are nucleon states, the next singularity comes from the exchange of three pions. This is a complex system which has internal degrees of freedom, i.e. the pions can move around their centre of mass. This singularity is then not a pole but a cut which starts at $t = 9m_\pi^2$. Writing (for nucleons)

$$\langle B(p_2) | D^{\pm} | A(p_1) \rangle = d(q^2)\, \bar{u}(p_2)\, i\gamma_5\, \tau^{\pm} u(p_1) \tag{I.6}$$

we assume an unsubtracted dispersion relation for $d(q^2)$:

$$d(q^2) = \text{pion pole} + \frac{1}{\pi}\int_{9m_\pi^2}^{\infty} dt \, \frac{\rho(t)}{t - q^2} \tag{I.7}$$

The spectral function $\rho(t)$ is a complicated object, but if the momentum transfer q^2 is not too large, say $|q^2| < m_\pi^2$, the denominator of the dispersion integrand is large $(t - q^2) > 8m_\pi^2$; it is then reasonable to assume that the contribution of the dispersion integral is negligible and to retain only the pion pole. Thus when A and B are nucleons we may assume that the matrix element is dominated by the pion pole, not only at the pole but in a neighbouring region. With the expression of the πN vertex (for charged pions),

$$\langle B(p_2)| j_\pi^{\pm} | A(p_1)\rangle = \sqrt{2}\, g_r(q^2)\, \bar{u}(p_2)\, i\gamma_5 \tau^{\pm} u(p_1) \tag{I.8}$$

where j_π is the pion source $(\Box_x + m_\pi^2)\, \phi\,(x) = j_\pi(x)$, and $g_r(q^2)$ is the renormalized π-N coupling form factor normalized to $g_r(m_\pi^2) = g_r = 13.4$, this dominance is expressed as

$$d(q^2) = \frac{\sqrt{2}\, f_\pi m_\pi^2 g_r}{m_\pi^2 - q^2} \tag{I.9}$$

This formula, which is the essence of the pole dominance version of PCAC, is approximate since we have neglected the contribution of the cut, but it is a relation between measurable quantities $d(q^2)$ and g_r, and the validity of this approximation may then be tested. This test, as we will see later, is made in the form of the Goldberger-Treiman relation[4)], which is found to be valid to about 10%.

Let us now turn to nuclei and examine how the pole dominance assumption works. Notice that in formula (I.9) written for nucleons, the residue of the pole is taken to be a fixed quantity. All the variation in the momentum transfer q^2 is contained in the denominator, which would imply a slow and universal variation for $q^2 \ll m_\pi^2$. This, however, cannot possibly be valid for nuclei. The reason is that nuclei have size, and one would expect that aside from the dependence on $(m_\pi^2 - q^2)^{-1}$, there would be a variation as a function of q^2 dependent on nuclear size which would be important even when q^2 is a small and space-like quantity. In fact, the μ capture in nuclei provides the form factors in the space-like region, and shows that for large nuclei the variation is indeed very rapid. The axial form factor is expected to go like the charge form factor:

$$F_A(q^2) = F_A(0)\left[1 - \frac{|\vec{q}|^2 \langle r^2 \rangle}{6}\right] \tag{I.10}$$

where $\sqrt{\langle r^2 \rangle}$ is the r.m.s. radius. The reason for this behaviour is the singularity structure of the matrix element. In nuclei the next singularity after the pion pole is not the three-pion cut. Because the nucleus has a small binding energy, it can decay virtually into a nucleon plus another nucleon or nucleus. This introduces the anomalous threshold singularity, usually represented by the triangle graph of Fig. 3. The position of this cut depends on the binding energy but it is located somewhere around $2m_\pi^2$ or $3m_\pi^2$. This singularity is not much further away than the pion pole and it is not justified to neglect its influence.

In order to incorporate this feature, let us go back to the pole dominance expression written in terms of j_π:

$$\langle B | D^\pm | A \rangle = f_\pi m_\pi^2 \frac{\langle B | j_\pi^\pm | A \rangle}{m_\pi^2 - q^2} \tag{I.11}$$

Now instead of taking the residue of the pole to be a fixed quantity as one does for nucleons, we allow a variation as a function of q^2 in the pion source matrix element if A and B are nuclear states:

$$\langle B | D^\pm | A \rangle = f_\pi m_\pi^2 \frac{\langle B | j_\pi^\pm | A \rangle_{q^2}}{m_\pi^2 - q^2} \tag{I.12}$$

where we have indicated the q^2 dependence explicitly in the matrix element. This dependence reflects the spatial extension of the pion source. This relation can be condensed in the following proportionality relation of Gell-Mann and Lévy[11] between D and the pion field $\phi^\pm$ (see Appendix A for the definition of $\phi^\pm$):

$$D^\pm(x) = f_\pi m_\pi^2 \, \phi^\pm(x) \tag{I.13}$$

which is equivalent to the previous one (I.12). This is the field theoretic version of the PCAC.

In such a formula, the full singularity structure is in principle taken into account, but we may enquire about the meaning of such a relation since on one side we have a measurable quantity $d(q^2)$ but on the right-hand side, the pion field or the pion source is a physical quantity only at $q^2 = m_\pi^2$. Without a model for the pion source, this relation would just be a definition of the extrapolating pion field. But with a model for the pion source, this relation acquires predictive power. In nuclear physics, this model is obtained from an optical potential. Of course this relation is again approximate since the model in itself is an approximation, but the approximation is here of a different nature from that of the pole dominance hypothesis. It should be emphasized that some sort of model (i.e. the optical model) is inevitable at this point, since the pole dominance assumption is in flagrant contradiction with experiments. For instance, such an assumption would imply that the scattering of pions by nuclei be independent of nuclear size, which is clearly incorrect.

II. SOFT-PION THEOREMS

1. EXAMPLES

We start with the celebrated Goldberger-Treiman[4)] relation. Let us consider the matrix element of the divergence of the axial current D between nucleon states, $\langle p(p_2)|D^+|n(p_1)\rangle$. From the explicit form of $\langle p|A_\lambda^+|n\rangle$, (I.2), we have

$$\langle p(p_2)|D^+|n(p_1)\rangle = \bar{u}_p(p_2)\left[2m_N g_A(q^2) + q^2 h(q^2)\right] i\gamma_5 \tau^+ u_n(p_1) \tag{II.1}$$

On the other hand, the PCAC [Eq. (I.13)] tells us that

$$\langle p(p_2)|D^+|n(p_1)\rangle = f_\pi m_\pi^2 \frac{\sqrt{2}\, g_r(q^2)}{m_\pi^2 - q^2}\, \bar{u}_p(p_2)\, i\gamma_5 \tau^+ u_n(p_1) \tag{II.2}$$

Equating the two expressions at zero momentum transfer gives the Goldberger-Treiman relation between the strong pion-nucleon coupling constant $g_r(0)$ and the constants of weak interactions f_π and g_A:

$$2m_N\, g_A(0) = \sqrt{2}\, f_\pi\, g_r(0) \tag{II.3}$$

This relation is exactly valid at zero momentum transfer, i.e. for pions of zero mass. It is a typical example of a soft-pion theorem which gives an exact relation between a strong interaction quantity involving a pion of zero mass and weak-interaction constants (or amplitudes). However, as with all the soft-pion theorems, it does not provide a direct relation between physical quantities. Before it can be applied to physical quantities, approximations or extrapolations have to be made.

Another example is the photoproduction of soft pions or the inverse process, the radiative capture of pions:

$$\pi^\pm + i \rightleftarrows \gamma + f \tag{II.4}$$

where i and f stand for nucleon or nucleon states. More explicitly i is a proton (neutron) and f a neutron (proton) if we are dealing with $\pi^-(\pi^+)$. With our normalization convention, the invariant amplitude for the pion capture process $\langle f(p_f)\gamma(k)|j_\pi^\pm|i(p_i)\rangle$ is related to the S-matrix by

$$S = i(2\pi)^4\delta^4(k + p_f - q - p_i)\langle f\gamma|j_\pi^\pm(0)|i\rangle \tag{II.5}$$

There are several ways of getting the soft-pion amplitude, which are all equivalent[12-14]. Here we use the quickest method, which makes use of the modification of the PCAC in the presence of the electromagnetic field through the minimal coupling principle. The reduction technique which is used in the latter sections is summarized in Appendix C.

The minimal coupling amounts to the usual replacement of ∂^μ by $\partial^\mu \pm ie\,\mathcal{A}^\mu$, where $\mathcal{A}^\mu$ is the e.m. field and the ± sign applies to $\pi^\mp$ annihilation or $\pi^\pm$ creation. Thus the PCAC now reads[6]:

$$f_\pi m_\pi^2\,\phi^\pm(x) = \left(\partial^\mu \mp ie\,\mathcal{A}^\mu(x)\right)A_\mu^\pm(x) \tag{II.6}$$

Taking then the matrix element between the states $|f\gamma\rangle$ and $|i\rangle$ gives

$$\frac{f_\pi m_\pi^2}{m_\pi^2 - q^2}\langle f\gamma|j_\pi^\pm|i\rangle = i\,q^\mu\langle f\gamma|A_\mu^\pm|i\rangle \mp ie\langle f\gamma|\mathcal{A}^\mu A_\mu^\pm|i\rangle \tag{II.7}$$

In varying q^μ, we shall always keep the spatial momentum $\underset{\sim}{q}$ of the pion to be zero $\underset{\sim}{q} = 0$ and vary the time component q_0 only, where $q_0 = p_{f0} + k_0 - p_{i0}$. The soft-pion limit is obtained by setting $q_0 = 0$. In this limit, the left-hand side of Eq. (II.7) becomes the invariant amplitude for soft pions multiplied by the constant f_π: $f_\pi\langle f\gamma|j_\pi^\pm|i\rangle$. The second term on the right-hand side is trivial. Since we work to first order in the e.m. coupling constant we can replace in $\langle f\gamma|\mathcal{A}^\mu A_\mu|i\rangle$ the e.m. field by the outgoing one which just annihilates the outgoing photon, and

$$\langle f\gamma|\mathcal{A}^\mu A_\mu^\pm|i\rangle = \langle f|\epsilon^\mu A_\mu^\pm|i\rangle \tag{II.8}$$

As for the first term, one would be tempted to say that it would vanish as q_0 goes to zero. However, we have to be more careful since $\langle f\gamma|A_\mu^\pm|i\rangle$ may diverge in that limit. The most singular terms are the Born terms shown by Fig. 4. Clearly, when q^μ goes to zero, the intermediate particle goes on the mass-shell and hence the propagator blows up. Since the soft-pion limit is a mathematical tool, one can decide how to choose this limit. Two possibilities exist; we illustrate them using the example of the nucleon case.

i) The neutron-proton mass difference is kept finite and the pion energy $q_0 \to 0$. In this case there is no contribution of the nucleon intermediate state because of the multiplying factor q_0.

ii) The mass difference $m_p - m_n$ is first taken to be zero in which case there is then a pole contribution to the soft-pion amplitude which is calculable from the Born graphs of Fig. 4. The figure 4c vanishes as $q^\mu = 0$. If we ignore the photon coupling to the nucleon through the anomalous magnetic momentum term, then only Fig. 4b (for π^-) or Fig. 4a (for π^+) contributes. For example for π^- the pole contribution is

$$- ie\, g_A(0)\, \bar{u}_f \not{q} \gamma_5 \tau^- \frac{(\not{p}_f - \not{q}) + m_N}{(p_f - q)^2 - m_N^2 + i\varepsilon} \not{\epsilon}\, u_i$$

$$\xrightarrow{q_\mu \to 0} - ie\, g_A(0)\, \bar{u}_f \not{\epsilon} \gamma_5 u_i + ie\, g_A(0)\, \bar{u}_f \frac{m_N \not{q} \not{\epsilon} \gamma_5}{p_f \cdot q} u_i$$

The first part cancels the term (II 10a) and the total "soft-pion" amplitude is

ii) $$\mathcal{M}^\pm(0) = \mp ie \frac{g_A(0)}{f_\pi} \left(\bar{u}_f \gamma_0 \not{\epsilon} \gamma_5 \tau^\pm u_i\right) \frac{m_N}{E_f}$$

This expression differs from the one obtained in case (i):

i) $$\mathcal{M}^\pm(0) = \mp ie \frac{g_A(0)}{f_\pi} \bar{u}_f \not{\epsilon} \gamma_5 \tau^\pm u_i$$

in particular by the presence of the γ_0 matrix.

The procedure (ii) is usually employed for the nucleon since it should provide a more realistic amplitude in the sense that it is closer to the physical amplitude [since clearly the physical case corresponds to $m_\pi \gg (m_p - m_n)$]. The distinction between the two limits is significant when the difference between the physical and the soft-pion amplitudes is ignored, which is often done in the nucleon case.

However, it is not possible to ignore this difference in the nuclear case, and one has to calculate the relation between the physical and the soft-pion amplitudes. It should be noted that in this case the apparent ambiguity in the choice of the soft-pion limit is not reflected in the physical amplitude.

In the first procedure the contribution of the nucleon (or nuclear) intermediate state which does not appear in the soft-pion limit appears in the physical amplitude as a correction to the soft-pion limit. This is this procedure which has been adopted here in the treatment of the nuclear problem. Thus the soft-pion amplitude is simply (defining $M^{\pm} = \langle f\gamma | j_{\pi}^{\pm} | i \rangle$; see Appendix C):

$$\mathcal{M}^{\pm}(0) = \mp \frac{ie}{f_{\pi}} \langle f | \varepsilon^{\mu} . A_{\mu}^{\pm} | i \rangle \tag{II.9}$$

where we use the notation $M(q_0)$ for the invariant amplitude exhibiting the q_0 dependence. Note that in the soft-pion limit the pseudoscalar term cannot contribute, since it is proportional to q^{μ}.

If the states i and f are nucleonic states we can use the Goldberger-Treiman relation $(g_A/f_{\pi}) = (g_r)/(\sqrt{2}\, m_N)$ to obtain the soft-pion amplitude

$$\mathcal{M}^{\pm}(0) = \mp ie \frac{g_r}{\sqrt{2}\, m_N} \bar{u}_f \, \varepsilon^{\mu} \gamma_{\mu} \gamma_5 \, \tau^{\pm} u_i \tag{II.10}$$

or in the non-relativistic limit

$$\pm ie \frac{g_r}{\sqrt{2}\, m_N} \chi_f \, \underset{\sim}{\sigma} \cdot \underset{\sim}{\varepsilon} \, \tau^{\pm} \chi_i$$

where χ_f and χ_i are two-component Pauli spinors. This is the Kroll-Ruderman theorem[15]. It can also be obtained as a soft-photon theorem. In this case the amplitude is the sum of the three Born graphs but with the pseudoscalar π-N coupling. Equation (II.10) is equivalent to the nucleon-antinucleon pair or seagull term.

The expression obtained above relates the radiative pion capture to the axial current matrix element of the weak processes[16-18]. Thus it connects this process to the Gamow-Teller matrix elements in μ-capture $(\mu^- + i \rightarrow f + \nu_{\mu})$ or in β-decay $(i \rightarrow f + e + \nu)$. The momentum transfer involved in each case is different and so the comparison between them requires a little amount of extrapolation.

Nevertheless, this is a very interesting result linking a strong interaction amplitude to a weak amplitude. Tests of this relation will be discussed in a later section.

Historically this link (II.9) was first observed[19,20] in the impulse approximation. In order to appreciate the significance of Eq. (II.9) as compared with the impulse approximation, let us recall what the impulse approximation consists of. In this picture, the pion absorption and the subsequent radiation are assumed to take place on a single nucleon which behaves essentially as a free one except for the correlations taken into the nuclear wave function. In other words, the transition operator H_{eff} is of a single-particle nature, which is obtained by taking the sum over the number of nucleons in a nucleus with the expression (II.10),

$$\mathcal{H}_{\text{eff}} = \pm\, ie \frac{g_r}{\sqrt{2}\, m_N} \sum_{i=1}^{A} \tau_i^{\pm}\; \underset{\sim}{\sigma_i} \cdot \underset{\sim}{\varepsilon}\; \delta(\underset{\sim}{r} - \underset{\sim}{r_i}) \tag{II.11}$$

and the initial and final wave functions are correlated in the usual way, i.e. through nuclear potential. One knows, however, that there is an exchange current contribution to formula (II.11) [21]. Therefore such a relation in the impulse approximation strictly holds when exchange currents are ignored. The relation (II.9), on the other hand, holds in general since it involves on the left-hand side a complete radiative pion capture amplitude and on the right-hand side a full axial current. We see that a relation of such a type acquires a deep significance when i and f are nuclear states.

The third example of a soft-pion result is the elastic scattering of pions. The derivation is rather lenghty and hence we will only quote the results obtained by Weinberg[8] and Tomozawa[9]. When both the incident and the scattered pions become soft, the amplitude for elastic scattering becomes universal. This reminds us very much of the similar result for soft photons. In both cases we find universality, since the target is seen by the soft pion or the soft photon as a point object. The scattering length depends, in that limit, only on the isospin of the target. The difference of the scattering lengths for $\pi^{\pm}$ vanishes linearly in q_0 and is inversely proportional to the pion decay constant squared f_π^2:

$$a_{\pi^-} - a_{\pi^+} = 2(Z-N)\, a^- = 2(Z-N)\frac{q_0}{4\pi f_\pi^2} \tag{II.12}$$

where N-Z is the neutron excess of the target and a^- stands for charge antisymmetric combination. The sum for π^- and π^+ amplitudes gives the symmetric amplitude a^+

which is of order m_π^2, and is generally considered to be very small. This value will be discussed in detail in Section III. If one takes the sum $a_{\pi^-} + a_{\pi^+}$ to be zero, then these two results can be condensed into one formula:

$$\hat{a} = \frac{q_0}{4\pi f_\pi^2} \underset{\sim}{T} \cdot \underset{\sim}{t} \tag{II.13}$$

where T and t are the target and pion isospin operators respectively, and $\hat{a}$ is taken as an operator which acts on particular isospin states. This is the Weinberg-Tomozawa relation. The derivation of this expression requires not only PCAC but also the equal time commutation relations of current algebra, because it involves two soft pions.

To summarize these examples, what is so remarkable about the soft-pion result is the simplicity and the universality of the interaction. The elastic scattering amplitude depends only on the target isospin, while the photoproduction amplitude depends only on the spin-isospin structure. They are expressible in terms of weak or e.m. amplitudes.

2. HOW WELL DO SOFT-PION RESULTS DESCRIBE REALITY ?

We have derived nice and simple expressions for processes involving soft pions. They are exact results for zero-mass pions but they are off-shell quantities and they do not apply as such to the physical pion. Now a natural question is; what is retained of the simplicity of the interaction when we reach the physical mass?

The simplest thing that one can do is to forget that they have been derived for pions of zero mass and to apply them to the physical pion. Now how good this procedure is would depend upon what sort of target is interacting with the pion. Clearly all this is a question of scale*).

Let us first consider the case when the target is a nucleon. Recall that the Goldberger-Treiman relation for a pion of zero mass is given by

$$g_r(0) = \frac{\sqrt{2}\, m_N\, g_A(0)}{f_\pi} \tag{II.14}$$

Now if we replace the off-shell coupling constant $g_r(0)$ by the on-shell one $g_r = g_r(m_\pi^2)$ we then get a relation between physical quantities:

*) Pursuing in the same picture we used at the beginning, one should use sand to map the shape of one's hand, not marbles. But marbles are fine enough for giants' hands and sand is too coarse for Tom Thumb.

$$g_r = \frac{\sqrt{2}\, m_N\, g_A(0)}{f_\pi} \tag{II.15}$$

With $f_\pi = 0.94\ m_\pi$, $g_A = 1.23$ we obtain $g_r = 12.4$, while the experimental value is $g_r = 13.4$. The agreement is within 10%.

Consider now the Weinberg-Tomozawa formula, Eq. (II.13). The scattering lengths are obtained from the extrapolation to zero energy of the phase shifts obtained from the analysis of the scattering experiments. The different analyses agree on the isospin antisymmetric amplitude a^- (related to $a_{\pi-} - a_{\pi+}$). The experimental value of $(1 + m_\pi/m_N)a^-$ is close to $0.10\ m_\pi^{-1}$, in good agreement with the soft-pion value $L \equiv m_\pi/4\pi f_\pi^2 = 0.09\ m_\pi^{-1}$. For the isospin symmetric amplitude $a^+ = \frac{1}{2}(a_{\pi-} + a_{\pi+})$ different analyses disagree not only on the magnitude but also on the sign [a^+ ranging from -0.014 to $+0.02\ m_\pi^{-1}$ [22]]. Nevertheless there is a concensus of opinion that a^+ is noticeably smaller than a^- as predicted by the soft-pion result. The soft-pion amplitudes for the threshold photoproduction of charged pions agree well with experiments. The neutral pion production vanishes in the soft-pion limit, again in qualitative agreement with the experiment where the production is found to be considerably reduced compared to charged pions. There is thus evidence that the amplitude extrapolates smoothly between the zero mass and the physical mass when the target is a nucleon.

Consider now the nuclear case. There are practically no scattering experiments of low-energy pions on nuclei from which scattering lengths may be obtained. But there is an indirect knowledge of these scattering lengths from the energy shifts of the atomic levels of π mesic atoms[23]. For a pion bound in a Bohr orbit, the strong interaction modifies the energy from the Bohr value by an amount ΔE, the energy shift. The 1s level shift is related to the s-wave phase shift since both represent the amount by which the external wave function is perturbed, one in the negative energy region, the other in the positive energy one. When the shift is small, the relation is[24]

$$\Delta E_{1s} / E_{1s} = -4\, a / B \tag{II.16}$$

where B is the Bohr radius and a is the π^--nucleus scattering length. From this relation, the measured shift of 20 keV in ^{16}O (T = 0) corresponds to the scattering length $a^+ \approx -0.40\ m_\pi^{-1}$. This value is far from the small value predicted by the soft-pion relation. Similarly the isospin odd amplitudes may be obtained from the energy shifts for nuclei of non-zero isospin. The values thus obtained are also not universal. For example, for nucleon numbers A = 11 and 18, we get

$(a^-)_{A=11} = 0.057\ m_\pi^{-1}$ and $(a^-)_{A=18} = 0.047\ m_\pi^{-1}$ which are smaller than the soft-pion value $0.09\ m_\pi^{-1}$. A general trend in nuclei is that the isospin-odd amplitude a^- is decreased and the isospin-even amplitude increased (in magnitude) compared to the soft-pion predictions. This is an indication that the smoothness condition which is necessary for applying soft-pion theorems to physical amplitudes is not valid for nuclear target. The discrepancies in other processes are also sometimes pronounced as we shall see later.

It was suspected at the time when the soft-pion theorems were introduced that these soft-pion techniques would be useless in nuclei because owing to the complexity of nuclear structure, the simplicity of the interaction would be lost when the physical pion mass is reached. It is now known from the work of Ericson et al.[25,26] and Figureau[27] that this fear is unfounded. They showed that one can in fact find a fairly simple and physically reasonable relation between the soft pion and the physical amplitudes. One of the major purposes of the following sections is to show that this is possible. In the remainder of this section, we illustrate in a sketchy way what one must do to go from the soft-pion point to the physical point.

Consider, for example, the radiative pion capture. Clearly what we need is an exact relation between the physical amplitude and the soft-pion amplitude, which one can evaluate at least approximately. Now the invariant amplitude is $\epsilon^\mu M_\mu^\pm(q_0) = \langle f\gamma | j_\pi^\pm(o) | i\rangle$ working always with the pion at rest $\underset{\sim}{q} = 0$. The reduction technique which is described in the Appendices can be used to take the photon out, in which case we always get an equal-time commutator which in this case is

$$\begin{aligned} &\mp \frac{ie}{f_\pi} \int d^4x \ \delta(x_0) < f \ [\epsilon . V^3(0), A_0^\pm(x)] | i > \\ &= \mp \frac{ie}{f_\pi} < f | \ \epsilon . A^\pm(0) | i > \end{aligned} \qquad \text{(II.17)}$$

This is the soft-pion amplitude that we have already obtained before. There are correction terms which are explicitly proportional to q_0 or higher order in q_0. Some of these terms are in the form of commutators which are unknown. These unknown commutators either vanish on their own in some models or are believed to be small on the basis of experience. They are explicitly written down in the Appendices. The remaining correction terms, when q_0 is set equal to m_π, represent the corrections arising when one goes from $q_0 = 0$ to the physical value. In the case of the radiative pion absorption, they have the form

$$e\, m_\pi \sum_n (2\pi)^3\, \delta(\underset{\sim}{p}_n - \underset{\sim}{p}_i)\, \frac{\langle f|\, \varepsilon^\mu J^{em}_\mu(0)\, |n\rangle\langle n|\, j^\pm_\pi(0)\, |i\rangle}{(E_n - E_i)(m_\pi - E_n + E_i)} - \text{c.t.} \qquad \text{(II.18)}$$

where the sum runs over all allowed intermediate states, c.t. stands for the crossed term (obtained by $J^{em}_\mu \rightleftarrows j^\pm_\pi$ with appropriate kinematics), and E_i the total energy of the state (i.e. a nucleon of momentum $\underset{\sim}{p}$ has $E_p = \sqrt{m_N^2 + \underset{\sim}{p}^2}$). Now the remaining task is to evaluate this sum rule. Since an exact evaluation of such a sum is not possible, the main problem is then to find a way in which the sum can be saturated rapidly with a few intermediate states. In evaluating the sum in nuclei, one may use the information available on nucleons as to what states should be used to saturate the sum rule.

In the <u>nucleon case</u>, Fubini and Furlan (FF) [28] and de Alfaro and Rossetti[29] have derived an elegant method which increases the convergence of the sum and makes it possible to saturate it with low-lying states only, namely nucleon states and nucleon states with one pion. Their idea is to sum two amplitudes. For instance, in π-nucleon scattering, they sum and subtract the amplitudes for negative and positive pions to obtain respectively the isospin symmetric and isospin antisymmetric combinations which have different transformation properties under crossing.

Similarly, for the radiative capture they add and subtract the amplitudes $\mathcal{M}$ and $\bar{\mathcal{M}}$ for the two processes

$$\begin{aligned} \mathcal{M} &: \quad \pi^-(q) + p(p_i) \rightarrow \gamma(k) + n(p) \\ \bar{\mathcal{M}} &: \quad \gamma(k) + p(p_i) \rightarrow \pi^+(q) + n(p) \end{aligned} \qquad \text{(II.19)}$$

Here, unlike the scattering, the crossing property is somewhat delicate, so a more detailed explanation is required.

Let us define the Mandelstam variables for the two processes

$$\begin{aligned} s &= (p_i + q)^2 , \quad & t &= (p_f - p_i)^2 , \quad & u &= (p_f - q)^2 \\ \bar{s} &= (p_f + q)^2 , \quad & \bar{t} &= (p_f - p_i)^2 , \quad & \bar{u} &= (p_i - q)^2 \end{aligned} \qquad \text{(II.20)}$$

Now for $\underset{\sim}{q} = 0$, and in the Breit frame where $\underset{\sim}{p}_i + \underset{\sim}{p}_f = 0$, we have

$$s = \bar{s} \quad , \quad t = \bar{t} \quad , \quad u = \bar{u} \tag{II.21}$$

From the crossing relation (assuming time-reversal invariance)

$$\bar{\mathcal{M}}(s,t,u) = \mathcal{M}^*(u,t,s) \tag{II.22}$$

we find using Eqs. (II.21) and (II.22)

$$\mathcal{M}(s,t,u) \pm \bar{\mathcal{M}}(s,t,u) = \mathcal{M}(s,t,u) \pm \mathcal{M}^*(u,t,s) \tag{II.23}$$

Thus in the sum only the crossing even terms (i.e. even under $s \leftrightarrow u$ or equivalently $k \to -k$ and $q \to -q$) survive, while in the difference only the odd terms remain. In the original derivation of Fubini and Furlan, these authors take advantage of such relations and saturate certain equal-time commutators directly. These various methods are equivalent. An explicit demonstration of such equivalence will be given in Section III for the elastic scattering.

In the nuclear problem, the FF method of combining two amplitudes is well adapted to the scattering process. However, for the photoproduction transition it can be used only when the initial and final nuclear states $|i\rangle$ and $|f\rangle$ are members of an isomultiplet. Only in this case the crossing relation relates the physical to the unphysical region. This restricts the transitions that we can treat in nuclei and we will not make use of this method. Instead, we will work with one single amplitude, as obtained from the reduction technique.

III. ELASTIC PION-NUCLEAR SCATTERING

A pion elastically scattered by a nucleus B (note that B stands in general for a nucleus of any quantum number), as $\pi^\alpha + B \to \pi^\beta + B$, may be scattered in the same isospin state ($\alpha = \beta$) or it may undergo single or double charge exchange. The third process occurs only for an isospin of the target ≥ 1 and it has a very small probability. If the double charge exchange is ignored, there exist two amplitudes T^- and T^+, which are respectively odd and even under crossing ($\pi^- \rightleftarrows \pi^+$):

$$\begin{aligned} (Z-N)\,T^- &= \frac{1}{2}\left[T_{\pi^-} - T_{\pi^+}\right] \\ T^+ &= \frac{1}{2}\left[T_{\pi^-} + T_{\pi^+}\right] \end{aligned} \tag{III.1}$$

where T^- and T^+ are called the charge antisymmetric (or alternatively charge exchange or isospin odd) and charge symmetric (or isospin even) amplitudes, Z and N are proton and neutron numbers, and $T_{\pi^\pm}$ the scattering amplitude for $\pi^\pm$ on the nucleus B.

