PION-NUCLEON SCATTERING THEORY

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Introduction

These lectures contain a selection of the topics which may be of interest to experimenters or theorists who are concerned with the new, and projected, very accurate low energy πN scattering experiments.

The selection could have been different, but there certainly was not time for everything

The topics are:

I Low Energy πN Parameters

II Coulomb Corrections

III The Use of Fixed t Dispersion Relations (Accurate Low Energy Phase Shifts)

IV Dynamics of Low Energy πN Scattering

V Information on σ.

The standard notation of πN dispersion theory is assumed. Much of the notation, and the elementary theory, can be found in "The Dynamics of Elementary Particles and the Pion-Nucleon Interaction," Courses A and B, Nordita Lecture Notes, Copenhagen, 1968 - 1970.

I Low Energy πN Parameters

I. Introduction

We shall discuss three topics which are relevant to high accuracy low energy πN experiments: i) parametrization of the S-wave phases, ii) sum rules and the Pomeranchuk theorem, iii) the determination of the coupling constant f^2.

I.1 Parametrization of the S-wave Phases

Years ago Cini et al. (1) pointed out that the expansion

$$\alpha_i = a_i q + b_i q^3 + c_i q^5 + \cdots \qquad (i = 1,3)$$

is not a good way to fit the S-wave πN phases α_i (i = 1,3). These series converge poorly, and using them is an easy way to find bad values for the scattering lengths a_1, a_3. Cini et al. also pointed out that crossing symmetry gives a strong hint about a good parametrization. Making use also of the dominance of N^*_{33} at low energies, we can in fact find simple and useful formulae (2).

First we have to specify some symbols *). Let q_L, ω_L and q, ω be the momenta and energies of the pion in the lab. system and the c.m.s. respectively. Then

$$\left.\begin{aligned} q_L/q &= W/M \\ \omega_L &= \frac{EW - M^2}{M} \end{aligned}\right\} \qquad (1)$$

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(1) M. Cini, R. Gatto, E.L. Goldwasser and M. Ruderman, Nouvo Cimento 10, 242, (1958)

(2) J. Hamilton and W.S. Woolcock, Rev. Mod. Phys 35 p. 737 (1963)

*) See for example ref. (2) for notation and equations.

where W and E are the total energy and the nucleon's energy in the c.m.s. We shall use μ, M for the (charged) pion and nucleon masses, and we often put $\mu = 1$ (M = 6.7). Also the usual invariants are

$$\left.\begin{aligned} s &= M^2 + \mu^2 + 2M\omega_L \\ t &= -2q^2(1-\cos\theta) \\ s+t+u &= 2M^2 + 2\mu^2 \end{aligned}\right\} \qquad (2)$$

where θ is the $\pi N \to \pi N$ scattering angle in the c.m.s.

The forward scattering amplitudes are $f_L(\omega_L)$ and $f(\omega)$ in the lab. and c.m.s. and

$$f_L/q_L = f/q \qquad (3)$$

In terms of the invariant scattering amplitudes A(s,t), B(s,t), we have

$$f_L = \frac{1}{4\pi}(A + \omega_L B) \; , \quad f = \frac{M}{4\pi W}(A + \omega_L B) \qquad (4)$$

Charge Notation and Crossing

We shall assume that the correct Coulomb corrections and the related small corrections for mass differences etc. have been made to the experimental data, and we shall assume (as seems to be the case) that charge independence is then obeyed to within a few percent. In what follows we shall use charge independent notation.

The $\pi N \to \pi N$ amplitudes are then written either as $f^{(3)}, f^{(1)},$ where the superscript denotes the isospin T = 3/2, 1/2, or as $f^{(+)}, f^{(-)}$ where (+) and (-) correspond to isospin T = 0 and T = 1, respectively, in the t-channel. Also

$$f^{(+)} = \frac{1}{3} f^{(1)} + \frac{2}{3} f^{(3)}$$
$$f^{(-)} = \frac{1}{3} \left(f^{(1)} - f^{(3)}\right)$$

Crossing is the exchange s $\leftrightarrow$ u, t $\rightarrow$ t, and we have the symmetry

$$\left.\begin{aligned} A^{(\pm)}(s,t) &= \pm A^{(\pm)}(u,t) \\ B^{(\pm)}(s,t) &= \mp B^{(\pm)}(u,t) \end{aligned}\right\} \qquad (5)$$

For forward scattering $t = 0$, and by eq. (2)

$$u = M^2 + \mu^2 - 2M\omega_L , \qquad (6)$$

so by eq. (4)

$$f_L^{(\pm)}(\omega_L) = \pm f_L^{(\pm)}(-\omega_L) \qquad (6a)$$

The forward amplitude in the c.m.s. is given by the p.w. series

$$f(\omega) = f_{0\frac{1}{2}}(s) + \left(2f_{1\frac{3}{2}}(s) + f_{1\frac{1}{2}}(s)\right) + \left(3f_{2\frac{5}{2}}(s) + 2f_{2\frac{3}{2}}(s)\right) + \ldots \qquad (7)$$

where $f_{\ell J}$ $(J = \ell \pm \frac{1}{2})$ are the partial wave amplitudes. Thus eq. (6a) suggest that when the energy is so low that the P-waves can be neglected, a useful parametrization might be

$$\frac{W}{M+\mu}\;\frac{\sin(2\alpha_1)+2\sin(2\alpha_3)}{2q} = (a_1+2a_3)\left[1+O(q_L^2)\right] \tag{8a}$$

$$\frac{W/\mu}{M+\mu}\;\frac{\sin(2\alpha_1)-\sin(2\alpha_3)}{2q} = \omega_L(a_1-a_3)\left[1+O(q_L^2)\right] \tag{8b}$$

where a_3, a_1 are the S-wave scattering lengths for T = 3/2,1/2.

<u>Use of Forward D.R.</u>

We can put eqs. (8) in a firm basis and also considerably improve them by using the once subtracted forward dispersion relations

$$f_L^{(+)}(\omega_L) = f_L^{(+)}(\mu) + \frac{f^2}{M(1-\mu^2/4M^2)}\,\frac{q_L^2}{\omega_L^2-\mu^4/4M^2}$$

$$+\frac{q_L^2}{4\pi^2}\int_\mu^\infty \frac{d\omega'}{q'}\,\sigma^{(+)}(\omega')\left[\frac{1}{\omega'-\omega_L}+\frac{1}{\omega'+\omega_L}\right] \tag{9a}$$

$$f_L^{(-)}(\omega_L) = \frac{\omega_L}{\mu}f_L^{(-)}(\mu) - \frac{2f^2}{\mu^2}\,\frac{1}{1-\mu^2/4M^2}\,\frac{\omega_L q_L^2}{\omega_L^2-\mu^4/4M^2}$$

$$+\frac{q_L^2}{4\pi^2}\int_\mu^\infty \frac{d\omega'}{q'}\,\sigma^{(-)}(\omega')\left[\frac{1}{\omega'-\omega_L}-\frac{1}{\omega'+\omega_L}\right] \tag{9b}$$

Here ω', q' are lab. system variables, and $\sigma^{(\pm)} = 1/2\,(\sigma_- \pm \sigma_+)$ where $\sigma_\pm$ are the total $\pi^\pm p$ cross-sections. Also

$$\left.\begin{aligned} f_L^{(+)}(\mu) &= (1+\mu/M)\left(\tfrac{1}{3}a_1+\tfrac{2}{3}a_3\right) \\ f_L^{(-)}(\mu) &= (1+\mu/M)\,\tfrac{1}{3}(a_1-a_3) \end{aligned}\right\} \tag{10}$$

The units are such that f^2 = 0.081.

Using eq. (7) and rewriting, eq. (9a) gives

$$\frac{\sin(2\alpha_1) + 2\sin(2\alpha_3)}{2q} \frac{W}{M+\mu} = (a_1 + 2a_3) + q_L^2\, C^{(+)}(\omega_L) \qquad (11a)$$

and eq. (9b) gives

$$\frac{\sin(2\alpha_1) - \sin(2\alpha_3)}{2q} \frac{W\mu}{M+\mu} = (a_1 - a_3)\omega_L + q_L^2\, C^{(-)}(\omega_L) \qquad (11b)$$

where

$$C^{(+)}(\omega_L) = \frac{3}{4\pi^2(1+\mu/M)} P\int_\mu^\infty \frac{d\omega'}{q'} \sigma^{(+)}(\omega')\left(\frac{1}{\omega'-\omega_L} + \frac{1}{\omega'+\omega_L}\right)$$

$$+ \frac{3f^2}{(M+\mu)(1-\mu^2/4M^2)(\omega_L^2-\mu^4/4M^2)}$$

$$- \frac{M}{W(1+\mu/M)} \mathrm{Re}\,(p_{11} + 2p_{13} + 2p_{31} + 4p_{33}) + (D\text{-waves}) \qquad (12a)$$

$$C^{(-)}(\omega_L) = \frac{3}{4\pi^2(1+\mu/M)} P\int_\mu^\infty \frac{d\omega'}{q'} \sigma^{(-)}(\omega')\left(\frac{1}{\omega'-\omega_L} - \frac{1}{\omega'+\omega_L}\right)$$

$$- \frac{6f^2\omega_L}{(1+\mu/M)(1-\mu^2/4M^2)(\omega_L^2-\mu^4/4M^2)}$$

$$- \frac{M}{W(1+\mu/M)} \mathrm{Re}\,(p_{11} + 2p_{13} - p_{31} - 2p_{33}) + (D\text{-waves}) \qquad (12b)$$

Here $p_{2T,2J}$ are the <u>reduced</u> P-wave p.w.a given by

$$q^2\, \mathrm{Re}\, p_{2T,2J} = \frac{\sin(2\alpha_{2T,2J})}{2q}$$

where $\alpha_{2T,2J}$ are the P-wave phase shifts (T = 1/2, 3/2; J = 1/2, 3/2).

We might expect some simplifying features in the coefficients $C^{(\pm)}(\omega_L)$. The reason is that the S-wave amplitudes have almost no long range N-exchange interaction whereas the P-wave amplitudes at low energies are dominated by the long range N-exchange interaction (see § IV below). If now we evaluate eqs. (12) roughly by:

(i) Assuming $\sigma^{(\pm)}$ are dominated by N^*_{33}, i.e. putting
$\sigma_+ = 8\pi q \, \mathrm{Im}\, p_{33}$, $\sigma_- = \frac{1}{3}\sigma_+$.

(ii) Using the Chew-Low expressions*) for $\mathrm{Re}\, p_{2T,2J}$,
then we find that $C^{(\pm)}(\omega_L)$ should show litte dependence on ω_L up to, say, 100 MeV lab. pion energy. The terms which have rapid energy dependence have cancelled out in eqs. (12). A preliminary evaluation of $C^{(\pm)}(\omega_L)$ by Woolcock [2] confirms this result.

I.2 Fitting the S-wave Data

Isospin 3/2

Estimates [2][3] give $C^{(+)}(\mu) \simeq C^{(-)}(\mu) \simeq$ -0.09 (units $\hbar$ = c = μ = 1). Since $C^{(\pm)}(\omega_L)$ are expected to vary slowly with ω_L, $C^{(+)}(\omega_L) - C^{(-)}(\omega_L)$ should be small up to 50 MeV or higher. Eqs. (11) give

$$X = a_3 - \frac{1}{3}(a_1 - a_3)(\omega_L - 1) + \frac{1}{3} q_L^2 (C^{(+)}(\omega_L) - C^{(-)}(\omega_L)) \qquad (13)$$

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*) See ref. (2) p. 767-769 for the details

(3) V.K. Samaranayake and W.S. Woolcock, University College London Preprint (1969), submitted to Lund Conference (1969)

where

$$X = \frac{W}{M+\mu} \frac{\sin(2\alpha_3)}{2q} \qquad (13a)$$

Since the coefficient of the q_L^2 term is small the plot of X against ω_L should be almost a straight line in the energy range up to 50 MeV lab. energy. This has been verified [(4)], and it made possible a fairly accurate determination of the scattering length a_3 ($a_3 = -0.091 \pm 0.005$).

With improved experimental data eq. (13) should be a useful form for parametrizing α_3. Writing $q_L^2 = \omega_L^2 - 1$, we see that the slope of the plot $X(\omega_L)$ near threshold will be

$$\left.\frac{dX}{d\omega_L}\right|_\mu = -\frac{1}{3}(a_1 - a_3) + \frac{2}{3}\left(C^{(+)}(\mu) - C^{(-)}(\mu)\right) \qquad (13b)$$

This quantity is of interest in another relation (cf. § I.3 below).

The basic dynamical reason why X has a very smooth low energy form is that the T = 3/2 S-wave scattering is mainly caused by a strong short range repulsion. There is some long range attraction due to σ-exchange and some medium range repulsion due to ρ-exchange, but the short range repulsion dominates (cf. § IV.3 below).

