3.3 <u>GEOMETRICAL THEORY OF SCOTCHLITE</u>

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The principal features of the reflective characteristics of "Scotchlite" can be explained by means of geometrical optics, i.e., ray tracing. This ray-tracing was performed by René Descartes who was concerned with explaining the meteorological phenomenon of the rainbow. Descartes undertook what may well be one of the first "Monte Carlo" calculations on record. In a paper entitled "De l'Arc-En-Ciel"(1) he published the results of tracing 10,000 rays through the successive processes of refraction reflection and refraction upon entering, traversing, and leaving a spherical droplet of water of index of refraction n = 250/187 = 1.337. Fig. 1 shows his diagrams, the rainbow being described there in Fig.19. Ray tracing is shown in Fig.22, where EF is the incident direction and FKNP is the path of the refracted and reflected ray.

The results of Descartes' computer output are summarized in Fig.2. Fig.3 shows the angular distribution in differential form presented in a modern way. A sharp "resonance" is seen at 41°, 30 min. which is also the cutoff value of the angular distribution. Fig.4 shows the paths of several rays as they traverse the water drop. It is seen that successive rays leave the drop at increasing angles until the 41 30 "resonance" angle is reached. At this point the angle begins to decrease again so that here the angle $41^{\circ}30'$ is the maximum attainable scattering angle.

Scotchlite is a reflective sheeting produced commercially by the 3M Company. This product has the property of directing a large fraction of the light reflected by it back toward the vicinity of the light source. If the angular distribution for the reflected light is sharp compared with the scattered light distribution from bubbles in liquid hydrogen, it becomes possible to observe these bubbles as dark objects against a bright

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(1) Oeuvres de Descartes, Tome V^eme^, Discours VIII^eme^, De-l'Arc-en-Ciel.

background when they are viewed from a point near the light source. This distribution, peaked in the forward direction, initially falls very sharply diminishing by a factor e^{-1} every seven degrees or so. Thus, if most of the light being returned by the Scotchlite towards the camera aperture is directed within an angular range -- small or comparable to seven degrees, an intervening bubble will scatter substantially more light out of the camera aperture than into it, thereby providing a dark bubble image against a bright background.

The optical elements of Scotchlite consist of small beads of transparent material having high refractive index. The beads are coated on one hemisphere with a metallic reflective coating, and are attached on their coated side to a backing sheet. In Scotchlite sheeting, the beads are packed closely so as to effectively provide a continuous surface having the reflective characteristics of the beads which were used.

Consider a bead having an index of refraction relative to its surroundings of n. Let the axis of the bead be defined by the line traversing the bead axially in a direction parallel to the direction of the incident light as shown in Fig. 5. Rays striking the bead near the north pole ($\phi = 0$) will be reflected in the region of the south pole of the bead and emerge with the deflection angle e. We may recall from geometrical optics that for $n = 2$ paraxial rays will come to a focus at the south pole of the bead, and will therefore emerge with $e = 0$. If $n < 2$, we find that the "paraxial rays" are reflected at points R, not the south pole, resulting in a finite deflection angle. Fig.5 shows just such a ray.

The "spherical aberration" due to the beads' surface plays an increasingly strong role as ϕ increases, and if $n > \sqrt{2}$ we find that for some value of ϕ the point R coincides with the south pole and e is zero. For $n = \sqrt{2}$ this condition is met when $\phi = 90^\circ$; see Fig.6.

In order to maximize the amount of light being returned with small e, it is desirable to usefully employ regions of ϕ not in the paraxial range. In Fig.7 we plot e vs. ϕ for a bead of relative index 1.90. For this index we see that $e \lesssim 1.3^{\circ}$ for all values of $\phi \leq 53^{\circ}$. This represents 64°/o of the projected area presented by the spherical bead. It is interesting to observe that Scotchlite FE-582 is produced with beads having an index near 1.9. The small size of the beads dictates smearing of the geometrical distribution by several tenths of a degree due to physical optics effects. Thus, the index 1.9 may very well correspond to an optimum index for maximizing the returned light near $e = 0$.