In the limit when both the incident and the scattered pions become soft, these amplitudes have simple expressions, given by the expectation values of equal-time commutators. For the physical mass,

$$\begin{aligned} (Z-N)\,T^-(m_\pi) = (Z-N)\,T^-(0) &- \frac{1}{m_\pi f_\pi^2}\int d^3x \, \langle B|[D^+(0), D^-(\underset{\sim}{x},0)]|B\rangle \\ &- \left\{ m_\pi^3 \sum_n (2\pi)^3 \delta(\underset{\sim}{p_n} - \underset{\sim}{p_B}) \frac{\langle B|j_\pi^+(0)|n\rangle\langle n|j_\pi^-(0)|B\rangle}{(E_n - E_B)^2\left[m_\pi^2 - (E_n - E_B)^2\right]} - c.t. \right\} \end{aligned} \tag{III.2}$$

and similarly

$$\begin{aligned} T^+(m_\pi) = T^+(0) &+ \frac{i}{m_\pi^2 f_\pi^2}\int d^3x \, \langle B|[D^+(0), \dot{D}^-(\underset{\sim}{x},0)]|B\rangle \\ &- \left\{ m_\pi^2 \sum_n (2\pi)^3 \delta(\underset{\sim}{p_n} - \underset{\sim}{p_B}) \frac{\langle B|j_\pi^-(0)|n\rangle\langle n|j_\pi^+(0)|B\rangle}{(E_n - E_B)\left[m_\pi^2 - (E_n - E_B)^2\right]} - c.t. \right\} \end{aligned} \tag{III.3}$$

where we shall understand that only the "connected" matrix element be taken (see Appendix B for this point), and where the soft-pion expressions are given by

$$(Z-N)\, T^-(0) = \frac{m_\pi}{f_\pi^2} \langle B | [Q_A^+(0), A_0^-(0)] | B \rangle$$
$$T^+(0) = -\frac{i}{f_\pi^2} \langle B | [Q_A^+(0), \dot{A}_0^-(0)] | B \rangle \tag{III.4}$$

The sum rules which relate the physical amplitudes to the soft pion expressions are derived with the reduction techniques.

As we have promised before, we shall now demonstrate how one can get the same result starting from a commutator as originally formulated by Fubini and Furlan. This we do for the isospin antisymmetric case, since the isospin symmetric amplitude can be obtained in exactly the same way.

Consider the equal time commutator of two axial charges $[Q_A^+(0), Q_A^-(0)]$. The axial charge $Q_A(x_0) = \int d^3x\, A_0\, (\underset{\sim}{x}, x_0)$ is not a constant of motion since the axial current is not conserved and it depends on time. From now on we shall suppress the time dependence for the sake of simplicity in writing.

This commutator is known from current algebra and it gives a vector charge:

$$[Q_A^+, Q_A^-] = 2\, Q_V^3 \;.$$

The vector charge Q_V^3 is $\int d^3x\, V_0^3(\underset{\sim}{x})$, where V_μ^3 is the isovector piece of the electromagnetic current $J_\mu^{em} = S_\mu + V_\mu^3$ (S_μ is the isoscalar e.m. current). The expectation value of the commutator between the nuclear (ground) state $|B\rangle$ at rest is then proportionnal to the proton excess Z - N. An insertion in the commutator of a complete set of states $\Sigma_n |n\rangle\langle n| = 1$ gives the following sum rule:

$$\langle B | [Q_A^+ , Q_A^-] | B \rangle = (2\pi)^3 \delta(\underset{\sim}{p}_{B'} - \underset{\sim}{p}_B)\, 2M (Z-N)$$
$$= \sum_n \left[\langle B | Q_A^+ | n \rangle \langle n | Q_A^- | B \rangle - \langle B | Q_A^- | n \rangle \langle n | Q_A^+ | B \rangle \right. \tag{III.5}$$

where $\underset{\sim}{p}_B$ and $\underset{\sim}{p}_{B'}$ are the initial and final momenta of the nucleus B ($\underset{\sim}{p}_{B'} = \underset{\sim}{p}_B = 0$). The factor 2M, where M is the nuclear mass, arises from the normalization of the nuclear states, i.e.

$$\langle p | p' \rangle = (2\pi)^3\, 2 p_0\, \delta(\underset{\sim}{p} - \underset{\sim}{p}') \tag{III.6}$$

This sum rule has no resemblance whatsoever to the relation between the physical and the soft-pion amplitudes since the amplitude for nuclear scattering does not appear at all. One has first to bring this sum rule into a form which displays the amplitudes explicitly. To do this we use the PCAC relation, but this time in a reverse order. In the usual reduction techniques, we replaced the pion field by the divergence of the axial current; here we start instead with the axial charge and transform it into a pion field. In order to do so, we relate the matrix element of the axial charge to that of the divergence D. The axial charge matrix element may also be written as

$$\langle B | Q_A^\alpha(x_0) | n \rangle = \int d^3x \, \langle B | A_0^\alpha(\underset{\sim}{x}, x_0) | n \rangle = \frac{1}{i(E_B - E_n)} \int d^3x \, \langle B | \frac{\partial}{\partial x_0} A_0^\alpha(\underset{\sim}{x}, x_0) | n \rangle \tag{III.7}$$

We assume that the axial current is localized, which means physically that there is no current flow at infinity. The space integral

$$\int d^3x \, \langle B | \underset{\sim}{\nabla} \cdot \underset{\sim}{A} | n \rangle$$

therefore vanishes and we can add it to the previous relation to obtain the relation

$$\begin{aligned} \langle B | Q_A^\alpha(x_0) | n \rangle &= \frac{1}{i(E_B - E_n)} \int d^3x \, \langle B | D^\alpha(\underset{\sim}{x}, x_0) | n \rangle \\ &= \frac{(2\pi)^3 \, \delta(\underset{\sim}{p}_B - \underset{\sim}{p}_n)}{i(E_B - E_n)} \langle B | D^\alpha(0, x_0) | n \rangle \end{aligned} \tag{III.8}$$

By applying the PCAC, the divergence D is transformed into a pion field. The sum on the right-hand side of Eq. (III.5) can thus be visualized as a sum over the processes where a pion is absorbed leading to all possible final states. This pion is in general not on its mass-shell. Its spatial momentum is zero because of the presence of the factor $\delta(\underset{\sim}{p}_n - \underset{\sim}{p}_B)$, and its energy is the energy transfer between the states $|B\rangle$ and $|n\rangle$. In certain cases, however, this pion can be on the mass-shell. This happens when the energy difference $E_n - E_B$ is equal to the pion mass $E_n - E_B = m_\pi$. This can occur in two ways:

i) the intermediate state is a (highly excited) nuclear state without a pion; for instance, two fast nucleons are emitted in opposite directions, as occurs in the pair-absorption process of the physical pion; or

ii) the intermediate state is a nuclear state with one pion $|n\rangle = |B'\pi\rangle$. The condition $E_n - M = m_\pi$ is satisfied when the intermediate nucleus is the ground state and the pion is at rest.

But whenever the pion is on the mass shell the matrix element of D is infinite and there is therefore a difficulty due to an apparent singularity which develops in the r.h.s. of Eq. (III.5). The intricacy associated with this question may best be appreciated by reading it in Ref. 28. Let us simply point out that this difficulty just reflects, in a dispersion theoretic language the necessity of a subtraction constant, therefore a model dependance (see also the Appendix B for a brief discussion on this matter). This problem can be handled in the following way[28]: Consider the matrix element of the equal time commutator (integrated over space) given by

$$(2\pi)^3\delta(\underset{\sim}{p}_{B'}-\underset{\sim}{p}_B)m_\pi^2 f_\pi^4 C = \int d^3x\, d^3x' \langle B|[D^+(\underset{\sim}{x},x_0), D^-(\underset{\sim}{x}',x_0)]|B\rangle \tag{III.9}$$

This commutator is then expanded again with a complete set of states

$$\begin{aligned}(2\pi)^3\, &\delta(\underset{\sim}{p}_{B'}-\underset{\sim}{p}_B)\, m_\pi^2 f_\pi^4\, C \\ &= \sum_n \int d^3x\, d^3x' \langle B|\, D^+(\underset{\sim}{x},0)|n\rangle\langle n|\, D^-(\underset{\sim}{x}',0)|B\rangle - \text{c.t.}\end{aligned} \tag{III.10}$$

From the relation between Q and D, the sum on the right-hand side can also be written as

$$\begin{aligned}(2\pi)^3\, &\delta(\underset{\sim}{p}_{B'}-\underset{\sim}{p}_B)\, m_\pi^2 f_\pi^4\, C \\ &= \sum_n (E_B-E_n)^2 \langle B|\, Q_A^+|n\rangle\langle n|\, Q_A^-\,|B\rangle - \text{c.t.}\end{aligned} \tag{III.11}$$

We subtract Eq. (III.11) divided by m_π^2 from Eq. (III.5). Then all the terms of the original sum are multiplied by the factor $1 - (E_n - E_B)^2/(m_\pi^2)$ which is zero when the energy difference is the pion mass, and we have now a new sum rule:

$$\langle B|[Q_A^+, Q_A^-]|B\rangle - \frac{1}{m_\pi^2}\int d^3x\, d^3x' \langle B|[D^+(\underset{\sim}{x},0), D^-(\underset{\sim}{x}',0)|B\rangle$$

$$= \int d^3x\, d^3x' \sum_n \left[\frac{1}{(E_n - E_B)^2} - \frac{1}{m_\pi^2}\right] \langle B|D^+(\underset{\sim}{x},0)|n\rangle\langle n|D^-(\underset{\sim}{x}',0)|B\rangle - \text{c.t.} \tag{III.12}$$

We emphasize that the removal of the difficulty has been done at the price of introducing a new commutator which is not given by the current algebra, on which we have to make model-dependent assumptions.

The threshold scattering amplitude is contained in the part of the sum where the intermediate state is the nuclear ground state with one pion. We would like to separate it explicitly. To do so we observe that in the matrix element $\langle B|D|B\pi\rangle$ there is a disconnected piece describing the matrix element for the nucleus B travelling without interacting with the pion, and the pion decaying into the vacuum via D. The rest is called the connected piece. If on the right-hand side of Eq. (III.12), we combine a connected piece for one matrix element and a disconnected piece for the other, this constitues the semi-disconnected part. In the semi-disconnected quantity the momentum transfer of the connected piece $\int d^3x \langle B\pi|D(\underset{\sim}{x}, 0)|B\rangle$ is such that $(E_n - M)^2 = m_\pi^2$ (remember that here the intermediate nucleus stays at rest and that the intermediate pion is then also at rest since there is no three-momentum transfer between the initial and intermediate states). So we are exactly at the pion pole for the matrix element of D. The residue of the pole $\langle B\pi|j_\pi|B\rangle$ is the threshold amplitude for the scattering of the pion on the nucleus B. The pion propagator $[m_\pi^2 - (E_n - E_B)^2]^{-1}$ which diverges is cancelled by the factor

$$\left[\frac{1}{(E_n - E_B)^2} - \frac{1}{m_\pi^2}\right]$$

introduced by the subtraction.

One can consider the semi-disconnected piece either in the direct term or in the crossed term. There thus appear the two threshold amplitudes for π^- and π^+ scattering. The initial commutator is chosen in order to get the isospin-odd combination. The sum rule for the physical amplitude $T = T(m_\pi)$ reads

$$\frac{T_{\pi^-} - T_{\pi^+}}{2} = \frac{2 m_\pi M}{f_\pi^2}(Z - N) - m_\pi^3 C \tag{III.13}$$

$$+ m_\pi^3 \sum_n (2\pi)^3 \delta(p_n - p_B) \frac{\langle B | j_\pi^+(0) | n \rangle \langle n | j_\pi^-(0) | B \rangle}{(E_n - M)^2 [(E_n - M)^2 - m_\pi^2]} - \text{c.t.}$$

This is just the same as Eq. (III.2) obtained from the reduction method.

1. THE SOFT-PION LIMIT

The results have been given before. They are just the equal-time commutators:

$$T^-(o) = \frac{2 m_\pi M}{f_\pi^2} \tag{III.14}$$

$$T^+(o) = -\frac{i}{f_\pi^2} \langle B | [Q_A^+ , \dot{A}_o^-(o)] | B \rangle$$

The Weinberg relation follows from the above if one neglects $T^+(o)$ and uses the relation $a = T/8\pi(M + m_\pi)$:

$$\left(1 + \frac{m_\pi}{M}\right) a^- = L \tag{III.15}$$

$$L = \frac{m_\pi}{4\pi f_\pi^2} .$$

We shall return to $T^+(o)$ later in this section, since it is an important quantity.

2. CORRECTIONS TO THE SOFT-PION LIMIT

We now discuss the correction terms in detail.

The only obvious limitation to the number of states that contribute to the saturation of the sums Eqs. (III.2) and (III.3) is due to the negative parity of the axial charge. Only the states which have a parity opposite to that of the ground state B can contribute.

As usual it is necessary, in the practical estimate of the sum, to suitably truncate it. In doing this truncation, we are guided by the results of Fubini-Furlan in the nucleon case. For a nucleon target, an excellent saturation is obtained with a small number of states, i.e. the nucleon with one pion. The states further away in energy than one pion mass play almost no role. It is likely that in the nuclear case the saturation will also be obtained with low-lying states. Whether, in analogy with the nucleon case, the nuclear states with one pion are sufficient is not obvious *a priori*. One may worry about the nuclear excitation spectrum: there is no equivalence for the nucleon case, since then the nucleon itself as the intermediate state is forbidden by parity and its excited states have large energy. But the nucleus has a very dense excitation spectrum at very low excitation energies. There was always great concern that this spectrum could completely alter the balance between the different terms of the sum, and would affect the relation between the physical and the soft-pion amplitudes in a radical and intricate way. This question was settled by the work of d'Auria et al.[30] who showed that this excitation spectrum was in fact unimportant. In the following, we shall keep it and calculate its contribution.

Retaining only the nuclear intermediate states plus states with nucleus and one pion, Eq. (III.2) has the form

$$
\begin{aligned}
(Z-N)\,T^-(m_\pi) &= (Z-N)\,T^-(0) - m_\pi^3\, C \\
&\quad - \left\{ \sum_{B'} \frac{m_\pi}{f_\pi^2}\left[1 - \frac{(E_{B'}-M)^2}{m_\pi^2}\right] \langle B|\, Q_A^+ \,|B'\rangle\langle B'|\, A_0^-(0)\,|B\rangle - \text{c.t.} \right\} \qquad \text{(III.16)} \\
&\quad + \left\{ \frac{m_\pi^3}{(2\pi)^3} \int \frac{d^3q}{2\omega_q\, 2E_q}\, \frac{\langle B(0)|\, j_\pi^+(0)\,|B_\alpha(-q)\,\pi^\alpha(q)\rangle\langle B_\alpha(-q)\,\pi^\alpha(q)|\, j_\pi^-(0)\,|B(0)\rangle}{(\omega_q + E_q - M)^2\left[(\omega_q + E_q - M)^2 - m_\pi^2\right]} - \text{c.t.} \right\} \\
&\quad + \left\{ \frac{m_\pi^3}{(2\pi)^3} \sum_{B' \neq B} \int \frac{d^3q}{2\omega_q\, 2E_{B'}(q)}\, \frac{\langle B(0)|\, j_\pi^+(0)\,|B'(-q)\,\pi^\alpha(q)\rangle\langle B'(-q)\,\pi^\alpha(q)|\, j_\pi^-(0)\,|B(0)\rangle}{(\omega_q + E_{B'}(q) - M)^2\left[(\omega_q + E_{B'}(q) - M)^2 - m_\pi^2\right]} - \text{c.t.} \right\}
\end{aligned}
$$

where α is the charge state of the intermediate pion which is to be summed over, B^a is the ground state B or its isobaric analogue depending if the intermediate scattering takes place without or with charge exchange, q is the intermediate pion momentum (indicated on the parenthesis), ω_q its energy $\sqrt{q^2 + m_\pi^2}$, $E_q = \sqrt{q^2 + M^2}$

the energy of the state B with momentum q, and $E_B'(q)$ the energy of the nuclear state B′ of a momentum q. We have not written down T^+ since it has the same structure.

The pion represented by the source j_π is not on the mass-shell. Its spatial momentum is zero (there is no spatial momentum transfer between the states $|B\rangle$ and $|n\rangle$), and its energy is the energy transfer $E_n - M$. It is off the mass-shell by the amount $(E_n - M)^2 - m_\pi^2$. Because of the appearance of these off-shell effects, this sum rule does not reduce to a relation between physical quantities. The evaluation of the sum depends on the detailed structure of the source function, which is a dynamical problem.

To proceed further, one has two possibilities. The first one is to make approximations which reduce the sum rule (III.16) to a relation between physical quantities. The other choice is to introduce a specific model of the source function. The first procedure may be more straightforward but it is less attractive for two reasons. First, the physical quantities in question, namely the π-nuclear scattering amplitudes, are not available from experiments. Moreover, without a model for the source function, one has no means of estimating the errors involved in the approximations. We shall then adopt the second method.

We have separated in Eq. (III.16) two types of rescattering terms: the coherent rescattering where the nucleus remains in its ground state $|n\rangle = |B\pi\rangle$; and the incoherent rescattering where it is excited $|n\rangle = |B'\pi\rangle$ where $B' \neq B$. The reason for this separation is that for the elastic scattering, i.e. in the matrix element $\langle B|j_\pi|B\pi\rangle$, we have a natural candidate for a model of the pion source function. The model we use is the optical potential which we describe below.

Consider the Schrödinger equation in the presence of a potential V_{opt}:

$$\nabla^2 \phi + 2 m_\pi E \phi = 2 m_\pi V_{opt} \phi \tag{III.17}$$

where ϕ is the pion wave function. This equation corresponds to the non-relativistic version of the Klein-Gordon equation, and on the right-hand side the quantity $2m_\pi V_{opt}\phi$ represents, in the non-relativistic domain, the source function j_π. We could therefore insert this value in the matrix element of the sum (III.16) to evaluate the coherent rescattering. It is, however, unnecessary and disadvantageous to do so because we can instead show the similarity between the dependence of the relativistic mass dispersion relation on the off-mass shell matrix elements and that of the non-relativistic potential dispersion relation on the off-energy shell matrix elements. Thus in the non-relativistic limit this formalism leads to the distorted-wave Born approximation in which the Born amplitude will be shown to be essentially the soft-pion amplitude with some small corrections.

To display this similarity we just have to write explicitly the two dispersion relations. The mass dispersion relation is already given in Eq. (III.16). Performing the summation over the possible charges of the intermediate pion, we get

$$T^- = T^-_B + \frac{m_\pi^3}{(2\pi)^3 2M}\int \frac{d^3q}{q^2\,\omega_q^3}\, T^-_{oq}\left[T^-_{qo} + 2T^+_{qo}\right]$$

(III.18)

$$T^+ = T^+_B + \frac{m_\pi^2}{(2\pi)^3 2M}\int \frac{d^3q}{q^2\,\omega_q^2}\left[2|T^-_{oq}|^2 + |T^+_{oq}|^2\right]$$

where T_B is what remains of the r.h.s. of Eq. (III.16) when the coherent rescattering has been separated out; and $T^{\pm}_{oq}$ is the iso-symmetric or antisymmetric part of

$$T^{\alpha\beta}_{oq} = \langle B(0)|\, j^\alpha_\pi(0)\,|B(-q)\,\pi^\beta(q)\rangle$$

the off-shell amplitude for π-nuclear scattering. We have neglected in Eq. (III.18) the recoil energy of the nucleus so that $E_q \approx M$.

Turning to potential scattering, if we define the scattering amplitude as

$$f = \frac{1}{2ik}\sum_\ell \left(e^{2i\delta_\ell} - 1\right) P_\ell(\cos\theta) \qquad \text{(III.19)}$$

then for pions of zero momentum we have

$$\left(1 + \frac{m_\pi}{M}\right) f_{oo} = -2\pi^2 \langle \chi_o|\, 2m_\pi V_{opt}\,|\Psi_o^{(+)}\rangle \qquad \text{(III.20)}$$

Here we normalize the wave functions by $\langle p|p'\rangle = \delta^3(\underset{\sim}{p} - \underset{\sim}{p}')$; the subscripts in f denote the initial and final momenta of pions; χ_0 is the plane wave state; and Ψ_0^+ the outgoing wave for pion of zero momentum. χ_0 and Ψ_0^+ are respectively solutions of the Schrödinger equations

$$H_0\chi_0 = E_0\chi_0 \qquad \text{and} \qquad (E_0 - H_0 + i\varepsilon)\,\Psi_0^+ = V_{opt}\,\Psi_0^+ ,$$

where H_0 is the Hamiltonian for the free pion. The pion kinetic energy E_0, which will be taken to be zero in our discussion, is written here for the sake of clarity. The π-nucleus interaction potential is denoted by V_{opt} so that the total Hamiltonian is $H = H_0 + V_{opt}$. The outgoing wave Ψ^+ also satisfies the following Lippmann-Schwinger equation

$$\Psi_0^{(+)} = \chi_0 + \frac{1}{E_0 - H_0 + i\varepsilon} V_{opt} \Psi_0^{(+)} \tag{III.21}$$

which can also be written in the form

$$\Psi_0^{(+)} = \chi_0 + \frac{1}{E_0 - H + i\varepsilon} V_{opt} \chi_0 \tag{III.22}$$

Using (III.22) the scattering amplitude is thus

$$\left(1 + \frac{m_\pi}{M}\right) f_{00} = -4\pi^2 m_\pi \langle \chi_0 | V_{opt} | \chi_0 \rangle - 4\pi^2 m_\pi \langle \chi_0 | V_{opt} \frac{1}{E_0 - H + i\varepsilon} V_{opt} | \chi_0 \rangle \tag{III.23}$$

To see the analogy with the mass dispersion relation, we introduce a complete set of outgoing states into the second term on the r.h.s. of Eq. (III.23):

$$\sum_q | \Psi_q^{(+)} \rangle \langle \Psi_q^{(+)} | = 1 \tag{III.24}$$

which leads to the following form

$$\left(1 + \frac{m_\pi}{M}\right) a = \left(1 + \frac{m_\pi}{M}\right) a_{Born} + \frac{1}{2\pi^2}\left(1 + \frac{m_\pi}{M}\right)^2 \int \frac{d^3q}{q^2} |f_{0q}|^2 \tag{III.25}$$

where $a \equiv f_{00}$, and

$$\left(1 + \frac{m_\pi}{M}\right) a_{Born} = \langle \chi_0 | -4\pi^2 m_\pi V_{opt} | \chi_0 \rangle$$

$$\left(1 + \frac{m_\pi}{M}\right) f_{0q} = \langle \chi_0 | -4\pi^2 m_\pi V_{opt} | \Psi_q^{(+)} \rangle \tag{III.26}$$

The form (III.25) displays explicitly the Born part and the dispersive integral which represents the pion wave distortion effect.

To bring the two dispersion relations [i.e. (III.18) and (III.25)] into identical forms, we relate the invariant relativistic amplitude to the non-relativistic one. In the non-relativistic domain,

$$T^{\pm} = 8\pi(M + m_\pi) f^{\pm} \tag{III.27}$$

Now if the pion is taken to be non-relativistic in the mass dispersion relations, that is, if we take $\omega_q \approx m_\pi$, Eq. (III.18) can be written as

$$\left(1 + \frac{m_\pi}{M}\right) a^- = \left(1 + \frac{m_\pi}{M}\right) a^-_B + \frac{1}{2\pi^2}\left(1 + \frac{m_\pi}{M}\right)^2 \int \frac{d^3q}{q^2} f^-_{0q}\left[f^-_{q0} + 2 f^+_{q0}\right] \tag{III.28}$$

where

$$a^-_B = \frac{T^-_B}{8\pi(M + m_\pi)}$$

and similarly for a^+:

$$\left(1 + \frac{m_\pi}{M}\right) a^+ = \left(1 + \frac{m_\pi}{M}\right) a^+_B + \frac{1}{2\pi^2}\left(1 + \frac{m_\pi}{M}\right)^2 \int \frac{d^3q}{q^2}\left[2\,|f^-_{0q}|^2 + |f^+_{0q}|^2\right]$$

These equations are identical in structure to Eq. (III.25), which can be seen in the following way. Consider for simplicity the target isospin ½ (and Z - N = ±1). Then the π-nuclear system can have total isospin ½ or 3/2. We denote the corresponding amplitudes by f_1 and f_3. Since

$$\begin{aligned} (Z - N) f^- &= \frac{f_{\pi^-} - f_{\pi^+}}{2} = (Z - N)\frac{f_1 - f_3}{3} \\ f^+ &= \frac{f_{\pi^-} + f_{\pi^+}}{2} = \frac{f_1 + 2 f_3}{3} \end{aligned} \tag{III.29}$$

we have

$$\begin{aligned} f^-(f^- + 2 f^+) &= \frac{1}{3}(f_1^2 - f_3^2) \\ 2(f^-)^2 + (f^+)^2 &= \frac{1}{3}(f_1^2 + 2 f_3^2) \end{aligned} \tag{III.30}$$

Then Eq. (III.28) can be written in one form:

$$\left(1+\frac{m_\pi}{M}\right)a_T = \left(1+\frac{m_\pi}{M}\right)(a_B)_T + \frac{1}{2\pi^2}\left(1+\frac{m_\pi}{M}\right)^2\int\frac{d^3q}{q^2}\left|(\beta_{oq})_T\right|^2 \tag{III.31}$$

for T = 1 or 3 corresponding to a ½ or 3/2 channel. This verifies our assertion. This identity can be shown for a target of any isospin[31].

It transpires from the above discussion that, within the approximations made thus far, we can identify $a_B^\pm$ of Eq. (III.28) to $a_{Born}^\pm$ of the potential model of Eq. (III.25). It is then natural to take the rest of the mass-correction terms to be associated with the pion wave distortion.

Before proceeding further, we emphasize that an equivalence between the potential picture and the mass dispersion formula is obtained when $\omega_q = \sqrt{q^2 + m_\pi^2}$ under the dispersion integral is taken to be m_π^2. This is obviously dangerous unless the integrand decreases rapidly as q is increased. It turns out, however, that because of the large nuclear size the integrand indeed becomes small for q large, and the error incurred in such an integral is small, the variation being roughly 10%-20% of the integral itself.

It is tempting to assume that in a_B the soft-pion term dominates, and hence the soft-pion term is just the Born contribution,

$$a^\pm_{Born} = a^\pm(o)$$

However, this identification is only tentative at this stage because an important step is still missing. Identifying a(o) with a_{Born} would amount to approximating the sum rules (III.2) and (III.3) by truncated ones in which we keep only the coherent rescattering as a correction to the soft-pion limit. *A priori* it is not clear whether this approximation is a good one.

That the soft-pion limit does not involve any rescattering is obvious from the expression of the soft-pion amplitude (strictly speaking one should take the expectation value in a particular isospin state):

$$\hat{a}(q_0) = \frac{q_0}{4\pi f_\pi^2}\,\underset{\sim}{T}\cdot\underset{\sim}{t}$$

which vanishes as $q_0 \to 0$. The soft pion interacts very weakly, somewhat like a neutrino, i.e. it interacts only once. When we switch on the pion mass, many scatterings occur before the pion leaves the target. The identification of a(o) with a_{Born} implies, then, that the only relevant rescattering processes are those for which the nucleus remains always in its ground state. If the nucleus is to stay in the ground state, the momentum $q = |\underset{\sim}{q}|$ that it can absorb is restricted to small values, say, $q \lesssim 1/R$ the inverse nuclear radius. All the processes involving larger momentum transfers are ignored in this picture. *A priori* there is no reason why their effect should not be as important as that of the coherent rescattering. Said differently, when the pion transfers small momenta, it can explore only the gross feature of the nucleus, i.e. the size. For larger momentum transfers, it explores smaller details of the target, such as the granular structure of the nucleus or the pair correlation. We do not know *a priori* whether it is a good approximation to neglect these last effects and to treat the scattering by the nucleus in the same way as the scattering by a piece of amorphous matter. The pion can also be absorbed and re-emitted, and we have also to discuss how this absorption changes the scattering properties.

These points are discussed in the following, where we study the effects of the remaining terms of the sum rule. We shall do this simultaneously for the isospin odd and even amplitudes since the techniques and the physical interpretations are similar, although the aim and the conclusions are quite different in the two cases. Our aim is to show first that the mass extrapolation scheme that we proposed works very well for the isospin-odd amplitude. Here the soft-pion limit is known from a commutator of current algebra. Therefore comparing the soft-pion plus the corrections to experimental values, we can gauge the validity of the procedure. On the other hand, in the isospin-symmetric case, the commutator which gives the soft-pion amplitude, is not known and is of considerable fundamental interest, as we shall discuss later. Since the correction terms here are similar to those of the isospin-odd amplitude, we use the same procedure as before to extract the isospin-even soft-pion amplitude from the measured values of the physical amplitudes.