Isospin 1/2

Eqs. (11) give

$$Z = a_1 + \frac{2}{3}(a_1 - a_3)(\omega_L - 1) + \frac{1}{3} q_L^2 \left(C^{(+)}(\omega_L) + 2C^{(-)}(\omega_L)\right) \qquad (14)$$

(4) J. Hamilton, Phys. Letters 20, 687 (1966)

where
$$Z = \frac{W}{M+\mu}\,\frac{\sin 2\alpha_1}{2q} \tag{14a}$$

We can write

$$Z = a_1 + (\omega_L - 1)\,\tfrac{2}{3}\left\{(a_1 - a_3) + C^{(+)}(\omega_L) + 2C^{(-)}(\omega_L)\right\} + \tfrac{1}{3}(\omega_L - 1)^2\left(C^{(+)}(\omega_L) + 2C^{(-)}(\omega_L)\right) \tag{14b}$$

Since $a_1 - a_3 \simeq 0.275$ and $C^{(+)}(\mu) + 2C^{(-)}(\mu) \simeq -0.27$, the coefficient of the linear term $(\omega_L - 1)$ is small whereas the coefficient of the quadratic term is not small (4). For this reason it is in practice not easy to determine a_1 accurately. Certainly a linear fit to Z or α_1, will lead to considerable errors. One should use a quadratic form in $(\omega_L - 1)$ for fitting Z at low energies.

I.3 The $C^{(\pm)}$ Relations

Now that accurate values of the cross-sections $\sigma_\pm$ are available down to low energies, we should bear in mind the possibility of using accurate P-wave phase shifts in order to evaluate $C^{(\pm)}(\omega_L)$ by eqs. (12) (ignoring the D-waves).

In particular for $\omega_L = \mu$ we have (2)

$$C^{(+)}(\mu) + \frac{1}{\zeta^2}(a_{11} + 2a_{13} + 2a_{31} + 4a_{33}) - \frac{3f^2}{M\zeta(1 - 1/4M^2)^2} = \frac{3}{2\pi^2\zeta}\int_1^\infty d\omega'\,\frac{\omega'}{q'}\,\frac{\sigma^{(+)}(\omega')}{\omega'^2 - 1} \tag{15a}$$

$$C^{(-)}(\mu) + \frac{1}{\zeta^2}(a_{11} + 2a_{13} - a_{31} - 2a_{33}) + \frac{6f^2}{\zeta(1 - 1/4M^2)^2} = \frac{3}{2\pi^2\zeta}\int_1^\infty \frac{d\omega'}{q'}\,\frac{\sigma^{(-)}(\omega')}{\omega'^2 - 1} \tag{15b}$$

where $\zeta = 1 + 1/M$ and ω', q' are lab. variables. Here $a_{2T,2J}$ are the P-wave scattering lengths.

With these relations we could connect the S-wave parametrization of § I.2 with the P-wave scattering lengths. Various other useful relations for $a_{2T,2J}$ are given by H and W (2).

I.4 Sum Rules and the Pomeranchuk Theorem

There is a famous sum rule (5)

$$\frac{1}{\mu}(1+\mu/M)\,\frac{2}{3}(a_1-a_3) = \frac{4f^2}{\mu^2}\frac{1}{1-\mu^2/4M^2} + \frac{1}{2\pi^2}\int_\mu^\infty \frac{d\omega'}{q'}\left(\sigma_-(\omega')-\sigma_+(\omega')\right) \qquad (16)$$

It is derived by assuming *) (i) the convergence of the integral, (ii) $\mathrm{Re} f_L^{(-)}(\omega_L)/\omega_L \to 0$ as $\omega_L \to \infty$, (iii) the condition $\sigma^{(-)}(\omega_L)\ln\omega_L \to 0$ as $\omega_L\to\infty$, and a smoothness condition on $\sigma^{(-)}(\omega_L)$ for $\omega_L\to\infty$. These enable us to write the unsubtracted D.R.

$$\mathrm{Re} f_L^{(-)}(\omega_L) = \frac{2f^2\omega_L}{\omega_L^2-\mu^4/4M^2} + \frac{\omega_L}{2\pi^2}P\int_\mu^\infty \frac{q'd\omega'}{\omega'^2-\omega_L^2}\sigma^{(-)}(\omega') \qquad (16a)$$

(here ω', q' are lab. variables). Now putting $\omega_L=\mu$ gives eq. (16).

If the Pomeranchuk theorem is untrue and $\sigma^{(-)}(\omega)\not\to 0$ as $\omega\to\infty$, eqs. (16) and (16a) are invalid. But it is easy to modify eq. (16a) so as to get a result. Using Fig. 1

(5) M.L. Goldberger, H. Miyazawa and R. Oehme, Phys. Rev. 99, 988 (1955)

*) See, for example, Course A, p. 166 - 169

we write

$$f_L^{(-)}(\omega) = \frac{1}{2\pi i}\int_{C_1} d\omega' \, \frac{f_L^{(-)}(\omega')}{\omega'-\omega}$$

$$= \frac{1}{2\pi i}\int_{(\Gamma_R + C_2 + C_3 + C_4)} d\omega' \, \frac{f_L^{(-)}(\omega')}{\omega'-\omega}$$

Letting ω tend to the real axis (from above) gives the new D.R.

$$\mathrm{Re}\, f_L^{(-)}(\omega_L) = \frac{2f^2\omega_L}{\omega_L^2 - \mu^4/4M^2} + \frac{\omega_L}{2\pi^2} P\int_\mu^R \frac{q' d\omega'}{\omega'^2-\omega_L^2}\, \sigma^{(-)}(\omega')$$

$$+ \frac{1}{2\pi i}\int_{\Gamma_R} d\omega' \, \frac{f_L^{(-)}(\omega')}{\omega'-\omega_L} \qquad (17a)$$

where R is some large fixed energy and $\mu \leq \omega_L < R$.

Now letting $\omega_L \to \mu$ gives the modified sum rule [(6)]

$$\frac{1}{\mu}(1+\mu/M)\,\frac{2}{3}(a_1 - a_3) = \frac{4f^2}{\mu^2}\,\frac{1}{1-\mu^2/4M^2}$$

$$+ \frac{1}{2\pi^2}\int_\mu^R \frac{d\omega'}{q'}\,(\sigma_-(\omega') - \sigma_+(\omega')) + \frac{1}{\pi\mu i}\int_{\Gamma_R} d\omega' \, \frac{f_L^{(-)}(\omega')}{\omega'-\mu} \qquad (17b)$$

Choosing R sufficiently large, we can use some asymptotic analytic form for $f_L^{(-)}(\omega)$ to give $f_L^{(-)}(\omega)$ on Γ_R . This asymptotic form can violate Pomeranchuk's theorem, if we wish.

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(6) W.S. Lam and T.N. Truong, Phys. Letters, 31B, 307 (1970)

Moreover if we now let R go to ∞, the divergence in the dispersion integral is cancelled by the integral over Γ_R, provided $f_L^{(-)}(\omega)$ obeys certain very general conditions as $|\omega| \to \infty$, (for example, we exclude an essential singularity in $f_L^{(-)}(\omega)$ as $|\omega| \to \infty$).

There is a rough way of seeing that the very high energy behaviour (defined as $\omega_L \geq$ 30 GeV) indicated by the Serpukhov data [7] will have little effect on the value of $(a_1 - a_3)$ deduced from eq. (17b). Choose R = 10 GeV. Then it is reasonable to expect that if $f_L^{(-)}(\omega)$ is fitted to what is known about this amplitude from 5 GeV to 30 GeV, the values of $f_L^{(-)}(\omega)$ on Γ_R will be only a little influenced by the particular form chosen for $f_L^{(-)}(\omega)$ in the very high energy region. This means that $(a_1 - a_3)$ is little altered.

Detailed calculations by Lam and Truong [6] have shown that this is indeed true. These authors also emphasize that conversely, we cannot use the sum rule in eq. (6) to obtain information on the behaviour of $\sigma^{(-)}(\omega)$ at very high energies.

I.5 The Coupling Constant f^2

In 1960 Spearman [8] showed by fitting forward $\pi^{\pm}p$ dispersion relations that the $\pi^{\pm}p$ coupling constant f^2 must lie in the range $0.075 < f^2 < 0.085$. Using the forward dispersion relation for $B_+(s, t=0)$ and effectively fitting the energy dependence of the Born term (nucleon pole) gave [2] $f^2 = 0.081 \pm 0.003$. In this method the dominant contribution

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(7) J.V. Allaby et al., Phys. Letters 30B, 500 (1969)
(8) T.D. Spearman, Nuovo Cimento 15, 147 (1960)

to the dispersion integral is from $0 < \omega_L - \mu \leq$ 300 MeV, and in effect one is getting f^2 from the N^*_{33} resonance. There have been several later analysis of the $f_\pm(\omega)$ forward relations by Samarnayake and Woolcock [(3),(9)], the latest giving $f^2 = 0.0815 \pm 0.002$. Höhler et al. [(10)] get a similar result. Better data on forward scattering and cross-sections are needed, as well as improved theoretical methods, if we are to achieve much higher accuracy.

The presence of Pomeranchuk violating terms in the amplitudes at high energies may cause changes in some (but not all) of the above calculations of f^2. However, Elvekjær,[(11)] using methods similar to those in § I.4 above, has found that the changes in the above values of f^2 due to taking account of the Serpukhov data[(7)], should be well within the errors quoted above. Of course it is desirable that any new techniques for calculating f^2 should suppress possible trouble from Pomeranchuk violating terms.

Analytic Continuation

The practical methods of analytic continuation developed by Cutkosky and Deo[(12)], and by Ciulli[(13)], should be used. Y.A. Chao and E. Pietarinen[(14)] have already used

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(9) V.K. Samaranayake and W.S. Woolcock, Phys. Rev. Letters 15, 936 (1965)

(10) G. Höhler et al., ZS für Phys. 229, 217 (1969)

(11) F. Elvekjaer (Nordita, unpublished (1970))

(12) R. Cutkosky and B.B. Deo, Phys. Rev. 174, 1859 (1968); Phys. Rev. Letters. 20, 1272 (1968)

(13) S. Ciulli, Nuovo Cimento, 61A, 787 (1969); 62A, 301 (1969)

(14) Y.A. Chao and E. Pietarinen, Phys. Rev. Letters 26, 1060 (1971)

such techniques to good effect in the more difficult case of KN scattering.

Let us speculate on how a modern determination of f^2 would be carried out *). Suppose we use the forward lab. amplitude $f_L^{(+)}(\omega)$ (here ω = lab. energy). By eq. (6a) $f_L^{(+)}(\omega)$ is an even function of ω, and we shall write it as $g(\omega^2)$. The singularities of $g(\omega^2)$ are the nucleon pole at $\omega^2 = \mu^4/4M^2$ (cf. eq. (9a)) and the cut $\mu^2 \leq \omega^2 \leq \infty$. By eq. (6a), $g(\omega^2)$ is regular on $-\infty \leq \omega^2 \leq 0$.

Suppose that we have fairly accurate measurements of $Re\, f_L^{(+)}(\omega)$ over the range $\mu \leq \omega \leq \omega_I$, and of $\sigma^{(+)}(\omega)$ over the (larger) range $\mu \leq \omega \leq \omega_{II}$. Now we use a conformal transformation $z(\omega^2)$ which tranforms the whole ω^2-plane into the interior of the ellipse in the z-plane, as shown in Fig. 2. This transformation is chosen so that the upper and lower sides of the line $\omega_{II}^2 < \omega^2 < \infty$ map on to the upper and lower halves of the ellipse. Also the line $-\infty \leq \omega^2 \leq \omega_{II}^2$ maps on to the major axis of the ellipse $(-z_\alpha \leq z \leq z_\alpha)$. It can be arranged that $\omega^2 = \mu^2$ and $\omega^2 = \omega_I^2$ transform to $z = -1$ and $z = +1$ respectively.

Now write g as a function of z and call it $g(z)$. Clearly $g(z)$ is regular inside the ellipse, except for the simple Born pole at $z = z_B$, and the cut $-1 \leq z \leq z_\alpha$. Thus the discrepancy function

$$\Delta(z) = g(z) - \frac{1}{\pi} \int_{-1}^{z_\alpha} \frac{Im\, g(z')\, dz'}{z' - z} \qquad (18)$$

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*) I am indebted to E. Pietarinen for a discussion on this topic

is regular inside the ellipse except for the Born pole. We should use the function

$$F(z) = (z+z_\alpha)(z-z_B)\,\Delta(z) \tag{18a}$$

so as to remove the pole at z_B ; also the factor $(z + z_\alpha)$ will suppress the effects of the singularity at z_α and therefore of any violation of Pomeranchuk's theorem.

The function $F(z)$ will have singularities on the ellipse, but it is regular inside the ellipse. Thus we can use a Legendre expansion

$$F(z) = \sum_{n=1}^{\infty} a_n\, P_n(z) \tag{19}$$

to give $F(z)$ at any point inside the ellipse. The next problem is to determine the expansion coefficients a_n.

We know $F(z)$ fairly accurately on $-1 \leq z \leq 1$ from the physical measurements, and we want to find $F(z_B)$, since that is proportional to f^2. The errors in our knowledge of $F(z)$ on $-1 \leq z \leq 1$, will however have the effect that we cannot determine sensible values of the higher coefficients a_n in the expansion in eq. (19). This is what is called a <u>stability problem</u>.

In order to get the optimum expansion[12][13] we must cut off the series in eq. (19) at $n = N$ where N depends inter alia on the size of the errors in the measured values of $F(z)$. (For an elementary account of this procedure see ref. [15]). Thus we use

$$F(z) = \sum_{n=1}^{N} a_n\, P_n(z) \tag{19a}$$

(15) J. Hamilton, "New Methods in the Analysis of π-N Scattering" in Springer Tracts in Modern Physics, Vol. 57 (1971)

This finite series gives the approximate continuation of $F(z)$ from $-1 \leq z \leq 1$. The smaller the errors in the data, the larger is N.

