HIGH-FREQUENCY NON-FERRITE CAVITIES

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Abstract
High-frequency, high-Q cavities are mostly used in electron accelerators. The electrons radiate and lose part of their energy, therefore they need relatively high RF voltages to remain on their stable orbit. Single cell and multicell cavities operating either in a standing wave mode or a travelling wave mode are presented.

1 ACCELERATORS: THE NEED FOR ELECTRIC FIELDS

Consider a rectangular coordinate system moving along the particle trajectory 's' (Fig. 1),

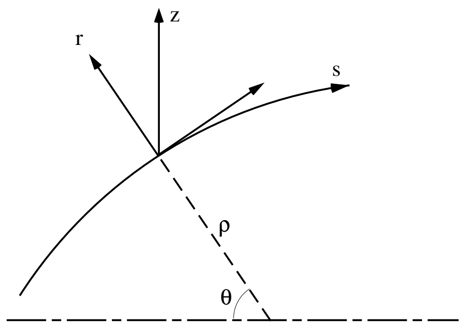

Fig. 1: Curvilinear coordinate system along the particle trajectory

and assume that the fields have single components:

$$\begin{aligned} \vec{E} &\rightarrow E_\theta \\ \vec{B} &\rightarrow B_z \end{aligned} \tag{1}$$

The Newton-Lorentz force,

$$\frac{d\vec{p}}{dt} = e\vec{E} + e\vec{v} \times \vec{B} \ , \tag{2}$$

can be expanded as follows:

$$\frac{d(mv_\theta)}{dt}.\vec{u}_\theta - m\frac{v_\theta^2}{\rho}.\vec{u}_r = eE_\theta.\vec{u}_\theta + ev_\theta B_z.\vec{u}_r \ , \tag{3}$$

where ρ is the local radius of curvature of the trajectory, and $\vec{u}_\theta$ and $\vec{u}_r$ are the unit vectors—tangent and perpendicular, respectively—to the trajectory. By identification one gets:

$$
\begin{aligned}
\frac{dp_\theta}{dt} &= eE_\theta \\
p_\theta/e &= B_z\rho \ ,
\end{aligned}
\tag{4}
$$

showing that the electric field provides energy and momentum while the magnetic field bends the particle trajectory. This result can be generalized to more complicated field patterns.

2 WHY RADIO FREQUENCY (RF) ELECTRIC FIELD ?

Assume a chain of electrodes (Fig. 2) fed by a high-voltage (HV) electrostatic generator.

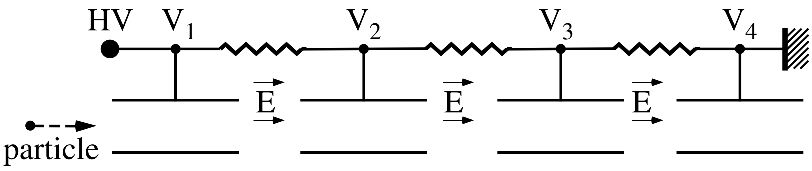

Fig. 2: Electrostatic acceleration

Since the DC voltage is shared between the electrodes in order to provide an electric field in each gap, the generator voltage is the sum of the partial voltages:

$$V_{total} = \sum_i V_i \ . \tag{5}$$

Such a system will hence be limited by electrical breakdown at the HV terminal (as will the energy of the electrostatic accelerator).

If we assume now that the electrodes are fed from an RF generator as shown on Fig. 3, the electric field has reverse polarities in consecutive gaps.

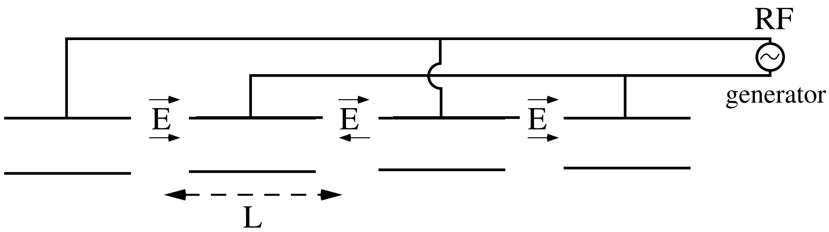

Fig. 3: Radio frequency (RF) acceleration

If the synchronism condition

$$L = \frac{vT_{RF}}{2} \tag{6}$$

is fulfilled, where v is the particle velocity, L the distance between consecutive gaps and T_{RF} the RF period, then for a given voltage V_{RF} all the gaps will accelerate the particle and provide energy. If the particle velocity increases the drift tube length grows between successive gaps, and at very high velocities it is worth considering higher RF frequencies.

3 ENERGY GAIN

In relativistic dynamics the total energy and momentum satisfy

$$\begin{aligned} E &= E_0 + W \\ E^2 &= E_0^2 + p^2c^2 \ , \end{aligned} \tag{7}$$

where E_0 is the rest energy and W the kinetic energy. Differentiating these expressions gives

$$dE = dW = vdp \ . \tag{8}$$

From the Lorentz force

$$\frac{dp}{dt} = eE_z \ , \tag{9}$$

where E_z is the electric field along the linear particle trajectory[1]

$$dW = eE_z dz \tag{10}$$

one gets:

$$W = e \int E_z dz = \text{eV} \tag{11}$$

where in practice V would represent the gap voltage.

4 ELECTRIC FIELD FROM ELECTROMAGNETIC WAVES

As mentioned above, higher kinetic energies require higher operating frequencies to limit the drift tube length. However, according to the accelerating concept described above, the current flowing on the surface of the drift tubes will generate a radiated power in the free space, which will increase linearly with frequency. Therefore it is advisable to enclose the gap in a cavity that will hold the electromagnetic (EM) energy (Fig. 4).