3. REMAINING CORRECTIONS

The difference between T_B and $T(o)$ contains an equal-time commutator:

$$\int d^3x \, \langle B|[D^+(0), D^-(\underset{\sim}{x},0)]|B\rangle \qquad \text{for } T^-$$

$$\int d^3x \, \langle B|[D^+(0), \dot{D}^-(\underset{\sim}{x},0)]|B\rangle \qquad \text{for } T^+ .$$

In a model where D is proportional to the canonical pion field, these expectation values vanish (recall that it is understood that the vacuum expectation values have to be subtracted off). We make the assumption of the vanishing of these quantities. For the antisymmetric amplitude, the fact that it does not give an appreciable contribution in the nucleon case is a justification of this assumption in the nuclear case.

Apart from these commutators, the relation between T_B and $T(o)$ involves several sums. Since the coherent rescattering process has been already considered, and in fact is calculable directly from the Schrödinger equation with an optical potential, it is natural to proceed with the incoherent rescattering processes where the nucleus is excited in the intermediate state. This is a part of terms which make up the Born term (in addition to the soft-pion amplitude). The question to be answered here is what happens when the pion rescatters and the momentum transferred to a nucleon is too large to be absorbed by the nucleus in its ground state. The incoherent rescattering contribution to the quantity a_B is:

$$(Z-N)\left(1+\frac{m_\pi}{M}\right)\left(\delta a^-\right)_{B/I.R.} = \frac{m_\pi^3}{8\pi M}\sum_{B'\neq B}\int\frac{d^3q}{2\omega_q\, 2E_{B'}(q)}\,\frac{\langle B(o)|j_\pi^+(o)|B'(-q)\,\pi^\alpha(q)\rangle\langle B'(-q)\,\pi^\alpha(q)|j_\pi^-(o)|B(o)\rangle}{\left(\omega_q+E_{B'}(q)-M\right)^2\left[\left(\omega_q+E_{B'}(q)-M\right)^2-m_\pi^2\right]} - c.t.$$

(III.32)

$$\left(1+\frac{m_\pi}{M}\right)\left(\delta a^+\right)_{B/IR} = \frac{m_\pi^2}{8\pi M}\sum_{B'\neq B}\int\frac{d^3q}{2\omega_q\, 2E_{B'}(q)}\,\frac{\langle B(o)|j_\pi^+(o)|B'(-q)\,\pi^\alpha(q)\rangle\langle B'(-q)\,\pi^\alpha(q)|j_\pi^-(o)|B(o)\rangle}{\left(\omega_q+E_{B'}(q)-M\right)\left[\left(\omega_q+E_{B'}(q)-M\right)^2-m_\pi^2\right]} + c.t.$$

where the intermediate pion isospin indices α are to be summed over, and where $E_{B'}(q)$ is the energy of the excited state B' with a momentum q. Let us first see how the inelastic excitations are taken into account in potential scattering. In potential theory one may describe the elastic as well as the inelastic scattering of the pions by some potential V. It is a non-diagonal potential which acts not only on the pionic variables but also on the nucleonic variables, and it can connect the nuclear ground state to the excited states. Its use to describe the elastic scattering would require the solution of a system of coupled channel equations. To simplify the matter it is a common procedure to use an optical potential V_{opt} which incorporates the effect of the excited states so that the inelastic excitations do not appear explicitly. Then the scattering equation simply becomes

$$T = V_{opt} + V_{opt}\,\frac{P_o}{E_o - H_o + i\varepsilon}\,T \qquad \text{(III.33)}$$

where H_0 is the Hamiltonian for the free particles and P_0 is a projection operator which projects onto the ground state. The relation between the optical V_{opt} and the non-diagonal one V is

$$V_{opt} = V + V \frac{P_E}{E_0 - H_0 + i\varepsilon} V_{opt}. \qquad \text{(III.34)}$$

where P_E projects onto the excited states ($P_E + P_0 = 1$). Indeed if the pion could be considered as non-relativistic in the incoherent rescattering processes, the relation between a_B and a(o) would be exactly the same as the relation between V_{opt} and V [and the relation between $a(m_\pi)$ and a(o) the same as the one between T and V]. However, only V_{opt} and not V is directly accessible from the experiments, and moreover the non-relativistic approximation is not a good one for the incoherent rescattering.

In order to evaluate the sum over B′ in Eq. (III.32) we make the static approximation which greatly simplifies the expressions. In this approximation, the nucleons are taken as massive, so that they move with very slow velocities and only the spatial correlations play a role. The excitation energies, which in a Fermi gas model are inversely proportional to the nucleon mass, are neglected, which amounts to taking all excited states to be degenerate with the ground state and $E_{B'}(q) \approx M$. The energy denominators are then independent of the particular state $|B'\rangle$ and closure may then be used*).

What remains to be calculated is the matrix element

$$\langle B(o) \,|\, j_\pi^+(o) \,|\, B'(-q)\, \pi(q) \rangle$$

This is somewhat delicate. There is no way of evaluating this directly and we need to make an impulse approximation. Even then it would not be clear *a priori* whether each nucleon interacts with the pion like a free particle or like a quasi-particle renormalized by some nuclear interactions. What it boils down to is whether one should use a π-nucleon amplitude or a π-nucleon amplitude renormalized in some way for describing pion scattering off a nucleon in nuclei. In this respect, we will be guided by the fact that in the non-relativistic limit, the relation between a_B and a(o) should reduce to a relation between V_{opt} and V. Just as the soft-pion technique is an expansion in the pion mass, we may consider that the potential V is of first order in the pion mass. Then it is sufficient to evaluate V_{opt} to second order in the pion mass, i.e.

$$V_{opt} \approx V + V \frac{P_E}{E_0 - H_0 + i\varepsilon} V \qquad \text{(III.35)}$$

*) In actual calculations, this static approximation is not really necessary. One could take into account the excitation energy which of course makes an analytic calculation impossible. Numerically it makes little difference even in quantitative scale. We shall discuss the errors incurred in the procedure.

This implies that we are justified to take in the matrix element $\langle B|j_\pi|B'\pi\rangle$ a sum of (unrenormalized) single-nucleon contributions. Neglecting off-shell effects in the π-N amplitude as well as its momentum dependence, we can write

$$[8\pi(M+m_\pi)]^{-1}\langle B(0)|j_\pi^+(0)|B'(-q)\pi^-(q)\rangle = \int d^3r\, e^{i\underset{\sim}{q}\cdot\underset{\sim}{r}}\langle B|\sum_i (a_N^+ + a_N^-\tau_i^3)\,\delta(\underset{\sim}{r}-\underset{\sim}{r}_i)|B'\rangle \tag{III.36}$$

$$[8\pi(M+m_\pi)]^{-1}\langle B(0)|j_\pi^-(0)|B'(-q)\pi^0(q)\rangle = \int d^3r\, e^{i\underset{\sim}{q}\cdot\underset{\sim}{r}}\langle B|\sum_i \sqrt{2}\,a_N^-\,\tau_i^-\,\delta(\underset{\sim}{r}-\underset{\sim}{r}_i)|B'\rangle$$

where a_N is the free π-N scattering length.

With these expressions and using closure, one sees that the sum of Eq. (III.32) involves two summations on the nucleon labels j and k. Symbolically,

$$\begin{aligned}&\langle B|\sum_{j,k}|B\rangle - \langle B|\sum_j|B\rangle\langle B|\sum_k|B\rangle \\ &= \langle B|\sum_{j=k}|B\rangle + \langle B|\sum_{j\neq k}|B\rangle - \langle B|\sum_j|B\rangle\langle B|\sum_k|B\rangle\end{aligned} \tag{III.37}$$

On the left-hand side of this equation, the intermediate ground state is subtracted out since only the intermediate states $B' \neq B$ should be taken into account. On the right-hand side, the first term is usually referred to as the "self-correlation" term, and the rest the pair correlation effects. If there were no pair correlations (such as Pauli and short-range effects), only the self-correlation term would survive. The self-correlation term is easy to calculate and one finds

$$\begin{aligned}\left(1+\frac{m_\pi}{M}\right)\left(\delta a_B^-\right)_{S.C.} &= \frac{m_\pi^3}{2\pi^2}\left(1+\frac{m_\pi}{m_N}\right)^2 a_N^-\left(a_N^- + 2a_N^+\right)\int\frac{d^3q}{q^2\omega_q^3} \\ &= \frac{2}{\pi}\left(1+\frac{m_\pi}{m_N}\right)^2 m_\pi\, a_N^-\left(a_N^- + 2a_N^+\right)\end{aligned} \tag{III.38}$$

$$\begin{aligned}\left(1+\frac{m_\pi}{M}\right)\left(\delta a_B^+\right)_{S.C.} &= A\,\frac{m_\pi^2}{2\pi^2}\left(1+\frac{m_\pi}{m_N}\right)^2\left[2(a_N^-)^2+(a_N^+)^2\right]\int\frac{d^3q}{q^2\omega_q^2} \\ &= \left(1+\frac{m_\pi}{m_N}\right)^2 m_\pi\, A\left[2(a_N^-)^2+(a_N^+)^2\right]\end{aligned}$$

when A is the nucleon number.

With $a_N^+ \approx 0$ and $[1 + (m_\pi/m_N)]\, a_N^- \approx 0.10\ m_\pi^{-1}$, experimental values, we find that the self-correlation effects are rather small:

$$\left(1+\frac{m_\pi}{M}\right)\left(\delta a_B^-\right)_{s.c.} \approx 0.006\ m_\pi^{-1}$$

$$\left(1+\frac{m_\pi}{M}\right)\left(\delta a_B^+\right)_{s.c.} \approx 0.02A\ \ m_\pi^{-1} \tag{III.39}$$

For the charge symmetric amplitude, however, the presumed smallness of $a^+(o)$ suggested by the vanishing of a_N^+ has the consequence that all corrections have to be considered.

In the charge exchange, instead, the self-correlation is only a 6% effect and it might even be ignored. It is nevertheless essential in order to understand the link with the multiple scattering treatment. The result that is obtained at this stage is the following, ignoring the rest of the corrections:

$$a^-(o) + \left(\delta a_B^-\right)_{s.c.} = a_N^- = a_B^- \tag{III.40}$$

since the self-correlation effect is clearly nothing but the rescattering on a single nucleon. Thus we find the result that if we were to ignore the correlations, the charge exchange potential would be proportional to the π-N amplitude. This approximation can be identified with the first-order approximation in the multiple scattering theory where the potential is given in terms of the single-particle density $\rho(r)$ as

$$2m_\pi V(r) = -4\pi\left(1+\frac{m_\pi}{M}\right) a_N\, \rho(r) \tag{III.41}$$

We consider the pair correlation term $\langle B|\Sigma_{j\neq k}|B\rangle - \langle B|\Sigma_j|B\rangle\langle B|\Sigma_k|B\rangle$. As is well known, the pair correlation suppresses the contribution obtained in its absence, the suppression being larger the smaller the momentum transfer. As we shall see below, the effect is not very significant for the isospin-odd amplitude, but is important for the isospin-even amplitude.

Let us define the correlation functions

$$G_{\alpha\beta}(r,r') = \langle B|\sum_{j\neq k} \delta(\underset{\sim}{r}-\underset{\sim}{r}_j)\,\tau_j^\alpha\, \delta(\underset{\sim}{r}'-\underset{\sim}{r}_k)\,\tau_k^\beta|B\rangle$$

$$- \langle B|\sum_j \delta(\underset{\sim}{r}-\underset{\sim}{r}_j)\,\tau_j^\alpha|B_a\rangle\langle B_a|\sum_k \delta(\underset{\sim}{r}'-\underset{\sim}{r}_k)\,\tau_k^\beta|B\rangle \tag{III.42}$$

where the isospin indices α and β can take the values +, −, or 3. They can also take the value o, for which case τ should be replaced by a unit operator. B_a is the ground state B if α or β is 0 or 3, and the isobaric analogue of B if α or β is + or −. The pair-correlation contribution to a_B is related to the Fourier transforms of the functions G by

$$(Z-N)\left(1+\frac{m_\pi}{M}\right)\left(\delta a_B^-\right)_{P.C.} = \frac{m_\pi^3}{2\pi^2}\left(1+\frac{m_\pi}{m_N}\right)^2\int d^3q\, d^3r\, d^3r'\, \frac{e^{i\underset{\sim}{q}(\underset{\sim}{r}-\underset{\sim}{r}')}}{q^2\,\omega_q^3}$$

$$\times\left\{ (a_N^-)^2\left[G_{+-}(r,r') - G_{-+}(r,r')\right] + a_N^- a_N^+\left[G_{03}(r,r') + G_{30}(r,r')\right]\right\} \quad \text{(III.43)}$$

$$\left(1+\frac{m_\pi}{M}\right)\left(\delta a_B^+\right)_{P.C} = \frac{m_\pi^2}{2\pi^2}\left(1+\frac{m_\pi}{m_N}\right)^2\int d^3q\, d^3r\, d^3r'\, \frac{e^{i\underset{\sim}{q}(\underset{\sim}{r}-\underset{\sim}{r}')}}{q^2\,\omega_q^2}$$

$$\times\left\{ (a_N^+)^2 G_{00}(r,r') + (a_N^-)^2\left[G_{33}(r,r') + G_{+-}(r,r') + G_{-+}(r,r')\right]\right\}$$

The structure of these correlation functions is well known. Consider, for instance, the part which depends on G_{-+}. Here a neutron is transformed into a proton and back into a neutron. In the Fermi gas model where only the Pauli correlation is taken into account, the Fourier transform is proportional to the neutron number times the fraction of the overlapping volume of two Fermi spheres displaced by the vector $\underset{\sim}{q}$; i.e.

$$\int d^3r\, d^3r'\, \frac{\sin q|\underset{\sim}{r}-\underset{\sim}{r}'|}{q\,|\underset{\sim}{r}-\underset{\sim}{r}'|}\, G_{-+}(r,r') = -N\left[1-\frac{3}{2}\frac{q}{2p_F}+\frac{1}{2}\left(\frac{q}{2p_F}\right)^3\right] \qquad q<2p_F \qquad \text{(III.44)}$$

$$= 0 \qquad q>2p_F$$

where P_f is the Fermi momentum. This function is also used in μ^- capture in nuclei . Similarly, for the Fourier transform of the function G_{+-},

$$\int d^3r\, d^3r'\, \frac{\sin q|\underset{\sim}{r}-\underset{\sim}{r}'|}{q\,|\underset{\sim}{r}-\underset{\sim}{r}'|}\, G_{+-}(r,r') = -Z\left[1-\frac{3}{2}\frac{q}{2p_F}+\frac{1}{2}\left(\frac{q}{2p_F}\right)^3\right] \qquad q<2p_F \qquad \text{(III.45)}$$

$$= 0 \qquad q>2p_F$$

The other Fourier transforms are obtained in a similar way. Collecting them together, we find

$$\left(1+\frac{m_\pi}{M}\right)\left(\delta a_B^-\right)_{P.C.} = -\frac{2m_\pi^3}{\pi}\left(1+\frac{m_\pi}{m_N}\right)^2 a_N^-\left(a_N^- + 2a_N^+\right)\int_0^{2p_F}\frac{dq}{\omega_q^3}\left[1-\frac{3}{2}\frac{q}{2p_F}+\frac{1}{2}\left(\frac{q}{2p_F}\right)^3\right] \tag{III.46}$$

$$\left(1+\frac{m_\pi}{M}\right)\left(\delta a_B^+\right)_{P.C.} = -A\,\frac{2m_\pi^2}{\pi}\left(1+\frac{m_\pi}{m_N}\right)^2\left[\left(a_N^+\right)^2+2\left(a_N^-\right)^2\right]\int_0^{2p_F}\frac{dq}{\omega_q^2}\left[1-\frac{3}{2}\frac{q}{2p_F}+\frac{1}{2}\left(\frac{q}{2p_F}\right)^3\right]$$

The role of the exclusion principle is quite transparent in these expressions. Whereas the rescattering effects contained in the self-correlation part allow the intermediate nucleon to take any momentum transfers, the rescattering occurring as for a free nucleon, the pair-correlation puts on this recoil the restrictions imposed by the exclusion principle. The total effect of the inelastic excitations should thus be reduced compared to the nucleon case:

$$\left(1+\frac{m_\pi}{M}\right)\left(\delta a_B^-\right)_{I.R.} = \frac{2m_\pi^3}{\pi}\left(1+\frac{m_\pi}{M}\right) a_N^-\left(a_N^- + 2a_N^+\right) \times\left[\int_0^{2p_F}\frac{dq}{\omega_q^3}\left[\frac{3}{2}\frac{q}{2p_F}-\frac{1}{2}\left(\frac{q}{2p_F}\right)^3\right]+\int_{2p_F}^{\infty}\frac{dq}{\omega_q^3}\right] \tag{III.47}$$

and similarly for a^+. The cut-off function

$$f(z) = \frac{3}{2}\,z - \frac{1}{2}\,z^3 \qquad (z = q/2p_F)$$

increases from $o(z = o)$ to $1(z = 1)$. The Fermi momentum $p_F \approx 1.95\ m_\pi$ is considerably larger than the pion mass. But since the integrals of Eq. (III.47) get their major contribution from regions $q \lesssim m_\pi$, there is a large suppression due to the exclusion principle. We find that it reduces the incoherent rescattering effect to the values

$$\left(1+\frac{m_\pi}{M}\right)\left(\delta a_B^-\right)_{I.R.} = 0.002\ \ m_\pi^{-1} \tag{III.48}$$

$$\left(1+\frac{m_\pi}{M}\right)\left(\delta a_B^+\right)_{I.R.} = 0.0095\ A\ \ m_\pi^{-1}$$

The over-all effect of the inelastic excitations is thus very small. This makes the soft-pion amplitude a better approximation than the π-N amplitude to the Born amplitude. In the charge exchange case the quantities $a^-(o)$ and a_N^- are nearly equal and the distinction is somewhat academic, but this remark is important for the application which will be made in the charge symmetric case.

Notice that the static approximation overestimates the influence of the inelastic excitations, since the energy denominators $E_{B'}(q) + \omega_q - M$ have been underestimated by the approximation $\varepsilon_{B'} = E_{B'} - M \approx 0$. If instead we replace $E_{B'}$ by some average value $\bar{\varepsilon}_{B'}$, the influence of the incoherent rescattering is reduced. For example, for $\bar{\varepsilon}_{B'} = 20$ MeV the contribution to a^+ is

$$\left(1 + \frac{m_\pi}{M}\right)\left(\delta a_B^+\right)_{\mathrm{I.R.}} = 0.0078\,A \quad m_\pi^{-1} \qquad \text{instead of} \qquad 0.0095\,A \quad m_\pi^{-1}$$

3.1 Pion absorption

We still have undetermined the contribution to the sum of the terms where the intermediate states are the nuclear states not accompanied by a pion. The matrix element involved, $\langle B|Q_A|B'\rangle$, represents the absorption of a virtual pion.

When the excitation energy $(E_B - M) = \varepsilon_{B'}$ is of the order of the pion mass, the pion is nearly physical and the absorption occurs predominantly by a two-nucleon process, the one-nucleon absorption being strongly suppressed by the energy-momentum conservation. For low excitation energies, i.e. $\varepsilon_{B'} \ll m_\pi$, the absorption is a one-body process. One may therefore distinguish approximately the two regions of excitation, the low-energy region where the dominant process is the one-nucleon absorption and the high-energy region where the many-body mechanism dominates.

The contribution of the first region has been estimated by d'Auria et al.[30]. They make three assumptions, all of which follow from considering only low-lying states and excluding highly excited ones:

i) the axial charge is taken to be the sum of single-nucleon contributions, $Q_A = \Sigma_i\, q_A(i)$, where $q_A(i)$ is the charge for a nucleon i treated as free;

ii) the multiplicative factor arising from the subtraction

$$1 - \left(\frac{\varepsilon_{B'}}{m_\pi}\right)^2$$

is taken to be unity;

iii) the axial charge for the nucleon is approximated by $q_A^{\pm} = (g_A/m_N)\underset{\sim}{\sigma}\cdot\underset{\sim}{p}\tau^{\pm}$ where p is the nucleon momentum (average of the initial and final values) with a constant form factor g_A, which amounts to neglecting the pseudoscalar part in the expression of the axial current (for free nucleons)

$$\langle p(p_2)|A_0^+|n(p_1)\rangle = \bar{u}_p(p_2)\left[g_A\gamma_0\gamma_5 + 2m_N\, g_A\frac{q_0}{m_\pi^2-q^2}\gamma_5\right]\tau^+ u_n(p_1) \qquad \text{(III.49)}$$

Since q_0 is the energy transfer which is here much smaller than the pion mass, the pseudoscalar term may be neglected. With these three assumptions and the closure approximation, they estimate for the charge exchange process the contribution of the single nucleon absorption

$$\left(1+\frac{m_\pi}{M}\right)\left(\delta a_B^-\right)_{\substack{\text{single nucleon}\\\text{absorption}}} = -\frac{m_\pi}{4\pi f_\pi^2}\,2g_A^2\frac{\langle\varepsilon\rangle}{m_N} \approx -0.004\ m_\pi^{-1} \qquad \text{(III.50)}$$

where $\langle\varepsilon\rangle$ is the average kinetic energy

$$\langle\varepsilon\rangle = \frac{3}{5}\frac{P_f{}^2}{2m_N} \quad .$$

For the charge symmetric amplitude, one more approximation is required. This is due to the fact that a straightforward closure is not possible because of an extra energy factor [see Eqs. (III.2) and (III.3)]. d'Auria et al. assume that there is only one state B_0 which is strongly coupled to the ground state B. The result is (assuming the reasonable value $E_{B_0} - E_B \approx 25$ MeV):

$$\left(1+\frac{m_\pi}{M}\right)\left(\delta a_B^+\right)_{\substack{\text{single nucleon}\\\text{absorption}}} \approx -\frac{m_\pi}{4\pi f_\pi^2}\,g_A^2\,\frac{(E_{B_0}-E_B)}{m_\pi}\cdot\frac{\langle\varepsilon\rangle}{m_N} \approx -0.001\ m_\pi^{-1} \qquad \text{(III.51)}$$

Although the approximations made are rough, this contribution can be safely ignored.

Finally, we come to the two-nucleon contributions. Unfortunately this is, for the moment, the weakest point in the mass extrapolation. It is necessary to describe microscopically the two-nucleon absorption mechanism. Even if one has information about the absorption rate for the physical pion, this merely gives a normalization at one point, i.e. at the physical point. But to perform the off-shell extrapolation requires the knowledge of the reaction mechanism. The problem is actually under investigation with a preliminary indication that the effect is small[32].

Some previous crude estimates[33,34] indicated that this may be a small contribution. It is reasonable to believe that the magnitude of this effect is not larger than the imaginary part of the amplitude, in which case we guess that

$$\left|\left(\delta a_B^-\right)_{\substack{\text{pair}\\ \text{absorption}}}\right| \lesssim 0.004\ m_\pi^{-1} \qquad \left|\left(\delta a_B^+\right)_{\substack{\text{pair}\\ \text{absorption}}}\right| \lesssim 0.009\ m_\pi^{-1}$$

The neglect of the two-nucleon absorption constitutes actually the largest uncertainty in the calculation, and it is very desirable to have a reliable estimate of this effect.

4. COMPARISONS WITH EXPERIMENTS

4.1 Charge exchange amplitude

Here the soft-pion amplitude is known:

$$\left(1+\frac{m_\pi}{M}\right) a^-(0) = L = 0.09\ m_\pi^{-1} \tag{III.52}$$

Adding the corrections discussed above, we find the theoretical value of the Born amplitude

$$\left(1+\frac{m_\pi}{M}\right) a^-_{\text{Born}} = 0.088\ m_\pi^{-1} \tag{III.53}$$

To compare to an "experiment", we use the π-nuclear optical potential of Krell and Ericson[35] which is obtained by the analysis of π-mesic data. Given the optical potential $V_{\text{opt}}(\underset{\sim}{x})$, a^-_{Born} is then the charge antisymmetric piece of the integral

$$-\frac{m_\pi}{2\pi}\int d^3x\ V_{\text{opt}}(\underset{\sim}{x})$$

and Krell-Ericson analysis leads to the "experimental" value

$$\left(1+\frac{m_\pi}{M}\right) a^-_{\text{Born}} = 0.092\, m_\pi^{-1} \tag{III.54}$$

The theoretical value (III.53) is in excellent agreement with this. This agreement gives confidence that the mass extrapolation has been done correctly, and in particular it seems to indicate that there is no very significant contribution from the two-nucleon absorption. The method may now be applied to other cases where such a test is not possible.

Before making the application to the charge symmetric amplitude, it is worth discussing in more detail how the soft-pion method is related to the more familiar treatment of multiple scattering.

4.2 Link with the multiple scattering formalism

The aim of a multiple scattering theory is to relate the nuclear amplitude to the amplitudes for the individual nucleons. We are here in a unique situation to make the connection between the two techniques. The reason is that the basic commutator which is involved in the Fubini-Furlan method gives a vector charge. Its matrix element is, from the e.m. current conservation, exactly equal to the sum of the individual nucleon contributions. There are no exchange effects contained in such a matrix element. An exact correspondence between the two methods is therefore possible.

In a multiple scattering method, the potential is, in first approximation, proportional to the π-N amplitude

$$2m_\pi V(r) = -4\pi\left(1+\frac{m_\pi}{m_N}\right)a_N\,\rho(r) \tag{III.55}$$

Corrections to this first-order approximation arise from the finite structure of matter. The presence of one nucleon influences the local density of the other nucleons, and the pion experiences this structure by virtually exciting the nucleus. In multiple scattering, this effect is the effective field correction[36] which is shown below to be equivalent to the pair-correlation term of the soft-pion technique. This correction accounts for the difference between the average field in the nuclear medium and the local effective field which excites a scatterer.

The hole profile around a scatterer at the point r is described by the nuclear pair-correlation function g(r′-r). Since one nucleon has been removed by the presence of the scatterer at the point r, the normalization condition for the function g is

$$\int d^3r\,\rho(\underline{r}')\,g(\underline{r}'-\underline{r}) = -1 \tag{III.56}$$

The difference between the effective and the average field is the field radiated by the hole. In the long wavelength limit

$$\phi_{\text{eff}} = \phi_{\text{av.}} - a\left\langle\frac{1}{r}\right\rangle\phi_{\text{eff}}. \tag{III.57}$$

where ⟨1/r⟩ is the average value of 1/r over the correlation function*).

$$\left\langle\frac{1}{r}\right\rangle = -\int d^3u\,\frac{1}{u}\,\rho(r+u)\,g(u)$$

The replacement of the average field by the effective field amounts to replacing π-N scattering length by an effective value

*) More precisely, it is a ground-state expectation value of the operator $\sum_{i\neq j}(1/|\underline{r}_i-\underline{r}_j|)$.

$$a_{eff} = a - a^2 \langle \frac{1}{r} \rangle \tag{III.58}$$

With the soft-pion technique, we found that with the self-correlation alone, the optical potential was proportional to the π-N amplitude. And the inclusion of the pair term modified this result by an amount

$$\left(\delta a_B\right)_{P.C.} = \frac{m_\pi^3}{2\pi^2}\left(1 + \frac{m_\pi}{m_N}\right)\left(a_N^-\right)^2 \int d^3q\, d^3r\, d^3r' \, \frac{e^{iq(\underline{r} - \underline{r}')}}{q^2\, \omega_q^3}\, G(r, r') \tag{III.59}$$

where the function G(r, r′) has been defined previously [Eq. (III.42)]. Here isospin indices are omitted to simplify the formulae. The function G is related to the hole profile function g(r−r′) by

$$G(r, r') = \rho(r)\, \rho(r')\, g(r - r') \tag{III.60}$$

In the non-relativistic limit $\varepsilon_q = m_\pi$, and with

$$\int d^3q \, \frac{e^{iq\underline{u}}}{q^2}\, \rho(r+u)\, g(u) = -2\pi^2 \langle \frac{1}{r} \rangle \tag{III.61}$$

the modification introduced by the pair correlation is thus

$$\left(\delta a_B\right)_{P.C.} = - a^2 \langle \frac{1}{r} \rangle \tag{III.62}$$

which is the same as the effective field correction. There is therefore a complete equivalence between the two theories concerning the effect of the nuclear polarization. However, in multiple scattering the starting point is the π-N amplitude where the rescattering effect on the nucleon is already built in and the role of the pair correlation is not as clear as in the soft-pion technique.

Finally, to make complete the link between the two approaches, we consider the Fermi motion corrective term which arises in the soft-pion method from the absorption of a pion on a single nucleon exciting low-lying states. A similar term is present in the multiple scattering picture[36]. Whether or not the origin of such a contribution is the same is not clear, but numerically they agree closely.