Using eq. (19a) we can find $F(z_B)$, and so f^2. Since z_B is close to -1, the continuation should work well. The error in f^2, as determined by eq. (19a), is easily estimated[(12)(13)].

II Coulomb Corrections

We shall only give a brief discussion of the principles and the main limitations of the methods at present used to obtain Coulomb, and related, corrections in low energy πN scattering.

II.1 Coulomb Scattering Notation*)

Consider the non-relativistic Coulomb scattering of two particles of charge $+e$. The radial wave equation for angular momentum ℓ is

$$\left\{ \frac{d^2}{dr^2} + q^2 - \frac{\ell(\ell+1)}{r^2} - \frac{2\gamma q}{r} \right\} f(r) = 0 \qquad (20)$$

where q is the momentum in the c.m.s. and

$$\gamma = \frac{1}{qa}$$

where a = $\hbar^2/me^2$ is a Bohr radius and m is the reduced mass. For $\pi^+ p$, a = 158 units ($\hbar = \mu = c = 1$) and is the Bohr

*) There are many clear accounts. See for example Chap. XI and Appendix I of ref. (16)

(16) A. Messiah, Quantum Mechanics, Vol. I, (North-Holland, Amsterdam (1961))

radius of the $\pi^- p$ mesic atom. In almost all cases of practical importance in πN scattering $\gamma \ll 1$.

The regular solution of eq. (20) is $F_\ell(\gamma, \rho)$ (where $\rho = qr$) and its asymptotic form is

$$F_\ell(\gamma,\rho) \underset{\rho \to \infty}{\sim} \sin\left(\rho - \gamma \ln(2\rho) + \sigma_\ell - \tfrac{1}{2}\ell\pi\right) \tag{20a}$$

where $\sigma_\ell = \arg \Gamma(\ell + 1 + i\gamma)$. We can also write

$$F_\ell(\gamma,\rho) = \frac{1}{2i}\left[e^{i\sigma_\ell} u_\ell^{(+)}(\gamma,\rho) - e^{-i\sigma_\ell} u_\ell^{(-)}(\gamma,\rho)\right] \tag{21}$$

where $u_\ell^{(\pm)}$ are solutions of eq. (20) which have the asymptotic forms

$$u_\ell^{(\pm)}(\gamma,\rho) \underset{\rho \to \infty}{\sim} \exp\left[\pm i\left(\rho - \gamma \ln(2\rho) - \tfrac{1}{2}\ell\pi\right)\right] \tag{21a}$$

The wave function $\psi_c(\underset{\sim}{x})$ corresponding to an incoming (distorted) plane wave can be written

$$\psi_c(\underset{\sim}{x}) = \frac{1}{2qr} \sum_{\ell=0}^{\infty} (2\ell+1)\, i^{(\ell+1)} \left[u_\ell^{(-)} - e^{2i\sigma_\ell} u_\ell^{(+)}\right] P_\ell(\cos\theta) \tag{22}$$

where θ is the scattering angle.

Suppose that in addition to the Coulomb interaction there is a hadronic interaction described by a potential of finite range (range $\lesssim 1F$). Suppose also that spin orbital effects are ignored for simplicity. In place of eq. (20) we have

$$\left(\frac{d^2}{dr^2} + q^2 - \frac{\ell(\ell+1)}{r^2} - \frac{2\gamma q}{r} - \frac{2m}{\hbar^2}V(r)\right) f_\ell(r) = 0 \tag{23}$$

The asymptotic form of the regular solution of this equation is

$$f_\ell(r) \underset{\rho \to \infty}{\sim} \sin(\rho - \gamma \ln(2\rho) + \sigma_\ell - \tfrac{1}{2}\ell\pi + \delta_\ell) \qquad (23a)$$

Here δ_ℓ is the phase produced by the hadronic potential in the presence of the Coulomb potential. If $V(r) \equiv 0$ then $\delta_\ell = 0$.

Analogous to eq. (22) we can write the asymptotic form of the wave function as

$$\psi_{Nc}(\underset{\sim}{x}) \underset{|\underset{\sim}{x}| \to \infty}{\sim} \frac{1}{2qr} \sum_{\ell=0}^{\infty} (2\ell+1) i^{(\ell+1)} [u_\ell^{(-)} - e^{2i\delta_\ell} e^{2i\sigma_\ell} u_\ell^{(+)}] \, P_\ell(\cos\theta) \qquad (24)$$

Hence
$$\psi_{Nc}(\underset{\sim}{x}) \underset{\rho \to \infty}{\sim}$$

$$\psi_c(\underset{\sim}{x}) - \frac{1}{2qr} \sum_{\ell=0}^{\infty} (2\ell+1) \, i^{(\ell+1)} e^{2i\sigma_\ell} (e^{2i\delta_\ell} - 1) \, u_\ell^{(+)}(\gamma,\rho) \, P_\ell(\cos\theta) \qquad (25)$$

Thus the differential cross-section for scattering is

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2 \qquad (26)$$

where

$$f(\theta) = f_c(\theta) + f'(\theta), \qquad (26a)$$

and $f_c(\theta)$ is the Coulomb amplitude:

$$f_c(\theta) = \frac{(-\gamma)}{2q\sin^2(\theta/2)} \exp[2i\sigma_0 - i\gamma \ln(\sin^2(\theta/2))] \qquad (26b)$$

while

$$f'(\theta) = \frac{1}{2iq} \sum_{\ell=0}^{\infty} (2\ell+1)\, e^{2i\sigma_\ell} (e^{2i\delta_\ell} - 1)\, P_\ell(\cos\theta) \tag{26c}$$

The hadronic scattering is given by $f'(\theta)$.

<u>Coulomb Interferance</u>

Eqs. (26) show that

$$\frac{d\sigma}{d\Omega} = |f_c(\theta)|^2 + 2\,\mathrm{Re}\,\{f_c^*(\theta) f'(\theta)\} + |f'(\theta)|^2 \tag{27}$$

The term $|f_c(\theta)|^2$ gives the pure Coulomb scattering and $|f'(\theta)|^2$ gives the pure hadronic scattering. The other term is the Coulomb interference, and since $f_c(\theta)$ is known, this term can be used to give extra information on $f'(\theta)$ over what is obtained from the $|f'(\theta)|^2$ term. On account of the very different ranges of the interactions, $|f_c(\theta)|^2$ dominates near the forward direction whereas $|f'(\theta)|^2$ dominates at the larger angles. (Fig 3)

<u>II.2 Coulomb Corrections</u>

Now we look at the main problem. The phase δ_ℓ defined by eqs. (23), (23a) is the phase which is measured (eqs. (26)). However δ_ℓ is not identical with the purely hadronic phase $\delta_{\ell N}$ which would be defined by the regular solution $f_{\ell N}(r)$ of

$$\left(\frac{d^2}{dr^2} + q^2 - \frac{\ell(\ell+1)}{r^2} - \frac{2m}{\hbar^2} V(r) \right) f_{\ell N}(r) = 0 \tag{28}$$

and the asymptotic condition

$$f_{\ell N}(r) \underset{\rho\to\infty}{\sim} \sin(\rho - \tfrac{1}{2}\ell\pi + \delta_{\ell N}) \tag{28a}$$

In order to investigate theoretical questions we need to know the hadronic phase $\delta_{\ell N}$, therefore we must find the correction $(\delta_\ell - \delta_{\ell N})$. For example, the isospin T = 3/2 phases $\delta_\ell^{(3/2)}$ measured in $\pi^+ p \to \pi^+ p$ and in $\pi^- p \to \pi^- p$ will not be identical, but (assuming charge independence) there is only one value $\delta_{\ell N}^{(3/2)}$. Since charge independence is a good symmetry in πN physics, we shall use it in conjunction with the hadronic phases $\delta_{\ell N}^{(T)}$.

We shall not repeat the history of the Coulomb corrections - there are good reviews available(17). As an illustration, we shall examine the S-wave $\pi^+ p$ amplitude. From eqs. (23) and (28) we have (for $\ell = 0$),

$$\frac{d}{dr}\left\{ f_0(r)\,\frac{df_{0N}(r)}{dr} - f_{0N}(r)\,\frac{df_0(r)}{dr} \right\} = -\frac{2\gamma q}{r}\, f_0(r)\, f_{0N}(r) \qquad (29)$$

Suppose that the hadronic interaction is negligible for $r \geq \bar{r}$, and also assume that $q^2 a \bar{r} \gg 1$. Since $\bar{r} \simeq 1F$ this last condition is very reasonable. (Remember that T_π = 5 MeV lab is $q \simeq 0.25$). Then the asymptotic forms in eqs. (23a) and (28a) are valid for $r \geq \bar{r}$. On integrating eq. (29) we have

$$q \sin(-\gamma \ln(2\rho) + \sigma_0 + \delta_0 - \delta_{0N}) + \frac{\gamma}{r}\cos(\rho - \gamma \ln(2\rho) + \sigma_0 + \delta_0)$$

$$= -2\gamma q \int_0^r dr'\, \frac{f_0(r') f_{0N}(r')}{r'} \qquad (29a)$$

(17) G.L. Oades, article in Springer Tracts in Modern Physics, Vol. 55 (1971).

This holds for $r \geq \bar{r}$. We can split the integral into the intervals $(0, \bar{r})$, $(\bar{r}, r)$ where r is large, and in the last interval use eqs. (23a), (28a). Now rewriting and letting $r \to \infty$ gives

$$\sin(-\gamma \ln(2q\bar{r}) + \sigma_0 + \delta_0 - \delta_{0N}) = -2\gamma \int_0^{\bar{r}} dr' \frac{f_0(r') f_{0N}(r')}{r'}$$
$$+ \gamma \int_{\bar{r}}^{\infty} \frac{dr'}{r'} \cos(2\rho' - \gamma \ln(2\rho') + \sigma_0 + \delta_0 + \delta_{0N}) \qquad (30)$$

This is the basic Coulomb correction equation, using the non-relativistic method. It gives $(\delta_0 - \delta_{0N})$ independent of the "separation radius" $\bar{r}$. The corrections for P-waves etc., and for $\pi^- p \to \pi^- p$, $\pi^- p \to \pi^0 n$, can be found by developments of the same method[17].

For $\gamma \ll 1$, $\sigma_0 = -\gamma C$ where C = 0.557 is Euler's constant. Hence by eq. (30)

$$\delta_0 - \delta_{0N} = O(\gamma)$$

If we write

$$\eta = -\gamma \ln(2q\bar{r}) + \sigma_0 + \delta_0 - \delta_{0N},$$

a good approximation to eq. (30) is

$$\eta = -2\gamma \int_0^{\bar{r}} dr' \frac{f_0(r') f_{0N}(r')}{r'} + \gamma \int_{\bar{r}}^{\infty} \frac{dr'}{r'} \cos(2\rho' + \eta + 2\delta_{0N}) \qquad (30a)$$

This equation gives

$$\frac{\partial}{\partial \bar{r}} (\delta_0 - \delta_{0N}) = \frac{1}{\bar{r}} O(\gamma^3)$$

Working to $O(\gamma)$, eq. (30a) yields

$$\delta_0 - \delta_{0N} = -2\gamma \int_0^{\bar{r}} dr' \frac{f_0(r') f_{0N}(r')}{r'}$$

$$+ \gamma \left\{ \ln(2q\bar{r}) + C - \cos(2\delta_{0N}) Ci(2q\bar{r}) + \sin(2\delta_{0N}) si(2q\bar{r}) \right\} \qquad (31)$$

The first term on the right of eq. (31) is the inner Coulomb correction(18). The second term is van Hove's (outer)(19) Coulomb correction. In the first term the factor $1/r'$ can be replaced by the actual potential - which is finite as $r' \to 0$

II.3 Comments

The method we have outlined has obvious defects:

a) It is non-relativistic, and van Hove(19) suggests that this can be remedied in part on replacing γ by $\gamma(1-\beta^2)^{1/2}$ where $c\beta$ is the relative velocity of the pion and nucleon in the c.m.s. Other corrections, such as magnetic moment terms, have been included by various authors*).

b) It depends on using a model potential for the hadronic interaction. Of the various parts of the low energy π-N interaction (cf. § IV below), the ρ -exchange, N-exchange, N*-exchange, terms cannot be represented by a potential, except in a very rough fashion. Moreover, the Schrödinger

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*) See ref. (17) for references

(18) J. Hamilton and W.S. Wolcock, Phys. Rev. 118, 291 (1960)

(19) L. van Hove, Phys. Rev. 88, 1358 (1952)

equation is not appropriate except at very low energies, and the Klein-Gordon equation encounters serious difficulties.

It is obvious that a much improved theoretical treatment is needed. Some promising attempts are mentioned in ref. (17).

Using eq. (31) with model potentials, a number of Coulomb corrections have been estimated. Ref. (20) contains some recent results. I am afraid that we have almost reached the situation where the theoretical uncertainty in the Coulomb corrections sometimes exceeds the accuracy of the best experiments.

Other Corrections

In $\pi^- p$ scattering it is necessary to allow for the channel $\pi^- p \to \gamma n$, and also to remember that the pions and the nucleons have different masses in $\pi^- p \to \pi^0 n$. These features affect the analysis of $\pi^- p$ scattering.

It is usual to include the kinematic corrections due to the mass differences when analysing the data. In an interesting paper Oades and Rasche[(21)] have started an examination of the dynamical effects of the mass differences. These corrections are not negligible, and further investigations of the mass difference effects are required.

