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UNITARITY RELATION FOR DEEP INELASTIC SCATTERING

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A B S T R A C T

Saturating the multiparticle states in the unitarity relation for virtual Compton scattering by two-particle states consisting of an infinite set of high mass vector mesons and a nucleon leads to a non-linear integral equation the solution of which determines the behaviour in the scaling limit and the non-forward scaling functions for deep inelastic scattering.

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The observed scaling behaviour of deep inelastic scattering obviously constitutes a constraint on the behaviour of the multiparticle final states in electroproduction. It is the aim of this work to suggest that unitarity together with a condition on the spectrum is at the origin of this constraint. Of course, what is needed is a multiparticle saturation of the unitarity relation for the virtual Compton amplitude.

In view of precocious scaling and Bloom-Gilman duality [1], it seems reasonable to assume that the final hadron states in deep inelastic scattering at not too large ω can be approximated by an infinite set of resonances. Unitarity constraints can be built in if one saturates the sum of intermediate states by those two-body channels in which the resonances occur. At least to a large extent at low ω they will be of the form : nucleon plus one of the vector mesons. Of course, there are additional contributions from other intermediate states. However, because of their complexity and the problem of double counting, these states will not be taken into account.

In a simplifying generalization of this picture, we assume that the final hadron states in deep inelastic scattering can be approximated by an infinite set of two-particle states consisting of a nucleon and a vector meson of increasing mass.

In order to make the approach as transparent as possible, we study the scattering of scalar photons on scalar nucleons of mass M. The absorptive part of the retarded Compton amplitude T^{ret} is given by *)

$$\mathrm{Im}\, T^{ret}(q^2,Q^2,s,t) = \frac{1}{2}\int d^4x\, e^{iqx} \langle p'|[J(x),J(0)]|p\rangle_C . \quad (1)$$

Here the kinematical variables are defined by $Q = (q+p'-p)$, $s = (q+p')^2$, $t = (p'-p)^2$, $u = (q-p)^2 = q^2 + Q^2 + 2M^2 - s - t$. The only saturation scheme which converts Eq. (1) into a feasible inelastic unitarity relation for T^{ret} consists in approximating the completeness sum by (see Figure)

$$1 \simeq \sum_n |N,\sigma_n\rangle\langle N,\sigma_n| \quad , \quad (2)$$

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*) The subscript "C" in Eq. (1) refers to the connected part.

where N denotes the scalar nucleon and σ_n an infinite set of scalar mesons. With a continuous distribution in the mass m of the scalar mesons σ_n, characterized by

$$n = \alpha(m^2) \quad , \tag{3}$$

Eq. (2) can be replaced by the continuum version ($\alpha' = d\alpha/dm^2$) *)

$$1 \simeq \int_{m_0^2}^{\infty} dm^2\, \alpha'(m^2) \int \frac{d^4k}{(2\pi)^3}\, \Theta(k_0 - M)\, \delta(k^2 - M^2) \int \frac{d^4\ell}{(2\pi)^3}\, \Theta(\ell_0 - m)\, \delta(\ell^2 - m^2) \tag{4}$$

$$\cdot \left| N(\vec{k}), \sigma(m, \vec{\ell}) \right\rangle \left\langle N(\vec{k}), \sigma(m, \vec{\ell}) \right| .$$

The lower limit m_0 of the mass spectrum of the scalar mesons remains an undetermined parameter, which drops out completely in the deep inelastic limit.

In analogy to vector meson dominance [2], we assume

$$\langle 0 |\, J(0)\, | \sigma(m, \vec{\ell}) \rangle = \gamma(m^2) \neq 0 \quad , \tag{5}$$

so that J(x) can serve as an interpolating field for the mesons σ [described by a scalar field $\phi(x, m^2)$]

$$J(x) = \int_{m_0^2}^{\infty} dm^2\, \alpha'(m^2)\, \gamma(m^2)\, \phi(x, m^2) \tag{6}$$

and

$$j(x, m^2) = (\Box + m^2)\, \phi(x, m^2) \tag{7}$$

as the source of the mesons.