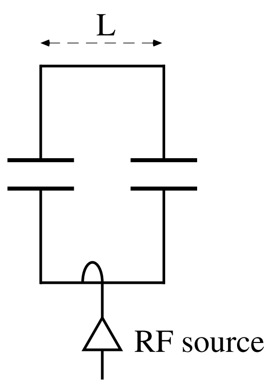

Fig. 4: Single-gap cavity

Such single-gap cavities, resonating at the operating RF frequency, can be powered and phased independently to ensure proper synchronism with respect to the particle velocity. The resonant standing

[1] In a linear accelerator z often refers to the longitudinal axis or direction of particle motion, while in a circular accelerator z can represent the vertical axis perpendicular to the plane of curvature.

wave (SW) mode is a transverse magnetic (TM) mode with a longitudinal electric field component, necessary for acceleration.

Efficient variants of the above scheme consist of placing the cavities adjacent to each other, either in a π-mode (Fig. 5(a)) or a 2π-mode (Fig. 5(b)).

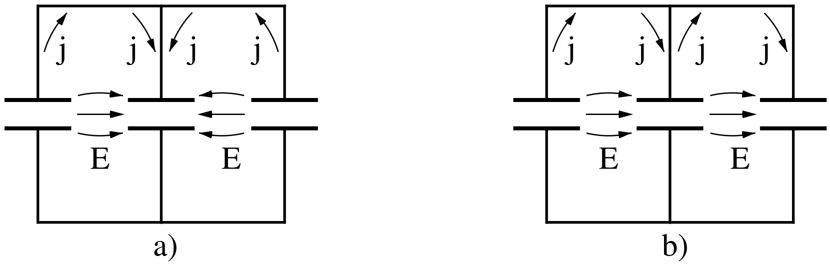

Fig. 5: Multigap cavities (E = electric field; j = wall current)

These modes can be obtained by suitable cavity geometry design leading, respectively, to the synchronism conditions

$$L = v\frac{T_{RF}}{2} \ \ (\pi\text{-mode}) \qquad\qquad L = vT_{RF} \ \ (2\pi\text{-mode}) \ ,$$

where L represents the distance between the gap centres.

5 THE PILL-BOX CAVITY

The most simple accelerating cavity is the pill-box cavity, which has cylindrical symmetry of radius a, and a relatively short active length l (Fig. 6)

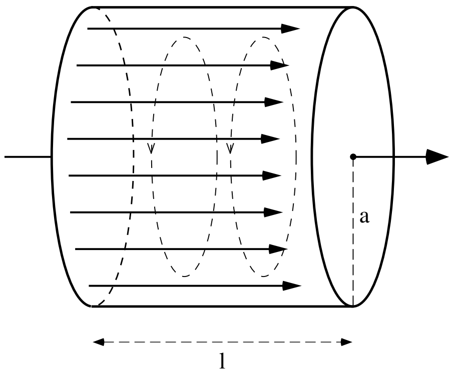

Fig. 6: Pill-box cavity with TM_{010} mode
$(\longrightarrow \ E_z \quad - - \rightarrow H_\theta)$

From Maxwell's equations

$$\begin{aligned} \nabla.\mathbf{E} &= 0 \\ \nabla.\mathbf{H} &= 0 \\ \nabla\times\mathbf{E} &= -\mu_0\frac{\partial\mathbf{H}}{\partial t} \\ \nabla\times\mathbf{H} &= \epsilon_0\frac{\partial\mathbf{E}}{\partial t} \ , \end{aligned} \tag{12}$$

one can derive the wave equation

$$\nabla^2\mathbf{A} - \epsilon_0\mu_0\frac{\partial^2\mathbf{A}}{\partial t^2} = 0 \quad (13)$$

where **A** represents either the electric field or the magnetic field vector.

The lowest frequency mode is the TM_{010} with only two components for the EM field, provided $\ell < 2a$

$$\left.\begin{array}{l} E_z = J_0(kr) \\ H_\theta = -\frac{j}{Z_0}J_1(kr) \end{array}\right\} e^{j\omega t} \ , \quad (14)$$

with

$$Z_0 = \left(\frac{\mu_0}{\epsilon_0}\right)^{1/2} = 377\ \Omega$$

$$k = \frac{2\pi}{\lambda} = \frac{\omega}{c} \ .$$

where λ is the free space wavelength and $\omega/2\pi$ is the RF frequency.

The amplitudes of the field components as function of the radial position are shown in Fig. 7. Since the longitudinal electric field must vanish at $r = a$, one gets $\lambda = 2.62a$ for the free space wavelength.

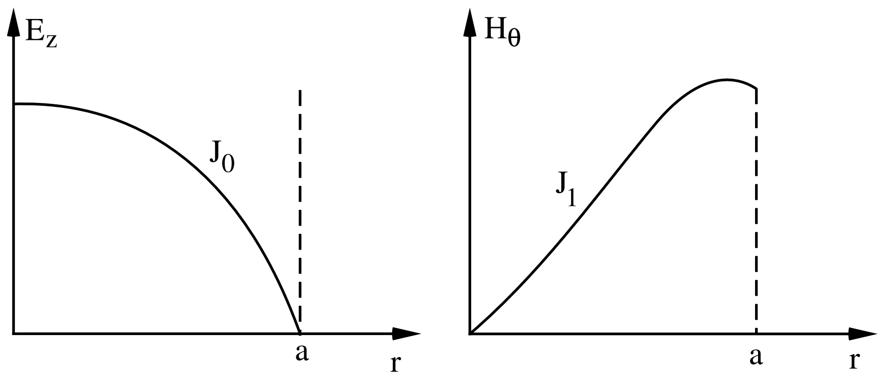

Fig. 7: Field amplitude with radial position

6 TRANSIT TIME FACTOR

In a pill-box cavity the longitudinal field amplitude is constant along the axis (Fig. 8), provided the beam tube is small enough.

$$E_z = E_0\ \cos\omega t\ = \ \frac{V}{g}\cos\omega t \quad (15)$$

where V is the gap voltage and g the gap length.