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(20) A.A. Carter et al., Nuclear Physics B26, 445 (1971)

(21) G.C. Oades and G. Rasche, Zürich Preprint (1970)

III The Use of Fixed t Dispersion Relations

(Accurate Low Energy Phase Shifts)

The first use of fixed t D.R. to improve our knowledge of πN phases was by C.G.L.N.[22]. However their method was severely limited by convergence problems[23]. In another form, a similar difficulty occurs in the general fixed t D.R.; the Legendre series over the $\pi N \to \pi N$ partial wave amplitudes can only be used*) to express the absorptive parts which appear in the fixed t D.R. for $-26\mu^2 < t < 4\mu^2$. Recently Steiner[24] has emphasized the same difficulty in relation to F.E.S.R. . Within these limits, fixed t D.R. have been much used+) to determine the small phase shifts from data on the large phase shifts.

Here we shall give a brief account of an ingenious method which has been developed and applied by Henry Nielsen[26].

III.1 Nielsen's Method

Henry Nielsen determines very accurate phase shifts in the low energy region, which is here defined as the range up to 270 MeV (lab. pion K.E.). He is interested in obtaining good values of the non-resonant phases, and in particular the

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*) See also ref. (23), +) See for example ref. (25)

(22) G. Chew, M. Goldberger, F.E. Low and Y. Nambu, Phys. Rev. 106, 1337 (1957)

(23) D.H. Lyth, Rev. Mod. Phys. 37, 709 (1965)

(24) F. Steiner, Phys. Letters 32B, 294 (1970)

(25) J. Baacke, G. Höhler and F. Steiner, ZS für Physik, 221 134 (1969)

(26) Henry Nielsen, Univ. of Aarhus preprint, 1971.

D-waves. In this low energy region there is some doubt about the accuracy of several of the partial waves in the CERN phases[27].

The input data is:

i) <u>up to 270 MeV ($60\mu^2 \leq s \leq 85\mu^2$)</u>

a) the π^+p total cross-section which is now very accurately determined[20].

b) the T = 3/2 S-wave phase α_3. This phase is smoothly varying and is easy to determine accurately. The CERN values[27] will be used, and allowance will be made for the effect of large inaccuracies in α_3.

c) the T = 3/2, P wave phase α_{31}. This is quite well enough known for the present purpose.

ii) <u>From 270 MeV to 2 GeV ($85\mu^2 \leq s \leq 250\mu^2$)</u>

The CERN phases[27] and inelasticities are used as input in this intermediate energy region. These values are accurate enough for the present purpose in this region.

iii) <u>Above 2 GeV</u>

An analytic continuation method is used to find the contribution of this high energy region to the D.R. in a fashion which is independent of models, subtraction constants, and the like.

The first step is to write the π^+p total cross-section in the low energy region ($60 \leq s \leq 85$) in the form

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(27) "CERN Experimental Solution" in UCRL-20030 (πN Scattering Data), Berkeley Report (1970)

$$\sigma^{(3/2)} = \frac{4\pi}{q^2}\left\{ \sin^2\alpha_3 + \sin^2\alpha_{31} + 2\sin^2\alpha_{33} + \cdots \right\} \tag{32}$$

Substituting the values of $\sigma^{(3/2)}$, α_3 and α_{31} gives α_{33} in the low energy region. The P_{31} partial cross-section is 1 - 2% of the P_{33} partial cross-section and only 10 - 20% of the S_{31} cross-section. Thus the P_{31} input is of adequate accuracy. If the S_{31} contribution to eq. (32) is increased or decreased by 20% the resulting value of α_{33} does not change by more than 0.°8. Such an increase or decrease in α_3 is far greater than the quoted errors on α_3 [27]. The D-wave contribution to eq. (32) is negligible.

Having α_3 and α_{33} in the low energy region, Henry Nielsen shows that the remaining partial waves in that energy region are uniquely determined by the fixed t D.R. (In fact this is also true for α_{31}). He makes the very reasonable assumption that there is negligible inelasticity in this region.

III.2 The High Energy Terms

The amplitudes B(s,t) and

$$A'(s,t) = A(s,t) + \frac{M(s-u)}{4M^2-t} B(s,t) \tag{33}$$

are used. The crossing symmetric variable is*)

$$\nu = \frac{1}{2M}(s-u) = \frac{1}{M}\left(s+\tfrac{1}{2}t-M^2-\mu^2\right)$$

$A'^{(+)}(\nu,t)$, $B^{(-)}(\nu,t)$ are even functions of ν and $A'^{(-)}(\nu,t)$, $B^{(+)}(\nu,t)$ are odd functions of ν .

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*) Beware of the factor 2 difference from some definitions of ν .

Neglecting subtractions, the fixed t D.R. are

$$Re\, A'^{(\pm)}(\nu,t) = K\frac{G^2}{M}\nu\left\{\frac{1}{\nu_0-\nu}\mp\frac{1}{\nu_0+\nu}\right\} + \frac{P}{\pi}\int_{\nu_{TH}}^{\infty} d\nu'\, Im\, A'^{(\pm)}(\nu',t)\left(\frac{1}{\nu'-\nu}\pm\frac{1}{\nu'+\nu}\right) \tag{34a}$$

$$Re\, B^{(\pm)}(\nu,t) = \frac{G^2}{M}\left\{\frac{1}{\nu_0-\nu}\mp\frac{1}{\nu_0+\nu}\right\} + \frac{P}{\pi}\int_{\nu_{TH}}^{\infty} d\nu'\, Im\, B^{(\pm)}(\nu',t)\left(\frac{1}{\nu'-\nu}\mp\frac{1}{\nu'+\nu}\right) \tag{34b}$$

with

$$\left.\begin{aligned} \nu_0 &= \frac{1}{M}(-\mu^2+\tfrac{1}{2}t) \\ \nu_{TH} &= 2\mu + t/2M \\ K &= \frac{2M^2}{4M^2-t} \end{aligned}\right\} \tag{35}$$

Let ν_1 correspond to $s = 85\mu^2$ and ν_P to $s = 250\mu^2$. For values of t in the range $-26\mu^2 \leq t \leq 0$, we have $\nu_P \simeq 30$, $\nu_1 \simeq 4 - 6$ and $0 < \nu_{TH} \leq 2$.

We shall use $F(\nu,t)$ to denote any of $A'^{(\pm)}$, $B^{(\pm)}$. We shall write

$$Re\, F(\nu,t) = F_B(\nu,t) + Re\, F_{L.E.}(\nu,t) + Re\, F_{I.E}(\nu,t) + Re\, F_{H.E.}(\nu,t) \tag{36}$$

where $F_B(\nu,t)$ is the nucleon pole term, and

$$Re\, F_{LE}(\nu,t) = \frac{P}{\pi}\int_{\nu_{TH}}^{\nu_1} Im\, F(\nu',t)\left[\frac{1}{\nu'-\nu}\pm\frac{1}{\nu'+\nu}\right]d\nu' \tag{37a}$$

$$Re\, F_{I.E}(\nu,t) = \frac{P}{\pi}\int_{\nu_1}^{\nu_P} Im\, F(\nu',t)\left[\frac{1}{\nu'-\nu}\pm\frac{1}{\nu'+\nu}\right]d\nu' \tag{37b}$$

We can if we wish regard $Re\, F_{HE}(\nu, t)$ as the contribution to $F(\nu, t)$ coming from a large circle $|\nu| = \nu_p$.

We should remember that in order to determine partial waves from $Re\, F(\nu, t)$ in the range u/p to $s = 85\mu^2$, we need to know $Re\, F(\nu, t)$ for $\nu_{TH} < \nu < \nu_1$, and $0 \geq t \gtrsim -16\mu^2$. We are therefore interested in the above equations for this range of t. For $\nu_1 < \nu < \nu_p$, $F(\nu, t)$ lies in the physical $\pi N \to \pi N$ region when $0 \geq t \gtrsim -16\mu^2$. Therefore $Re\, F(\nu, t)$ can be determined fairly accurately from the CERN phase shifts(27) in this region. Now eq. (36) is used to find $Re\, F_{H.E.}(\nu, t)$ for $\nu_1 < \nu < \nu_p$, $0 \geq t \gtrsim -16\mu^2$ in the following way.

The term $F_B(\nu, t)$ is readily evaluated, using $f^2 = 0.081$. Eqs. (37a), (37b) give $Re\, F_{LE}(\nu, t)$, $Re\, F_{I.E}(\nu, t)$ for $\nu_1 < \nu < \nu_p$, $0 \geq t \gtrsim -16\mu^2$. The values of $Im\, F(\nu', t)$ needed in eq. (37b) are given by a convergent Legendre series in terms of the CERN phase shifts(27). In evaluating eq. (37a) only*) $\sin^2 \alpha_3$ and $\sin^2 \alpha_{33}$ are used to give $Im\, F(\nu', t)$ $(\nu_{TH} < \nu' < \nu_1)$.

The functions $F_{H.E}(\nu, t)$ are analytic in the ν-plane with cuts $-\infty \leq \nu \leq -\nu_p$, $\nu_p \leq \nu \leq \infty$. Knowing $F_{H.E}(\nu, t)$ on $\nu_1 < \nu < \nu_p$ (and $-\nu_p < \nu < -\nu_1$), standard analytic continuation methods (cf. the refs. in § I.5) show how to find $F_{H.E}(\nu, t)$ on $\nu_{TH} < \nu < \nu_1$, $(0 \geq t \gtrsim -16\mu^2)$.

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*) There is a small error here due to not using all the phases, but this disappears on iteration.

The results of the analytic continuation method turn out to be close to what we would get by the simple device of fitting $A'^{(+)}_{H.E.}$ to the form $a + b\nu^2$, $A^{(-)}_{H.E}$ and $B^{(+)}_{H.E}$ to the forms $c\nu$, and $B^{(-)}_{H.E.}$ to a constant. As an example the results for $t = -10\mu^2$ are shown in Fig. 4.

The continuation to (ν_{TH}, ν_1) should be very accurate since this interval is far from the cuts of $F_{H.E}(\nu,t)$ and is small compared with the interval (ν_1, ν_p) where $F_{H.E}(\nu,t)$ is known.

It should be emphasized that this elegant method avoids many problems normally associated with the high energy region.

III.3 The Results

Now $Re\, F(\nu,t)$ can be evaluated in the low energy region $(\nu_{TH} < \nu < \nu_1,\ 0 \geq t \gtrsim -16\mu^2)$ using eqs. (36), (37a), (37b). In the first run the CERN values[27] of the phases are used as input in $\nu_{TH} < \nu < \nu_1$, except α_3 and α_{33} which are kept fixed (as described in § III.1). The S_{11} amplitude is deduced from S_{31} and $f_0^{(-)}(s)$; this is because $f_0^{(+)}(s)$ is poorly determined by the method on account of large cancellations among the fixed t D.R.'s in this case.

In this way $Re\, f_{\ell\pm}(s)$ are found in the low energy region. These values are then unitarized by going to the nearest point on the unitary circle. The distance to the unitary circle is always small. Now we have the phases (step 1).

Next, these step 1 phases (plus α_3 and α_{33}) are used as input and the procedure is repeated. This gives the step

2 phases, and further iteration does not alter them. Also it was shown that the same final results can be reached by putting all phases, except α_3 and α_{33} equal to zero in the initial run.

The input α_{33} values (from eq. (32)) are shown in Fig. 5. The resulting D_{15} and D_{35} phases are shown in Fig. 6, and S_{11} in Fig. 7. Notice that S_{11} is still poorly determined.*) The resulting P_{13} phase is almost identical with that obtained in the experiments of Bugg et al.(28) In a number of cases there are noticable deviations from the CERN values(27).

Henry Nielsen has pointed out that the D-wave results are particularly useful for precise investigations of the σ-exchange and ρ-exchange interactions in low energy πN scattering (cf. § V.1 below).

IV Dynamics of Low Energy πN Scattering

We shall give a brief description of the exchange forces which are responsible for low energy πN scattering. The nature of the interactions has been elucidated by analysis of the experimental data, and it has in various cases been confirmed by the predictions of low and medium energy partial wave amplitudes (p.w.a.) which are then made possible. For a more thorough survey see, for example, ref. (29).

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*) Of course this is mainly directly due to the large error assumed for S_{31} and the fact that S_{11} is given by $f_0^{(-)}(s)$ and S_{31}

(28) D.V. Bugg et al., Nuclear Phys. B26, 588 (1971)

(29) J. Hamilton, "Pion-Nucleon Interactions" in High Energy Physics, Vol. I (ed. E. Burhop) (Academic Press, N.Y. 1967)

It is hoped that more precise experimental results will provide further checks and refinements of this description. In particular, more work is required on the important σ-exchange process, and a better understanding of the P_{11} (Roper) resonance is desirable.

IV.1 Singularities of P.W.A.

By studying p.w.a. we can obtain a good description of the low energy πN dynamics because the left hand cuts (l.h.c.) of the p.w.a. give analogues of the potential description which is of such use in, for example, nuclear physics. It should however be emphasized at the outset that one cannot actually give potentials which represent the interactions - the πN problem is essentially relativistic. The left hand cut terms describe the dynamics.

It is worth remembering that the πN p.w.a. are essentially elastic up to 400 MeV pion lab. K.E., except for P_{11}. In P_{11} the channel $\pi N \rightarrow \sigma N$ is possible for σ in an S-state relative to N, so inelasticity appears from 300 MeV onwards, and it grows rapidly with energy. The only other candidates for inelasticity below 550 MeV are D_{13} and D_{33}. Here $\pi N \rightarrow \pi N^*_{33}$ is possible with π in an S-state relative to N^*_{33}. These amplitudes do indeed show some inelasticity in the region 400 - 500 MeV.