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*) We use the covariant normalization $\langle \sigma(m', \vec{\ell}') | \sigma(m, \vec{\ell}) \rangle = (2\pi)^3 \frac{2\ell_0}{\alpha'(m^2)}\, \delta^3(\vec{\ell} - \vec{\ell}')\, \delta(m^2 - m'^2)$.

Insertion of Eq. (4) into Eq. (1) then yields

$$\mathrm{Im}\, T^{ret}(q^2,Q^2,s,t) = \frac{1}{2\,(2\pi)^2}\int_{m_0^2}^{\infty} dm^2\,\alpha'(m^2)$$

$$\cdot\int d^4\ell\,\Theta(\ell_0-m)\,\delta(\ell^2-m^2)\,\Theta(q_0+p_0'-\ell_0-M)\,\delta\big((q+p'-\ell)^2-M^2\big)$$

(8)

$$\cdot\int d^4y\, e^{-i\ell y}\,\Theta(-y_0)\langle p'|[J(0),j(y,m^2)]|q+p'-\ell\rangle\int d^4z\, e^{i\ell z}\,\Theta(z_0)\langle q+p'-\ell|[j(z,m^2),J(0)]|p\rangle.$$

Generalized scalar meson dominance as expressed by Eq. (6) leads to the following relation

$$\mathrm{disc}_{q^2}\, T^{ret}(q^2,Q^2,s,t) =$$

$$\mathrm{disc}_{q^2}\left\{ i\int_{m_0^2}^{\infty} dm^2\,\alpha'(m^2)\gamma(m^2)\int d^4x\, e^{iqx}\,\Theta(x_0)\,\frac{\langle p'|[j(x,m^2),J(0)]|p\rangle}{m^2-q^2}\right\}$$

(9)

$$= i\pi\,\Theta(q^2-m_0^2)\,\alpha'(q^2)\,\gamma(q^2)\int d^4x\, e^{iqx}\,\Theta(x_0)\langle p'|[j(x,q^2),J(0)]|p\rangle.$$

After substitution of Eq. (9) into Eq. (8) one arrives at the unitarity relation

$$\mathrm{disc}_s\, T^{ret}(q^2,Q^2,s,t) =$$

$$\frac{2}{(2\pi)^4}\int_{m_0^2}^{\infty}\frac{dm^2}{\alpha'(m^2)\,\gamma^2(m^2)}\int d^4\ell\,\Theta(\ell_0-m)\,\delta(\ell^2-m^2)\,\Theta(q_0+p_0'-\ell_0-M)\,\delta\big((q+p'-\ell)^2-M^2\big)$$

(10)

$$\cdot\,\mathrm{disc}_{m^2}\, T^{ret}(q^2,m^2,s+i\varepsilon,t')\ \mathrm{disc}_{m^2}\, T^{ret}(m^2,Q^2,s-i\varepsilon,t''),$$

where we have denoted $t' = (q-\ell)^2$ and $t'' = (\ell-Q)^2$. Carrying out the $(d^4\ell)$ integration in the usual manner, Eq. (10) reduces to

$$\mathrm{disc}_s T^{ret}(q^2,Q^2,s,t) = \int_{m_0^2}^{\infty} dm^2 \, \frac{\rho(m^2,s)}{4} \, \Theta(s-(m+M)^2) \cdot \int d\Omega_{\vec{\ell}} \, \mathrm{disc}_{m^2} T^{ret}(q^2,m^2,s+i\varepsilon,t') \, \mathrm{disc}_{m^2} T^{ret}(m^2,Q^2,s-i\varepsilon,t'') \qquad (11)$$

with $\rho(m^2,s)$ given by

$$\rho(m^2,s) = \frac{1}{(2\pi)^4} \frac{1}{\alpha'(m^2)\,\gamma^2(m^2)} \frac{1}{s} \lambda(s,m^2,M^2) \qquad (12)$$

and

$$\lambda(x,y,z) = \left[x^2+y^2+z^2-2xy-2xz-2yz\right]^{1/2} . \qquad (13)$$

An alternative version of Eq. (11) is

$$\mathrm{disc}_s T^{ret}(q^2,Q^2,s,t) = \int_{m_0^2}^{\infty} dm^2 \, \rho(m^2,s)\,\Theta(s-(m+M)^2) \, \frac{2s^2}{\lambda^2(s,m^2,M^2)\,\lambda(s,q^2,M^2)\,\lambda(s,Q^2,M^2)} \qquad (14)$$