Similar curves can be drawn for different values of n. When $n = 2.0$, the curve is tangent at $\phi = 0$ and reaches no maximum. For $n = \sqrt{2}$ we find the whole curve above the axis, returning to $e = 0$ only at $\phi = 90^{\circ}$.

In Fig.7 the region of the curve corresponding to the range $10^{\circ} < \phi < 30^{\circ}$ results in reflected light in the range $0.9^{\circ} \leq e \leq 1.3^{\circ}$. This range in ϕ corresponds geometrically to $\approx$ 20°/o of the projected area of the bead. Thus, if physical optics effects did not wash out the structure, we would see a "rainbow" in the vicinity of $e = 1^{\circ}$ whose total radiant energy accounted for about 20°/o of the reflected light.

For dark field illumination of bubble chambers, the situation is somewhat changed. In this case it is desired to allow no light to enter the camera pupil from the Scotchlite background directly. The bubbles in the liquid hydrogen are now detected by the light which they scatter into the aperture of the photographic objective. To bring this about it is, of course, necessary to provide an intense beam of light directed "close" to the camera aperture as measured relative to the light scattering characteristics of hydrogen bubbles. In order to provide that as little light as possible enter the camera aperture directly from the background, it is necessary to have the intense beam of light directed "far" from the camera aperture as measured relative to the characteristics of this beam.

The above mentioned conditions can be approached using a Scotchlite reflector if the light source being used is:

a) close enough to the camera aperture so that the light returning towards it from the Scotchlite background need be scattered by a small angle to strike the aperture;

b) far enough from the camera aperture so that only a small intensity is directed into the aperture without having undergone any scattering.

The desired characteristics of the Scotchlite now require a sharp edge in the angular distribution in order to provide a high intensity of light directed near the camera aperture while at the same time allowing little or no light to actually enter the aperture.

We seek a reflective material where angular characteristics limit the reflected light to a cone of a reasonably small opening angle. In Fig.8 we have a family of curves similar to the curve in Fig.7 but drawn for different indices of refraction. In order to make the presentation more meaningful, the abscissa in this plot corresponds to the projected area of the disk instead of the co-latitude angle φ. We may interpret the curves of Fig.8 to indicate for example, that for beads of refractive index $n = 1.70$ about $90^{o}/o$ of the reflected light is returned within a cone of about 9^{o} half-angle.

The figure, of course, represents a kind of ideal geometrical Scotchlite. The characteristics of real Scotchlite differ somewhat from the sharp curves. We may expect the following:

a) The sharp curves should be made fuzzy due to the physical optics diffraction effects. The amount of spreading is dependent on the size of the beads being used but will in any event be about $1/2^{o}$ to 1^{o}.

b) The geometric curves will be somewhat "improved" in the vicinity of 0.6 to 1.0 of the projected area. This is due to the fact that the reflection upon entering the bead and upon leaving the bead reduces the intensity in an important way for rays striking the interface at an oblique angle.

c) The light "lost" to the cone by the mechanism described above as well as the light reflected by the "non-optical" surfaces between the spheres of course contributes to the general background intensity.

For a dark-field illumination system it is, of course, desirable to have as narrow a cone as is practical since the hydrogen scattering function has a 5° doubling rate. However, since it will not be possible to avoid background reflection, it will also not be useful to impose a $100^{\circ}/o$ criterion on the cone since narrower cones result from $95^{\circ}/o$ or $90^{\circ}/o$ criteria. In view of this it seems likely that a material with $n = 1.7$ or $n = 1.75$ may prove to give optimum contrast for bubbles in hydrogen against the general background.

We can see in Fig.8 that about $40^{\circ}/o$ of the light will be returned between the angles of 7° and 9° for beads of $n = 1.7$. For this material, then, the "rainbow" angle is in this vicinity, and only $10^{\circ}/o$ or less of the light will fall outside of this bright ring.

Fig.9 shows the estimated divergence function based on geometrical considerations for beads of index $n = 1.7$. It was derived from the curve in Slide 10 integrating in divisions of one degree. The peak near 9° is, of course, the rainbow peak. Fig.10 shows the rainbow angle as a function of bead index calculated on the basis of the geometrical theory.