Let us recall how the Fermi motion correction arises in this formalism. The π-nucleon interaction has an s- and a p-wave component. The amplitude is given by

$$f = a + c\, \underset{\sim}{k} \cdot \underset{\sim}{k}' \tag{III.63}$$

k and k$'$ are the momenta of the incident and scattered pion. In nuclei the motion of the nucleons induces a modification of the p-wave part which becomes the translationally invariant form

$$c\, m_\pi [v_\pi - v_N]\, m_\pi [v'_\pi - v_N] \tag{III.64}$$

where v_π, v_N are the pion and nucleon velocities respectively. This induces an s-wave interaction, and in the charge exchange potential a_N^- is to be replaced by

$$a_N^- - 2\, \frac{(m_\pi)^2}{m_N} \langle \varepsilon \rangle\, c_1 \tag{III.65}$$

where c_1 is a constant related to the p-wave scattering volumes $\alpha_{2t,2j}$ where t and j are the total isospin and spin of the π-N system

$$c_1 = \frac{1}{3}\left(2\alpha_{33} + \alpha_{31} - 2\alpha_{13} - \alpha_{11}\right) \tag{III.66}$$

Neglecting all scattering volumes but the dominant one α_{33}

$$c_1 \approx \frac{2}{3}\alpha_{33} \tag{III.67}$$

In the static model and the narrow resonance approximation, we have (ω_R = resonance energy in unit of m_π):

$$\alpha_{33} = \frac{4}{3}\, \frac{g_r^2}{4\pi\, m_\pi m_N^2} \left[1 - \frac{m_\pi}{\omega_r}\right]^{-1} \tag{III.68}$$

Using the Goldberger-Treiman relation $g_r = m_N g_A/\sqrt{2} f_\pi$, we see that the quantity a_N^- is modified by an amount

$$- \frac{m_\pi}{4\pi f_\pi^2} \quad 2 g_A^2 \frac{\langle \varepsilon \rangle}{m_N} \quad \frac{4}{9}\left[1 - \frac{m_\pi}{\omega_r}\right]^{-1} \tag{III.69}$$

With $\omega_R/m_\pi \approx 2$, the quantity $4/9[1 - (m_\pi/\omega_R)]$ is close to 1 and hence Eq. (III.69) is the same as the Fermi correction of the soft-pion technique.

There is thus a complete equivalence between the soft-pion technique and the multiple scattering method. The first provides in a natural way the optical potential up to the second order in the pion mass and gives a clear picture of the pair correlation effect.

We can now answer the following question: What will happen if the saturation of the sum rule is not completely attained with the low-lying states which have been considered? The answer to this is as follows. If it turns out that the saturation of the sum rule in the nucleon case requires extra contributions, they should also be present in the nuclear case. They are necessary in order to recover the expression of the π-N amplitude in the Born amplitude and therefore the equivalence with multiple scattering. There would be no modification of the physical pictures given here, but of course the predicted value of the Born amplitude would be somewhat modified. The good agreement between the experimental value of the π-N amplitude (a_N^-) and the predicted one with the Fubini-Furlan method seems to show no need for further contributions.

4.3 Charge symmetric amplitude

The charge symmetric soft-pion amplitude $a_N^+(o)$ (for nucleon target) is an extremely interesting quantity from the point of view of strong interaction dynamics[37]. The discussion on the interpretation of such a term would be a subject that is somewhat out of the scope of this review, and hence we shall only show how we can extract the π-nucleon amplitude $a_N^+(o)$ (using the confidence on the mass-extrapolation obtained above) from pion-nuclear scattering and compare it with the results obtained from π-N scattering.

We first derive $a^+(o)$ for nuclei from known physical amplitudes and the corrections we described above, and then extract from it the nucleon value $a_N^+(o)$. This procedure has been used by Huang[38], Gensini[39], and Ericson and Rho[40].

The expression we want is

$$T^+(o) = -\frac{i}{f_\pi^2} < B | [Q_A^+ , \dot{A}_0(o)] | B > \tag{III.70}$$

which may be written as

$$T^+(o) = \frac{1}{f_\pi^2} < B | [Q_A^+ , [H_{SB}(o), A_0^-(o)]] | B > \tag{III.71}$$

where H_{SB} is the part of the strong interaction Hamiltonian which breaks the conservation of the axial charge (known as the chiral symmetry). The knowledge of

$T^+(0)$ thus brings us information about the way in which the chiral symmetry is broken in nature[37], and can perhaps decide which symmetry-breaking schemes among many proposed so far is the correct one. In conformity with the notation in the literature, it is convenient to define the "σ-commutator term" for the nucleon σ_{NN} as

$$T_N^+(o) = -\frac{4m_N}{f_\pi^2}\sigma_{NN}$$
$$\sigma_{NN} = -\frac{1}{4m_N}\langle N|[Q_A^+,[H_{SB}(o), A_o^-(o)]]|N\rangle \qquad \text{(III.72)}$$

To distinguish the nucleon quantity from nuclear quantity, we shall always put the suffix N for the former.

One may wonder whether there is any advantage in obtaining σ_{NN} with nuclei rather than with nucleons. The answer in this case is affirmative. In fact there are several reasons why nuclei are more advantageous than nucleons despite the complexity of nuclear structure.

i) The input information used to extract the soft-pion amplitude is available. The threshold π-nuclear amplitudes are accurately estimated from the π-mesic data. This contrasts with the situation for nucleons, where different analyses give values of $a_N^+(m_\pi)$ ranging from $-0.014\ m_\pi^{-1}$ to $+0.02\ m_\pi^{-1}$.

ii) The incoherent rescattering correction to reach $a^+(o)$ from a^+_{Born} (which is known) is considerably suppressed in nuclei because of the Pauli exclusion principle. Thus the difference $|a^+_{Born} - a^+(o)|$ in nuclei is small compared to a^+_{Born}, in contrast to the π-N system where $|a_N^+(m_\pi) - a_N^+(o)|$ is large compared to the magnitude of $a_N^+(m_\pi)$.

This fact will prove to be of a great interest in resolving the confused status for σ_{NN} as obtained from the π-N scattering. The situation is as follows: The FF mass extrapolation was studied by von Hippel and Kim[41] for π-N scattering as well as K-N and π-Σ processes. They neglected the off-shell effects of the amplitudes in the mass extrapolation. Using the informations obtained from the elastic and inelastic processes for K-N and π-Σ systems, they deduced for the π-N system

$$a^+(o) = -0.03\ m_\pi^{-1} \quad \text{or} \quad \sigma_{NN} = 28\ \text{MeV} \qquad \text{(III.73)}$$

Recently, Cheng and Dashen[42] recalculated this quantity using a different method (i.e. partial wave dispersion relation for fixed momentum), and found an entirely different result:

$$a^+_N(0) = -0.12\ m_\pi^{-1} \qquad \sigma_{NN} = 110\ \text{MeV}. \tag{III.74}$$

On the other hand, Höhler et al.[43] made a similar analysis which is in disagreement with that of Cheng and Dashen:

$$a^+_N(0) = -0.040\ m_\pi^{-1} \quad \text{or} \quad \sigma_{NN} = 37\ \text{MeV}. \tag{III.75}$$

At the moment, there is no explanation for these large discrepancies. We may just mention that these differences greatly influence the interpretation of those numbers in terms of chiral symmetry-breaking mechanism. It is thus quite appealing to use the nuclei as a supplementary source of information.

The Krell-Ericson analysis[35] of the π-mesic data gives the Born amplitude

$$\left(1+\frac{m_\pi}{M}\right) a^+_{\text{Born}} = -0.035\ A \qquad m_\pi^{-1} \tag{III.76}$$

If we neglect the two-nucleon absorption contribution, the incoherent rescattering is the only correction appearing in the Born amplitude:

$$\left(1+\frac{m_\pi}{M}\right) a^+_{\text{Born}} = \left(1+\frac{m_\pi}{M}\right)\left[a^+(0) + \left(\delta a^+_B\right)_{I.R.}\right] \tag{III.77}$$

With

$$\left(1+\frac{m_\pi}{M}\right)\left(\delta a^+_B\right)_{I.R.} = +0.0095\ A \qquad m_\pi^{-1}$$

as we have already calculated we find from Eqs. (III.76) and (III.77):

$$\left(1+\frac{m_\pi}{M}\right) a^+(0) = -0.044\ A \qquad m_\pi^{-1} \tag{III.78}$$

If we make the plausible assumption that this quantity can be extrapolated linearly (in mass number) to A = 1, we find

$$a^+_N(0) = -\frac{0.044}{1+\dfrac{m_\pi}{m_N}} = -0.038\ m_\pi^{-1} \quad \text{or} \quad \sigma_{NN} = 34\ \text{MeV}. \tag{III.79}$$

which agrees with the results of Höhler et al. and von Hippel-Kim, but is in definite disagreement with the value of Cheng and Dashen. The result of Cheng and Dashen would imply a difference between the Born and the soft-pion amplitudes:

$$\left(1+\frac{m_\pi}{M}\right)\left[a^+_{Born}-a^+(0)\right]=\left(-0.035+0.135\right)A\ m_\pi^{-1}=0.10\,A\ m_\pi^{-1}$$

which is a very large number.

Let us examine what would be the probable errors in this calculation. In particular, would the error be large enough to accommodate the result of Cheng and Dashen? This is an important question if we are to convince the sceptics of the usefulness of π-nuclear scattering in this matter.

The first approximation which may be questioned is, as in the work of von Hippel and Kim, the neglect of the off-shell effects of the π-N amplitude. However, our extrapolation procedure is much less sensitive to these effects, because the exclusion principle cuts down considerably the contribution from the incoherent rescattering in which these effects appear. The fact that we find a result which is consistent with that obtained by Kim and von Hippel indicates that there is not a large sensitivity to the off-shell effects, for if there were one would expect fo find a significant difference between the π-N and the π-nuclear results. There is yet another indication that the off-shell effects probably do not play a prominent role. It is that in spite of the neglect of the off-shell behaviour we get excellent results for the charge antisymmetric case, both in π-N and π-nuclear scattering.

Although these arguments are not quantitative, the important point is that such effects would be less important in nuclei than nucleons.

The other approximations that have been made are the static approximation in the evaluation of the incoherent rescattering. Its validity has been discussed and it tends to overestimate $|a^+(o)|$ since it overestimates the incoherent rescattering integral, which is positive.

The same remark applies to the non-relativistic approximation $\varepsilon_q = m_\pi$ in the coherent rescattering integral. It is nevertheless a valid approximation (within 10-20%) since the form factor of the nucleus decreases rapidly with the momentum transfer q. Note that it again overestimates the magnitude of the coherent rescattering term and thus of $a^+(o)$.

There is also the neglect of the two-nuclear absorption which constitutes an uncertainty in the calculation. In order to reach the value of Cheng and Dashen, the effect of the absorption in a^+_B should be attractive (> 0) and as large as $\approx 0.09\ m_\pi^{-1}$, which is ten times the imaginary part of the amplitude. We consider this possibility to be very unlikely; the estimates that have been made, although rough, give instead a repulsive effect much smaller in magnitude than the number quoted.

Another source of uncertainty is our neglect of the subtraction constant proportional to $\int d^3x \langle B|[D^+(o), \dot{D}^-(x,o)]|B\rangle$ [see Eq. (IV.3) and (B.6)], the reason for neglecting it being that its contribution vanishes if D's are proportional to the canonical pion field operators. It is not clear whether this is justified. In the simplest model where the commutator does not vanish, namely the free quark model, the commutator turns out to be proportional to the σ-commutator[41]. Von Hippel and Kim estimate the multiplying coefficient to be small. However its exact value is quite uncertain. It could therefore happen that the σ-commutator and the subtraction terms nearly cancel each other, both being large in magnitude but giving a small resulting value. This would explain the discrepancy between our value and that of Cheng and Dashen. In the absence of any knowledge on the commutator in question, this possibility cannot be ruled out.

A final point*) which may not be serious in practice, but delicate in principle is the assumption we made to reach the <u>nucleon</u> value Eq. (III.79) from the <u>nuclear</u> value Eq. (III.78). The experiments together our calculation support that a simple A dependence emerges as

$$\left(1 + \frac{m_\pi}{M}\right) a^+(o) = A\left(1 + \frac{m_\pi}{m_N}\right)\xi \qquad \text{(III.80)}$$

Now our assumption is that $\xi \equiv a_N^+(o)$. A justification for this relation Eq. (III.80) is not as straightforward as for the antisymmetric case for which in the soft-pion limit, the nuclear value is rigorously related to (N-Z) times the nucleon value. In the antisymmetric case, one may consider the scattering amplitude to be given by the ρ-exchange, and since ρ is coupled to a conserved current, the amplitude becomes "universal" as we have emphasized before. On the other hand, in the language of σ-model[11], a^+ may be considered to be due to σ-exchange. Since σ is not coupled to a conserved current, the same universality argument cannot be invoked. What this amounts to saying is that there may be many-body effects (such as binding effects) preventing us from an unambiguous identification $\xi = a_N(o)$. In practice, such an effect may very well be negligible, this conjecture being based on our experiences with the Gamow-Teller coupling of β-decay in nuclei where the many-body effect is found to disturb the nucleon value only by a small amount although the current is not conserved[21]. This question certainly deserves further studies.

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*) G.E. Brown and B.W. Lee (private communication).

To summarize, we may safely conclude that apart from the subtraction constant, there cannot be a contribution from the above sources large enough to accomodate the Cheng and Dashen value. Our conclusion is also supported by several more recent analyses of the same quantity in different methods[44], all of which favor the symmetry breaking scheme of Gell-Mann, Oakes and Renner[45].

IV. RADIATIVE PION CAPTURE AND PION PHOTOPRODUCTION

1. MASS DISPERSION RELATION

The radiative pion absorption

$$\pi^{\pm} + i \to \gamma + f$$

and the photoproduction

$$\gamma + f \to \pi^{\pm} + i \quad ,$$

where i and f denote nuclear states, are related by detailed balance, and so we shall deal here only with the radiative capture process. As before, the invariant amplitude is written as $\epsilon^{\mu}M_{\mu}$, with ϵ^{μ} the proton polarization four-vector. Unless ambiguity arises, we shall also denote $M = \epsilon^{\mu}M_{\mu}$, and the pion-mass dependence will be given as $M(q_0)$ or $M_{\mu}(q_0)$, except for $q_0 = m_{\pi}$ for which we do not show it explicitly.

The physical amplitude is related to the soft-pion one by the following sum rule which is derived in detail in Appendix C:

$$\mathcal{M}_{\mu}^{\pm} = \mathcal{M}_{\mu}^{\pm}(0) + e\, m_{\pi} \sum_{n} (2\pi)^3 \delta(\underset{\sim}{p_n} - \underset{\sim}{p_i}) \frac{\langle f | J_{\mu}^{em}(0) | n \rangle \langle n | j_{\pi}^{\pm}(0) | i \rangle}{(E_n - E_i)[m_{\pi} + E_i - E_n]} - \text{c.t.} \qquad \text{(IV.1)}$$

The soft-pion amplitude is

$$\mathcal{M}_{\mu}^{\pm}(0) = \mp \frac{ie}{f_{\pi}} \langle f | A_{\mu}^{\pm}(0) | i \rangle \qquad \text{(IV.2)}$$

Recall that we are always working with the pion three-momentum equal to zero.

The mass extrapolation procedure is in essence exactly the same as the elastic π-nuclear scattering, though there are some differences in detail, so that we will pursue the discussion in parallel with the previous section. The FF extrapolation technique was applied to the nuclear case by Ericson and Figureau[26]. The importance of the rescattering terms was also emphasized by Fulcher and Eisenberg[46].

The choice of the states to be kept in the saturation of the sum in Eq. (IV.1) is guided by the analysis of the same sum rule extensively done for the nucleon. More states are needed here than for the elastic π-nuclear scattering. One has to include not only nuclear intermediate states and nuclear states plus a pion as in scattering, but also nuclear states with one vector meson as has been shown in the nucleon case[28,47,48]. Since we do not make use of the FF method, we also have to include pair terms when the intermediate state contains a nucleon-antinucleon pair.

A similar set of intermediate states gives a good saturation in the case of the photoproduction of a pion off a nucleon and reproduces the results of the FF extrapolation method.

In analogy with the elastic π-nuclear scattering, it is convenient to separate from the sum over n in Eq. (IV.1) the state consisting of the ground state $|i\rangle$ with a pion, i.e. $|i\pi\rangle$, which represents, in the same way as before, the important distortion effect of the pion wave. If we denote by $\mathcal{M}_B$ the remainder of the amplitude when this separation has been done, we have

$$\mathcal{M}^{\pm}_{\mu} = \mathcal{M}^{\pm}_{B\mu}$$

$$+\frac{e m_\pi}{(2\pi)^3}\int\frac{d^3q\, d^3p'_i}{2\omega_q 2E'_i}\Bigg\{\delta(\underset{\sim}{p}'_i+\underset{\sim}{q}-\underset{\sim}{p}_f)\frac{\langle f|J^{em}_\mu(0)|i_a(p'_i)\pi^\alpha(q)\rangle\langle i_a(p'_i)\pi^\alpha(q)|j^{\pm}_\pi(0)|i\rangle}{(E'_i+\omega_q-E_i)[m_\pi+E_i-E'_i-\omega_q]} \qquad \text{(IV.3)}$$

$$-\delta(\underset{\sim}{p}'_i+\underset{\sim}{q}-\underset{\sim}{p}_f)\frac{\langle f|j^{\pm}_\pi(0)|i_a(p'_i)\pi^\alpha(q)\rangle\langle i_a(p'_i)\pi^\alpha(q)|J^{em}_\mu(0)|i\rangle}{(E'_i+\omega_q-E_f)[m_\pi+E'_i+\omega_q-E_f]}\Bigg\}$$

where, depending on the value of α, i_a is the ground state i or its isobaric analogue.

Here $\underset{\sim}{P}_i{}'$ and $E_i{}'$ denote the momentum and energy of the intermediate state having the same structure as the ground state i, and $\underset{\sim}{q}$ and ω_q the momentum and energy of the pion ($\omega_q = \sqrt{\underset{\sim}{q}^2+m_\pi^2}$). As it is defined, $\mathcal{M}_B$ contains all other intermediate states, such as purely nuclear intermediate states and excited states plus a pion, etc. We shall write their explicit forms when we discuss each term in detail.

In order to simplify Eq. (IV.2) further, we note that the denominator $m_\pi + \omega_q + E_i' - E_f$ in the second integral is always larger than $2m_\pi$ (omitting the small energy difference $E_i' - E_f$), while that of the first integral $m_\pi + E_i - E_i' - \omega_q$ may take small values. It is therefore legitimate to omit the second integral in the sum rule.

Off-shell amplitudes again appear in the integral, but here we meet two different kinds; one, the pion scattering $\langle f(\underset{\sim}{p}_f)|j_\pi^\pm(o)|i(\underset{\sim}{p}_i-\underset{\sim}{q})\pi(\underset{\sim}{q})\rangle$ which is familiar from the last section; the other, the pion photoproduction amplitude $\langle i(\underset{\sim}{p}_f-\underset{\sim}{q})\pi(\underset{\sim}{q})|j_\mu^{em}(o)|i\rangle$ that we have not yet dealt with. Once more we resort to the potential scattering formalism to take care of the off-shell extrapolation. We shall show below that the mass extrapolation formula (IV.2) is equivalent to the non-relativistic Lippmann-Schwinger equation.

Let us denote the potential for the radiative capture by $V_{\pi^\pm\gamma}$, and the optical potential for π-nuclear interaction by $V_{\pi^\pm N}$. The transition amplitude is (suffix p stands for potential picture)

$$\mathcal{M}_p^\pm = \langle \chi_{\gamma f} | V_{\pi^\pm\gamma} | \Psi^{(+)} \rangle \tag{IV.4}$$

where $|\chi_{\gamma f}\rangle$ is the plane wave state for the γf system and $\Psi^{(+)}$ the incoming eigenstate for the πi system of the total Hamiltonian $H_0 + V_{\pi^\pm\gamma} + V_{\pi^\pm N}$. Note that in order to avoid the confusion with the charge states of pion appearing in $M^\pm$ and $\pi^\pm$, we put the boundary conditions $(\pm)$ within a parenthesis.

We introduce the π-nuclear states ϕ which are eigenfunctions of the strong interaction Hamiltonian

$$\left(H_o + V_{\pi^\pm N}\right) \Phi^{(\pm)} = E\, \Phi^{(\pm)} \tag{IV.5}$$

To first order in the e.m. coupling constant one may replace $\Psi^{(\pm)}$ by $\phi^{(\pm)}$ in the expression of the transition amplitude

$$\mathcal{M}_p^\pm = \langle \chi_{\gamma f} | V_{\pi^\pm\gamma} | \Phi^{(+)} \rangle \tag{IV.6}$$

Since the state $|\phi^{(\pm)}\rangle$ obeys the Lippmann-Schwinger equation

$$|\Phi^{(+)}\rangle = |\chi_{\pi^\pm N}\rangle + \frac{1}{E+i\epsilon-H_o} V_{\pi^\pm N} |\Phi^{(+)}\rangle = |\chi_{\pi^\pm N}\rangle + \frac{1}{E+i\epsilon-H_o-V_{\pi^\pm N}} V_{\pi^\pm N} |\chi_{\pi^\pm N}\rangle \tag{IV.7}$$

the transition amplitude becomes

$$\mathcal{M}_p^{\pm} = \langle \chi_{\gamma f} | V_{\pi^\pm \gamma} | \chi_{\pi^\pm N} \rangle + \langle \chi_{\gamma f} | V_{\pi^\pm \gamma} \frac{1}{E + i\varepsilon - H_0 - V_{\pi^\pm N}} V_{\pi^\pm N} | \chi_{\pi N} \rangle$$

$$= \mathcal{M}^{\pm}_{Born} + \int d^3q \, \frac{\langle \chi_{\gamma f} | V_{\pi^\pm \gamma} | \Phi_q^{(+)} \rangle \langle \Phi_q^{(+)} | V_{\pi^\pm N} | \chi_{\pi^\pm N} \rangle}{E + i\varepsilon - E_q} \tag{IV.8}$$

where assuming that $\Phi^{(+)}$ forms a complete set, we have used the relation $\Sigma_q |\Phi_q^{(+)}\rangle\langle\Phi_q^{(+)}| = 1$ to obtain the second form.

Recalling that the scattering amplitude in non-relativistic potential theory is

$$\left(1 + \frac{m_\pi}{M}\right) f^{\pm}_{q'q} = -4\pi^2 m_\pi \langle \chi_{q'} | V_{\pi^\pm N} | \Phi_q^{(+)} \rangle = \frac{T}{8\pi M} \tag{IV.9}$$

we find, setting $E = 0$, $E_q = q^2/2m_\pi$:

$$\mathcal{M}_p^{\pm} = \mathcal{M}^{\pm}_{Born} + \frac{1}{2\pi^2}\left(1 + \frac{m_\pi}{M}\right) \int \frac{d^3q}{q^2} \langle \chi_{\gamma f} | V_{\pi^\pm \gamma} | \Phi_q^{(+)} \rangle f^{\pm *}_{q0}$$

We may now compare this to Eq. (IV.3) which, when we set $E_i' \approx E_i \approx M$, leads to

$$\mathcal{M}^{\pm} = \mathcal{M}^{\pm}_B - \frac{e m_\pi^2}{4\pi^2}\left(1 + \frac{m_\pi}{M}\right) \int d^3q \, \frac{\left(1 + \frac{\omega_q}{m_\pi}\right)}{q^2 \omega_q^2} \langle f | \varepsilon . J^{em}(0) | i \pi^{\pm} \rangle f^{\pm *}_{q0} \tag{IV.10}$$

In non-relativistic approximation for the pion $\omega_q \approx m_\pi$ under the integral which was found to be good to 10-20%, we see that Eqs. (IV.9) and (IV.10) are equivalent.

We are thus allowed to make the identification

$$\mathcal{M}^{\pm}_{Born} = \mathcal{M}^{\pm}_B = \mathcal{M}^{\pm}(0) + \text{corrections} \tag{IV.11}$$

As in the elastic π-nuclear scattering, this is approximate because of the non-relativistic approximation made above, but it is a valid approximation.

CORRECTIONS TO THE SOFT-PION LIMIT

We discuss here the "corrections" of Eq. (IV.11). We shall see that they are much smaller in magnitude than the soft-pion value. For simplicity, we shall work in Coulomb gauge $\epsilon^0 = 0$, since for the pion on the mass-shell $M_\mu^\pm$ is gauge invariant, i.e. $k^\mu M_\mu^\pm = 0$.

There are two kinds of corrections: first, those corrections which are present also in the nucleon case. They appear in the nuclear case as additional corrections in the impulse approximation when the meson exchange and other nuclear renormalization effects are neglected in the soft-pion technique. A typical example of these is the ρ-meson contribution. The second type of correction is typical for the nuclear problem. It renormalizes the π-N amplitude in nuclear matter. An example is the effective field correction with which we are already familiar from the last section. It is, however, more convenient to discuss them according to the types of intermediate states which contribute.

2.1 Nuclear intermediate states

Consider the intermediate states not accompanied by a pion. Among these the most important correction arises from the states containing a nucleon-antinucleon pair. This is shown first for the nucleon case. One inserts as the intermediate state, the state consisting of a nucleon plus a nucleon-antinucleon pair $n = |NN\bar{N}\rangle$. The energy denominators $E_n - E_i$ and $E_n - E_i - m_\pi$ are large, approximately $2m_N$. In spite of these large values, these intermediate states play an appreciable role. The reason is the well-known strong coupling of the pion to the nucleon-antinucleon pair through the γ_5 coupling.

Consider for definiteness the reaction

$$\pi^+ + N \to \gamma + P \quad ,$$

where N and P are the neutron and proton, respectively. In calculating the matrix elements $\langle P|j_\mu^{e.m.}(o)|n\rangle\langle n|j_\pi^+(o)|N\rangle$ and the crossed term $\langle P|j_\pi^+(o)|n\rangle\langle n|j_\mu^{e.m.}(o)|N\rangle$, the intermediate state n having a nucleon-antinucleon pair can be $PN\bar{N}$ for the former and $NP\bar{P}$ for the latter. If we drop the term where j_μ^{em} couples to $N\bar{N}$, since it is only a small anomalous magnetic moment, then we have, using Eq. (IV.1) (i.e. only Fig. 5b contributes):

$$\left(\delta\pi\sigma^+_B\right)_{pair} = + \frac{e m_\pi}{(2m_N)^2}\langle 0|j_\pi^+(0)|N(p_i)\,\bar{P}(-p_i)\rangle\langle P(p_f)\,\bar{P}(-p_i)|J_\mu^{em}(0)|0\rangle \qquad \text{(IV.12)}$$

where we have retained only the dominant doubly-disconnected term (that is, a term in which a nucleon propagates freely while the electromagnetic current creates a pair and a pion coupling to a pair). Now summing over the antiproton polarization, we get explicitly

$$\left(\delta M_B^+\right)_{pair} = ie\,\frac{\sqrt{2}\,m_\pi}{(2m_N)^2}\,g_r\,\bar{u}_p(p_f)\,\varepsilon\cdot\gamma\,\gamma_5\,\tau^+\,u_N(p_i) \tag{IV.13}$$

Or, using the Goldberger-Treiman relation and in the non-relativistic limit, we get

$$\left(\delta M_B^+\right)_{pair} = -i\frac{m_\pi}{2m_N}\,\frac{e g_A}{f_\pi}\,\chi_f^+\,\underset{\sim}{\sigma}\cdot\underset{\sim}{\varepsilon}\,\tau^+\chi_i \tag{IV.14}$$

For the negative pion capture $\pi^- + P \to N + \gamma$, the expressions are the same as Eqs. (IV.13) and (IV.14) except for the τ^- operator which replaces the τ^+ operator.

Note that Fubini and Furlan did not need the pair term because in adding the amplitudes as we already discussed before, the pair term is damped, because of larger energy denominators.

One can easily extend the above results to nuclei. In the impulse approximation, Eq. (IV.14) is simply modified to

$$\left(\delta M_B^\pm\right)_{pair} = -i\,\frac{m_\pi}{2m_N}\,\frac{e g_A}{f_\pi}\,\langle f|\,\sum_j \underset{\sim}{\sigma_j}\cdot\underset{\sim}{\varepsilon}\,\tau_j^\pm\,\delta(\underset{\sim}{r_j})\,|i\rangle \tag{IV.15}$$

Another nuclear intermediate state which should be considered is the genuine nuclear state B′. Call it $(\delta M_{B'}^{\pm})_{\text{Fermi motion}}$

$$\left(\delta M_B^\pm\right)_{\substack{Fermi\\motion}} = -\frac{ie}{f_\pi}\sum_{B'}\left\{\frac{m_\pi - E_{B'} + E_f}{m_\pi}\langle f|Q_A^\pm|B'\rangle\langle B'|\varepsilon^\mu J_\mu^{em}(0)|i\rangle\right.$$

$$\left. -\frac{m_\pi - E_i + E_{B'}}{m_\pi}\langle f|\varepsilon^\mu J_\mu^{em}(0)|B'\rangle\langle B'|Q_A^\pm|i\rangle\right\} \tag{IV.16}$$

The states B′ can be low-lying states ($E_{B'} - E_{\substack{i\\f}} \ll m_\pi$) or highly excited ones ($E_{B'} - E_{\substack{i\\f}} \approx m_\pi$). In the first case $(m_\pi - E_{B'} + E_f)/m_\pi \approx 1$, the operators may be approximated by the following single nucleon operators

$$Q_A^{\pm} = \sum_j \frac{g_A}{m_N} \underset{\sim}{\sigma}_j \cdot \underset{\sim}{p}_j \, \tau_j^{\pm} \tag{IV.17}$$

$$\underset{\sim}{J}^{em}(\underset{\sim}{r}) = \sum_j \frac{f_1^s + f_1^v \tau_j^3}{2} \, \frac{\underset{\sim}{p}_j}{m_N} \, \delta(\underset{\sim}{r} - \underset{\sim}{r}_j) \tag{IV.18}$$

where f_1^S and f_1^V are the isoscalar and isovector electric form factors, and p is the nucleon momentum operator. Now consider the matrix element

$$\langle f(p_f) | \sum_j \underset{\sim}{\sigma}_j \cdot \underset{\sim}{p}_j \tau_j^{\pm} | B'(p_{B'}) \rangle \tag{IV.19}$$

Here p_j is the average momentum (i.e. half of the initial plus final nucleon momenta)

$$\underset{\sim}{p}_j = \frac{1}{2}[\overrightarrow{\underset{\sim}{p}_j} + \overleftarrow{\underset{\sim}{p}_j}] \tag{IV.20}$$

with the arrows indicating on which wave functions the operator is to act.