Consider a particle passing through the middle of the gap at $t = 0$, such that its position is $z = vt$. The total energy gained by the particle is

$$\begin{aligned} \Delta W &= \int_{-g/2}^{g/2} e\frac{V}{g}\cos\omega\frac{z}{v}dz \\ \Delta W &= eV\frac{sin\frac{\theta}{2}}{\frac{\theta}{2}} = eVT \ , \end{aligned} \quad (16)$$

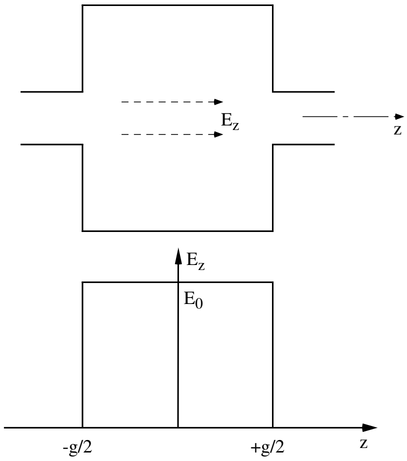

Fig. 8: Cavity gap and approximate field amplitude

where $\theta = \frac{\omega g}{v}$ is the transit angle and T is the transit time factor ($0 < T < 1$). For example, in the case of a 2π-mode structure with $g = \frac{L}{z}$ (see Fig. 3), one has $\theta = \pi$ and $T = 0.637$.

In the more general case where the electric field varies along the axis

$$E_z = E_z(z,t) \quad , \tag{17}$$

the energy gain can be expressed as

$$\Delta W = e\mathrm{Re}\int_0^g E_0(z)e^{j\omega t}dz \tag{18}$$

$$\omega t = \omega\frac{z}{v} - \psi_p \quad , \tag{19}$$

where ψ_p is the initial phase position of the particle. One can write

$$\Delta W = e\mathrm{Re}\left\{e^{-j\psi_p}\int_0^g E_0(z)e^{j\omega\frac{z}{v}}dz\right\} \tag{20}$$

or, expanding the complex integral in phase and amplitude

$$\Delta W = e\mathrm{Re}\left\{e^{-j\psi_p}e^{j\psi_i}|\int_0^g E_0(z)e^{j\omega\frac{z}{v}}dz|\right\} \quad . \tag{21}$$

Introducing $\phi = \psi_p - \psi_i$, the total energy gain becomes

$$\Delta W = e|\int_0^g E_0(z)e^{j\omega\frac{z}{v}}dz|cos\phi \tag{22}$$

while the transit time factor T, which corresponds to the maximum energy a particle can gain, becomes:

$$T = \frac{|\int E_0(z)e^{j\omega\frac{z}{v}}dz|}{\int E_0(z)dz} \quad . \tag{23}$$

This approach is particularly useful in case of multigap cavities, and also when using certain computer codes to design a cavity.

7 SHUNT IMPEDANCE

The shunt impedance R_s is a figure of merit that relates the accelerating voltage V in the gap to the power P_d dissipated in the cavity walls:

$$P_d = \frac{V^2}{R_s} \ . \tag{24}$$

An uncorrected shunt impedance Z is also defined through the peak voltage (the integral of the field envelope along the cavity axis) and then:

$$R_s = ZT^2 \ . \tag{25}$$

It is also usual, especially in the case of travelling wave accelerating sections, to introduce the power lost per unit length

$$\frac{dP_d}{dz} = -\frac{E_z^2}{r} \tag{26}$$

where E_z is the electric field along the axis and r the shunt impedance per unit length. The rate of power lost depends on i_w, the wall current (related to H), and on r_w the wall resistance per unit length,

$$\frac{dP_d}{dz} \propto i_w^2 r_w \ . \tag{27}$$

The wall resistance per unit length is equal to the resistivity divided by the area in which the current flows

$$r_w = \rho / 2\pi a\delta \ , \tag{28}$$

where a is the cavity radius

$$a \propto 1/\omega \ , \tag{29}$$

δ the skin depth

$$\delta = (2\rho/\omega\mu)^{1/2} \ , \tag{30}$$

and μ is the wall permeability of the material. The axial electric field is such that

$$E_z \propto i_w / a \ . \tag{31}$$

Combining the previous expressions shows that

$$r \propto \omega^{1/2} \ , \tag{32}$$

hence power efficiency will favour higher frequencies.

8 QUALITY FACTOR AND STORED ENERGY

The quality factor Q of a cavity is defined as

$$Q = \frac{\omega W_s}{P_d} \quad , \tag{33}$$

where W_s is the stored energy in the cavity volume. Another quantity of interest is the ratio

$$\frac{R_s}{Q} = \frac{V^2}{\omega W_s} \quad , \tag{34}$$

which only depends on the cavity geometry, and is directly accessible to measurement.