It is interesting to note that Descartes, too, found a "practical" application for varying the index of refraction of the medium producing the rainbow. In the paper cited earlier he suggests the construction of a complex fountain in which fluids of different indices of refraction can be projected into various regions of the air. By judicious arrangement of the nozzles it is possible to assemble different opening angles into a mystical figure appearing in the sky. He suggests a glowing column or a glowing cross: "... ouvrant et fermant à propos les trous de ces diverses fontaines, on pourra faire que ce qui paraîtra coloré ait la figure d'une croix, ou d'une colonne, ou de quelque autre telle chose qui donne sujet à l'admiration. Mais j'avoue qu'il y faudrait de l'adresse et de la dépense, afin de proportionner ces fontaines, et faire que les liqueurs y sautassent si haut que ses figures puissent être vues de fort loin par tout un peuple sans que l'artifice s'en découvrit.

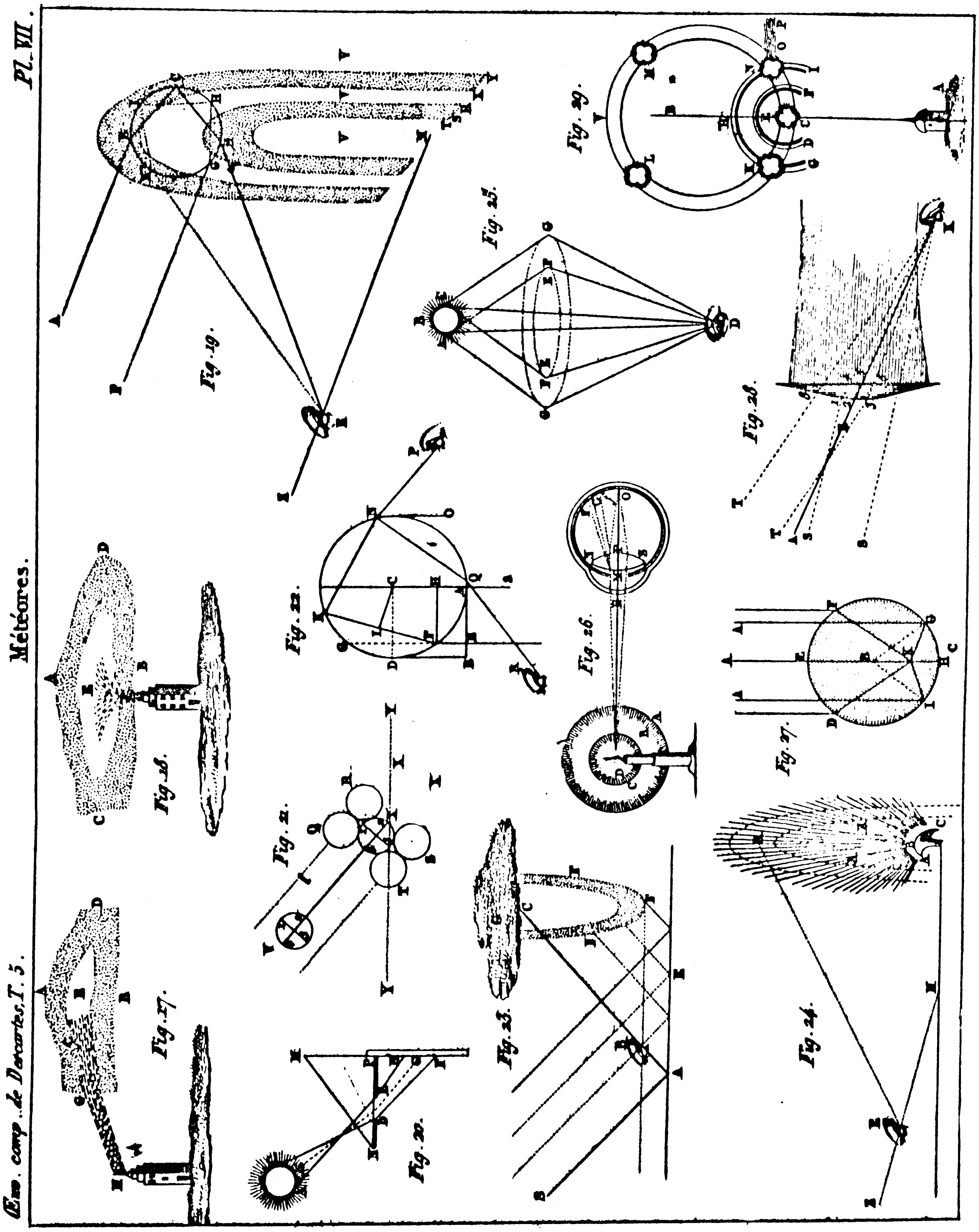


Fig. 1 - Ray Tracing Diagrams by René Descartes.

Fig. 2 - Descartes' Computer Output*