The wave function for the state $B'(p_{B'})$ is described by a plane wave for the c.m. momentum P and an internal wave function ϕ, a function of the relative coordinates ρ:

$$\Psi_{B'}(\underset{\sim}{r}_1, \underset{\sim}{r}_2, \cdots) = e^{i \underset{\sim}{P}_{B'} \cdot \underset{\sim}{R}} \phi(\underset{\sim}{\rho}_1, \underset{\sim}{\rho}_2, \cdots)$$

where the c.m. position is $\vec{R} = +(\Sigma_j r_j)/A$. The differential operator ∇_j acting on $\Psi_{B'}$ gives

$$\frac{1}{i}(\underset{\sim}{\nabla}_j)_{r_j} \Psi_{B'}(r_1, r_2 \cdots) = \frac{\underset{\sim}{P}_{B'}}{A} \Psi_{B'}(r_1, r_2 \ldots) + e^{i \underset{\sim}{P}_{B'} \cdot \underset{\sim}{R}} \frac{1}{i} (\underset{\sim}{\nabla}_j)_{\rho_j} \phi(\rho_1, \rho_2, \cdots) \tag{IV.21}$$

where $(\nabla j)\rho_j$ acts on the relative coordinate ρ_j. This relation expresses the well-known fact that the velocity of a particle is the sum of the c.m. velocity plus the velocity relative to the c.m. Since each velocity can be decomposed into two parts, the product has four terms. The terms with one c.m. velocity and one relative velocity average out to zero. The part with the two c.m. velocities does not give any contribution in the Coulomb gauge, since the product $\varepsilon^{\mu}\ \langle\ B(p_f)|j_{\mu}^{em}|i(p_i)\ \rangle$ is then proportional to $\varepsilon^{\mu}(p_i+p_f)_{\mu} = \varepsilon^{\mu}(2p_i+q-k)_{\mu}$, which vanishes in the Coulomb gauge and the lab. system, $\underset{\sim}{q} = \underset{\sim}{p}_i = 0$. The part with the two relative velocities, the remaining term, is estimated by closure:

$$\left(\delta \mathcal{M}_B^{\pm}\right)_{\substack{\text{Fermi}\\ \text{motion}}}$$

$$\simeq -\frac{ie}{f_\pi}\frac{g_A}{2m_N^2}\langle f|\sum_{j,k}\underset{\sim}{\sigma}_j\cdot(\underset{\sim}{p}_j)_{rel}\,\tau_j^{\pm}\, e^{i(\underset{\sim}{p}_f-\underset{\sim}{p}_i)\cdot\underset{\sim}{r}_k}\,(f_1^s+f_1^v\tau_k^3)\,\varepsilon\cdot(\underset{\sim}{p}_k)_{rel}+\text{c.t}|i\rangle$$

$$\simeq \pm\frac{ie}{f_\pi}\frac{2g_A}{3}\frac{\langle\varepsilon\rangle}{m_N}f_1^v\langle f|\sum_j\underset{\sim}{\sigma}_j\cdot\underset{\sim}{\varepsilon}\,\tau_j^{\pm}\,\delta(\underset{\sim}{r}_j)|i\rangle \qquad \text{(IV.22)}$$

This correction enhances the soft-pion amplitude only by a very small amount (≈ 1.5%).

Notice that the magnetic part of the e.m. current has not been taken into account because its contribution is of second order in the pion mass. This can be seen easily: it is proportional to the product of momenta $(\underset{\sim}{p}_{B'} + \underset{\sim}{p}_f)\cdot(\underset{\sim}{p}_{B'} - \underset{\sim}{p}_i) = 2\underset{\sim}{p}_f\cdot(\underset{\sim}{p}_f - \underset{\sim}{p}_i)$ which is in the lab system ($\underset{\sim}{p}_i = 0$, $\underset{\sim}{p}_f = -\underset{\sim}{k}$) equal to $2|\underset{\sim}{k}|^2 \approx 2m_\pi^2$.

As we have mentioned in the last section, the two-nucleon absorption correction is hard to calculate. For this reason, this piece of the contribution is left as an uncertainty in the calculation.

2.2 Incoherent rescattering

This contribution arises from intermediate excited states $B' \neq i$ accompanied by one pion, $|n\rangle = |B'\pi\rangle$, i.e. the pion rescatters before getting absorbed, exciting the nucleus in the intermediate state. The resulting modification of $\mathcal{M}_B$ is

$$\left(\delta\mathcal{M}_B^{\pm}\right)_{I.R.} = +\frac{e\,m_\pi}{(2\pi)^3}\sum_{B'\neq i}\int\frac{d^3q\,d^3p_{B'}}{2\omega_q\,2E_{B'}}\left\{\delta(\underset{\sim}{p}_{B'}+\underset{\sim}{q}-\underset{\sim}{p}_i)\frac{\langle f|\varepsilon\cdot J^{em}(0)|B'\pi^{\alpha}\rangle\langle B'\pi^{\alpha}|j_\pi^{\pm}(0)|i\rangle}{(E_{B'}+\omega_q-E_i)(m_\pi-\omega_q-E_{B'}+E_i)}\right. \qquad \text{(IV.23)}$$

$$\left.-\,\delta(\underset{\sim}{p}_{B'}+\underset{\sim}{q}-\underset{\sim}{p}_f)\frac{\langle f|j_\pi^{\pm}(0)|B'\pi^{\alpha}\rangle\langle B'\pi^{\alpha}|\varepsilon\cdot J^{em}(0)|i\rangle}{(E_{B'}+\omega_q-E_f)(m_\pi+\omega_q+E_{B'}-E_f)}\right\}$$

In principle the isospin index α has to be summed over. But since we are dealing with a small correction term we keep only the index $\alpha = +$ for M^+ and $\alpha = -$ for M^-, i.e. we treat the photoproduction of neutral pions as negligible compared to that of the charged pions. We make the same approximations as in the elastic π-nuclear case, namely the static approximation $E_{B'} = M$, and we take for the matrix elements of Eq. (IV.23) single-nucleon contributions:

$$8\pi(M+m_\pi)^{-1}\langle B'(p_i-q),\pi^\pm(q)|j_\pi^\pm(0)|i(p_i)\rangle = \int d^3r\, e^{-iq\cdot r}\langle B'|\sum_j(a_N^+ \mp a_N^- \tau_j^3)\delta(r-r_j)|i\rangle$$

$$\langle f(p_f)|J_\mu^{em}(0)|B'(p_i-q)\pi^\pm(q)\rangle = \pm\frac{ieg_A}{f_\pi}\int d^3r\, e^{iq\cdot r}\langle f|\sum_j(\sigma_j)_\mu \tau_j^\pm\delta(r-r_j)|B'\rangle \tag{IV.24}$$

As in the scattering process, the closure is made within the same approximation which overestimates the correction. Then the resulting expression can be decomposed into two parts: the first part is the self-correlation effect in which the pion rescatters on the same nucleon before being absorbed. Its expression is

$$\left(\delta M_B^\pm\right)_{S.C.} = \pm\frac{ieg_A}{f_\pi}m_\pi\left(1+\frac{m_\pi}{m_N}\right)(a_N^+ + a_N^-)\langle f|\sum_j \sigma_j\cdot\epsilon\,\tau_j^\pm\delta(r-r_j)|i\rangle\frac{2}{\pi}\int_0^\infty\frac{dq}{\omega_q^2} \tag{IV.25}$$

$$= \pm\frac{ieg_A}{f_\pi}m_\pi\left(1+\frac{m_\pi}{m_N}\right)(a_N^+ + a_N^-)\langle f|\sum_j \sigma_j\cdot\epsilon\,\tau_j^\pm\delta(r-r_j)|i\rangle$$

This correction alone would modify the soft-pion amplitude $M^\pm(0) \approx \pm10\%$. The second part arises from the pair-correlation. It is again estimated in the infinite Fermi gas model. Its effect added to the previous one modifies the integral of the Eq. (IV.25) with a cut-off factor which reflects the effect of the exclusion principle.

The integral $\int_0^\infty dq/\omega_q^2$ of Eq. (IV.25) is replaced by

$$\int_0^{2p_F}\frac{dq}{\omega_q^2}\left[\frac{3}{2}\frac{q}{2p_F}-\frac{1}{2}\left(\frac{q}{2p_F}\right)^3\right]+\int_{2p_F}^\infty\frac{dq}{\omega_q^2} \tag{IV.26}$$

The over-all effect of the inelastic excitations is then reduced by a factor $\simeq 2$ by the exclusion principle. Thus the net magnitude of the renormalization is of the order of 5%.

2.3 Vector mesons

It results from the studies[28,47,48] on the photoproduction from nucleon that a sizeable contribution arises from an intermediate state with a vector meson, i.e. $|n\rangle = |N\rho\rangle$. In analogy with the treatment of the $|N\rho\rangle$ intermediate state, one may retain only the semi-disconnected term of the form

$$\langle 0|\, J_\mu^{em} \,|\rho\rangle\langle \rho\, N|\, j_\pi \,|N\rangle \tag{IV.27}$$

This kind of correction is of the basic property of the photoproduction mechanism and hence should be taken into account in treating the nuclear process. We can do this simply in impulse approximation.

The resulting modification of the Born amplitude is

$$\left(\delta \mathcal{M}_B^{\pm}\right)_{\substack{\text{vector}\\ \text{meson}}} = \pm\, 0.1\, \frac{ie\, g_A}{f_\pi} \langle f|\sum_j \underset{\sim}{\sigma}_j \cdot \underset{\sim}{\varepsilon}\; \tau_j^{\pm}\, \delta(\underset{\sim}{r} - \underset{\sim}{r}_j)|i\rangle \tag{IV.28}$$

where the numerical factor 0.1 comes from the estimate of Verzegnassi[48]. In getting this number, many approximations were involved, but with this correction he reproduces very well the experimental values of the photoproduction amplitude on a nucleon. Even if the agreement is fortuitous and has an origin other than that of the vector meson contribution, we have no reason to question this result, since there has to be a correction such as (IV.28), whatever its origin is.

3. RESULTS AND CONCLUSIONS

Adding all the corrections, we finally find the following value of the Born amplitude (in the Coulomb gauge):

$$\mathcal{M}^{\pm}_{\text{Born}} = \mp \frac{ie}{f_\pi} \langle f|\, \overset{\mu}{\varepsilon}\, A_\mu |i\rangle \begin{pmatrix} +0.08 \\ -0.22 \end{pmatrix} \frac{ie\, g_A}{f_\pi} \langle f|\sum_j \underset{\sim}{\sigma}_j \cdot \underset{\sim}{\varepsilon}\; \tau_j^{\pm}\, \delta(\underset{\sim}{r}_j)|i\rangle \tag{IV.29}$$

where the first part represents the soft-pion amplitude, and the upper and lower numbers correspond respectively to the corrections for $\pi^{\pm}$ absorption. If the exchange effects are neglected in the axial current matrix element, this expression becomes

$$\mathcal{M}^{\pm}_{\text{Born}} = \begin{pmatrix} +1.08 \\ -1.22 \end{pmatrix} \frac{ieg_A}{f_\pi} \langle f \,|\, \sum_j \underset{\sim}{\sigma_j} \cdot \underset{\sim}{\varepsilon}\, \tau_j^{\pm}\, \delta(\underset{\sim}{r_j}) | i \rangle \tag{IV.30}$$

The corresponding effective Hamiltonian is slightly renormalized over the one derived from the naive impulse approximation, by the effect of the exclusion principle.

The meson exchange effects which are contained in the axial current matrix element $M(o)$ do not have to be evaluated, but they are automatically included if the matrix element is taken from other experimental sources. In such a way the many-body effects would be properly taken into account in the physical amplitude.

In order to calculate the value of the physical amplitude, one needs two pieces of information; firstly, the axial current matrix element for a momentum transfer corresponding to the process [roughly $t = (p_i - p_f)^2 \approx -m_\pi{}^2$]; secondly, one has to estimate the effect of the distortion. These two points are now discussed.

There exist some transitions for which the axial current is known for two values of the momentum transfer: i.e. at $t = 0$ from β-decay and at $t \approx -m_\mu^2$ from μ-capture. These provide the most interesting tests of the soft-pion expansions, as will become clear.

The form factor expansion of the axial current between the nuclear states i and f involves, in general, several form factors. In the Coulomb gauge the pseudo scalar form factor does not play any role in the radiative capture. However, in many cases one has to deal with more than one form factor. Delorme[49] has developed a general method based on the comparison with the impulse approximation in order to select the most important terms. For the $0^+ \to 1^+$ transitions, he finds that there is only one form factor F_A to be determined. It is associated with a part of the axial current which, in the impulse approximation, is dominated by the expression $\Sigma_j \underset{\sim}{\sigma}_j \tau_j^{\pm} \delta(\underset{\sim}{r} - \underset{\sim}{r}_j)$. This form factor is known at zero momentum transfer from β-decay. In order to extrapolate in momentum transfer t, one makes the assumption that this variation is the same as that of the magnetic form factor $F_M(t)$ for the transition between the state i and the isobaric analogue of the state f (this last form factor is known from electron scattering):

$$\frac{F_A(t)}{F_A(0)} = \frac{F_M(t)}{F_M(0)} \tag{IV.31}$$

This relation is approximately valid in the impulse approximation[49,50].

The validity of the assumption (IV.31) may be checked from the comparison of the calculated and measured μ-capture rates. To give an idea of how it works, we quote the results for the transition ^{6}Li (1^+, T=0) → ^{6}He (0^+, T=1). Delorme has used the assumption (IV.31) and the β-decay information, and found the capture rate Λ_{th} to be

$$\Lambda_{th} = (1.38 \pm 0.16)\ 10^3\ \mathrm{sec}^{-1}\ .$$

This is in good agreement with the experimental value[51]:

$$\Lambda_{exp} = (1.60\ {}^{+\,0.32}_{-\,0.12})\ 10^3\ \mathrm{sec}^{-1}\ .$$

As for the distortion effect, this is best calculated with the π-nucleon optical potential. For the pion in 1s orbit, and for light nuclei, the optical model wave function varies slowly over the nuclear dimension. In such a case it is reasonable to take the wave function to be a constant. Then the distortion effect would appear as a multiplicative factor, namely:

$$\mathcal{M}(m_\pi) = C_\pi^{1/2}\ \mathcal{M}(0) \tag{IV.32}$$

where C_π is an average over interaction volume of the ratio of the pion probability $|\phi|^2$ with/without the strong interaction potential.

In Figs. 6 and 7 we show the probabilities for pions bound in the 1s atomic orbit of the nuclei ^{12}C and ^{16}O. For the well-known reason that the strong interaction potential is repulsive in the s state, it produces a reduction of the probability, $C_\pi < 1$*). Strictly speaking, since the ratio of the probabilities with/without strong interaction depends on the particular point in the nucleus, an exact account for the distortion requires the knowledge of the distribution of the interaction in the nucleus, i.e. it introduces a model dependence. However, this dependence is less pronounced in light nuclei, as can be seen by the comparison of ^{12}C and ^{16}O.

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*) If only the rough magnitude of C_π is wanted, the following expression derived for a square-well[52] may be used:

$$C_\pi^{\frac{1}{2}} = \frac{1}{\left[1 + \frac{1}{3}\, 2m_\pi V R^2\right]^2}$$

where R is the equivalent square-well radius of the nucleus ($R^2 = 5/3 \langle r^2\rangle$) and the potential barrier V is V ≈ 15 MeV.

A quantitative test of the relation between π and μ capture should be made at a first stage on light nuclei ($A \lesssim 12$). At a later stage this comparison in heavier nuclei could possibly produce information on the axial current distribution.

Some experiments on radiative pion capture have been performed recently.

However, the difficulty of isolating low-energy pions (s-wave) has severely restricted the cases where the relation could be quantitatively tested. For instance, there exist measurements of the branching ratio $R = \Gamma_{\pi\gamma}/\Gamma_{tot}$ where the radiative capture leads to a specific final state, and Γ_{tot} stands for the total rate of the pion absorption through all channels (mainly two-nucleon absorption).

If the pions were all captured in the 1s orbit, one would easily deduce the value of $\Gamma_{\pi\gamma}$ since the total absorption rate is known from the width of the 1s level. Unfortunately, a large fraction of the pions is absorbed from the atomic p-state even in light nuclei (typically of the order of 68% in ^{6}Li). The capture from the p-state is not well described by the effective Hamiltonian [Eq. (IV.32) with Eq. (IV.30)]. The p-wave function varies much over the interaction volume; because of this variation, terms proportional to pion momentum are no longer negligible. So one has to resort to other methods to calculate this quantity. Delorme does this using the Chew-Low-Goldberger-Nambu photopion production amplitude for the p-orbit capture and Eq. (IV.32) for the s-orbit capture. For

$$\pi^- + {}^6\mathrm{Li} \rightarrow {}^6\mathrm{He} + \gamma$$

he finds R to be 0.75×10^{-2}, to be compared with the experiment[51] $R_{exp} = (1.0 \pm 0.1)\ 10^{-2}$. The general agreement is rather good but the absorption from the 2p state is responsible for about half of this number, and no definite conclusion can be drawn about the agreement for the 1s state alone.

Actually the best convincing case for the relation between π and μ capture is the Panofsky ratio in He^3:

$$P = \frac{\pi^- + {}^3\mathrm{He} \rightarrow \pi^0 + {}^3\mathrm{H}}{\pi^- + {}^3\mathrm{He} \rightarrow \gamma + {}^3\mathrm{H}}$$

which is measured[53] to be 2.28 ± 0.2. The doublet ^{3}He and ^{3}H having the same quantum numbers as the proton and neutron, the form factor expansion of the axial current is the same as for the nucleon. For our purpose, only the axial form factor needs to be determined. Its value is known[54] at $t = 0$ from β-decay, $F_A(o) = -1.21$, and at $t = -0.27\ \mathrm{fm}^{-2}$ from μ-capture, $F_A(-0.27)/F_A(o) = 0.88$. A linear extrapolation to the value $t = -0.47\ \mathrm{fm}^{-2}$ of π-capture gives $F_A(-0.47)/F_A(o) = 0.80$.

The distortion effect is negligible for negative pions, because there is an accidental cancellation between the isospin-independent and isospin-dependent parts of the potential. From these informations the radiative capture probability for π^- bound in the 1s orbit of ^{3}He is calculated to be $\Gamma_{1s} = 4.1 \times 10^{15}$ sec^{-1}. Correspondingly the threshold photoproduction cross-sections are

$$\frac{d\sigma}{d\Omega} = A \quad \frac{|\underset{\sim}{q}|}{|\underset{\sim}{k}|} \quad ,$$

where $\underset{\sim}{q}$ and $\underset{\sim}{k}$ are the pion and photon momenta, with $A_{\pi^-} = 1.8$ μb. The absorption from the 2p state should not be important in such a light nucleus. But even if it is not negligible, the Panofsky ratio is not sensitive to the absorption from the 2p state because both the radiative capture and the charge exchange probabilities are small in this state.

The value of the charge exchange probability depends slightly (≈ 10%) on the way in which it is estimated, i.e. whether one uses the impulse approximation or the soft-pion method. The impulse approximation leads to a Panofsky ratio P = 2.1 while the soft-pion method gives P = 1.9. The good agreement of these numbers with the measured value P = 2.28 confirms the validity of the approach used, but needless to say, a direct measurement of the radiative capture would be extremely valuable. Although other quantitative tests should be made, one can foresee that the radiative capture will become a tool for measuring the axial form factors of nuclei.

In the soft-pion approach as expounded here, one uses the global treatment of nuclei where no mention is made of the composite structure of the nucleus (apart from some small correction terms) and one can avoid the explicit appearance of the nuclear wave functions. Of course the information about the wave functions is implicitly contained in the axial form factors, and the extent that the nuclear structure enters into the picture is the same as in other methods. The way one should look at the soft-pion approach of this section has been, up to now, as follows: firstly, we would like to ensure that the mass extrapolation has been done properly, then to predict transition rates and compare them with experiments. In contructing the models for extrapolation such as (IV.31), one may resort to impulse approximation, but this has to be checked *a posteriori*. If the tests show that the method works, then it will be feasible to derive the axial form factors from measurements. Then as an ultimate goal, one could attempt to describe these form factors by means of microscopic nuclear structure theory. We are not yet at this stage either experimentally or theoretically.

We would like to mention one application which has been discussed rather extensively in the literature. It is to learn something about the giant resonance multiplets which may be excited by the pion capture, μ-capture, photo absorption, or electron scattering. This aims at a particular aspect of nuclear structure, and therefore can be (or perhaps should be) done in impulse approximation.

The operator in the pion capture (from s-orbit) process is

$$\mathcal{H}_{eff} \propto \sum_i \underset{\sim}{\sigma}_i \cdot \underset{\sim}{\varepsilon}\, \tau_i^{\pm}\, e^{i \underset{\sim}{k} \cdot \underset{\sim}{r}_i} \tag{IV.33}$$

where $\vec{k}$ is the photon momentum. Making the usual multiple expansion of the plane wave $e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}_i}$ and keeping only the leading order term in each multipole, the dipole term is proportional to

$$\sum_i (\underset{\sim}{\sigma}_i \cdot \underset{\sim}{\varepsilon})(\underset{\sim}{k} \cdot \underset{\sim}{r}_i)\, \tau_i^{\pm}$$

This operator would lead to the analogue states of the multiplet giant resonance states of the nucleus.

The big advantage of the pions over μ-capture, whose axial current can also excite similar states, is that the γ-ray emitted may be observed while the neutrino is not observable.

Such an experiment has been performed by several groups, in particular by Bistirlich et al.[55)] on ^{12}C. The latter measure the energy of the γ spectrum following the absorption of a negative pion stopped. They observe three peaks (Fig. 8). One peak corresponds to the ground state of ^{12}B. They interpret the two other peaks to excitation of giant dipole resonance states.

From the theoretical point of view, again the capture from higher orbits obscures the role of the operator. One cannot expect that a state would be coupled to the ground state only through (IV.33), but other operators may destroy the coherence one would like to have. This problem is an open one and may probably be resolved as we accumulate more data. We can nevertheless say that the radiative capture of pions may prove to be a promising tool for the studies of various interesting excitation modes.

V. PION PRODUCTION IN NUCLEON-NUCLEON AND NUCLEON-NUCLEUS COLLISIONS

1. SOFT-PION LIMIT AND MASS DISPERSION RELATION

In the reactions treated so far, the dominant mode for pion interaction with nuclei was essentially a single-particle process. Thus the soft-pion limit which accounts for the bulk of the contributions contained no information as to how two or more nucleons in a nucleus interact. It is only through corrections to the soft-pion limit that the many-body structure showed up. Although the corrections are found to be important, the crucial point is that the effect of a two or more nucleon correlation turns out not to be important.

In this section, we turn to a process -- production of a pion near threshold in a nuclear interaction -- which cannot proceed in the absence of correlations. The correlation is necessary because otherwise the energy momentum cannot be conserved. Even in the soft-pion limit, a pion cannot be emitted from a nucleon, as is well known from the Adler consistency condition[6], unless it interacts with at least one other nucleon. Therefore the soft-pion limit is expected to play a role quite different from its role in the previous reactions. The processes which are interesting experimentally as well as theoretically are of two types:

$$p + p \rightarrow \pi^0 + p + p \tag{V.1a}$$

$$p + p \rightarrow \pi^+ + p + n \tag{V.1b}$$

$$p + p \rightarrow \pi^+ + D \tag{V.1c}$$

$$p + A(Z,N) \rightarrow \pi^0 + \text{anything} \tag{V.2a}$$

$$p + A(Z,N) \rightarrow \pi^0 + A'(Z+1,N) \tag{V.2b}$$

$$p + A(Z,N) \rightarrow \pi^+ + \text{anything} \tag{V.2c}$$

$$p + A(Z,N) \rightarrow \pi^+ + A''(Z,N+1) \tag{V.2d}$$

where "anything" stands for all final states, meaning that only the produced pion is measured. Both reactions (V.1) and (V.2) are of great interest from the point of view of soft-pion theorems. Furthermore, both will provide answers to some of the fundamental questions of nuclear physics, perhaps in a slightly different fashion. The reaction (V.1) is analogous to the N-N bremsstrahlung and hence is expected to provide similar information about the N-N interaction. Because of the pion mass, the N-N amplitude appearing in the former would be considerably off-shell even at threshold. Thus the off-shell property of the N-N interaction would play a more prominent role in the former than in the latter. The second set of reactions, in particular (V.2b) and (V.2d), would provide valuable information about many-body correlations. Since the probability of a high-energy nucleon sticking to a bound state in a final nucleus would be very minute in the absence of short-range correlation, the experimental results of such reactions (which are already available) can be extremely useful for shedding some light on this subject once the reaction mechanism is well understood.

Since the subject is in a relatively early stage, little progress has been made in such a direction until now. However, much work has been done to understand the basic reaction mechanism in terms of soft- and hard-pion theories. Although up to now no entirely new information has emerged from these theories, these researches lend greater justification to some of the old approaches widely used in the past, and provide grounds for further improvements.

The theory needed for these processes is quite similar to that used in the previous sections. In an earlier approach to this problem, Beder[56], and Schillaci, Silbar and Young[57] applied directly the Adler-Dothan soft-pion theorem[7] generalized from the well-known Low theorem to the threshold production of a single pion $NN \to NN\pi$. The defect of this approach -- ambiguities in extrapolating back to the real pion mass -- was later remedied by Banerjee et al.[58] by means of the Fubini-Furlan mass dispersion relation. Banerjee et al. succeeded in obtaining an unambiguous extrapolation procedure, and also in clarifying the differences of the soft-pion limits in the Adler-Dothan and the Fubini-Furlan approaches. In this article, we follow the latter.

For the sake of generality, let us represent both processes (V.1) and (V.2) in one form:

$$N(p) + i(p_i) \rightarrow \pi^{\alpha}(q) + f(p_f) \tag{V.3}$$

where N is the projectile nucleon with four-momentum p, π^{α} a pion in charge state α (a = 1, 2, 3), i the initial nucleon or nucleus, f the final state of two uncorrelated nucleons or a nucleus or fragments [for processes (V.2a) and (V2.c)].

We shall also denote whenever possible the initial system (Ni) by I to simplify the writing. The S-matrix for this reaction is denoted by

$$S^{\alpha}_{i\to f} = {}_{out}\langle \pi^{\alpha} f | N i \rangle_{in} = {}_{out}\langle \pi^{\alpha} f | I \rangle_{in} \tag{V.4}$$

If we reduce out only the pion, and use the PCAC, then we get

$$S^{\alpha}_{i\to f} = i(2\pi)^4 \delta^4(p_f + q - p_i - p)\mathcal{M}^{\alpha} \tag{V.5}$$

$$\mathcal{M}^{\alpha} = \frac{m_\pi^2 - q^2}{F_\pi m_\pi^2} q^{\lambda} \langle f | A^{\alpha}_{\lambda}(0) | I \rangle \tag{V.6}$$

Let us now consider the soft-pion limit in the invariant amplitude (V.6), i.e.

$$\mathcal{M}^{\alpha}_{soft} = \frac{1}{F_\pi} \lim_{q^{\lambda}\to 0} q^{\lambda} \langle f | A^{\alpha}_{\lambda}(0) | I \rangle \tag{V.7}$$

Clearly only the terms in $\langle f|A^a_\lambda(o)|I\rangle$ which diverge as $q_\lambda \to 0$ will survive, and these are just those graphs where the A^a_λ acts on the external legs. This is the content of the Adler-Dothan theorem which is a version of Low theorem applicable to pions. It is an important point of Low's theorem that only the on-shell nucleon-nucleon amplitude is required. Beder[56)] used the theorem in this way, but the original derivation of the soft-pion results by Schillaci et al.[57)] was somewhat different. They introduced instead a fictitious current or a "spurion" (in particular, the electromagnetic current) of zero four-momentum, reduced out both the pion and the spurion, and then related this spurion amplitude in the soft-pion limit to the production amplitude. As the authors point out, the result is just the same as (V.7). The main motivation for such a technique is to use the known equal-time commutation relations arising from the Ward identity. In a later paper, Young[59)] uses this technique for the mass extrapolation, in a spirit quite similar to the Fubini-Furlan analyses of photopion production off a nucleon.