The stored energy in the cavity volume is

$$W_s = \frac{\mu}{2} \int_V |H|^2 dV = \frac{\epsilon}{2} \int_V |E|^2 dV \quad , \tag{35}$$

while the power loss in the walls is

$$P_d = \frac{1}{2} \int_S R_w |H|^2 dS \quad . \tag{36}$$

The surface resistance R_w for a layer of unit area and width δ (skin depth) is

$$R_w = \frac{1}{\sigma\delta} \tag{37}$$

$$\delta = \frac{1}{\sqrt{\pi\mu\sigma f}} \tag{38}$$

with σ the conductivity and f the frequency. Hence:

$$P_d = \frac{\pi\mu\delta f}{2} \int_S |H|^2 dS \quad , \tag{39}$$

and finally:

$$Q = \frac{2}{\delta} \frac{\int_V |H|^2 dV}{\int_S |H|^2 dS} = \frac{2}{\delta} K \frac{V}{S} \quad , \tag{40}$$

where K is a form factor for a given cavity geometry.

Considering the TM_{010} mode in a pill-box cavity,

$$\int_V H_\theta^2 dV = \ell \int_0^a J_1^2(kr) 2\pi r dr \tag{41}$$

$$\int_S H_\theta^2 dS = 2 \int_0^a J_1^2(kr) 2\pi r dr + 2\pi a \ell J_1^2(ka) \quad , \tag{42}$$

one gets

$$\frac{1}{Q} = \frac{\delta}{\ell} \frac{\int_0^a J_1^2(kr) r dr + \frac{a\ell}{2} J_1^2(ka)}{\int_0^a J_1^2(kr) r dr} \quad . \tag{43}$$

From the relation

$$\int_0^a J_1^2(kr) r dr = \frac{a^2}{2} J_1^2(ka) \tag{44}$$

one gets

$$Q = \frac{\ell}{\delta} \frac{a}{a+\ell} \quad \propto \omega^{-1/2} \quad . \tag{45}$$

For a pill-box cavity, at 3 GHz, one gets from Section 5 radius $a = 3.8$ cm. With $\delta = 10^{-6}$ m (copper) and $\ell = 5$ cm one gets $Q = 21590$. This shows that a simple pill-box cavity, with no lossy material in the volume, has a rather high Q value. An increase in the Q value can be obtained by proper shaping of the volume to minimize losses for a given stored energy. Following the same procedure one also gets

$$\frac{r}{Q} = \frac{V^2}{\omega W_s \ell} = 2.58 \mu f \quad \propto \omega \tag{46}$$

with

$$r = \frac{R_s}{\ell} \quad . \tag{47}$$

9 FILLING TIME

It is shown from the definition of Q that the energy is dissipated at a rate proportional to the stored energy

$$P_d = -\frac{dW_s}{dt} = \frac{\omega}{Q} W_s \quad . \tag{48}$$

If the cavity is initially filled with electromagnetic energy, that stored energy will decay as follows:

$$W_s = W_{so} e^{-\frac{t}{\tau}} \quad , \tag{49}$$

where

$$\tau = \frac{Q}{\omega} \tag{50}$$

Since the stored energy is proportional to the square of the electric field, then the field will decay with a time constant 2τ. If the cavity is fed by an external RF source, then the stored energy will build up like

$$W_s = W_{so}[1 - e^{-\frac{t}{2\tau}}]^2 \quad , \tag{51}$$

and τ is called the filling time of the cavity. If the cavity is coupled to an external load, Q should be replaced by the loaded Q, Q_L, to take account of the additional losses.

10 CAVITY EQUIVALENT CIRCUIT

Solutions of Maxwell's equations show that when the cavity resonates on a given mode, such as the TM_{010}, the time average energy stored in the electric field equals the time average energy stored in the magnetic field:

$$W_{se} = W_{sm} \tag{52}$$

$$\frac{\epsilon}{4} \int_V |E|^2 dV = \frac{\mu}{4} \int_V |H|^2 dV \ , \tag{53}$$

and within an RF period the energy oscillates between electric and magnetic. This is the case for a lumped RLC parallel circuit at resonance (Fig. 9).

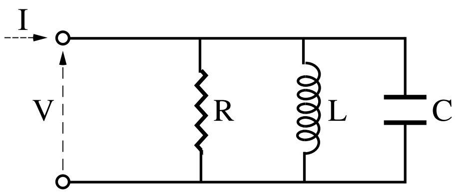

Fig. 9: Resonant circuit

The time average energy stored in the electric field (capacitor) is

$$W_{se} = \frac{1}{4} C V V^* \tag{54}$$

while the time average energy stored in the magnetic field (inductance) is

$$W_{sm} = \frac{1}{4} L I_L I_L^* \ , \tag{55}$$

where I_L is the current in the inductance L.

At resonance

$$\begin{aligned} \omega_0 &= (LC)^{-1/2} \\ V &= \omega_0 L I_L \end{aligned} \tag{56}$$

and the time average energy stored in the system is

$$W_s = W_{se} + W_{sm} = \frac{1}{2} C V V^* \ . \tag{57}$$

Since the power loss in the 'equivalent' resistance is

$$P_d = \frac{1}{2} G V V^* \quad \text{with } G = R^{-1} \tag{58}$$

the quality factor of the circuit is

$$Q = \frac{\omega_0 C}{G} = \omega_0 R C = \frac{R}{\omega_0 L} \ . \tag{59}$$

From the knowledge of ω_0, Q and R/Q, it is possible to build up the equivalent resonant circuit of a cavity.