La Ligne HF	La Ligne CI	L'Arc FG	L'Arc FK	L'Angle ONP	L'Angle SQR
1000	748	168,30	171,25	5,40	165,45
2000	1496	156,55	162,48	11,19	151,29
3000	2244	145, 4	154, 4	17,56	136,' 8
4000	2992	132,50	145,10	22,30	122, 4
5000	3740	120	136, 4	27,52	108,12
6000	4488	106,16	126,40	32,56	93,44
7000	5236	91, 8	116,51	37,26	79,25
8000	5984	73,44	106,30	40,44	65,46
9000	6752	51,41	95,22	40,57	54,25
10000	7480	0	83,10	13,60	69,30
8000	5984	73,44	106,30	40,44	65,46
8100	6058	71,48	105,25	40,58	64,37
8200	6133	69,50	104,20	41,10	63,10
8300	6208	67,48	103,14	41,20	62,54
8400	6283	65,44	102, 9	41,26	61,43
8500	6358	63,34	101, 2	41,30	60,32
8600	6432	61,22	99,56	41,30	58,26
8700	6507	59, 4	98,48	41,28	57,20
8800	6582	56,42	97,40	41,22	56,18
8900	6657	54,16	96,32	41,12	55,20
9000	6732	51,41	95,22	40,57	54,25
9100	6806	49, 0	94,12	40,36	53,36
9200	6881	46, 8	93, 2	40, 4	52,58
9300	6956	43, 8	91,51	39,26	52,25
9400	7031	39,54	90,38	38,38	52, 0
9500	7106	36,24	89,26	3 ,32	51,54
9600	7180	32,30	88,12	36, 6	52, 6
9700	7255	28, 8	86,58	34,12	52,46
9800	7330	22,57	85,43	31,31	54,12

*This table was taken from pages 278-279 of the referenced book (see Ref. 1).