The singularities of a $\pi N \rightarrow \pi N$ p.w.a. $f_{\ell\pm}(s)$ are shown in Fig. 8. The main features of the l.h.c. are:

(i) the cut $(M - \mu^2/M)^2 \leq s \leq M^2 + 2\mu^2$ gives the long range part of the nucleon exchange (N-exchange) term.

(ii) the cut $0 < s < (M-\mu)^2$ gives the exchange of excited baryons, and in the cases we shall consider, it is mainly N^*_{33}-exchange (Δ -exchange) which matters

(iii) the circle $|s| = M^2 - \mu^2$ gives the t-channel exchanges. These are σ -exchange and ρ -exchange. (cf. Fig. 9)

We should make a few more technical comments. The cuts which are near the physical threshold $s_0 = (M+\mu)^2$ give the longer range parts of the interaction (range $\simeq 1F$). The cuts, or parts of cuts, further off from s_0 give medium and short range parts of the interaction. By short range we will mean $\lesssim$ 0.2 F.

On the circle, we can only calculate the effect of σ -exchange or ρ -exchange for the arc $|\arg s| \lesssim 66^\circ$. This is because the Legendre series expansions used to relate the t-channel phenomena to the s-channel l.h.c. are not valid beyond this arc. It is useful to lump together the contributions from the remainder of the circle, $|\arg s| > 66^\circ$, and from the cut $-\infty < s < 0$. We call them the short range interaction.

As to the t-channel contributions, the T = 0 effects (e.g. σ -exchange) contribute only to the p.w.a $f^{(+)}_{\ell\pm}(s)$ for $\pi N \to \pi N$, and the T = 1 effects (e.g. ρ -exchange) contribute only to $f^{(-)}_{\ell\pm}(s)$.

IV.2 Reduced P.W.A.

Because we do not know much about the short range interactions it is very useful to suppress their effects as much as possible. This can be done for orbital angular momentum $\ell \geq 1$, by making use of the centrifugal barrier. In place of $f_{\ell\pm}(s)$ we write the partial wave D.R. for

$$F_{\ell\pm}(s) = f_{\ell\pm}(s)/q^2 \tag{38}$$

By requiring $F_{\ell\pm}(s_0)$ to be finite we have built in the centrifugal effect. Now $|q^2|$ is large on all cuts which are far from s_0, so the l.h.c. contributions Im $F_{\ell\pm}(s)$ are altered relative to the Im $f_{\ell\pm}(s)$ in an interesting way. The ratio of the contributions from the short range cuts to those from the longer range cuts is much smaller in the case of $F_{\ell\pm}(s)$ than when using $f_{\ell\pm}(s)$. The larger is ℓ, the more marked is this effect.

This procedure has made it possible to predict the non-resonant P, D and F waves in the low energy region(29). From information derived from an analysis*) of the experimental S-wave π-N data, the σ-exchange and ρ-exchange contributions to the l.h.c. of $F_{\ell\pm}(s)$ were determined. The N-exchange and Δ-exchange terms are also readily found. Now, relying on the fact that the shorter range contributions are well suppressed, the non-resonant amplitudes can be estimated.

Furthermore, a unitary sum rule can be used to give a fair estimate of the small short range interaction which was ignored in the first approximation, and an iteration method can give an estimate of the physical integral (rescattering)

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*) See § IV.3 and V.I below

term. In this way[29] the non-resonant amplitudes can be fairly accurately predicted up to 500 MeV (or higher), and the results agree well with the experimental phases.

In order to give some idea of the behaviour of the various l.h.c. interaction terms, we show in Fig. 10. the l.h.c. contributions to $F_{\ell\pm}(s)$ for the amplitudes P_{13} and D_{35}.

IV.3 Special Cases

We shall briefly discuss some special cases which for one reason or another fall outside the scheme we have just discussed. They are the S-waves and the P_{11}-amplitude.

S-Waves

The low energy behaviour of S_{31} and S_{11} are mainly determined by i) σ-exchange and ρ-exchange, ii) a very strong short range repulsion. There is a negligible amount of long range N-exchange and not much Δ-exchange (or other excited baryon exchange).

The strong short range repulsion (SR) has a range $\simeq$ 0.2 F, and it occurs in the amplitude $f_0^{(+)}(s)$ but not in $f_0^{(-)}(s)$. In terms of πN isospin, we have

$$\left.\begin{aligned} f_0^{(\frac{1}{2})}(s) &= f_0^{(+)}(s) + 2f_0^{(-)}(s) \\ f_0^{(\frac{3}{2})}(s) &= f_0^{(+)}(s) - f_0^{(-)}(s) \end{aligned}\right\} \tag{39}$$

The σ-exchange gives a strong long range attraction in $f_0^{(+)}(s)$, and ρ-exchange gives a medium range effect in

$f_0^{(-)}(s)$ so that $f_0^{(-)}(s)$ is positive in the physical region.

Thus we can say schematically that the forces are

T = 1/2: (SR repulsion) + (σ attraction) +2(ρ attraction)
T = 3/2: (SR repulsion) + (σ attraction) + (ρ repulsion)

The method of deducing these various interactions from the data on the S-wave phases makes use of the fact that the ranges of the interactions differ, and it especially depends on using crossing symmetry. Crossing symmetry can give us the values of $f_0^{(\pm)}(s)$ on the cut $0 < s < (M-\mu)^2$ (cf. Fig. 8). Since this is inside the circle, and the physical cut is outside, we can more or less isolate*) the circle contribution to $f_0^{(\pm)}(s)$ and so get the σ and ρ-exchange interactions(29). Now-a-days this is done by using modern analytic continuation methods(30). More accurate phase shifts would be very valuable in improving this method still further.

The Amplitude P_{11}

In this case we would have to consider (for low energies) the channels

$$\pi N \rightarrow \pi N$$
$$\pi N \rightarrow \sigma N$$
$$\sigma N \rightarrow \sigma N$$

Multi-channel calculations are not easy, but the worst aspect

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*) See § V.I below for more details, and a similar game with D-waves

(30) H. Nielsen, J. Lyng Petersen and E. Pietarinen, Nuclear Phys. B22, 525 (1970)

is that σ is such a wide "resonance". It is probably fair to say that there has not been a really good dynamical description of this amplitude, although on the other hand none of the observed features are difficult to understand in qualitative terms.

V Information on σ

There are at present three main methods for obtaining information on low-energy pion-pion scattering in the T = 0 J = 0 state. They depend on using: i) πN p.w.a., ii) πN backward scattering, iii) the process $\pi N \to \pi\pi N$. Each method involves analytic continuation. Experiments of high accuracy by low energy machines can contribute to the improvements of all three methods. Here we briefly survey these methods.

V.1 The Use of $\pi N \to \pi N$ p.w.a.*)

The singularities of a πN p.w.a. $f_{\ell\pm}(s)$, or $F_{\ell\pm}(s)$ are shown in Fig. 8. From the scattering data we can find $f_{\ell\pm}(s)$ on the physical cut $s_0 \leq s \leq \infty$, and with the aid of crossing symmetry we can also find $f_{\ell\pm}(s)$ on the cut $0 < s \leq (M-\mu)^2$.

Now consider the discrepancy function

$$\Delta^{(+)}(s) = f_{\ell\pm}^{(+)}(s) - \frac{1}{\pi}\int_{s_0}^{\infty} ds' \frac{\mathrm{Im}\, f_{\ell\pm}^{(+)}(s')}{s'-s} - \frac{1}{\pi}\int_{0}^{(M-\mu)^2} ds' \frac{\mathrm{Im}\, f_{\ell\pm}^{(+)}(s')}{s'-s} - \frac{1}{\pi}\int_{(M-\mu^2/M)^2}^{M^2+2\mu^2} ds' \frac{\mathrm{Im}\, f_{\ell\pm}^{(+)}(s')}{s'-s} \qquad (40)$$

The singularities of $\Delta^{(+)}(s)$ are the cuts $|s| = M^2-\mu^2$ and

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*) See for example ref. (29) and ref. (30)

the line $-\infty \leq s \leq 0$. It is regular elsewhere.

By using the experimental data we can determine $\Delta^{(+)}(s)$ accurately on the low energy and medium energy part of the line $s_0 \leq s \leq \infty$, and on a large part of the line $0 < s \leq (M-\mu)^2$. Analytic continuation of $\Delta^{(+)}(s)$ from these two lines towards the circle $|s| = M^2-\mu^2$ gives the discontinuity in $\Delta^{(+)}(s)$ across the circle[30].

The discontinuity in $\Delta^{(+)}(s)$ across the circle is closely related to the absorptive part of the amplitude for the t-channel process

$$\pi\pi \to N\bar{N} \tag{41}$$

in the isospin state T = 0 . The analytic continuation will work best for the "front" of the circle (say, $|\arg s| \lesssim 66^\circ$). On that arc the discontinuity in $\Delta^{(+)}(s)$ is predominatly due to the absorptive part $\mathrm{Im}\, f_+^0(t)$ of the helicity amplitude $f_+^0(t)$ for the T = 0, J = 0^+ channel of the process in eq. (41). On the arc in question the range is $4\mu^2 \leq t \lesssim 50\mu^2$, and $\mathrm{Im}\, f_+^0(t)$ has been determined[30] over this range, starting from the S-wave πN amplitude $f_0^{(+)}(s)$.

It is then possible to find $\mathrm{Re}\, f_+^0(t)$ over the same range by solving the partial wave D.R. for $f_+^0(t)$. The most recent way[30] of doing this employs an analytic continuation method to avoid possible difficulties from unknown far away contributions to the D.R. for $f_+^0(t)$. (this is the same sort of dèvice as was used in § III.2 above).

Finally, extended unitarity[31] gives

$$f_{+}^{0}(t) = |f_{+}^{0}(t)|\, e^{i\delta_{0}^{0}(t)} \tag{42}$$

where $4\mu^2 \leq t \leq 16\mu^2$ and $\delta_0^0(t)$ is the T = 0 , J = 0 , $\pi\pi \to \pi\pi$ phase. Since there is little evidence of the process $2\pi \to 4\pi$ up to much higher energies, eq. (42) is assumed to hold for our range $4\mu^2 \leq t \lesssim 50\mu^2$. Now it is possible to deduce $\delta_0^0(t)$ for $4\mu^2 \leq t \lesssim 50\mu^2$.

This whole procedure has been carried out by Henry Nielsen, J.L. Petersen and E. Pietarinen[30] starting from S-wave πN-data. The resulting phase $\delta_0^0(t)$ does not pass through $\pi/2$, but it is consistent with a very broad absorptive part in the T = 0 , J = 0 , $\pi\pi \to \pi\pi$ scattering amplitude. This agrees fairly well with what is usually called the σ-meson ($m_\sigma \simeq$ 700 MeV, $\Gamma_\sigma \gg$ 100 MeV).

We shall not discuss the estimates of accuracy of the analytic continuations etc. (see ref. (30) for details, and ref. (15) for an elementary account of this work). It should however be emphasized that much more accurate low energy πN phases*) would be very welcome. They would make it possible to repeat the analysis with increased precision.

<u>D-Wave Discrepancies</u>

An interesting application of the low energy πN D-wave phases resulting from the fixed t D.R. described in § III

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*) Especially α_1

(31) F. Mandelstam, Phys. Rev. Letters 4, 84 (1960)

above has been made by Henry Nielsen[26]. He uses them to derive discrepancy functions $\Delta^{(+)}_{2\pm}(s)$; these are defined as in eq. (40), but using the reduced D-wave amplitudes $F^{(+)}_{2\pm}(s)$. The values of $\Delta^{(+)}_{2-}(s)$ are shown in Fig. 11. The dots are the values deduced form the CERN phases[27]. The crosses are the values deduced from Nielsen's new phases (they are only shown where they differ from the dots).

Because of the factor $(q)^{-4}$ in the definition of $F^{(+)}_{2\pm}(s)$, the importance of the regions near $(M\pm\mu)^2$ on $(M+\mu)^2 \leq s \leq \infty$ and $0 \leq s \leq (M-\mu)^2$ is enhanced. Also the portion of the cut $|s| = M^2-\mu^2$ near $s = M^2-\mu^2$ has enhanced importance (compared with the S-wave case discussed above). Therefore this D-wave discrepancy should give a greater possibility of distinguishing between the various T = 0, J = 0, $\pi\pi \to \pi\pi$ phases which have been proposed by many authors. For example, the solid line in Fig. 11 shows the value of $\Delta^{(+)}_{2-}(s)$ calculated from the values of $\mathrm{Im}\, f^0_+(t)$ given in ref. (30). [This solid line should still be adjusted to include a small short range term].

<u>V.2 Backward $\pi N \to \pi N$</u>[30]

Atkinson[32] showed that the backward $\pi N \to \pi N$ amplitude and the backward $\pi\pi \to N\bar{N}$ amplitude are essentially the same analytic function. Putting $\cos\theta = -1$ in eq. (2) gives $t = -4q^2$, so the backward $\pi N \to \pi N$ region is $-\infty \leq t \leq 0$. For $\pi\pi \to N\bar{N}$ the (pseudo) physical range is $4\mu^2 \leq t \leq \infty$.