We emphasize that in the soft-pion limit, the proper choice of the on-shell N-N amplitude as allowed by the Adler-Dothan theorem is actually quite ambiguous in the case of pion emission in contradistinction to that of the photon emission. The reason is that the pion mass m_π = 140 MeV is too large for an extrapolation with the smoothness required in the N-N amplitude. Consequently it is not clear at what energy the on-shell t-matrix should be evaluated. Near threshold, this becomes really serious, because of the rapid variation with energy of the elastic t-matrix.

The method of Banerjee et al.[58] avoids this ambiguity and leads to an unambiguous pion-mass extrapolation (with a certain price to pay). For this purpose they reduce out both the nucleon N and the pion, obtaining

$$\mathcal{M}^{\alpha}(q) = \frac{-i}{F_\pi m_\pi^2} \int d^4x \, e^{iq\cdot x} (\Box + m_\pi^2)\, \theta(x_0) \langle f | [\partial^\lambda A_\lambda^\alpha(x), \bar{J}_N(0)] | i \rangle \, u(p) \tag{V.8}$$

where $\bar{j}_N(x) = \bar{\psi}(x)(-i\overleftarrow{\not\partial} - M)$ and $\bar{\psi}(x)$ is the nucleon field operator. The Ward identity follows from formula (V.8) in moving the ∂^λ to the left of the step function $\theta(x_0)$:

$$\begin{aligned}\mathcal{M}^{\alpha}(q) = -i \frac{m_\pi^2 - q^2}{F_\pi m_\pi^2} \Big\{ & -\int d^4x \, e^{iq\cdot x} \langle f | [A_0^\alpha(x), \bar{J}_N(0)] \delta(x_0) | i \rangle \, u(p) \\ & - i q^\lambda \int d^4x \, e^{iq\cdot x} \theta(x_0) \langle f | [A_\lambda^\alpha(x), \bar{J}_N(0)] | i \rangle \, u(p) \Big\}\end{aligned} \tag{V.9}$$

Specializing now to the frame in which the pion is at rest ($|\underset{\sim}{q}| \to 0$), one finds

$$\begin{aligned}\mathcal{M}^{\alpha}(q_0) = i \frac{m_\pi^2 - q_0^2}{F_\pi m_\pi^2} \Big\{ & \langle f | [Q_A^\alpha(0), \bar{J}_N(0)] | i \rangle \, u(p) \\ & - q_0 \sum_n (2\pi)^3 \delta(\underset{\sim}{p_n} - \underset{\sim}{p_i}) \frac{\langle f | \bar{J}_N(0) | n \rangle \langle n | A_0^\alpha(0) | i \rangle u(p)}{q_0 + E_n - E_i + i\delta} - \text{c.t.} \Big\}\end{aligned} \tag{V.10}$$

where the second term comes from the usual time integration. The physical amplitude is simply $M^{\alpha}(m_{\pi})$. And the soft-pion limit is obtained by setting $q_0 \to 0$ with, however, the important restriction that when $q_0 = 0$, the reduced-out nucleon is taken off-shell at the same time so that

$$(\text{mass of reduced-out nucleon})^2 = (p_i - p_f)^2 = [p - (m_\pi, 0)]^2 \tag{V.11}$$

This is an important difference from the Adler-Dothan's soft-pion limit.

Although not as well established as the commutators we encountered in the previous sections, the commutator in Eq. (V.10) is nevertheless a known object,

$$[Q_A^{\alpha}(0), J_N(0)] = [J_N(0) + 2M\overline{\Psi}(0)]\gamma_5 \frac{\tau^{\alpha}}{2} \tag{V.12}$$

This commutator is satisfied in the σ-model[11], yields the Weinberg-Tomozawa formula for the π-N scattering length, and is generally believed to be more general than is warranted by the model[58].

In the limit $q_0 \to 0$, the second term of Eq. (V.10) survives only for the intermediate states $|n\rangle$ for which $E_n = E_i$ or $E_n = E_f$. This simply corresponds to the nucleon pole terms, and may be represented by Figs. 9a, b, c for the case in which $|f\rangle$ is a two-nucleon state and $|i\rangle$ a one-nucleon state. Because of the special role given to the reduced-out nucleon, the equal-time commutator (ETC) term looks quite different from the rest of the pole terms. In other words, in a rigorous sense, it cannot be represented by Fig. 9d. The reason is simply this. Since both the projectile nucleon and the emitted pion are reduced out, taking the pion off the mass-shell forces the nucleon to go off-shell as well, while the other external nucleons remain on the mass-shell. This is in contrast with the Adler-Dothan approach in which the pion is emitted from the external legs in a symmetric way. It turns out, however, that the commutator term can be manipulated, with the help of some reasonable approximations, into a form similar to the rest of the terms, thus being represented by Fig. 9d on an equal footing with Figs. 9a-c. This is what Banerjee et al. do. It requires considerable algebra to show this, so we shall not go through it here.

Under certain circumstances, some of the graphs in Fig. 9 can be ignored with a considerable simplification in the soft-pion limit; namely π^0 cannot radiate from a state $|i\rangle$ with zero spin unless there is an opposite-parity spin-zero state degenerate with $|i\rangle$. If the deuteron is the final state f, it cannot emit π^0 because of the isospin conservation. Furthermore, if the pion is produced near threshold, the outgoing nucleons carry negligible three-momentum. Thus the post-emission is expected to be considerably suppressed. From such various considerations, one then concludes that a good approximation in the soft-pion limit is to keep only the pre-emission terms

$$\mathcal{M}^{\alpha}_{soft} \simeq i \frac{1}{F_\pi} \langle f | [\dot{Q}^{\alpha}_A(0), \bar{J}_N(0)] | i \rangle u(p) + \text{other preemission terms.} \tag{V.13}$$

Now given the soft-pion limit, how can one extrapolate it to the on-mass-shell quantity? This is the same problem which we encountered in the previous sections. In analogy with the photoproduction and elastic scattering, one may start with Eq. (V.10), saturate it with a few intermediate states, and then devise a method of extrapolation by comparing it with a potential scattering formalism. In practice, however, this is not easy. The reason is that unlike in previous cases, the pion-rescattering process is not the most important correction. It is rather a two-nucleon intermediate state (a two-nucleon cut) in sum over n which is expected to be dominant. Consequently, approximating the sum with the two-nucleon cut does not lead to anything resembling the integral equation one would like to have. In contrast to the other processes, the soft-pion amplitude has nothing to do with a Born term in a series.

It turns out to be more profitable to rearrange the formula a little differently. Instead of taking out ∂^λ through the step function in Eq. (V.8), one may now move the operator ($\Box + m_\pi^2$) inside the time-ordering. This procedure has been used in Appendices B and C. The result is

$$\mathcal{M}^{\alpha}(q_0) = \sum_n (2\pi)^3 \delta(\underset{\sim}{p}_f - \underset{\sim}{p}_n) \frac{\langle f | j^{\alpha}_\pi(0) | n \rangle \langle n | \bar{J}_N(0) | i \rangle u(p)}{q_0 + E_f - E_n + i\delta} - \text{c.t.}$$

$$- \frac{i}{F_\pi m_\pi^2} \int d^3x \langle f | [\dot{D}^{\alpha}(0, \underset{\sim}{x}), \bar{J}_N(0)] | i \rangle u(p) \tag{V.14}$$

where the commutator $\int d^3x\, [D^\alpha(x), \bar{j}_N(o)]\, \delta(x_0)$ vanishing in the σ-model has simply been dropped. Now the object of the game is to relate Eq. (V.14) to a non-relativistic amplitude T [where $S = i(2\pi)\, \delta\, (E_f - E_I)\, T$],

$$T = \langle \phi_f^{(-)} | V_\pi | \Psi_I^{(+)} \rangle \tag{V.15}$$

where

$$(K + U) \phi_f^{(-)} = \varepsilon_f \phi_f^{(-)}$$

$$H \Psi_I^{(+)} = \varepsilon_I \Psi_I^{(+)}$$

with the Hamiltonian

$$H = K + U + V_\pi + \cdots$$

Here U is the piece describing nuclear interaction, V_π the pion-production potential and ... stands for other complicated interactions such as NNπ interaction. One may further simplify Eq. (V.15) by ignoring the complicated pieces (...) of the Hamiltonian, and write

$$T = \langle \phi_f^{(-)} | V_\pi | \phi_I^{(+)} \rangle \tag{V.16}$$

which is the familiar DWBA. To make a correspondence with Eq. (V.16), Banerjee et al. retain in the first term of Eq. (V.14) only the two-nucleon intermediate states, more precisely the leading singularities*). They then write [note that $T = T(m_\pi)$]

$$T(q_0) = C + \langle \phi_f^{(-)} | V_\pi \frac{1}{\varepsilon(q_0) - K - U + i\delta} U | \chi_I \rangle \tag{V.17}$$

with the definitions:

$$\langle \phi_n^{(+)} | U | \chi_I \rangle = (2\pi)^3 \delta(\underset{\sim}{p}_n - \underset{\sim}{p}_i - \underset{\sim}{p}) \; \langle n | J_N(0) | i \rangle \, u(p)$$

$$\langle \phi_f^{(-)} | V_\pi | \phi_n^{(+)} \rangle = (2\pi)^3 \delta(\underset{\sim}{p}_f - \underset{\sim}{p}_n) \; \langle f | j_\pi^\alpha(0) | n \rangle . \tag{V.18}$$

$$C = (2\pi)^3 \delta(\underset{\sim}{p}_f - \underset{\sim}{p}_i - \underset{\sim}{p}) \frac{-i}{F_\pi m_\pi^2} \int d^3x \langle f | [\dot{D}^\alpha(0, \underset{\sim}{x}), J_N(0)] | i \rangle \, u(p)$$

*) In Ref. 58, the authors take only the right-hand cut ignoring the c.t. term (which gives rise to the left-hand cut). This procedure was pointed out by the authors themselves to be erroneous, since near the soft-pion point, the left-hand cut is not negligible. In fact, their mass dispersion through the potential scattering formalism, i.e. the DWBA, is correct only when the right- and left-hand cut singularities are correctly kept. In sum, their final result is correct.

and

$$\epsilon(q_0) = E_f + q_0 - M_f$$
$$\epsilon_n = E_n - M_n \tag{V.19}$$

where M_f and M_n are the rest masses of the final and intermediate states, respectively (i.e. for two-nucleon states, both are just $2M_N$). Since at $q_0 = m_\pi$ Eq. (V.17) must be the same as the physical amplitude Eq. (V.16), by comparing the two, one can identify C to be the Born term

$$C = \langle \phi_f^{(-)} | V_\pi | \chi_I \rangle \tag{V.20}$$

Thus within the approximation of keeping only the leading singularities, the suitable way of going to the on-the-mass-shell amplitude is found to be the distorted wave-Born approximation Eq. (V.16). It is clear that in order to introduce pion distortion, one needs to go beyond the DWBA.

What now remains is an explicit form for V_π. This has to be deduced from the only known quantity -- the soft-pion limit Eq. (V.13). This was done by a clever manipulation:

$$T_{iN\to\pi f}(q_0 = m_\pi) = \sum_n T_{n\to\pi f}(q_0 = 0) \langle \chi_n | \frac{m_\pi}{\epsilon(q_0=0) - K + i\delta} | \phi_I^{(+)} \rangle \tag{V.21}$$

from which Banerjee et al. obtain [using the soft-pion limit $T(q_0 = 0)$ and some lenghty algebra]:

$$V_\pi^\alpha = \frac{g_A}{F_\pi} \sum_n \frac{m_\pi}{M} \frac{\tau^\alpha}{2} \underset{\sim}{\sigma}_n \cdot \underset{\sim}{\nabla}_n \tag{V.22}$$

where n runs over all <u>nucleons</u> involved in the transition, and $\underset{\sim}{\nabla}_n$ stands for the gradient operator acting on n^{th} nucleon. This is the well-known operator obtained from the pseudovector coupling. This operator had been used without much justification before the advent of the PCAC and current algebra. What is significant in the derivation discussed here is that the coupling is now fixed in a model-independent way.

A somewhat more transparent physical picture of the role of the soft-pion limit emerges if one examines Eq. (V.20). The soft-pion amplitude (V.13) for the pre-emission processes is found to describe the situation where the initial-state interaction is turned off. In other words, this amplitude would be an accurate approximation if the initial-state interaction is negligible. This can be seen in the following way. Since $(E_i - E_f)/m_\pi = 1$ from energy conservation, we have

$$T(m_\pi) = \frac{1}{m_\pi} \langle \phi_f^{(-)} | (E_I - E_f) V_\pi | \phi_I^{(+)} \rangle \qquad \text{(V.23)}$$
$$= - \frac{1}{m_\pi} \langle \phi_f^{(-)} | [u, V_\pi] | \phi_I^{(+)} \rangle$$

which shows that the complete amplitude contains both pre- and post-emission processes. Now suppose that the initial state interaction is absent. Then

$$T_{\chi_I} = \frac{1}{m_\pi} \langle \phi_f^{(-)} | (E_I - E_f) V_\pi | \chi_I \rangle \qquad \text{(V.24)}$$
$$= - \frac{1}{m_\pi} \langle \phi_f^{(-)} | u V_\pi | \chi_I \rangle$$

Thus the post-emission disappears. In non-relativistic form, the soft-pion limit (V.13) with its off-shell property is equivalent to Eq. (V.24).

One obvious question which may be raised here is: How good is the approximation of keeping only the two-nucleon cut or equivalently how reliable is the DWBA? For the π^0 production near threshold, there are fairly sound theoretical grounds that the two-nucleon-one-pion cut, for instance, would be not too important, but for charged pion production, the validity of the DWBA is highly doubtful. Formally such a contribution appears as a correction to the DWBA. In practice, a systematic method to incorporate it is yet to be found.

2. COMPARISON WITH EXPERIMENTS

Beder[56]), and Schillaci et al.[57]) have applied the Adler-Dothan soft-pion theorem to the threshold pion production in $pp \to np\pi^+$ for the laboratory energy between 300 MeV and 350 MeV. Schillaci and Silbar[60a]) also examined its applicability to the threshold reaction $pp \to d\pi^+$. They further extended[60b]) the analysis of the unbound system to higher lab. energies (500-800 MeV) and to non-zero pion three-momentum in the over-all centre-of-mass frame.

As shown in Table 1, taken from Ref. 57, the threshold production for $pp \to np\pi^+$ at $\sim$ 300 MeV agrees well with experiment. However, for higher energies the deviation becomes considerable. One can see this at T_{lab} = 740 MeV [60], the result of which is given in Fig. 10. As for the bound system, the theorem does not work at all. There is a serious ambiguity in interpreting the Adler-Dothan theorem for the deuteron, so that the meaning of the Schillaci-Silbar calculation is not clear. In any event, the off-shell extrapolation is not expected to be smooth, because of the rapid variation in the deuteron form factor in momentum transfer. This would mean that neither the soft-pion theorem à la Adler-Dothan nor the one à la Fubini-Furlan, including the two-nucleon cuts, gives a correct description of the process. There is strong evidence that the $NN\pi$ cut in Eq. (V.15) would in fact be dominant: Koltun and Reitan[61] and Lazard et al.[62] had already shown before the use of the soft-pion techniques that the contribution from Fig. 11 is far greater than the external emission graphs.

The reason for the discrepancy observed for the unbound transition at T_{lab} = 740 MeV is presumably due to the neglect of $\Delta(1236)$, which would be included in the $NN\pi$ cut. The physical reason why this may be so is that even if the pion is at rest, the recoiling nucleon can be in relative p-state with respect to the pion in the c.m. frame, and of course the p-wave is strongly dominated by the resonance Δ [for further details on this question, see Drechsel and Weber[63]]. Way above the threshold, this is a well-known fact; what is surprising is that the influence is felt even very near the threshold.

The neutral pion production is somewhat simpler to treat, since here the pion-nucleon rescattering correction is expected to be negligible for low pion kinetic energy. This fact has been utilized by Zollman[64], who has looked at the process $pp \to pp\pi^0$ in DWBA for the energy range from 275 MeV to 300 MeV. The aim is to detect the difference of the various N-N potentials, free of the p-wave complication. Since those potentials are manufactured to fit on-shell N-N data, the difference, if any, would be indicative of the off-shell property of the potentials. But the calculation shows that the differences appear only at a quantitative level, not on a qualitative scale, and hence in practice it would be difficult to distinguish the finer details.

The application to complex nuclei (to the best of our knowledge) has been limited to the reaction of the type (V.2a). In particular, Banerjee et al.[65] and Shuster[66] have considered the reaction

$$p + {}^{12}C \to \pi^0 + \text{anything} \qquad (V.25)$$

at proton laboratory energy between 200 MeV and 600 MeV. This is a sum rule type of experiment, since all final states (bound and unbound) are summed over. Neglecting the final-state interaction of the pion and the post-emission graphs, and using the distorted wave function available from other sources, they obtain a fair agreement with experiments at lower incident energies, i.e. $\lesssim$ 300 MeV. The calculation is compared with experiments in Fig. 12. Notice that there is a systematically increasing disagreement at higher energy, the theory undershooting the experiment. The discrepancy might be due essentially to the increasing importance of $\Delta(1236)$. In any event, the description is liable to fail at high energies. An analogous reaction with charged pion emission would certainly be badly approximated by the DWBA, as the $NN\pi$ cut would be non-negligible at all energies.

Whereas the sum-rule type of experiments such as (V.25) may shed some light on the dynamics of the many-body system only indirectly, the processes (V.2b) and (V.2d) would be of greater interest from the point of view of nuclear structure. The momentum conservation requires a fairly high momentum component in the initial wave function, so that the reaction proceeds mainly through the short-range correlation. There are some experiments[67,68] available for $p + {}^{12}C \rightarrow {}^{13}C + \pi^{+}$ with ${}^{13}C$ in its ground state as well as excited states, but no theoretical analysis has been done yet. Together with its inverse reaction, the pion capture, this would be an interesting problem for the future.

VI. CONCLUSION

The motivation for considering the soft-pion limit in the nuclear processes can be summarized as follows. In this limit, the interaction of pions with nuclei becomes of a simpler nature. For instance, no rescattering occurs in the elastic π-nuclear scattering and in the radiative capture process. Also in that limit the pion production in nucleon-nuclear collision occurs through the external emission only. Contrary to the situation for a nucleon target, the complications which are introduced by the physical mass cannot be ignored in the nuclear case, and the great simplicity of the interaction is of course lost for the physical amplitude.

Nevertheless, one gains important information from the simple interaction which results in zero mass limit, and it is the basic interaction Hamiltonian for the processes considered.

Now how to derive the physical amplitude from this basic Hamiltonian depends upon the process. We have found that this is rather easy for the elastic π-nuclear scattering and the radiative capture. The difference between the soft-pion and the physical amplitudes amounts mainly to that between a plane-wave pion and a distorted-wave pion in the potential scattering picture. In very light nuclei this last procedure amounts to introducing a multiplicative factor ($<$ 1) which is constant over the interaction volume. The physical amplitude is thus obtained in a model-independent way by multiplying the soft-pion one by this factor. In heavier nuclei the mass extrapolation cannot be made in such a model-independent way, and the knowledge of the spatial distribution of the interaction is required.

It is quite remarkable that the granular structure of the nucleus, which manifests itself by excitation spectra, plays such an unimportant role in spite of the smallness of the excitation energies compared to the pion mass. The reason can be understood in terms of the scale associated with this effect. The dimensionless parameter which fixes the magnitude of this effect is $m_\pi d$, the ratio of the typical correlation distance d to the pion Compton wavelength m_π^{-1}. It is not, as it was commonly thought to be, the ratio of the pion mass to the average excitation energy ϵ (i.e. m_π/ϵ), which is a very large quantity. If this large ratio were the relevant parameter, an expansion in pion mass would be meaningless.

That the expansion parameter is indeed small and that the mass extrapolation makes sense is tested nicely by the charge antisymmetric π-nuclear amplitude for which the physical amplitudes are known (either theoretically or experimentally measured). This test is successful within the precision of the existing experiments. However, to make this test more severe it would be valuable to have more accurate measurements of the isospin effects in the 1s energy shifts in π-mesic atoms.

Conservely to the usual procedure of obtaining the physical amplitude from the soft-pion one, one may use the extrapolation technique to extract the soft-pion limit starting from known quantities, this soft-pion limit being closely related to some fundamental interaction. This is found to be the case with the charge symmetrie amplitude, the soft-pion limit of which is the σ-commutator which provides information on the chiral symmetry-breaking mechanism. In the radiative capture process, one may also be able to use the mass extrapolation to determine the nuclear axial current form factor in the space-like region.

To sharpen the arguments we have presented so far, especially for the σ-commutator, it would be highly desirable to have a reliable estimate of the two-nucleon absorption, an uncertainty in the mass extrapolation procedure. Also in treating incoherent rescattering in the scattering and photoproduction processes, it would be interesting to examine the possible role of collective nuclear excitations so as to avoid static approximation. This approximation leads to an overestimate of the rescattering correction. But to pin down the numbers accurately, in particular for the σ-commutator, this would be insufficient

The mass extrapolation problem for the pion production in nucleon-nuclear collision has not yet reached the same level of precision as the other processes considered. It is much more complicated. Nevertheless, it is a significant result that the soft-pion limit provides the basic production operator. What would be needed is the method to treat the two-nucleon-one-pion cut which has been neglected here, but clearly would be important for charged pion production. The application to the nuclear structure studies, such as the nucleon-nucleon correlation and nuclear spectroscopy, presents a considerable amount of work to be done.

Finally there is a case where in our opinion the mass extrapolation problem has not yet found a satisfactory answer. It is the mass extrapolation involved with the Goldberger-Treiman relation as applied to nuclei. As we have seen, there exists an exact relation between the pionic vertex and the axial current matrix element for a vanishing momentum transfer ($q \equiv p_f - p_i$):

$$\lim_{q^\mu \to 0} i q^\mu \langle f | A_\mu^\pm(0) | i \rangle = f_\pi \langle f | \partial_\pi^\pm(0) | i \rangle$$

where f and i are nuclear or nucleonic states. In the nucleonic states, or for nuclei of nucleonic quantum number, this relation is expressed as

$$g_r(o) = \frac{\sqrt{2}\, m_N\, g_A(o)}{f_\pi}$$

with M replacing m_N if it is a nucleus.

The problem for the nucleon case is that while $g_A(o)$, $g_r(m_\pi^2)$, and f_π are known experimentally, $g_r(o)$ is not a measured quantity. Therefore it is necessary to make an extrapolation from $q^2 = m_\pi^2$ to $q^2 = 0$. For nuclei, this is a lot more serious since there we have yet no measurements of $g_r(m_\pi^2)$, and furthermore the extrapolation is not expected to be as smooth as in the nucleon case. In addition to this, the behaviour of the pseudoscalar form factor as q^2 moves from zero to $-m_\mu^2$ is relevant in muon capture, and one would like to know how this extrapolation can be done.

The method that is quite suitable for the nucleon and perhaps for light nuclei is to write a once-subtracted dispersion relation for the pionic vertex

$$K(q^2) = g_r(q^2) / g_r(m_\pi^2)$$

as

$$K(q^2) = K(0) + \frac{q^2}{\pi} \int dq'^2 \, \frac{\mathrm{Im}\, K(q'^2)}{q'^2(q'^2 - q^2)}$$

For the nucleons, the beginning of the cut is located at $q^2 = 9m_\pi^2$, but in nuclei there are anomalous threshold singularities, so the cut starts already at rather low values of q'^2.

The attempt to understand the nucleon value $\Delta = K(m_\pi^2) - K(o) = 1 - K(o) \simeq 0.05 - 0.10$ using this dispersion relation has been mostly unsuccessful[69]. Yet the extrapolation is clearly a smooth one. The situation in nuclei is certainly much worse. The extension of the pionic source is expected to produce a strong momentum dependence in the pionic form factor and K(o) should deviate appreciably from 1. This effect may be taken into account in light nuclei by including the anamalous threshold cut, and in fact such a calculation for the three-nucleon system (^{3}He, ^{3}H) is being carried out by Jarlskog and Yndurain[70].

In heavier nuclei, however, the size effect in the pionic vertex may not arise solely from the triangular graph singularity, since there is no simple and direct relation between the nuclear size and the binding energy which determines the position of the triangular graph cut. It would therefore be an arduous task to estimate the correction 1 - K(o) from the dispersion relation of the form given above.

To resolve this problem satisfactorily for heavier nuclei, it seems that we need to have a better understanding of the role of the soft-pion limit for the π-nuclear vertex function $K(q^2)$. In particular, we would have to know what the simplification amounts to in a vertex when the soft-pion limit is taken. Recall that in the π-nuclear elastic scattering, the soft-pion amplitude becomes a universal quantity and may be taken to correspond to a Born amplitude (i.e. no rescattering). The extrapolation to the pion mass is found to have a nice physical interpretation of bringing in the rescattering effect. It would be desirable to formulate the extrapolation problem in the nuclear Goldberger-Treiman relation in a similar spirit*).

Acknowledgements

We are indebted to Dr. A. Figureau and Dr. M. Shuster for illuminating discussions and invaluable help in preparing this review and to Dr. S.K.M. Wong for checking the formulae and correcting some illusive errors.

We are also grateful to Dr. M. Krell and Dr. H. Schmitt for providing the pionic wave functions of ^{12}C.

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*) What one does for the muon capture in the absence of other methods is to write the pionic source in the impulse approximation (as a sum of individual nucleon contribution), and then to calculate the momentum dependence using nuclear wave functions.

APPENDIX A

A.1 NOTATIONS AND CONVENTIONS

1. Symbols

m_π = pion mass

m_N = nucleon mass

M = nuclear mass

$\omega_q = \sqrt{\underset{\sim}{q}^2 + m_\pi^2}$

$E_i = \sqrt{\underset{\sim}{p}_i^2 + M_i^2}$

$E_q = \sqrt{\underset{\sim}{q}^2 + \underset{\sim}{M}^2}$

f_π = charged pion decay constant $\simeq 0.94\ m_\pi$

$F_\pi = f_\pi/\sqrt{2}$

$g_A \equiv g_A(o) = 1.23$

g_r = renormalized π-N coupling constant $\simeq 13.4$

e = electric charge > 0, $\frac{e^2}{4\pi} \simeq \frac{1}{137}$

$V_\lambda^\alpha(x)$ = vector current

$J_\lambda^{e.m}(x)$ = e.m. current

ϕ_π^α = pion field operator

$A_\lambda^\alpha(x)$ = axial current

$j_\pi^\alpha(x)$ = pion source

$D^\alpha(x) = \partial^\lambda A_\lambda^\alpha(x)$.

2. Metric

The convention of Drell-Bjorken[10] is used, i.e.: $\chi^\mu = (\chi^0, \chi^1, \chi^2, \chi^3) = (t, \underset{\sim}{x})$, $\chi_\mu = (\chi_0, \chi_1, \chi_2, \chi_3) = (t, -\underset{\sim}{x})$, $\chi_\mu = g_{\mu\nu}\chi^\nu$. The γ-matrices satisfy the anti-commutation rule $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$. We also use the explicit forms of the γ matrices:

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \underset{\sim}{\gamma} = \begin{pmatrix} 0 & \underset{\sim}{\sigma} \\ -\underset{\sim}{\sigma} & 0 \end{pmatrix}, \qquad \gamma_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

3. Isospin indices

In dealing with pions, one is faced with two notations: One in which $\alpha, \beta, \ldots = 1, 2, 3$ or the other with +, −, 0 symbols. In the text we use as a rule the second convention, but it is useful to write down the relation

$$\pi^{\pm} = (\pi^1 \pm i\,\pi^2)/\sqrt{2}$$

$$\pi^0 = \pi^3$$

$$\phi_\pi^{\pm} = \frac{1}{\sqrt{2}}(\phi_\pi^1 \pm i\,\phi_\pi^2)$$

$$\jmath_\pi^{\pm} = \frac{1}{\sqrt{2}}(\jmath_\pi^1 \pm i\,\jmath_\pi^2)$$

For the weak current $J_\lambda^{\pm}$,

$$J_\lambda^{\pm} = J_\lambda^1 \pm i\, J_\lambda^2$$

with the definition ($\alpha = 1, 2, 3$)

$$\langle p' | J_\lambda^\alpha | p \rangle = \bar{u}(p')\frac{\sigma^\alpha}{2}[\ldots\ldots]u(p)$$

An amplitude for elastic π-N scattering may be given in several different ways: isospin symmetric or antisymmetric or in total space. Call the former by $a^{\pm}$, and the latter by a_1 or a_3 with the notation a_{2T}. The relation is

$$a^+ = (a_1 + 2a_3)/3 \qquad\qquad a_3 = a^+ - a^-$$

$$a^- = (a_1 - a_3)/3 \qquad\qquad a_1 = a^+ + 2a^-$$

These combinations apply to the invariant amplitude T or the amplitude f or the scattering length a.