11 INPUT IMPEDANCE

The impedance of the resonant circuit, as seen from the input, is

$$Z_{in} = \left(\frac{1}{R} + \frac{1}{j\omega L} + j\omega C\right)^{-1} \tag{60}$$

with

$$\omega = \omega_0 + \Delta\omega \ . \tag{61}$$

Expanding for small values of $\Delta\omega/\omega_0$ gives

$$Z_{in} = \frac{\omega_0^2 RL}{\omega_0^2 L + j2R\Delta\omega} \tag{62}$$

or

$$Z_{in} = \frac{R}{1 + j2Q\frac{\Delta\omega}{\omega_0}} \ . \tag{63}$$

If $Q = \omega_0/(2\Delta\omega)$ then $\Delta\omega$ corresponds to $0.707|Z_{in}|_{max}$, where $|Z_{in}|_{max} = R$. The relative bandwidth (BW) is defined as $2\Delta\omega/\omega_0$ so one can write

$$Q = 1/BW \ . \tag{64}$$

If R represents the losses in the resonant circuit then Q is the unloaded Q. If additional losses come from the coupling to an external load, represented by R_L in parallel, the quality factor becomes

$$\begin{aligned} Q_L &= \frac{R_t}{\omega_0 L} \\ R_t &= \frac{RR_L}{R+R_L} \ . \end{aligned} \tag{65}$$

Introducing the external Q, Q_e

$$Q_e = \frac{R_L}{\omega_0 L} \ , \tag{66}$$

one gets

$$\frac{1}{Q_L} = \frac{1}{Q} + \frac{1}{Q_e} \ . \tag{67}$$

12 MULTICELL STANDING WAVE CAVITIES

If the accelerator requires high accelerating voltages (such as electron linacs and storage rings) it is often more efficient to use multicell (coupled together) cavities rather than many single-cell cavities (e.g. pill-box cavities) powered separately.

Consider again a two-gap system enclosed in two metallic boxes and powered independently with proper phasing to provide a 2π phase difference between gaps (Fig. 10).

Since the circulating currents compensate each other in the common wall, this wall is useless in terms of boundary conditions for solving Maxwell equations. Hence a variant of this system consists of placing the drift tubes in a single resonant tank. The corresponding accelerating section is well-known as the Alvarez structure (Fig. 11) and is still used in proton linacs.

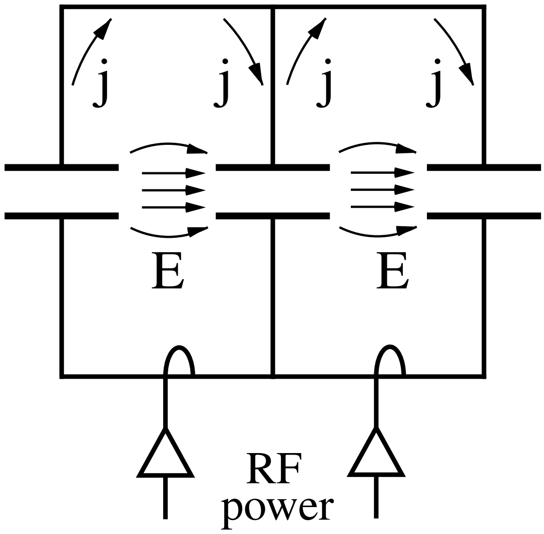

Fig. 10: 2π-mode in a two-cell cavity

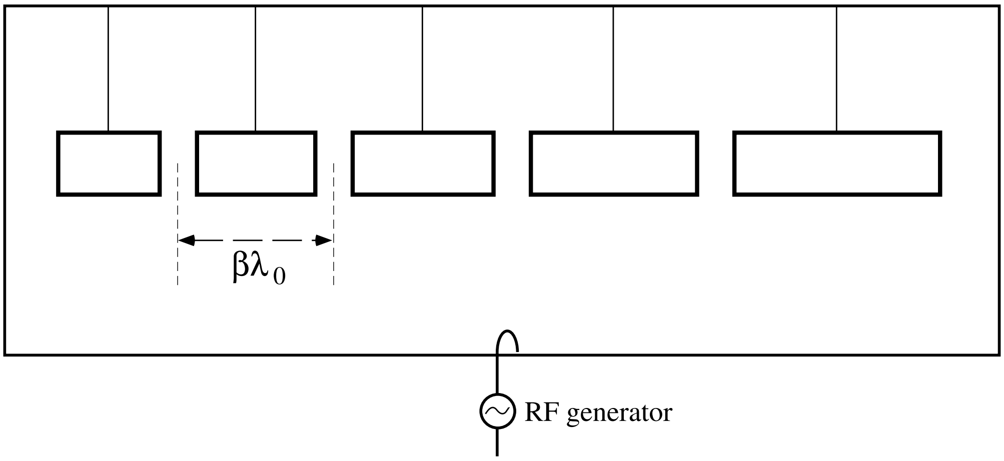

Fig. 11: Alvarez structure

There are other types of multicell standing wave (SW) structures used in modern accelerators, such as the side-coupled structure shown in Fig. 12, where the coupling between cells is reinforced using resonant cavities.