Let $F^{(+)}(t)$ $(-\infty \leq t \leq 0)$ be the backward πN amplitude in the (+) charge combination. If we know this well enough

(32) D. Atkinson, Phys. Rev. 128, 1908 (1962)

in the low and medium energy range, then by analytic continuation we can find $F^{(+)}(t)$ on $4\mu^2 \leq t \leq \infty$ (or at least on the lower part of that range). In fact we get both the real and the imaginary parts of $F^{(+)}(t)$

In a particular normalization[32], for $t \geq 4\mu^2$, we have

$$F^{(+)}(t) = -\frac{8\pi}{M p_t^2} \sum_{\substack{J=0 \\ (J\,\mathrm{even})}}^{\infty} (J+\tfrac{1}{2}) (p_t q_t)^J f_+^J(t) \tag{43}$$

where q_t, p_t are the pion and nucleon momenta in the c.m.s. for $\pi\pi \to N\bar{N}$. Also $f_+^J(t)$ is the + helicity amplitude for $\pi\pi \to N\bar{N}$ with angular momentum J.

Analogous to eq. (42) we have

$$f_+^J(t) = |f_+^J(t)|\, e^{i\,\delta_0^J(t)} \tag{42a}$$

for $4\mu^2 \leq t \lesssim 50\mu^2$. $\delta_0^J(t)$ is the $\pi\pi \to \pi\pi$ phase for the state $T = 0$ and angular momentum J. There is good reason to believe that $\delta_0^2(t)$ is small[33] in $4\mu^2 \leq t \lesssim 50\mu^2$. The same must be true for $\delta_0^J(t)$ with $J \geq 4$. Thus to a good approximation

$$\mathrm{Im}\, F^{(+)}(t) \simeq -\frac{4\pi}{M p_t^2}\, \mathrm{Im}\, f_+^0(t)\ ,\ (4\mu^2 \leq t \lesssim 50\mu^2) \tag{43a}$$

However we cannot do this for $\mathrm{Re}\, F^{(+)}(t)$; indeed in the case of $\mathrm{Re}\, F^{(+)}(t)$ the series in eq. (43) converges

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(33) G.C. Oades, Phys. Rev. 132, 1277 (1963)

slowly[30]. Let θ_t be the scattering angle in $\pi\pi \to N\bar{N}$. The nucleon pole occurs at

$$\cos\theta_t = i \frac{(\frac{1}{2}t - \mu^2)}{2(m^2 - t/4)^{1/2}(t/4 - \mu^2)^{1/2}}$$

For $4.2\mu^2 < t < 25\mu^2$ this is close to the physical range of $\cos\theta_t$. Thus in the Legendre series which gives eq. (43) we require to include terms up to J = 6 to get 10% accuracy.

The difficulty can be overcome by removing the nucleon pole term and treating it explicitly. In works earlier than ref. (30) this was not done, and the results they found for $\delta_0^0(t)$ are therefore suspect.

The remainder of the analysis is now straight forward. (for details see refs. (30) and (15)) $\mathrm{Re}\, f_+^0(t)$ is determined over $4\mu^2 \leq t \lesssim 50\mu^2$, and eqs. (42) and (43a) then give $\delta_0^0(t)$. The results are close to those obtained from the S-wave πN amplitude $f_0^{(+)}(s)$, as described in § V.1 above.

V.3 The Process $\pi N \to \pi\pi N$

This method - the famous Chew-Low extrapolation - lies somewhat outside the scope of these lectures. Fig. 12a shows the general process $\pi N \to \pi\pi N$, and by extrapolation in t, the nucleon's momentum transfer, one can isolate the process shown in Fig. 12b, because it has a pole at $t = \mu^2$. This extrapolation is to be carried out for each value of mass of the dipion system, so a very large amount of data is required. In order to keep the instability of the extrapolation under control one should use good analytic continuation techniques and error estimates[12][13]. A nice example of using such methods to analyse pion production at 2.8 GeV/c is the work of Baton et al.[34].

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(34) J.P. Baton, G. Laurens and J. Reignier, Phys. Letters 33B, 525 and 528 (1970)

Figure Captions

Fig. 1 Contours in the ω -plane. Γ_R consists of two large semicircles of radius R. C_2 consists of contours around the physical cut from $\omega = \mu$ to $\omega = R$ and around the crossed cut from $\omega = -R$ to $\omega = -\mu$. C_3 and C_4 are around the nucleon poles at $\omega = \pm \mu^2/2M$

Fig. 2 The conformal transformation of the whole ω^2 -plane into the interior of the ellipse in the z -plane. The transform of the pole $\omega^2 = \mu^4/4M^2$ is z_B.

Fig. 3 Schematic figure showing the several parts of eq. (27)

Fig. 4 The dots show $F_{H.E}$ (ν , $t = -10\mu^2$) calculated from eq. (36). The curves show the fits

Fig. 5 The phase α_{33} deduced from eq. (32) is shown by the curve. The dots are the CERN values, ref. (27)

Fig. 6 The curves show the predicted phases D_{15} and D_{35}. The dots and error bars are the CERN values, ref. (27)

Fig. 7 The phase α_1 . The error bars show the CERN values[27]. The solid line is Nielsen's prediction with the CERN values of α_3 as input. The dashed lines show the changes when the S_{31} contribution to $\sigma^{(3/2)}$ is changed by $\pm$ 20% (see text).

Fig. 8 The singularity structure of $f_{\ell\pm}(s)$ or $F_{\ell\pm}(s)$.

Fig. 9 The process N-exchange, Δ -exchange ρ -exchange, σ-exchange

Fig. 10 The left hand cut contributions to $F_{\ell\pm}(s)$ in the case of P_{13} and D_{35}. The values in the low energy region are shown

Fig. 11 The discrepancy $\Delta^{(+)}_{2-}(s)$

Fig. 12 a) The process $\pi N \to \pi\pi N$
b) The pion exchange part
t is the momentum transfer to the nucleon.

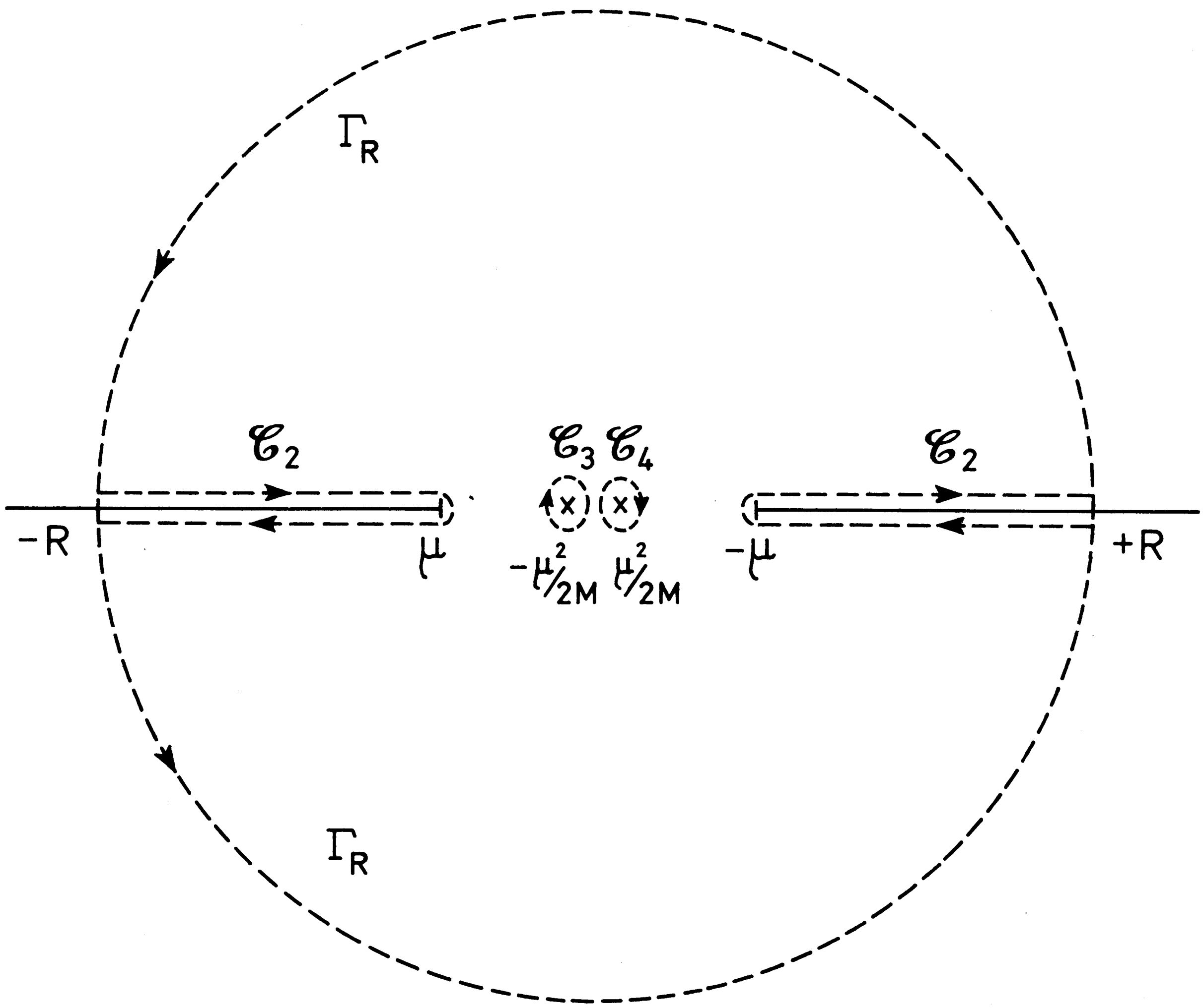
Γ_R
$\mathscr{C}_2$
$\mathscr{C}_3$ $\mathscr{C}_4$
$\mathscr{C}_2$
$-R$
μ
$-\mu^2/2M$ $\mu^2/2M$
$-\mu$
$+R$
Γ_R

Fig. 1

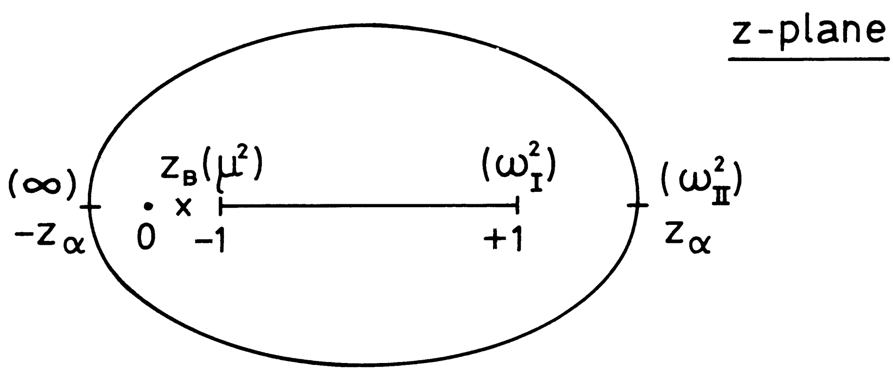
z-plane
(∞)
$-z_\alpha$
0
$z_B(\mu^2)$
-1
(ω_I^2)
+1
(ω_{II}^2)
z_α

Fig. 2

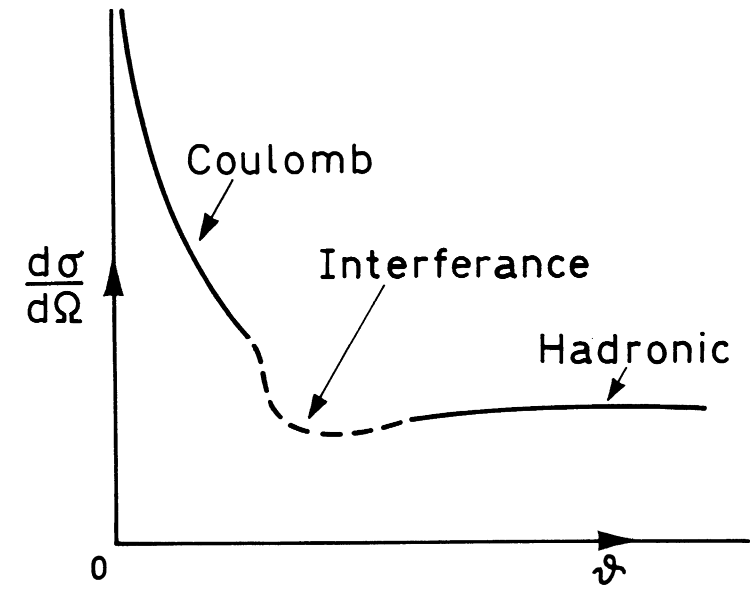
Coulomb
Interferance
Hadronic
$\frac{d\sigma}{d\Omega}$
0
ϑ