A.2 CURRENT COMMUTATION RULES

We need only the once-integrated ones. With

$$Q_A^\alpha(0) = \int d^3x\, A_0^\alpha(\underset{\sim}{x}, 0)$$

$$Q_V^\alpha(0) = \int d^3x\, V_0^\alpha(\underset{\sim}{x}, 0)$$

they are (at equal time $x_0 = 0$)

$$\left[V_\mu^\alpha(\underset{\sim}{x}),\ Q_V^\beta(0)\right] = i\ \epsilon_{\alpha\beta\gamma}\ V_\mu^\gamma(\underset{\sim}{x})$$

$$\left[V_\mu^\alpha(\underset{\sim}{x}),\ Q_A^\beta(0)\right] = i\ \epsilon_{\alpha\beta\gamma}\ A_\mu^\gamma(\underset{\sim}{x})$$

$$\left[A_\mu^\alpha(\underset{\sim}{x}),\ Q_V^\beta(0)\right] = i\ \epsilon_{\alpha\beta\gamma}\ A_\mu^\gamma(\underset{\sim}{x})$$

$$\left[A_\mu^\alpha(\underset{\sim}{x}),\ Q_A^\beta(0)\right] = i\ \epsilon_{\alpha\beta\gamma}\ V_\mu^\gamma(\underset{\sim}{x})$$

where $\epsilon_{123} = +1$ is the fully antisymmetric symbol.

APPENDIX B

FORMULAS FOR ELASTIC PION-NUCLEON (OR -NUCLEAR) SCATTERING

In this appendix we derive the main formulas used in Section III, using the reduction technique. The argument is not rigorous, but acceptable for our purpose. Consider the reaction

$$\pi^{\alpha}(q) + N(p_i) \to \pi^{\beta}(k) + N(p_f) \tag{B.1}$$

where we use the isospin indices for pions, α, β = 1, 2, 3, instead of the ($\pm$, 0) convention used in the main text, and momenta are indicated within the parenthesis. For convenience we confine ourselves to the target isospin ½ so that formulas will be applicable to π-nucleon system or π-nuclear system with the nucleus of isospin ½. Generalization to an arbitrary isospin is really straightforward, and except for the double charge exchange (i.e. $\pi^{\pm} \to \pi^{\mp}$), which is possible for nuclei of isospin larger than ½, the ensuing formulas hold with only small changes.

We shall confine ourselves to the elastic scattering,

$$q = k \quad , \quad p_i = p_f \tag{B.2}$$

and work with the pions at rest

$$\underset{\sim}{q} = \underset{\sim}{k} = 0 \tag{B.3}$$

Define the S-matrix and invariant T-matrix by

$$S^{\beta\alpha} = i\,(2\pi)^4\,\delta^4(k + p_f - q - p_i)\,T^{\beta\alpha}$$

and the isospin even (+), and isospin odd (-) amplitudes by

$$T^{\beta\alpha} = \frac{1}{2}[\tau_\beta, \tau_\alpha]_+ T^+ + \frac{1}{2}[\tau_\beta, \tau_\alpha]_- T^-$$

Then the threshold amplitudes (i.e. $q_0 = k_0 = m_\pi$) are given by

$$(\,Q_A = Q_A(0)\,)$$

$$\frac{1}{2}[\tau_\beta, \tau_\alpha]_- T^-(m_\pi) = -m_\pi^3 C + \frac{m_\pi}{F_\pi^2} \langle f | [Q_A^\beta, A_0^\alpha(0)] | i \rangle$$

$$- \left\{ m_\pi^3 \sum_n (2\pi)^3 \delta(\underset{\sim}{p}_n - \underset{\sim}{p}_i) \frac{\langle f | j_\pi^\beta(0) | n \rangle \langle n | j_\pi^\alpha(0) | i \rangle_c}{(E_n - E_i)^2 [m_\pi^2 - (E_n - E_i)^2]} - \text{c.t.} \right\} \tag{B.4}$$

$$\frac{1}{2}[\tau_\beta, \tau_\alpha]_+ T^+(m_\pi) = -i\, m_\pi^2 C' - \frac{i}{F_\pi^2} \langle f | [Q_A^\beta, \dot{A}_0^\alpha(0)] | i \rangle$$

$$- \left\{ m_\pi^2 \sum_n (2\pi)^3 \delta(\underset{\sim}{p}_n - \underset{\sim}{p}_i) \frac{\langle f | j_\pi^\beta(0) | n \rangle \langle n | j_\pi^\alpha(0) | i \rangle_c}{(E_n - E_i) [m_\pi^2 - (E_n - E_i)^2]} - \text{c.t.} \right\} \tag{B.5}$$

where the subscript c means that only the "connected" piece (terminology defined below) should be taken when the intermediate n is a πN state (fully disconnected pieces are to be ignored by the definition of the S-matrix), and where

$$C^{\beta\alpha} = \int d^3x \, \langle f | [\phi^\beta(0), \phi^\alpha(\underset{\sim}{x}, 0)] | i \rangle$$

$$C'^{\beta\alpha} = \int d^3x \, \langle f | [\dot{\phi}^\beta(0), \phi^\alpha(\underset{\sim}{x}, 0)] | i \rangle \tag{B.6}$$

Q_A is an axial charge, ϕ the pion field operator, j_π the pion source all defined in Appendix A, and the intermediate state $|n\rangle$ runs over all states allowed by the selection rules (such as ground or excited nuclear states + n pions, for $n \geq 0$). The constant F_π is related to the charged-pion decay constant by

$$\phi^\alpha(x) = \left(F_\pi m_\pi^2\right)^{-1} \partial_\lambda A_\lambda^\alpha(x) \tag{B.7}$$

The soft-pion limit is easily obtained by letting $q_0 \to 0$ in T [i.e. Eq. (B.13) below or taking the term lowest in power of m_π]

$$\frac{1}{2}\left[\tau_\beta, \tau_\alpha\right]_- T^{(-)}(q_0) \overset{q_0 \to 0}{=} \frac{q_0}{F_\pi^2} \langle f | [Q_A^\beta, A_0^\alpha(0)] | i \rangle \tag{B.8}$$

$$= i \frac{q_0}{F_\pi^2} \epsilon_{\beta\alpha\gamma} \langle f | V_0^\gamma(0) | i \rangle$$

$$\frac{1}{2}\left[\tau_\beta, \tau_\alpha\right]_+ T^{(+)}(q_0) \overset{q_0 \to 0}{=} -\frac{i}{F_\pi^2} \langle f | [Q_A^\beta(0), \dot{A}_0^\gamma(0)] | i \rangle \tag{B.9}$$

The second equality in Eq. (B.8) follows from the commutation relation (Appendix A).

We now prove Eqs. (B.4) and (B.5). The S-matrix in LSZ reduction formalism is given by

$$S_{fi}^{\beta\alpha} = i^2 \int d^4x\, d^4y\, e^{ik.x} e^{-iq.y} (\Box_x + m_\pi^2)(\Box_y + m_\pi^2)\, \theta(x_0 - y_0) \langle f | [\phi^\beta(x), \phi^\alpha(y)] | i \rangle \tag{B.10}$$

Integration over one of the variables, say y, gives

$$S_{fi}^{\beta\alpha} = i(2\pi)^4 \delta^4(k + p_f - q - p_i)\, i\,(m_\pi^2 - k^2)(m_\pi^2 - q^2) \int d^4x\, e^{ik.x}\, \theta(x_0) \langle f | [\phi^\beta(x), \phi^\alpha(0)] | i \rangle \tag{B.11}$$

hence from the definition of $T^{\beta\alpha}$, we get

$$T^{\beta\alpha} = i(m_\pi^2 - k^2)(m_\pi^2 - q^2)\int d^4x\, e^{ik.x}\,\theta(x_0)\langle f|[\phi^\beta(x),\phi^\alpha(0)]|i\rangle \qquad \text{(B.12)}$$

Substituting ϕ^α by $\partial_\lambda A_\lambda^\alpha$ [Eq. (B.7)], and integrating by parts with the help of identity $\partial/\partial x_0\ \theta(x_0) = \delta(x_0)$, one finds for $\vec{q} = \vec{k} = 0$, $q_0 = k_0$,

$$T^{\beta\alpha}(q_0) = i\,\frac{(m_\pi^2 - q_0^2)^2}{F_\pi^2 m_\pi^4}\int d^4x\, e^{iq_0x_0}\Big\{ q_0^2\,\theta(x_0)\langle f|[A_0^\beta(x), A_0^\alpha(0)]|i\rangle \qquad \text{(B.13)}$$

$$- \delta(x_0)\langle f|[A_0^\beta(x), \dot{A}_0^\alpha(0)]|i\rangle - iq_0\,\delta(x_0)\langle f|[A_0^\beta(x), A_0^\alpha(0)]|i\rangle\Big\}$$

One can simplify further by doing an integration over $\underset{\sim}{x}$ and introducing axial charge, but we need not do so for the moment. If we had integrated over the x variable in Eq. (B.10), we would have obtained somewhat different formula than that of Eq. (B.13), but it is obviously equivalent. One could symmetrize it as von Hippel and Kim did, but it is unnecessary for our purpose.

If one wants to go beyond the soft-pion limit, Eq. (B.13) is not a convenient form to work with, since the quantity inside the curly bracket must have a double pole at $q_0^2 = m_\pi^2$ in order to make $T^{\beta\alpha}$ non-vanishing, when the pions go on the mass-shell. It is necessary to extract this pole. We now proceed to reduce the double pole to a single pole. This we can do without introducing any model dependence. Consider the first term of Eq. (B.13). Multiplying by an energy momentum conservation δ function we have

$$(2\pi)^4\,\delta^4(p_f - p_i)\,(m_\pi^2 - q_0^2)^2\int d^4x\, e^{iq_0x_0}\, q_0^2\,\theta(x_0)\langle f|[A_0^\beta(x), A_0^\alpha(0)]|i\rangle$$

$$\text{(B.14)}$$

$$= q_0^2(m_\pi^2 - q_0^2)\int d^4x\, d^4y\, e^{iq_0(x_0-y_0)}(m_\pi^2 + \partial_{y_0}^2)\,\theta(x_0 - y_0)\langle f|[A_0^\beta(x), A_0^\alpha(y)]|i\rangle$$

Moving the operator $(m_\pi^2 + \partial_{y_0}^2)$ to the right, it becomes

$$(m_\pi^2 - q_0^2)^2 \int d^4x \, e^{iq_0x_0} q_0^2 \theta(x_0) \langle f | [A_0^\beta(x), A_0^\alpha(0)] | i \rangle$$

$$= (m_\pi^2 - q_0^2) \int d^4x \, e^{-iq_0x_0} q_0^2 \theta(-x_0) \langle f | [A_0^\beta(0), (m_\pi^2 + \partial_{x_0}^2) A_0^\alpha(x)] | i \rangle \tag{B.15}$$

$$- (m_\pi^2 - q_0^2) q_0^2 \langle f | [Q_A^\beta(0), \dot{A}_0^\alpha(0)] | i \rangle - i (m_\pi^2 - q_0^2) q_0^3 \langle f | [Q_A^\beta(0), A_0^\alpha(0)] | i \rangle$$

Combining Eqs. (B.13) and (B.15), we get

$$T^{\beta\alpha}(q_0) = i \frac{m_\pi^2 - q_0^2}{F_\pi^2 m_\pi^4} \Big\{ q_0^2 \int d^4x \, e^{-iq_0x_0} \theta(-x_0) \langle f | [A_0^\beta(0), (m_\pi^2 + \partial_{x_0}^2) A_0^\alpha(x)] | i \rangle \tag{B.16}$$

$$- m_\pi^2 \langle f | [Q_A^\beta(0), \dot{A}_0^\alpha(0)] | i \rangle - i q_0 m_\pi^2 \langle f | [Q_A^\beta(0), A_0^\alpha(0)] | i \rangle \Big\}$$

The integral in the first term inside the square bracket of Eq. (B.16) can be performed if one inserts a complete set $\sum_n |n\rangle\langle n|$ in the commutator. The result is

$$F_\pi m_\pi^2 q_0^2 \sum_n (2\pi)^3 \delta(\underset{\sim}{p}_n - \underset{\sim}{p}_i) \frac{\langle f | A_0^\beta | n \rangle \langle n | j_\pi^\alpha(0) | i \rangle}{(q_0 - E_n + E_i)(E_n - E_i)} - c.t. \tag{B.17}$$

where c.t. stands for the crossed term

$$A_0^\beta(0) \leftrightarrows j_\pi^\alpha(0) \quad , \quad E_n - E_i \leftrightarrows E_f - E_n$$

To obtain Eq. (B.17), we have used the relation

$$\int d^3x \langle n | (m_\pi^2 + \partial_{x_0}^2) A_0^\alpha(x) | m \rangle \tag{B.18}$$

$$= \frac{F_\pi m_\pi^2}{i(E_n - E_m)} \int d^3x \langle n | [(\Box_x + m_\pi^2) \phi^\alpha(x) | m \rangle = \frac{F_\pi m_\pi^2}{i(E_n - E_m)} \int d^3x \langle f | j_\pi^\alpha(x) | i \rangle$$

In order to extract the remaining pole in Eq. (B.16), we now are forced to introduce a model dependence. Consider first the antisymmetric combination which can be written as

$$\frac{\frac{1}{2}\left[T^{\beta\alpha}(q_0) - T^{\alpha\beta}(q_0)\right]}{q_0^2(m_\pi^2 - q_0^2)} = \frac{1}{q_0 m_\pi^2 F_\pi^2} \langle f | [Q_A^\beta(0), A_0^\alpha(0)] | i \rangle$$

(B.19)

$$- \left\{ q_0 \sum_n (2\pi)^3 \delta(\underset{\sim}{p_n} - \underset{\sim}{p_i}) \frac{\langle f | j_\pi^\beta(0) | n \rangle \langle n | \phi^\alpha(0) | i \rangle}{(E_i - E_n)^2 [q_0^2 - (E_n - E_i)^2]} - \text{c.t.} \right\}$$

Clearly the second term on the right-hand side of Eq. (B.19) has a term which blows up when $q_0 = m_\pi$. We would like to extract this term explicitly. To do so, we add to Eq. (B.19) the commutator C [Eq. (B.6)] which may be written as a sum rule

$$\frac{q_0}{m_\pi^2} C^{\beta\alpha} = \frac{q_0}{m_\pi^2} \sum_n (2\pi)^3 \delta(\underset{\sim}{p_n} - \underset{\sim}{p_i}) \frac{\langle f | j_\pi^\beta(0) | n \rangle \langle n | \varphi^\alpha(0) | i \rangle}{[m_\pi^2 - (E_n - E_i)^2]} - \text{c.t.} \qquad \text{(B.20)}$$

Then we get

$$\frac{\frac{1}{2}\left[T^{\beta\alpha}(q_0) - T^{\alpha\beta}(q_0)\right]}{q_0^2(m_\pi^2 - q_0^2)} = - \frac{q_0}{m_\pi^2} C^{\beta\alpha} + \frac{1}{q_0 m_\pi f_\pi^2} \langle f | [Q_A^\beta(0), A_0^\alpha(0)] | i \rangle$$

(B.21)

$$- \left\{ \frac{q_0}{m_\pi^2} \sum_n (2\pi)^3 \delta(\underset{\sim}{p_n} - \underset{\sim}{p_i}) \frac{\langle f | j_\pi^\beta(0) | n \rangle_c \langle n | j_\pi^\alpha(0) | i \rangle_c}{(E_i - E_n)^2 [m_\pi^2 - (E_n - E_i)^2]} - \text{c.t.} \right\} + \text{D.C.}$$

where as mentioned before the subscript C dictates only the connected piece to be taken when the intermediate state is of the πN system, and D.C. is the semi-disconnected part of the term

$$q_0 \sum_n (2\pi)^3 \delta(\underset{\sim}{p}_n - \underset{\sim}{p}_i)\left[\frac{1}{(E_n - E_i)^2\left[q_0^2 - (E_n - E_i)^2\right]} - \frac{1}{m_\pi^2\left[m_\pi^2 - (E_n - E_i)^2\right]}\right]\langle f|j_\pi^\beta(0)|n\rangle\langle n|\phi^\alpha(0)|i\rangle - \text{c.t.} \tag{B.22}$$

One may define the connected and disconnected terms rigorously by using the reduction technique. This was done by Fubini and Furlan[28]. Roughly what they amount to is as follows. If $|n\rangle = |N\pi\rangle$, then in a matrix element like $\langle N_f|G|N_i\pi\rangle$ for an arbitrary current G, one has a term in which N_i propagates freely, and the rest which does not. Thus we may have

$$\langle N_f|G|N_i\pi\rangle = \langle N_f|N_i\rangle\langle 0|G|\pi\rangle + \langle N_f|G|N_i\pi\rangle_c$$

Note that not all currents can give rise to non-vanishing disconnected terms. Here only the axial current can. The first is referred to as disconnected, and the second as connected. When a product of two of such matrix elements appears as in Eq. (B.22), we will then have a fully connected piece, two semi-connected (or semi-disconnected) pieces, and a fully disconnected piece. The fully disconnected piece is to be discarded by definition (i.e. it is the unit matrix in the S-matrix).

To calculate the semi-disconnected piece of Eq. (B.22), one has to be careful in dealing with the poles. In fact one cannot obtain it if one starts with the conditions (B.2) and (B.3). One has to impose these conditions only at the end of the manipulation. By a judicious handling of the terms, one will find that the second term in the square bracket of Eq. (B.22) does not give any pole term. The remaining term leads to

$$\text{D.C.} = \frac{1}{m_\pi^2}\,\frac{\frac{1}{2}\left[T^{\beta\alpha}(m_\pi) - T^{\alpha\beta}(m_\pi)\right]}{m_\pi^2 - q_0^2} \tag{B.23}$$

Thus setting $q_0 = m_\pi$ wherever it is harmless and cancelling the pole term we obtain from Eqs. (B.21) and (B.23) the result of Eq. (B.4):

The proof for Eq. (B.5) follows in the same way if one uses

$$\tfrac{1}{2}\left[T^{\beta\alpha}(q_0) + T^{\alpha\beta}(q_0)\right] / q_0^2(m_\pi^2 - q_0^2)$$

and the commutator C'. The details will be a good exercise for an eager reader.

APPENDIX C

FORMULAS FOR RADIATIVE PION ABSORPTION

Let us consider the reaction

$$\pi^{\alpha}(q) + i(p_i) \rightarrow \gamma(k) + f(p_f) \tag{C.1}$$

wherein the same notation as in Appendix B is used. We define the invariant amplitude M^{α} related to the S-matrix as

$$S_{fi}^{\alpha} = i(2\pi)^4 \delta^4(k + p_f - p_i - q) \mathcal{M}^{\alpha} \tag{C.2}$$

Then our theorem is that the threshold amplitude ($\vec{q} = 0$) is given by

$$\mathcal{M}^{\alpha} = \varepsilon^{\mu} \mathcal{M}_{\mu}^{\alpha}$$

$$\mathcal{M}_{\mu}^{\alpha} = \frac{e}{F_{\pi}} \varepsilon_{3\alpha\beta} \langle f | A_{\mu}^{\beta}(0) | i \rangle + q_0 e \sum_{n} (2\pi)^3 \delta(\underset{\sim}{p}_n - \underset{\sim}{p}_i) \frac{\langle f | J_{\mu}^{em}(0) | n \rangle \langle n | j_{\pi}^{\alpha}(0) | i \rangle}{(E_n - E_i)[q_0 - E_n + E_i]} - \text{c.t.} \tag{C.3}$$

where ε_{μ} is the photon polarization four-vector ($k_{\mu}\varepsilon^{\mu} = 0$). $\varepsilon_{\alpha\beta\gamma}$ is the completely antisymmetric tensor ($\epsilon_{123} = +1$), and J_{μ}^{em} is the electromagnetic current,

$$J_{\mu}^{em}(0) = S_{\mu}(0) + V_{\mu}^{3}(0) \tag{C.4}$$

a sum of isoscalar (S) and isovector (V, third component) components. The inverse reaction $\gamma(q) + i(p_i) \rightarrow \pi^{\alpha}(k) + f(p_f)$ can be treated in an analogous way; hence it will not be repeated here.

Equation (C.3) can be proven in two different ways. The first method is that used for the π-N scattering, the second method the minimal coupling principle. Both lead to the same results. Both are presented below to clarify the connection.

In LSZ-formalism, the amplitude is

$$\mathcal{M}_\mu^\alpha = -ie\int d^4x\, e^{-iq.x}\left(\Box_x + m_\pi^2\right)\theta(-x_0)\langle f|\left[J_\mu^{em}(0),\, \phi^\alpha(x)\right]|i\rangle \tag{C.5}$$

Using PCAC [Eq. (B.7)], integrating by parts, and setting $\vec{q} = 0$,

$$\begin{aligned}\mathcal{M}_\mu^\alpha = -\frac{e}{F_\pi m_\pi^2}\Big\{ & i(m_\pi^2 - q_0^2)\int d^4x\, \delta(x_0)\langle f|\left[J_\mu^{em}(0),\, A_0^\alpha(x)\right]|i\rangle \\ & - q_0\int d^4x\, e^{-iq_0x_0}\left(m_\pi^2 + \partial_{x_0}^2\right)\theta(-x_0)\langle f|\left[J_\mu^{em}(0),\, A_0^\alpha(x)\right]|i\rangle\Big\}\end{aligned} \tag{C.6}$$

In the second term, moving the operator $(m_\pi^2 + \partial^2_{x_0})$ to the right, one finds that the artificial zero $(m_\pi^2 - q_0^2)$ in the first term is eliminated, i.e.

$$\begin{aligned}\mathcal{M}_\mu^\alpha = -\frac{e}{F_\pi}\Big\{ & i\int d^4x\, \delta(x_0)\langle f|\left[V_\mu^3(0),\, A_0^\alpha(x)\right]|i\rangle \\ & + \frac{q_0}{m_\pi^2}\int d^4x\, \delta(x_0)\langle f|\left[J_\mu^{em}(0),\, \dot{A}_0^\alpha(x)\right]|i\rangle \\ & - \frac{q_0}{m_\pi^2}\int d^4x\, e^{-iq_0x_0}\theta(-x_0)\langle f|\left[J_\mu^{em}(0), (m_\pi^2 + \partial_{x_0}^2)A_0^\alpha(x)\right]|i\rangle\Big\}\end{aligned} \tag{C.7}$$

The first term in the curly bracket becomes, from the commutation rule of Appendix A:

$$i^2\epsilon_{3\alpha\beta}\langle f|A_\mu^\beta(0)|i\rangle \tag{C.8}$$

The second term which will be dropped is of order $q_0\delta$, where δ is the SU(2) $\times$ SU(2) symmetry breaking, so that it is most likely to be negligible, and the last term in the parenthesis becomes, after integrating over x:

$$-\left\{q_0\sum_n (2\pi)^3\,\delta(\underset{\sim}{p_n} - \underset{\sim}{p_i})\frac{\langle f|J_\mu^{em}(0)|n\rangle\langle n|j_\pi^\alpha(0)|i\rangle}{(E_n - E_i)(q_0 - E_n + E_i)} - c.t\right\} \tag{C.9}$$

which follows from using Eq. (B.18). Equation (C.7), (C.8) and (C.9) lead to Eq. (C.3). (A great care must be exercised in the expansion of the equal time commutators in order to insure that the disconnected parts are properly taken into account; otherwise one may be led to wrong expressions such as the Eq. 43 of Ref. 46 where the correction term does not vanish in the soft-pion limit.)

Let us now turn to an alternative derivation based on the "minimal electromagnetic coupling" principle, which states that in the presence of electromagnetic field $\mathcal{a}_\lambda(x)$, the divergence ∂_λ is modified to $\partial_\lambda \mp ie_\lambda$ for $\pi^\mp$ creation, with $e > 0$. According to this, then, the PCAC should be modified to

$$\partial^\mu A_\mu^\alpha(x) + e\,\epsilon_{3\alpha\beta}\,\mathcal{a}^\mu(x)\,A_\mu^\beta(x) = F_\pi m_\pi^2 \phi(x) \tag{C.10}$$

Taking the matrix elements of both sides between the initial state $|i\rangle$ and the outgoing state $\langle f\gamma|$,

$$ie\,q^\mu \langle f\gamma|A_\mu(0)|i\rangle + e\,\epsilon_{3\alpha\beta}\langle f\gamma|\mathcal{a}^\mu(0)\,A_\mu^\beta(0)|i\rangle = \frac{F_\pi m_\pi^2}{m_\pi^2 - q^2}\langle f\gamma|j_\pi^\alpha(0)|i\rangle \tag{C.11}$$

As we work in first order in e, the electromagnetic field is just the outgoing field $\mathcal{a}_\lambda^{\text{out}}$, so that it simply annihilates the photon in the final state:

$$\langle f\gamma|\mathcal{a}^\mu(0)\,A_\mu^\beta(0)|i\rangle = \epsilon^\mu \langle f|A_\mu^\beta(0)|i\rangle \tag{C.12}$$

Separating out a pole term (at $q^2 = m_\pi^2$) from the first term in Eq. (C.11), we obtain

$$\epsilon^\mu \mathcal{M}_\mu^\alpha = \langle f\gamma|j_\pi^\alpha(0)|i\rangle = \frac{1}{F_\pi}\left\{e\,\epsilon_{3\alpha\beta}\,\epsilon^\mu\langle f|A_\mu^\beta(0)|i\rangle + iq^\mu\overline{\langle f\gamma|A_\mu^\alpha(0)|i\rangle}\right\} \tag{C.13}$$

where the bar implies that a pion pole term has been separated off. This equation is valid for pions both on and off the mass-shell.

In order to give a precise meaning to the barred quantity, it is best to start with the matrix element $\langle f\gamma|A_\mu^\alpha(o)|i\rangle$. Reducing out the photon

$$\langle f\gamma|A_\mu^\alpha(0)|i\rangle = -i\int d^4x\, e^{ik.x}\Big\{e\theta(x_0)\langle f|[\epsilon.J^{em}(x), A_\mu^\alpha(0)]|i\rangle \tag{C.14}$$

$$-ik_0\,\delta(x_0)\langle f|[\epsilon.\mathcal{a}(x), A_\mu^\alpha(0)]|i\rangle + \delta(x_0)\langle f|[\epsilon.\dot{\mathcal{a}}(x), A_\mu^\alpha(0)]|i\rangle\Big\}$$

Let us assume, as is customary, that

$$\left[\epsilon . a(x), A_\mu(0)\right]_{x_0=0} = \left[\epsilon\, \dot{a}(x), A_\mu(0)\right]_{x_0=0} = 0 \qquad \text{(C.15)}$$

For the remaining term in Eq. (C.14), we have

$$\delta^4(p_f + k - p_i - q)\, i \int d^4x\, e^{ik.x}\, \theta(x_0) \langle f | [\epsilon . J^{em}(x), A_\mu^\alpha(0)] | i \rangle$$

$$= \delta^4(p_f + k - p_i - q)\, i \int d^4x\, e^{-iq.x}\, \theta(-x_0) \langle f | [\epsilon . J^{em}(0), A_\mu^\alpha(x)] | i \rangle \qquad \text{(C.16)}$$

$$= -\delta^4(p_f + k - p_i - q) \left\{ \sum_n (2\pi)^3 \delta(\underset{\sim}{p_n} - \underset{\sim}{p_i}) \frac{\langle f | \epsilon . J^{em}(0) | n \rangle \langle n | A_\mu^\alpha(0) | i \rangle}{(q_0 - E_n + E_i)} - \text{c.t.} \right\}$$

Now <u>eliminating</u> the pion pole term in Eq. (C.16) amounts to replacing $\langle n | A_\lambda^\alpha(o) | i \rangle$ by $\overline{\langle n | A_\mu^\alpha(o) | i \rangle}$, which through a generalized Goldberger-Treiman relation is given by

$$i(p_n - p_i)^\mu \overline{\langle n | A_\mu^\alpha(0) | i \rangle} = F_\pi \langle n | j_\pi^\alpha(0) | i \rangle \qquad \text{(C.17)}$$

where $|n\rangle$ is any state (i.e. nucleus + n pions). Equations (C.14), (C.16) and (C.17) yield

$$i\, q_0 \langle f \gamma | A_0^\alpha(0) | i \rangle = e q_0 F_\pi \sum_n (2\pi)^3 \delta(\underset{\sim}{p_n} - \underset{\sim}{p_i}) \frac{\langle f | \epsilon . J^{em}(0) | n \rangle \langle n | j_\pi^\alpha(0) | i \rangle}{(E_n - E_i)(q_0 - E_n + E_i)} - \text{c.t.} \qquad \text{(C.18)}$$

substituting Eq. (C.18) into Eq. (C.13) leads to Eq. (C.3).

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Table 1

Cross-section in barns: comparison between soft-pion theory and experiment for $pp \to pn\pi^+$,

T_{lab} (MeV)	SSY[a)]	M[b)]	R[c)]
300	11	10	10
305	17	13	17
310	24	17	28
315	31	31	42
320	39	41	66

a) Theoretical calculation of Schillaci, Silbar and Young (Ref. 57).

b) Experiment: analysis by Mandelstam [Proc. Roy. Soc. London A 244, 491 (1958)].

c) Experiment: analysis by Rosenfeld [Phys. Rev. 96, 139 (1954)].

Figure captions

Fig. 1 : External and internal emission graphs for the bremsstrahlung process. In the former, the photon is emitted from the external nucleon; in the latter it is emitted from the internal nucleons or mesons.

Fig. 2 : Matrix element of the divergence of the axial current (D) between hadronic states A and B. For A and B nucleon states, one-pion exchange dominates.

Fig. 3 : The triangle diagram representing the anomalous threshold singularity for the 3-nucleon system. For instance, it could be $^3H \to {}^3He$ transition, where the intermediate state is a deuteron and D acts such that $n \to p$.