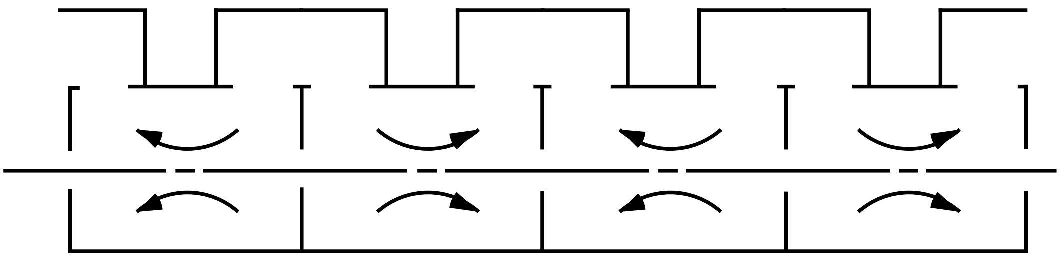

Fig. 12: Side-coupled structure

13 TRAVELLING-WAVE STRUCTURES

For ultra-relativistic particles (electrons, positrons) travelling-wave (TW) accelerating structures are preferred to SW accelerating structures because:

– a particle can travel on the crest of the wave;
– there is no need to care about the transit time factor;
– there is higher shunt impedance (forward wave only dissipates); and
– there is continuous acceleration (no drift).

However, proper acceleration with a TW cavity requires the phase velocity v_p of the wave to be equal to the particle velocity v ($\simeq c$).

Since standard waveguides (e.g. cylindrical) have $v_p > c$, the trick of lowering the phase velocity consists of loading the guide with iris. This has led to the concept of 'iris-loaded structures' (Fig. 13).

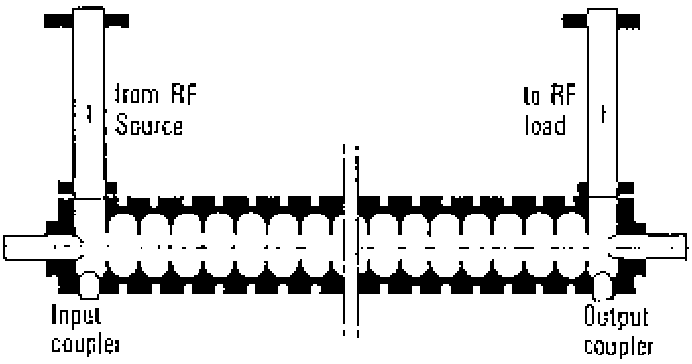

Fig. 13: Travelling-wave accelerating structure

Most electron linacs use these structures, operating on a $\frac{\pi}{2}$ or $\frac{2\pi}{3}$ mode, and they can be as long as 7 m.

14 THE TM_{01} MODE

In a cylindrical waveguide the simplest mode with a longitudinal electric field that can propagate is the TM_{01} mode:

$$\left.\begin{aligned} E_z &= E_0 J_0(k_c r) e^{-j\beta z} \\ E_r &= j\frac{\beta}{k_c} E_0 J_1(k_c r) e^{-j\beta z} \\ H_\theta &= \frac{1}{Z_0} j \frac{k}{k_c} E_0 J_1(k_c r) e^{-j\beta z} \end{aligned}\right\} e^{j\omega t} \ , \tag{68}$$

where $Z_0 = 377\ \Omega$ is the vacuum impedance and β is the propagation factor of the wave travelling in the + z direction and satisfying the relation

$$\beta^2 = k^2 - k_c^2 \tag{69}$$

with

$$\begin{aligned} k &= \frac{2\pi}{\lambda} = \frac{\omega}{c} \\ k_c &= \frac{2\pi}{\lambda_c} = \frac{\omega_c}{c} \ , \end{aligned} \tag{70}$$

where λ is the free space wavelength and ω_c the cut-off frequency. Since E_z must vanish at $r = a$, where a is the inner radius of the cylindrical waveguide, one has:

$$J_0(k_c a) = 0$$
$$k_c a = 2.4 \tag{71}$$

In order for the wave to propagate, β must be real and positive, hence

$$\begin{aligned} k^2 &> k_c^2 \\ \omega &> \omega_c \ , \end{aligned} \tag{72}$$

The velocity at which the wave propagates is just

$$v_p = \frac{\omega}{\beta} \tag{73}$$

and the guided wave length, λ_g, is such that

$$\beta = \frac{2\pi}{\lambda_g} = \frac{\omega}{v_p} \ . \tag{74}$$

15 PHASE VELOCITY AND GROUP VELOCITY

For propagating waves one has the relation

$$\frac{\omega^2}{v_p^2} = \frac{\omega^2}{c^2} - \frac{\omega_c^2}{c^2} \ , \tag{75}$$

showing that in a cylindrical waveguide

$$v_p > c \ . \tag{76}$$

In the Brillouin diagram this is represented by an hyperbola (Fig. 14).

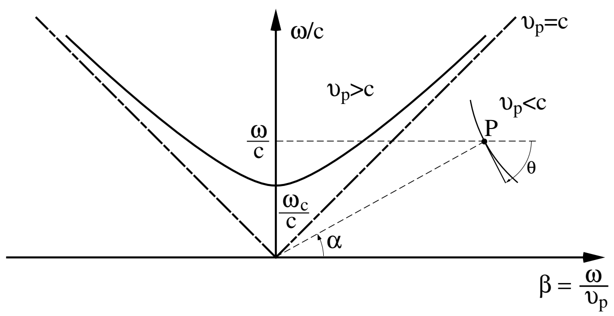

Fig. 14: Brillouin diagram

To lower the phase velocity the waveguide can be loaded with equally spaced disks (ultra-relativistic particles). In the Brillouin diagram a slow wave structure operates below the 45° line. If P(ω, β) represents the operating point, then

$$\text{tg}\alpha = \frac{v_p}{c} \tag{77}$$

$$\text{tg}\theta = \frac{d\left(\frac{\omega}{c}\right)}{d\left(\frac{\omega}{v_p}\right)} = \frac{1}{c}\frac{d\omega}{d\beta} = \frac{1}{c}v_g \ , \tag{78}$$

where $v_g = \left(\frac{d\beta}{d\omega}\right)^{-1}$ is called the group velocity.