Fig. 3

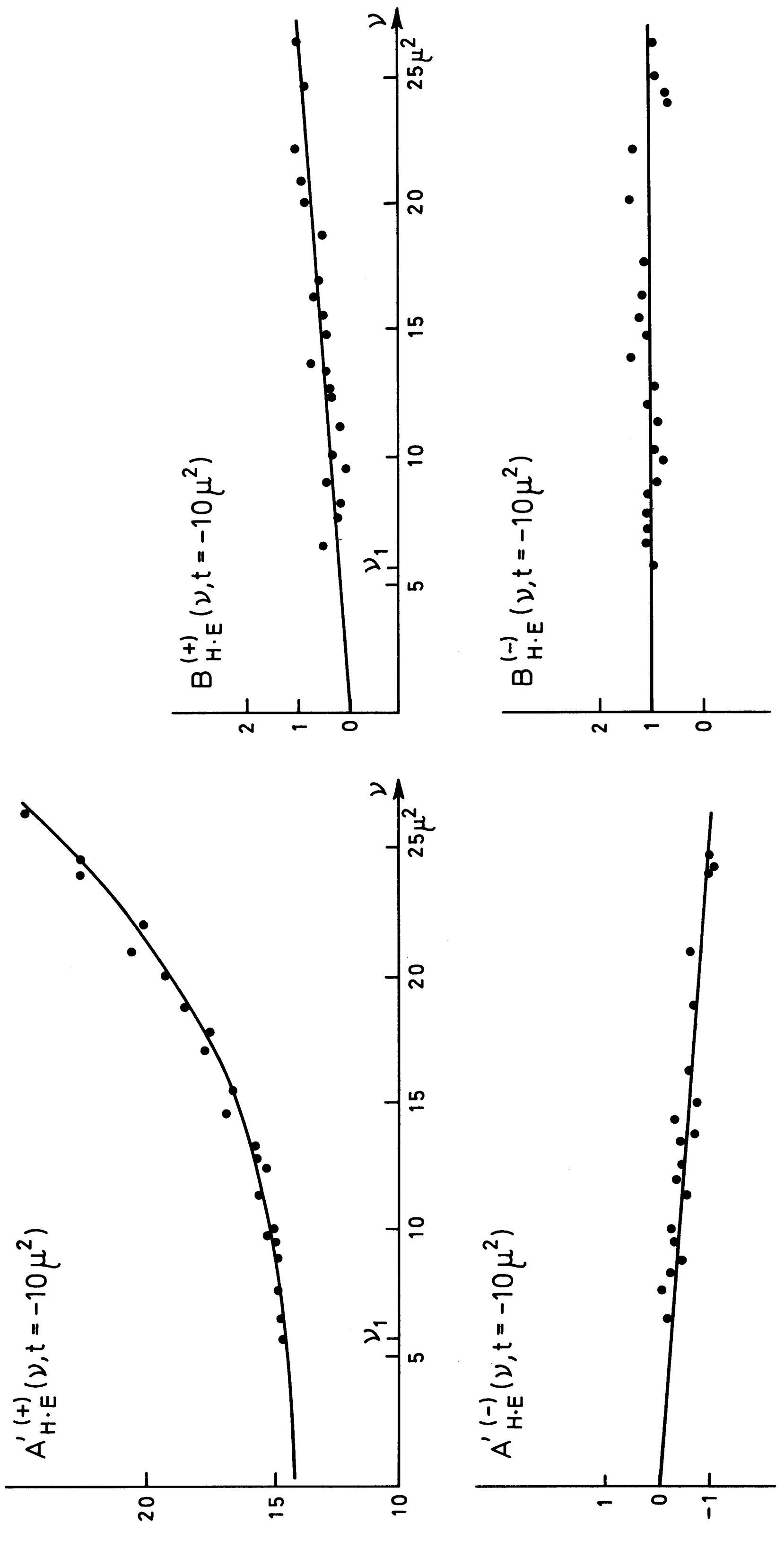
$A'^{(+)}_{H\cdot E}(\nu, t=-10\mu^2)$
$A'^{(-)}_{H\cdot E}(\nu, t=-10\mu^2)$
$B^{(+)}_{H\cdot E}(\nu, t=-10\mu^2)$
$B^{(-)}_{H\cdot E}(\nu, t=-10\mu^2)$
ν_1
$25\mu^2$
ν