$$\cdot \int_{t'_-}^{t'_+} dt' \int_{t''_-}^{t''_+} dt'' \, \frac{\Theta(K(z,z',z''))}{\sqrt{K(z,z',z'')}} \, \mathrm{disc}_{m^2} T^{ret}(q^2,m^2,s+i\varepsilon,t') \, \mathrm{disc}_{m^2} T^{ret}(m^2,Q^2,s-i\varepsilon,t''),$$

where $K(z,z',z'')$ is the kernel

$$K(z,z',z'') = 1 - z^2 - z'^2 - z''^2 + 2zz'z'' \tag{15}$$

and where the relations to the cosines of the angles $z = \cos(\vec{p},\vec{p}\,')$, $z' = \cos(\vec{p}\,',\vec{\ell})$, $z'' = \cos(\vec{p},\vec{\ell})$ are given by

$$\begin{aligned} t &= r(s,q^2,Q^2,M^2,z) \\ t' &= r(s,q^2,m^2,M^2,z') \\ t'' &= r(s,m^2,Q^2,M^2,z'') \quad , \end{aligned} \tag{16}$$

$$\begin{aligned} r(s,q^2,Q^2,M^2,z) = 2M^2 &- \frac{1}{2s}(s+M^2-q^2)(s+M^2-Q^2) \\ &+ \frac{1}{2s}\lambda(s,q^2,M^2)\lambda(s,Q^2,M^2)\; z \; . \end{aligned} \tag{17}$$

The limits $t_{\pm}$ of the t integrations in Eq. (14) are reached in the expressions (16) for t' and t'' through the values $z = \pm 1$, respectively.

We study the final unitarity relation, Eq. (14) in the Bjorken scaling limit, defined in non-forward direction by

$$\begin{aligned} \text{B-lim} = \lim \Big(q^2 \to \infty, Q^2 \to \infty, s \to \infty, &\frac{q^2}{s} = x \text{ fixed}, \\ &\frac{Q^2}{s} = y \text{ fixed}, \; t \text{ fixed} \Big) \; , \end{aligned} \tag{18}$$

and call the leading term of $T^{ret}(q^2,Q^2,s,t)$ in this limit $T_B^{ret}(q^2,Q^2,s,t)$. The leading term K_B of the kernel K in this limit turns out to be

$$K(z,z',z'') \xrightarrow[\text{B-lim}]{} \frac{4}{s^2} K_B(\tau,\tau',\tau'') \tag{19}$$

with

$$K_B(\tau,\tau',\tau'') = 2(\tau\tau' + \tau\tau'' + \tau'\tau'') - \tau^2 - \tau'^2 - \tau''^2 \quad , \tag{20}$$

where we have defined $(z = m^2/s)$:

$$\tau = \frac{1}{(1-x)(1-y)}\left[t + M^2 \frac{(x-y)^2}{(1-x)(1-y)}\right]$$

$$\tau' = \frac{1}{(1-x)(1-z)}\left[t' + M^2 \frac{(x-z)^2}{(1-x)(1-z)}\right] \tag{21}$$

$$\tau'' = \frac{1}{(1-z)(1-y)}\left[t'' + M^2 \frac{(z-y)^2}{(1-z)(1-y)}\right] .$$

The asymptotic form of the unitarity relation Eq. (14) in the B limit reads now *)

$$\mathrm{disc}_s \overline{T}_B^{ret}(q^2,Q^2,s,t) = \int_{m_0^2}^{\infty} \frac{dm^2}{s} \rho_B(m^2,s)\,\Theta(s-m^2)$$

$$\cdot \int_{-s}^{0} d\tau' \int_{-s}^{0} d\tau'' \frac{\Theta(K_B)}{\sqrt{K_B}} \mathrm{disc}_{m^2} \overline{T}_B^{ret}(q^2,m^2,s+i\varepsilon,t'(\tau'))\,\mathrm{disc}_{m^2} \overline{T}_B^{ret}(m^2,Q^2,s-i\varepsilon,t''(\tau'')), \tag{22}$$

where $\rho_B(m^2,s)$ is the leading term of $\rho(m^2,s)$ in the B limit :

$$\rho_B(m^2,s) = \frac{1}{(2\pi)^4} \frac{1}{\alpha'(m^2)\gamma^2(m^2)} \frac{s-m^2}{s} . \tag{23}$$