16 ENERGY FLOW VELOCITY

The average power that flows through a transverse cross section of a waveguide is given by the integral of the Poynting vector

$$P = \frac{1}{2}\mathcal{R}e\int_S (E_T \times H_T)dS \ . \tag{79}$$

For a TM mode the relation between the transverse field components is

$$\frac{E_T}{H_T} = Z_0\frac{\lambda}{\lambda_g} \tag{80}$$

hence

$$P = \frac{1}{2Z_0}\frac{k}{\beta}\int_S |E_T|^2 dS \ . \tag{81}$$

The energy stored in the magnetic field (purely transverse) per unit length is:

$$w_{sm} = \frac{\mu}{4}\int_S |H_T|^2 dS = \frac{\mu}{4}\frac{1}{Z_0^2}\frac{k^2}{\beta^2}\int_S |E_T|^2 dS \tag{82}$$

The energy stored in the electric field per unit length is equal to that of the magnetic field. Hence the total energy stored per unit length is

$$w_s = w_{se} + w_{sm} = 2w_{sm} \ . \tag{83}$$

The velocity of the energy flow is then

$$v_e = \frac{P}{w_s} = \frac{1}{\mu}Z_0\frac{\beta}{k} = \frac{\beta}{k}c \ . \tag{84}$$

Since

$$v_g = \left(\frac{d\beta}{d\omega}\right)^{-1} = \left[\frac{d\left(\frac{\omega^2}{c^2} - k_c^2\right)^{1/2}}{d\omega}\right]^{-1} \tag{85}$$

one finally gets

$$v_g = \frac{\beta c^2}{\omega} = \frac{\beta}{k}c = v_e \ , \tag{86}$$

showing that in the particular case discussed the group velocity corresponds to the energy flow velocity.

17 SPACE HARMONICS IN LOADED WAVEGUIDES

In an infinite periodic structure (Fig. 15) the wave equation must satisfy the periodic boundary conditions of the disk arrangement (equally spaced for ultra-relativistic particles). Floquet's theorem suggests solutions of the form

$$\begin{aligned} E(r,\theta,z) &= e^{-\gamma z}E_1(r,\theta,z) \\ H(r,\theta,z) &= e^{-\gamma z}H_1(r,\theta,z) \ , \end{aligned} \tag{87}$$

where E_1 and H_1 are periodic functions of z, with period d:

$$\begin{aligned} E_1(r,\theta,z+d) &= E_1(r,\theta,z) \\ H_1(r,\theta,z+d) &= H_1(r,\theta,z) \ . \end{aligned} \tag{88}$$

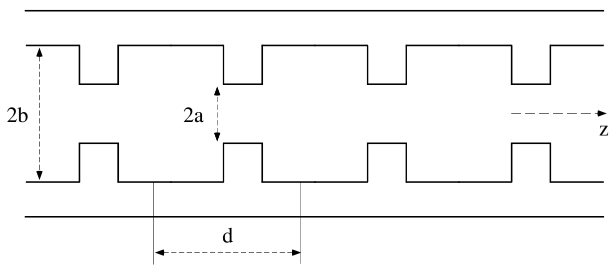

Fig. 15: Periodic iris loaded waveguide

The Fourier expansion of the field is

$$E_1(r,\theta,z) = \sum_{n=-\infty}^{+\infty} E_{1n}(r,\theta)e^{-j2n\pi z/d} \tag{89}$$

and since for lossless structures $\gamma = j\beta_0$ (with β_0 real), one can write

$$E(r,\theta,z) = \sum_{n=-\infty}^{+\infty} E_{1n}(r,\theta)e^{-j\beta_n z} \tag{90}$$

with

$$\beta_n = \beta_0 + 2n\pi/d \ . \tag{91}$$

The wave can be looked at as the sum of 'space harmonics', each harmonic having a different phase velocity,

$$v_{pn} = \frac{\omega}{\beta_0 + \frac{2\pi n}{d}} \ , \tag{92}$$

but the same group velocity

$$v_{gn} = \left(\frac{d\beta_n}{d\omega}\right)^{-1} = \left(\frac{d\beta_0}{d\omega}\right)^{-1} = v_{g0} \ . \tag{93}$$

Since the same value of the group velocity repeats with a distance $2\pi/d$ on the scale of the Brillouin diagram (see Fig. 16), the dispersion curve passes though minima and maxima. The waveguide is of the pass-band type.

In practice, with particle velocity c, the operating point P_0 should be on the 45° line. Most commonly $\pi/2$ and $2\pi/3$ modes are used, which correspond to $d = \lambda/4$ or $d = \lambda/3$.

A periodically loaded waveguide can be represented by a chain of coupled resonant circuits (Fig. 17). Each cell is represented by an equivalent RLC circuit while the electric coupling (iris) is represented by an additional capacitor C′. The analysis of such a network, in the absence of losses ($R = 0$), leads to the dispersion relation

$$\omega^2 = \omega^2_{\pi/2}(1 - K\cos\beta d) \ , \tag{94}$$

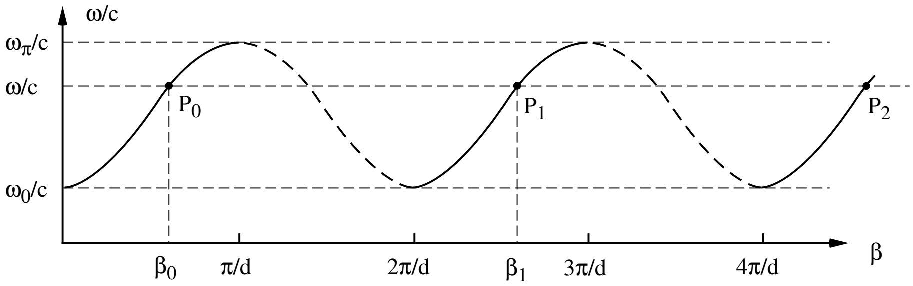

Fig. 16: Brillouin diagram for a slow wave structure

where $K < 1$ is the coupling factor.