Fig. 4

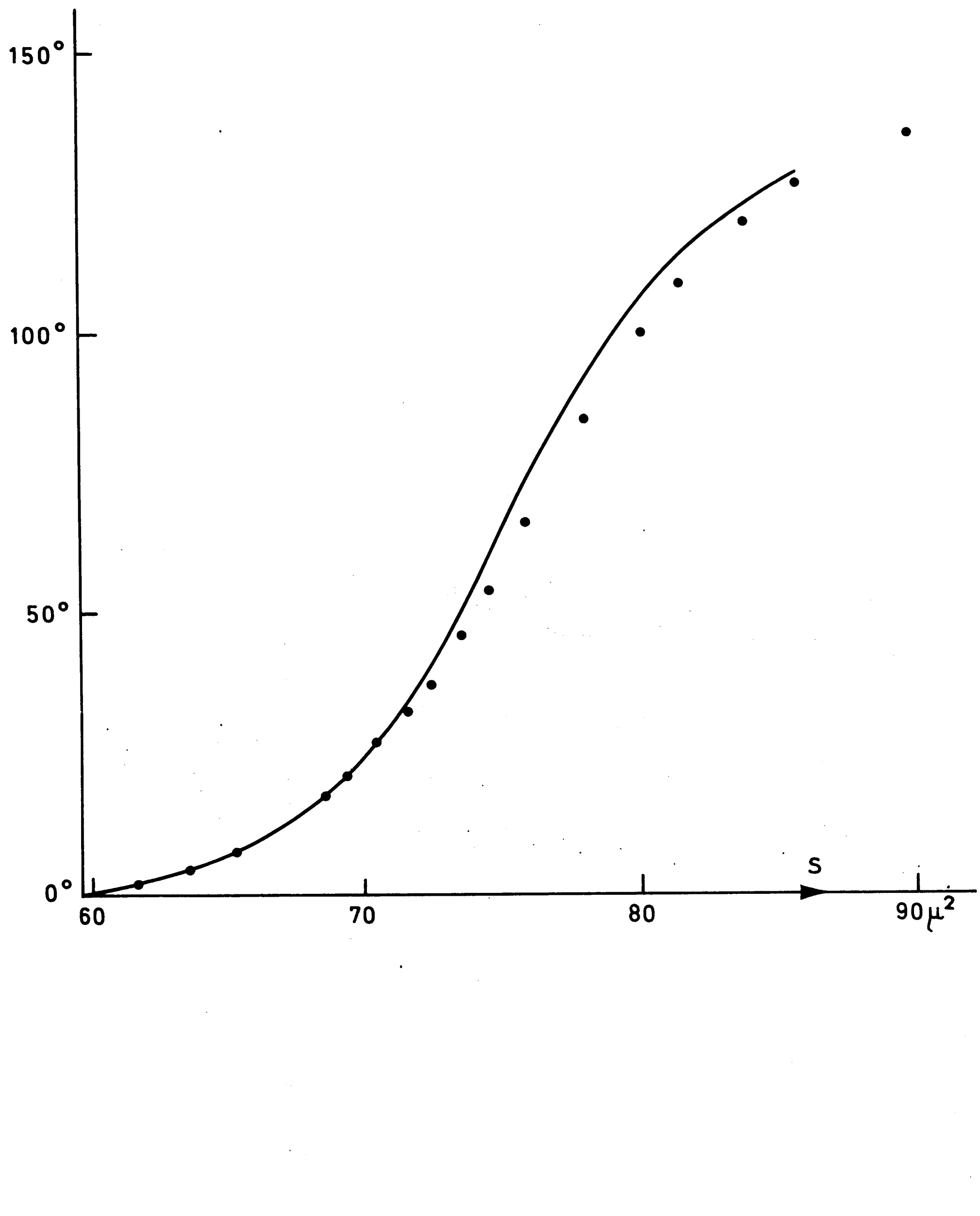

Fig. 5

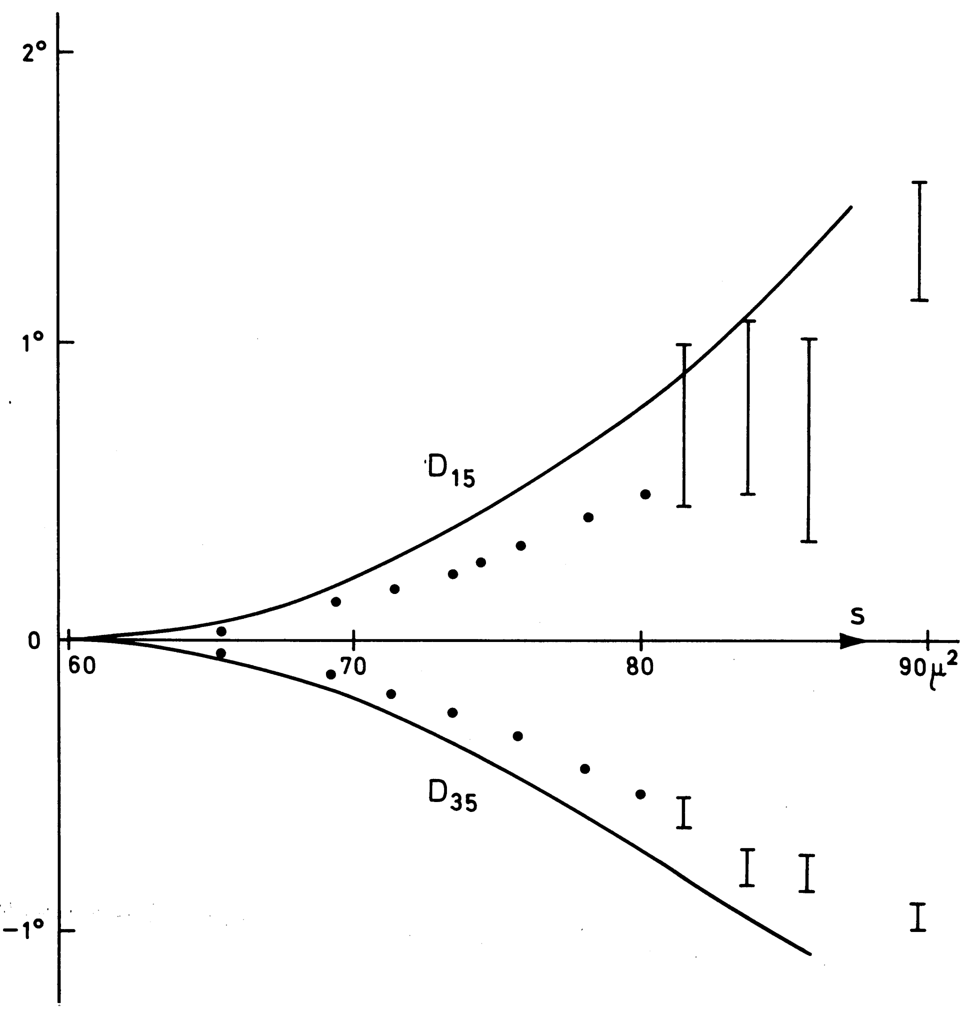

Fig. 6

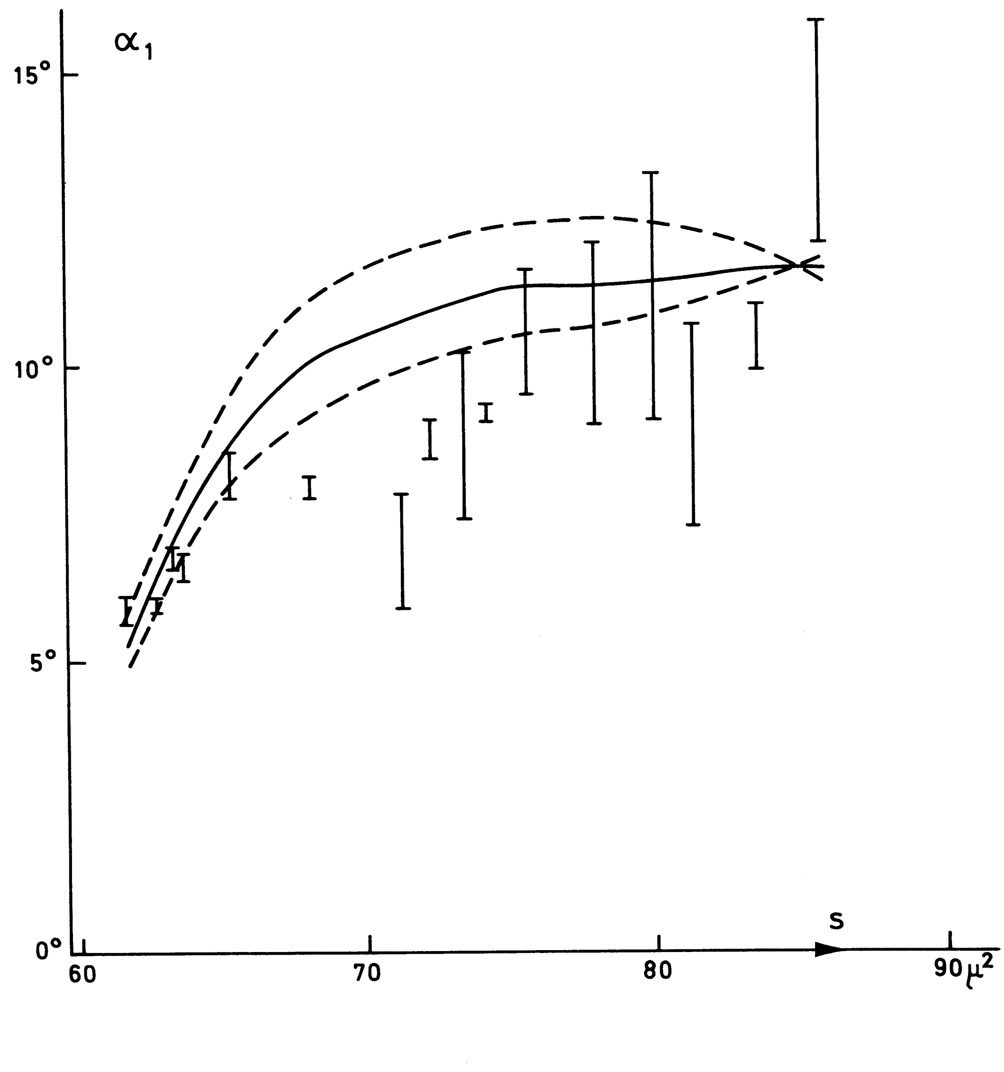

Fig. 7

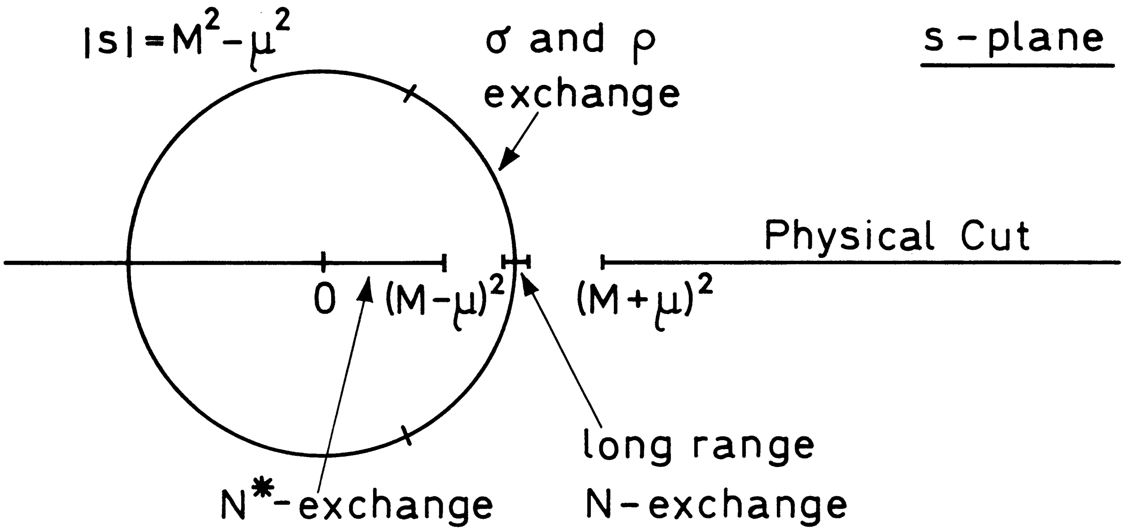
$|s| = M^2 - \mu^2$
σ and ρ
exchange
s - plane
Physical Cut
0
$(M-\mu)^2$
$(M+\mu)^2$
long range
N-exchange
N*- exchange

Fig. 8

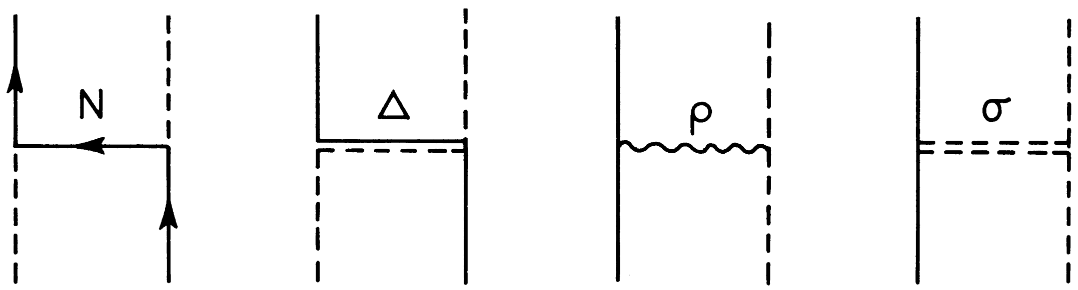
N
Δ
ρ
σ

Fig. 9

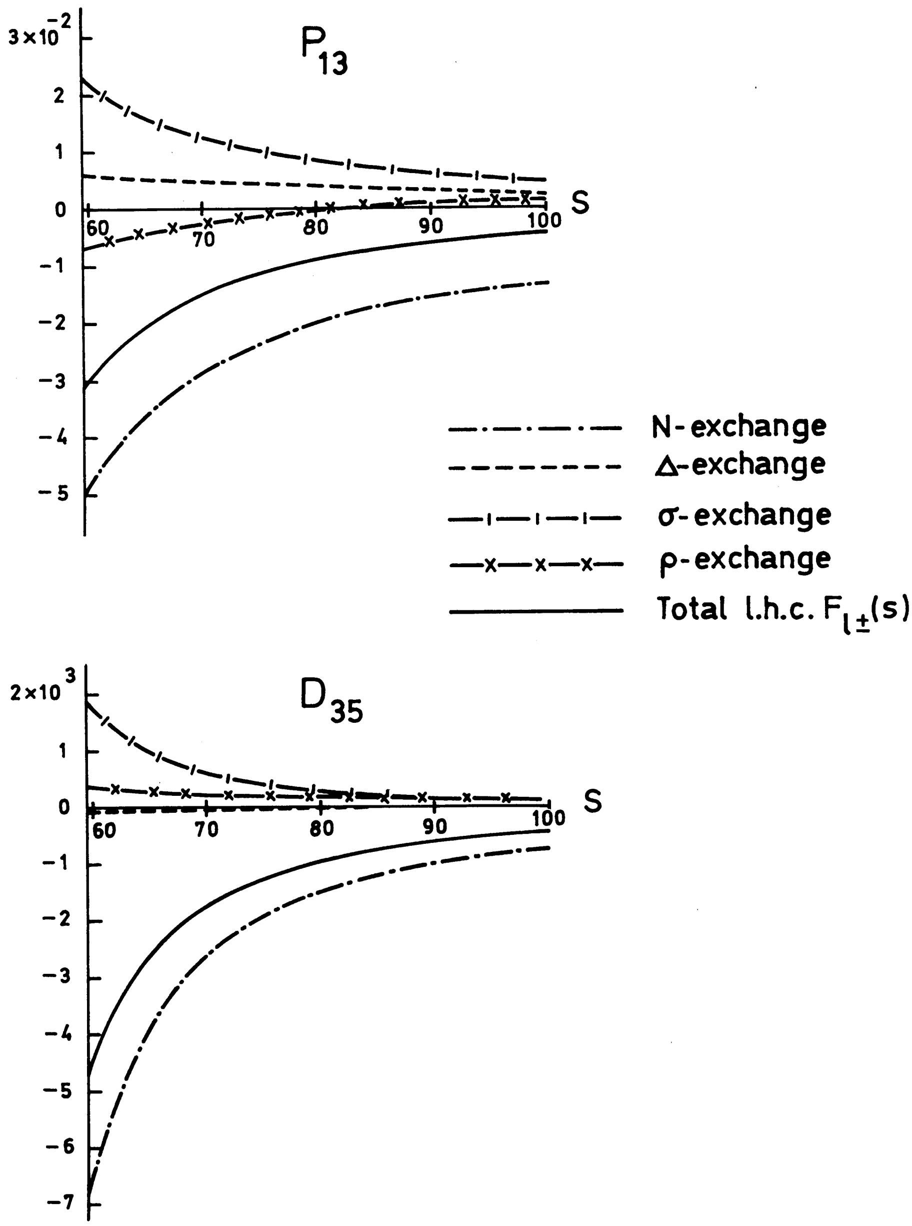
3×10⁻²
2
1
0
-1
-2
-3
-4
-5
P13
60
70
80
90
100
S
N-exchange
Δ-exchange
σ-exchange
ρ-exchange
Total l.h.c. F_{l±}(s)
2×10³
1
0
-1
-2
-3
-4
-5
-6
-7
D35
60
70
80
90
100
S

Fig. 10

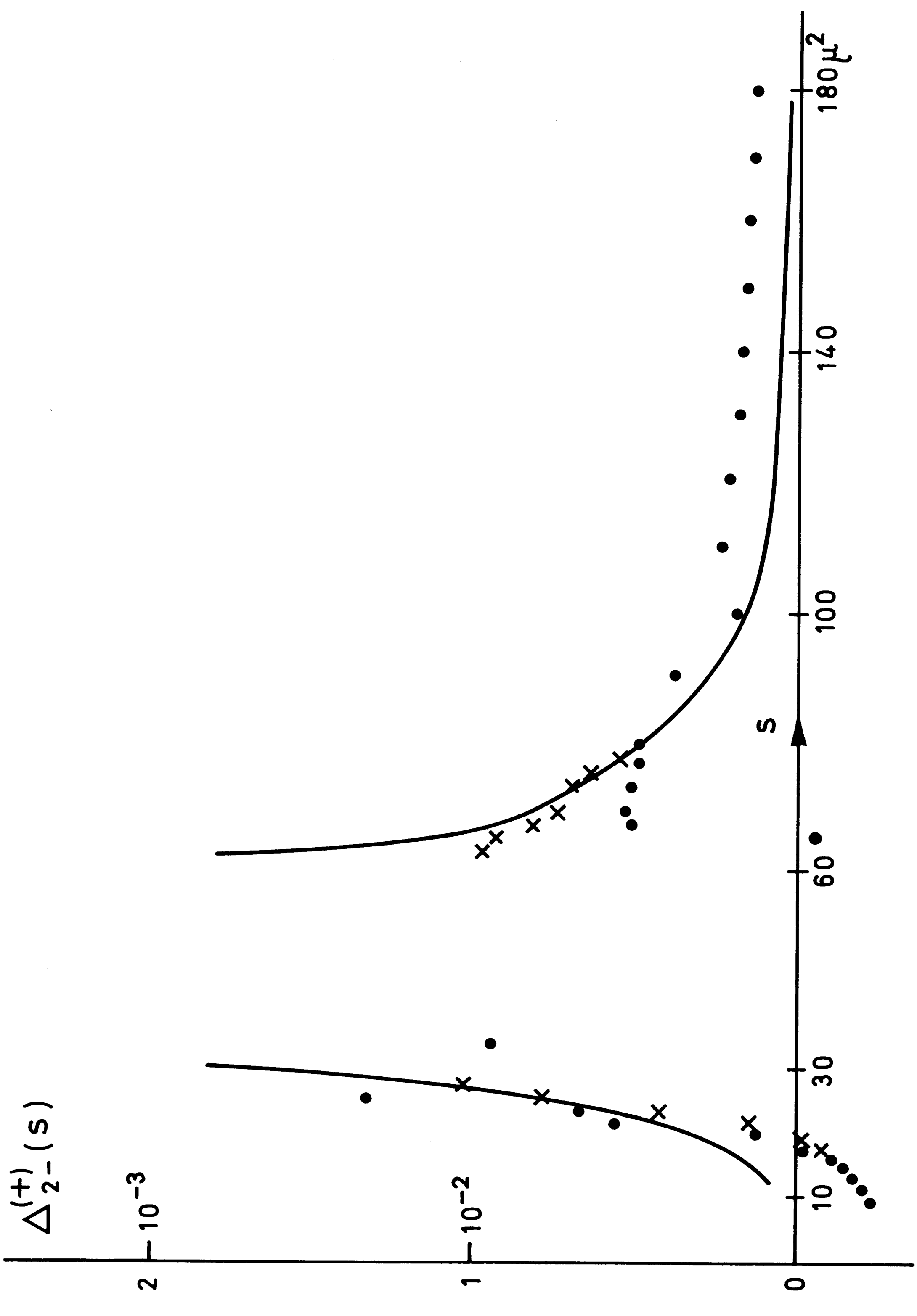

Fig. 11

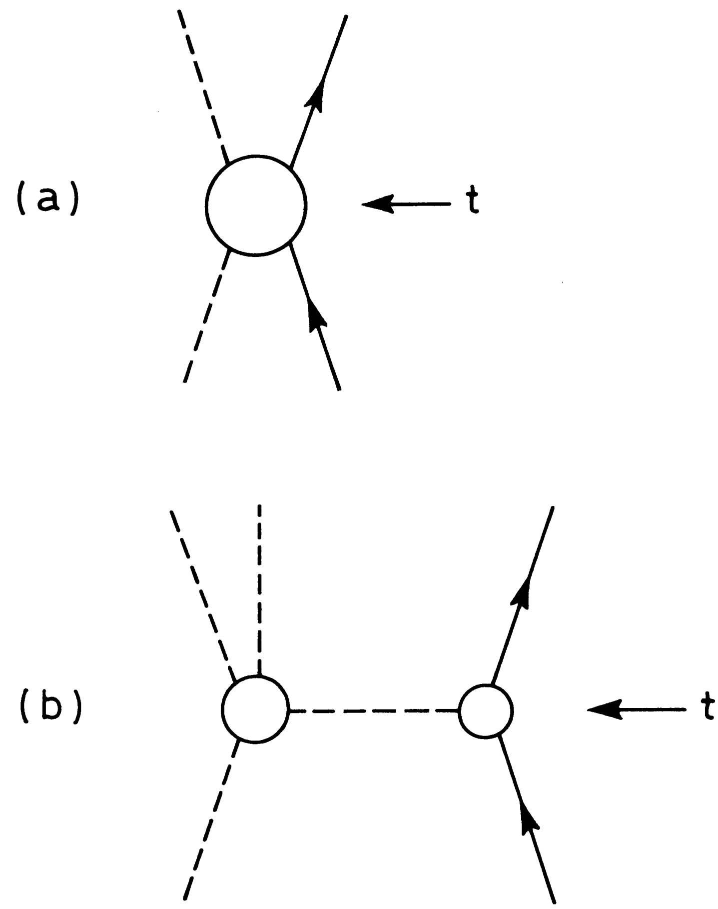

Fig. 12