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*) We should like to note that for the derivation of Eq. (22) we have assumed that the interchange of the B limit with the t integrations is allowed. A consequence of this interchange could be that possible anomalous singularities in the B limit are lost. For a discussion of the occurrence of such singularities in the scaling limit, see Ref. 3).

Denoting the double discontinuity of $T_B^{ret}(q^2,Q^2,s,t)$ with respect to the mass variables q^2 and Q^2 by $\bar{G}_B(q^2,Q^2,s,t)$:

$$\bar{G}_B(q^2,Q^2,s,t) = \mathrm{disc}_{q^2}\, \mathrm{disc}_{Q^2}\, T_B^{ret}(q^2,Q^2,s,t) \tag{24}$$

we have with the additional notation

$$G_B(q^2,Q^2,s,\tau) = \bar{G}_B(q^2,Q^2,s,t(\tau)) \tag{25}$$

the simpler relation

$$\mathrm{disc}_s\, G_B(q^2,Q^2,s,\tau) = \int_{m_0^2}^{\infty} \frac{dm^2}{s}\, \rho_B(m^2,s)\, \Theta(s-m^2)$$

$$\cdot \int_{-s}^{0} d\tau' \int_{-s}^{0} d\tau'' \frac{\Theta(K_B(\tau,\tau',\tau''))}{\sqrt{K_B(\tau,\tau',\tau'')}}\, G_B(q^2,m^2,s+i\varepsilon,\tau')\, G_B(m^2,Q^2,s-i\varepsilon,\tau''). \tag{26}$$

This equation is our final <u>deep inelastic unitarity relation</u>. We recall that essentially two assumptions were needed for its derivation :

(i) the final hadron states are sufficiently well exhausted by a continuous set of two-particle states, consisting of - in this simplified model - a scalar nucleon and a scalar meson of continuous mass spectrum [Eq. (4)] ;

(ii) a generalized scalar dominance hypothesis [Eq. (6)] which supposes that the scalar "electromagnetic" current of this model is dominated by the continuous set of scalar mesons in analogy to the generalized vector dominance hypothesis for the real electromagnetic current.

These dynamical assumptions bring about the non-linear integral equation character of the unitarity relation (26).

In a second paper published elsewhere [4], we shall construct the solution to the non-linear integral equation (26). It shows that under certain conditions on the spectrum of the scalar mesons scaling behaviour prevails in the deep inelastic region, as well for scattering as for annihilation. Since the solution is explicitly given, the details of the scaling function can be studied.

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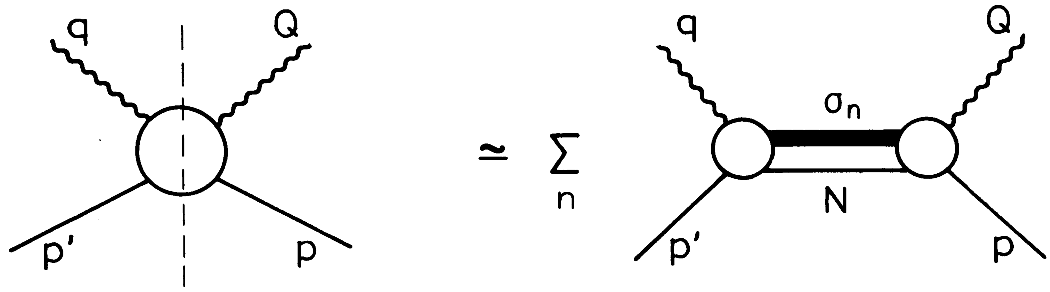
FIGURE CAPTION

Saturation of the unitarity relation.


q
Q
p'
p
≃ $\sum_n$
σ_n
N