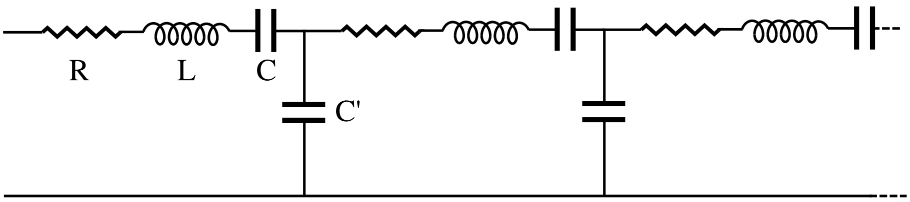

Fig. 17: Coupled resonant circuits

18 DEFLECTING CAVITIES

For the accelerating TM modes considered abhttp://mostinfo.net/hlam/test/ove, only the E_z component was non-zero on axis. However, higher order modes can show non-zero transverse components on axis as well, which can then deflect particles.

a) Standing-wave cavities

Consider the TM_{110} mode with components (θ, r, z):

$$\begin{aligned} E_z &= E_0 J_1(kr)\cos\theta &\approx E_0 kr\cos\theta \quad \text{(near axis)} \\ H_r &= -j\frac{E_0}{Z_0}\frac{J_1(kr)}{kr}\sin\theta &\approx -j\frac{E_0}{Z_0}\sin\theta \quad \text{(near axis)} \\ H_\theta &= -j\frac{E_0}{Z_0}J_1'(kr)\cos\theta &\approx 0 \qquad\qquad \text{(near axis)}\ . \end{aligned} \tag{95}$$

Since $r\cos\theta = x$ and if one assumes $\sin\theta = 1$ then H_r becomes H_y and

$$\begin{aligned} E_z &= \left(\frac{\partial E_z}{\partial x}\right) x & \text{(zero on axis)} \\ H_y &= -j\frac{E_0}{Z_0} = -\frac{j}{Z_0 k}\left(\frac{\partial E_z}{\partial x}\right) & (\neq 0 \text{ on axis})\ . \end{aligned} \tag{96}$$

Consider now a relativistic electron ($v \leq c$) traversing the cavity of length L on the axis. It gets a horizontal impulse from the H_y component,

$$\Delta p_x = -e\int_0^L (v_z \mu H_y)\frac{dz}{v_z} = j\frac{e}{\omega}\int_0^L \left(\frac{\partial E_z}{\partial x}\right) dz\ , \tag{97}$$

showing that a SW TM mode can deflect relativistic particles travelling on the axis. This transverse kick is related to the transverse gradient of the longitudinal electric field.

b) Travelling-wave cavities

Consider now a TW structure with transverse field components and a particle travelling parallel to the axis with $v \leq c$:

$$\begin{aligned} \mathcal{E}_\perp &= \mathbf{E}_\perp(x,y)e^{j(\omega t-\beta z)} \\ \mathcal{H}_\perp &= \mathbf{H}_\perp(x,y)e^{j(\omega t-\beta z)} \quad , \end{aligned} \tag{98}$$

where the phase velocity is $v_p = \omega/\beta$. The Newton Lorentz force perpendicular to the axis is

$$\mathbf{F}_\perp = e[\mathbf{E}_\perp + v\mu(\mathbf{u}\times\mathbf{H}_\perp)]e^{j(\omega t-\beta z+\phi_0)} \quad , \tag{99}$$

where **u** is the unit vector along the axis. Analysis of Maxwell solutions for TM modes gives the identity

$$\mu(\mathbf{u}\times\mathbf{H}_\perp) = -\frac{1}{v_p}\mathbf{E}_\perp + \frac{j}{\omega}\nabla_\perp \mathrm{E}_z \quad , \tag{100}$$

hence the force becomes

$$\mathbf{F}_\perp = e\left[\left(1-\frac{v}{v_p}\right)\mathbf{E}_\perp + j\frac{v}{\omega}\nabla_\perp \mathrm{E}_z\right]e^{j(\omega t-\beta z+\phi_0)} \quad . \tag{101}$$

If the particle is synchronous with the wave, $v = v_p$, and since $\omega t = \omega \frac{z}{v} = \beta z$ one gets:

$$\mathbf{F}_\perp = \frac{e}{\beta}\nabla_\perp E_z \quad , \tag{102}$$

with the particle on the crest of the wave. In principle there will be no transverse deflection from the TM_{01} mode. In practice, however, input and output couplers can give a local field asymmetry that can provide locally a non-zero $\nabla_\perp E_z$. Higher order modes can deflect particles and as a matter of fact they can be used as RF beam separators.

The relation between the transverse deflection and the transerve gradient of the longitudinal electric field is often referred as the Panofsky–Wenzel theorem. [2]

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[2] W.K.H. Panofsky, W.A. Wenzel, *Review of Scientic Instrument*, **27**, (1956) 967.