Fig. 4 : The most singular terms in the matrix element $\langle f\gamma|A_\nu^\pm|i\rangle$ as $q_0 \to 0$ considered in Eq. (II.7). The blob represents the place where the axial current (A_ν) acts, and the wiggle stands for the photon of momentum $k = (k_0, \underset{\sim}{k})$

Fig. 5 : Pair term ($N\bar{N}$) contributing m_π/m_N correction to Eq. (IV.12) for the case of π^+ capture

Fig. 6 : The pion probability $|\phi|^2$ in unit of fm^{-3} in 1s and 2p orbits for ^{12}C as a function of distance r from the nucleus. The subscripts undist. and dist. stand, respectively, for wave functions calculated without and with the strong interaction. This has been calculated by H. Schmitt and M. Krell (private communication) using the optical potential of Ref. 35.

Fig. 7 : The pion probability $|\phi|^2$ for 1s orbit for ^{16}O calculated by Krell and Ericson (Ref. 35): (I) without strong interaction; (II, III) with strong interaction, where (II) contains full nuclear potential (standard); and (III) with the parameter $b_0 = -0.03$, Im $(B_0) = 0.04$ (see Ref. 35 for the definition).

Fig. 8 : Radiative π^- capture results of Bistirlich et al. (Ref. 55):

a) Uncorrected energy spectrum of γ-rays from π^- capture in C. The smooth curve is a fit with three Breit-Wigner forms plus a contribution from the pole graph given there.

b) Energy spectrum for pions with a mean kinetic energy of 40 MeV.

c) Spectrum with the pole model subtracted. The solid line is the best fit, the dashed curve a calculation by Kelly and Überall [Nuclear Phys. A118, 302 (1968)].

Fig. 9 : External emission graphs in nucleon(p)-nucleon(p_i) or nucleon(p)-nucleus(p_i) collision (to be viewed from down to up). The oval block represents the full N-N t-matrix.

Fig. 10 : Comparison between theory and experiment for the π^+ production in proton-proton collisions (taken from Ref. 60). GSS stands for the soft-pion calculation of Grant et al. [Phys. Rev. 184, 1737 (1969)], and the experimental points are LASL, Haddock et al., and Hirt et al., quoted in Ref. 60.

Fig. 11 : A graph which is not taken into account in the hard-pion theory of Banerjee et al. (Ref. 58), but is known to be important (Refs. 60-62) for the process $p + p \rightarrow d + \pi^+$ (to be viewed from down to up).

Fig. 12 : Comparison between experiment [Dunaitsev and Prokoshkin, Nuclear Phys. 56, 300 (1964)] and theory (Ref. 66) for $^{12}C + p \rightarrow \pi^0$ + anything (taken from Ref. 66).

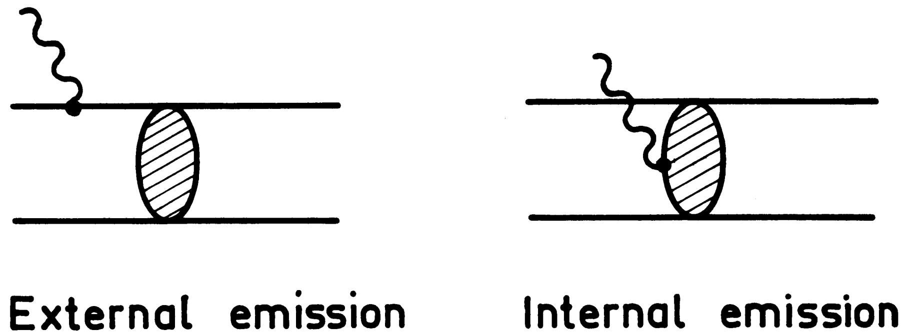


Fig. 1

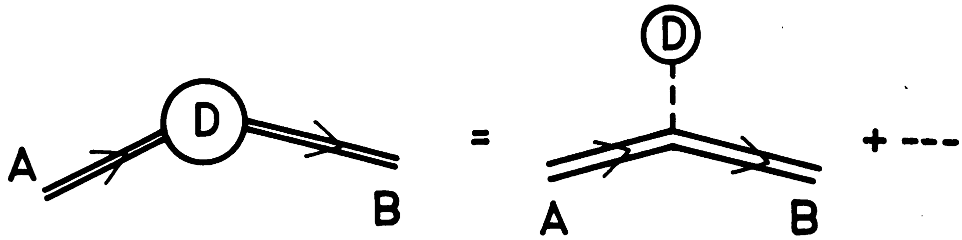


Fig. 2

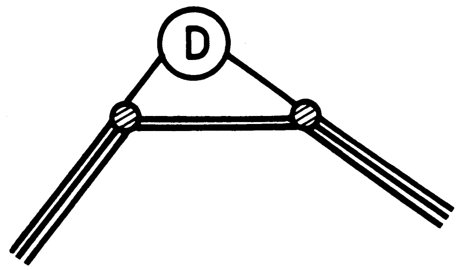


Fig. 3

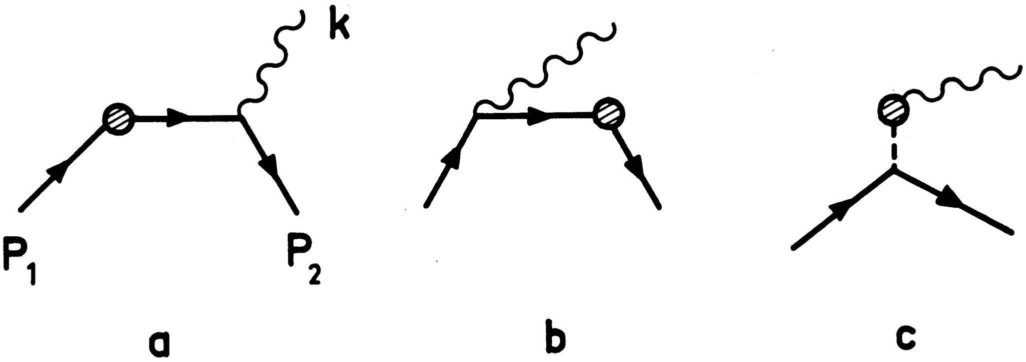


Fig. 4

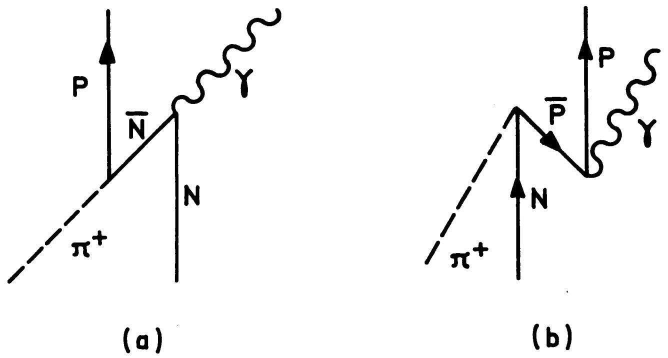
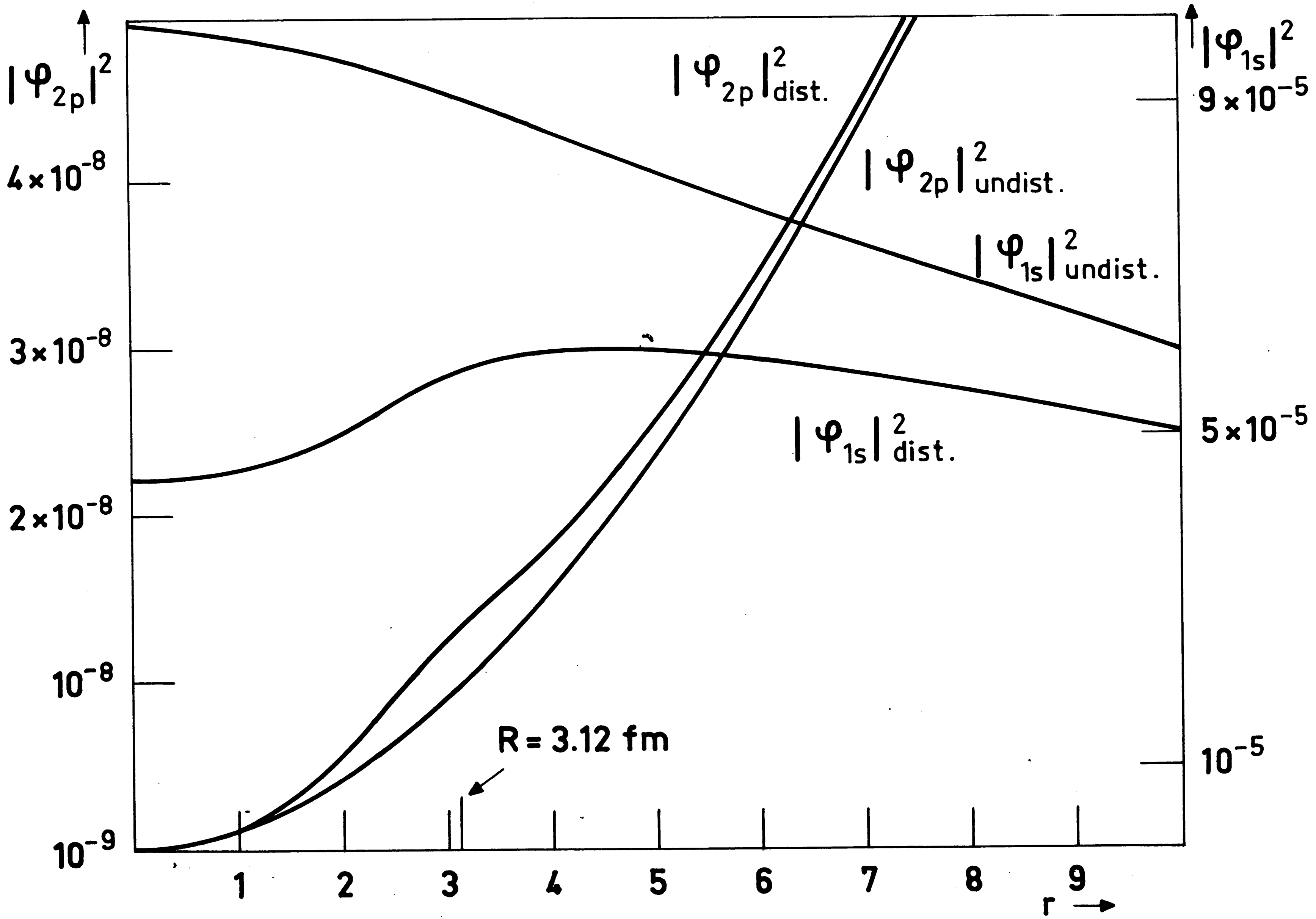


Fig. 5



$|\varphi_{2p}|^2$
4×10^{-8}
3×10^{-8}
2×10^{-8}
10^{-8}
10^{-9}
$|\varphi_{1s}|^2$
9×10^{-5}
5×10^{-5}
10^{-5}
$|\varphi_{2p}|^2_{dist.}$
$|\varphi_{2p}|^2_{undist.}$
$|\varphi_{1s}|^2_{undist.}$
$|\varphi_{1s}|^2_{dist.}$
R = 3.12 fm
1
2
3
4
5
6
7
8
9
r

Fig. 6

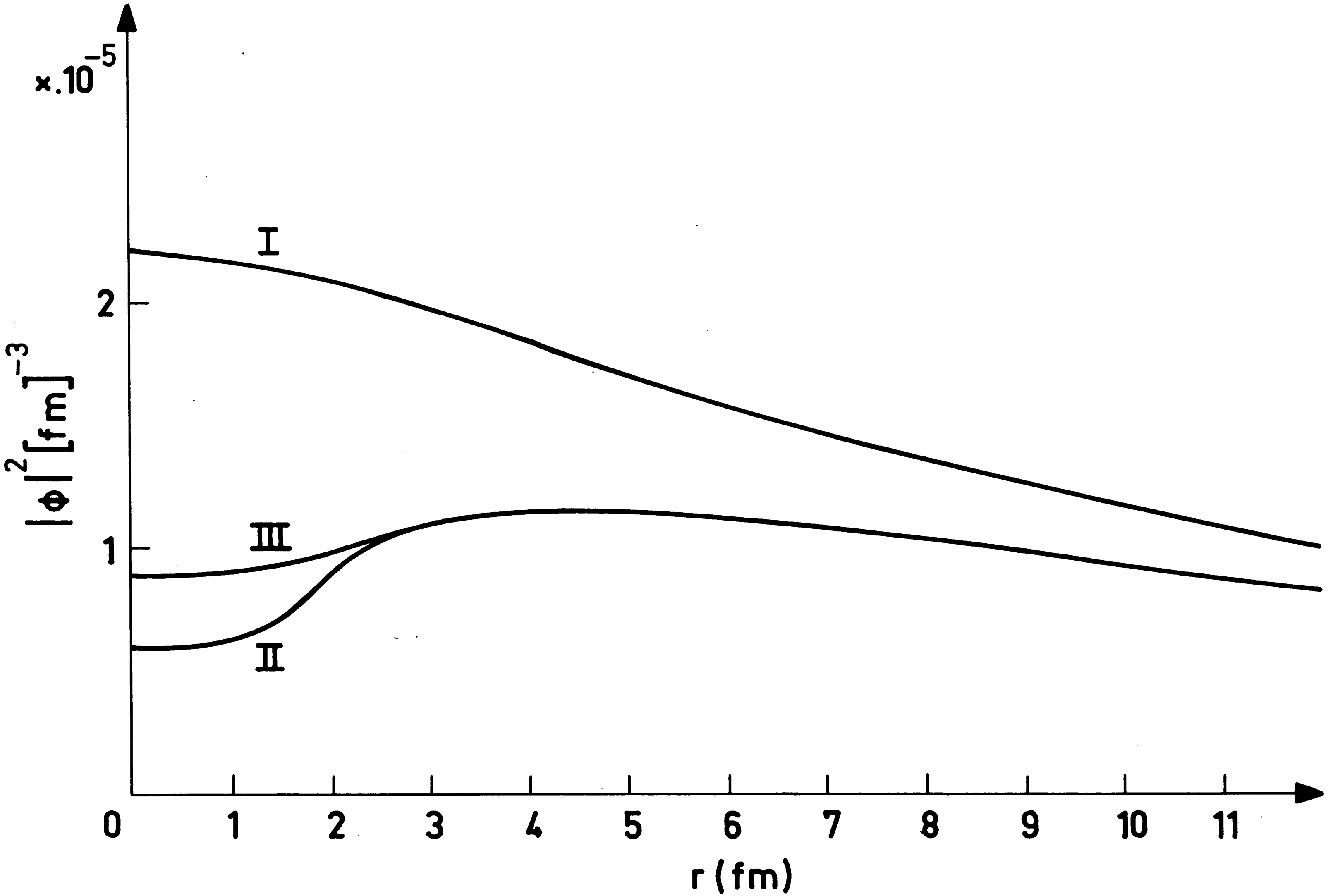
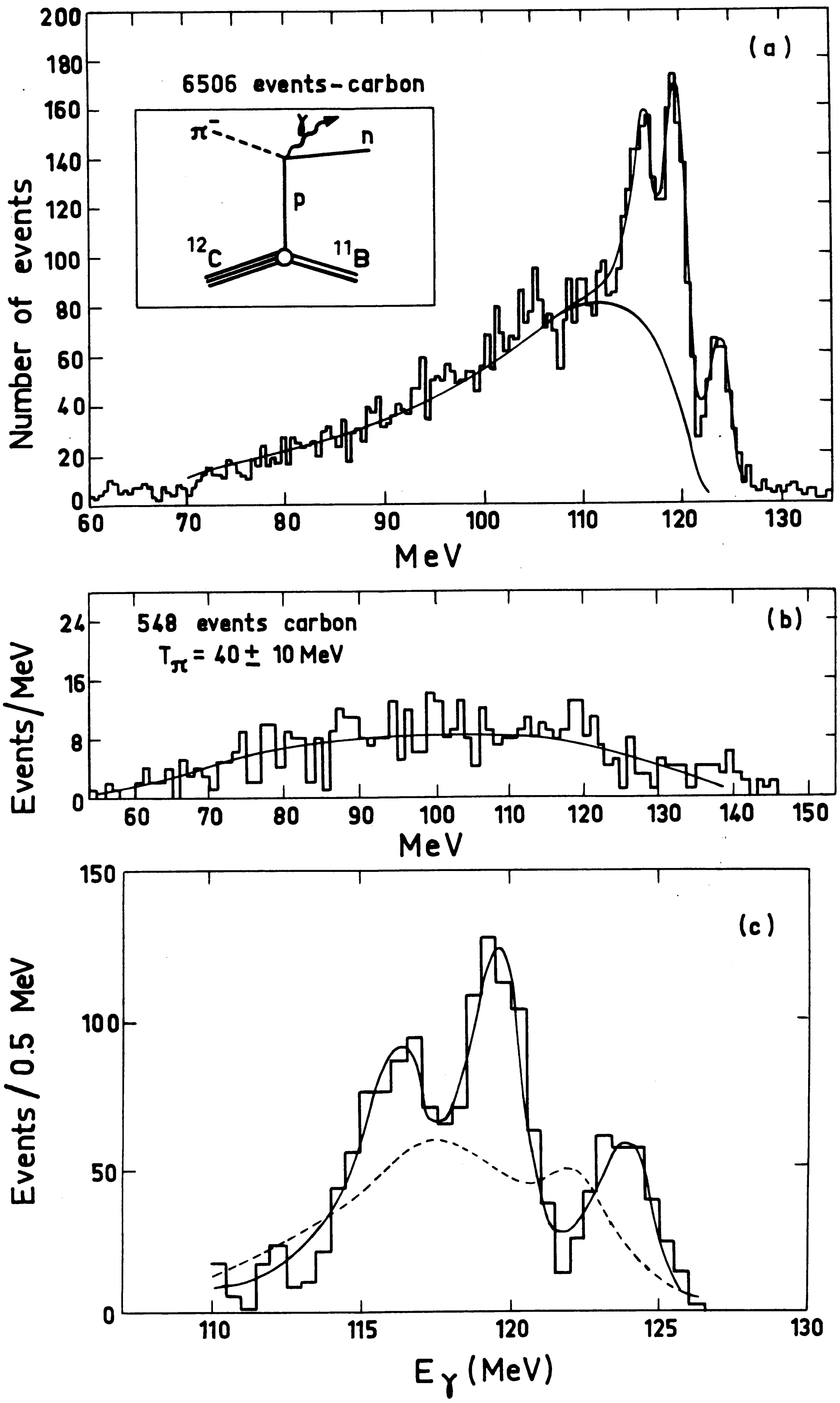


Fig. 7



(a)
6506 events-carbon
π⁻
γ
n
p
12C
11B
Number of events
MeV
(b)
548 events carbon
Tπ = 40 ± 10 MeV
Events/MeV
MeV
(c)
Events / 0.5 MeV
Eγ (MeV)

Fig. 8

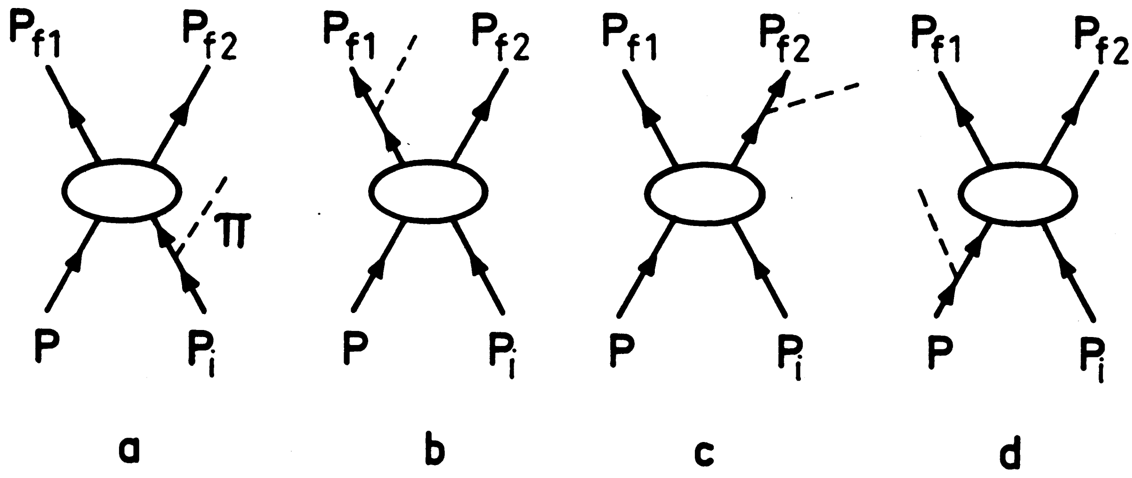


Fig. 9

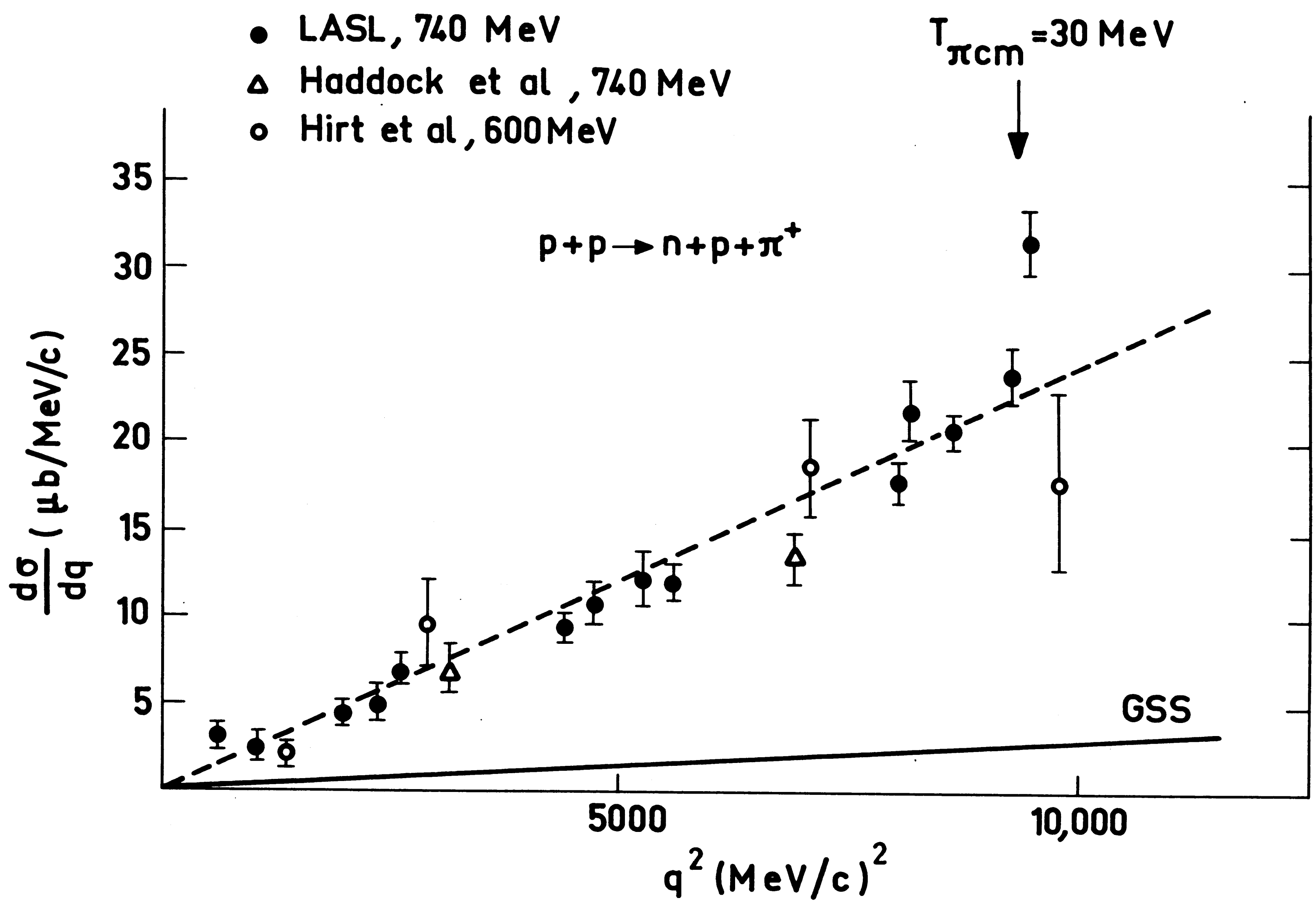

● LASL, 740 MeV
△ Haddock et al, 740 MeV
○ Hirt et al, 600MeV
$T_{\pi cm}$ =30 MeV
$p+p \rightarrow n+p+\pi^+$
$\frac{d\sigma}{dq}$ (μb/MeV/c)
35
30
25
20
15
10
5
GSS
5000
10,000
q^2 (MeV/c)^2

Fig. 10

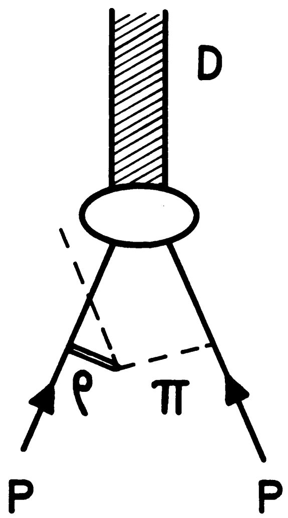
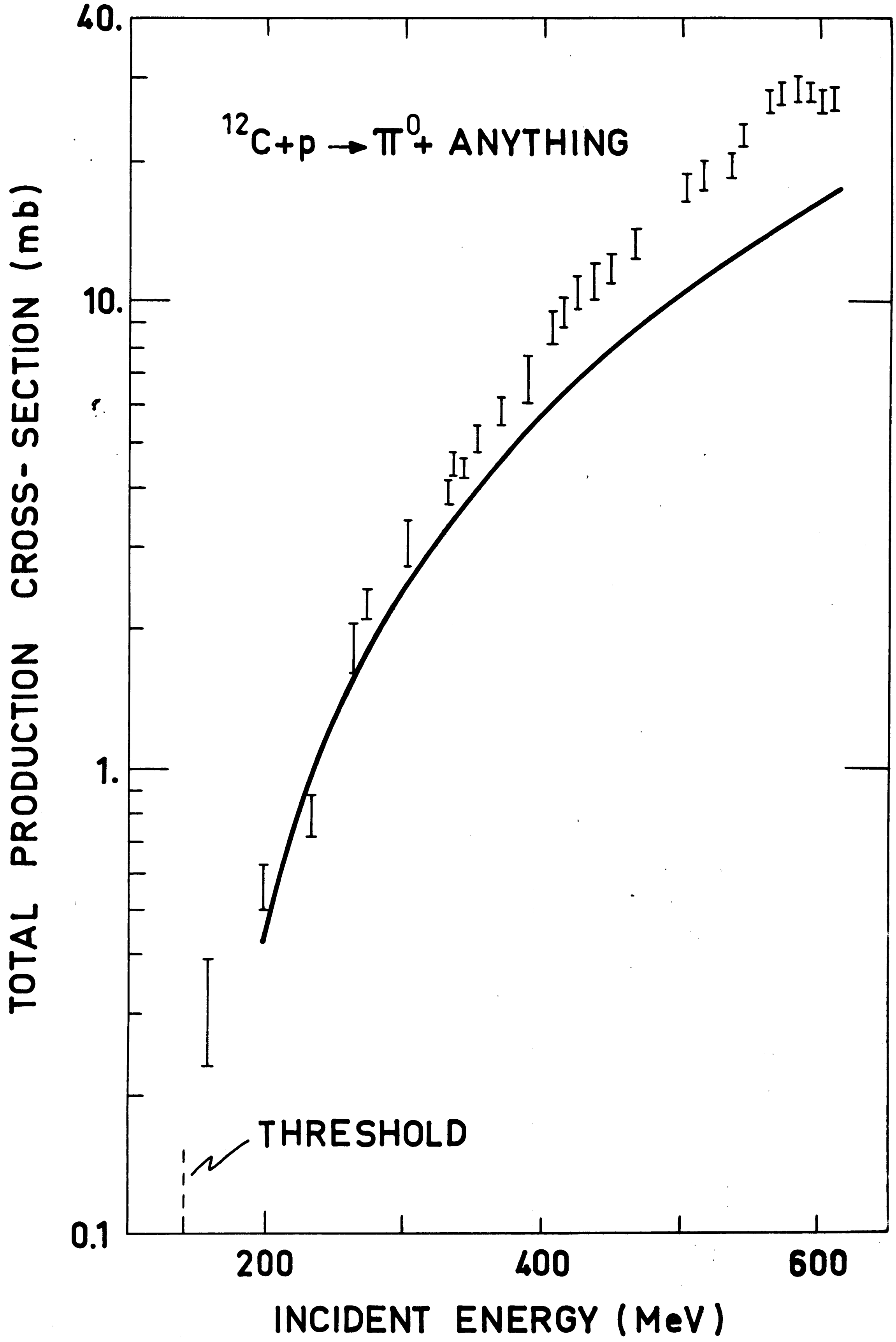


Fig. 11



40.
10.
1.
0.1
$^{12}C+p \rightarrow \pi^0+$ ANYTHING
TOTAL PRODUCTION CROSS-SECTION (mb)
THRESHOLD
200
400
600
INCIDENT ENERGY (MeV)

Fig. 12