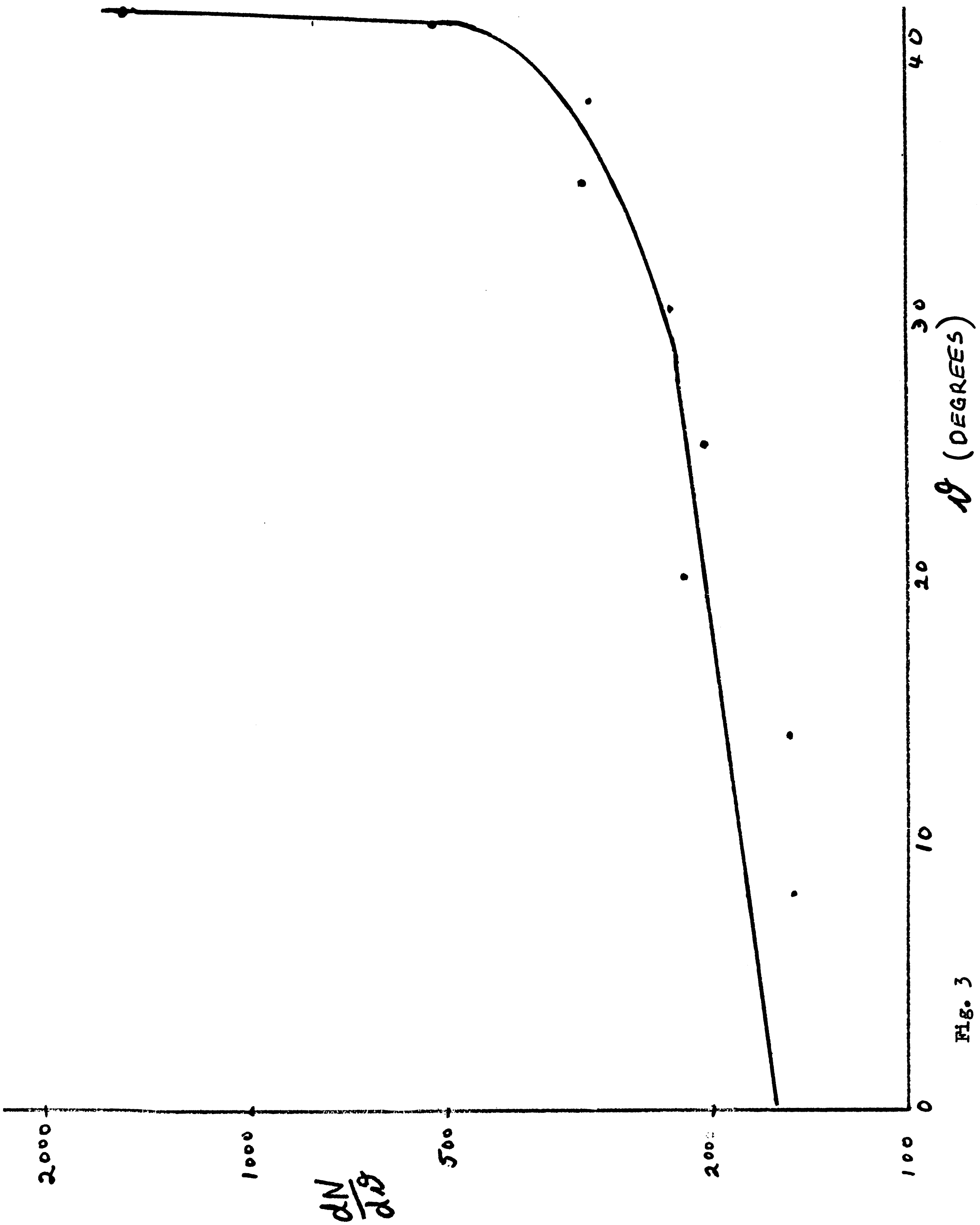


Fig. 3

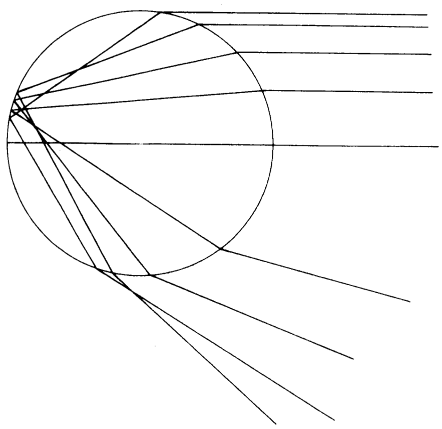


Fig. 4

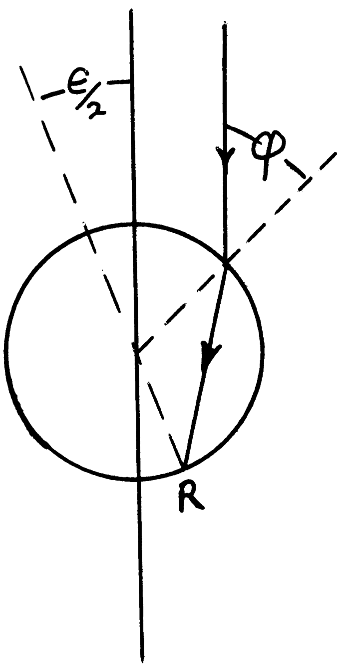


Fig. 5

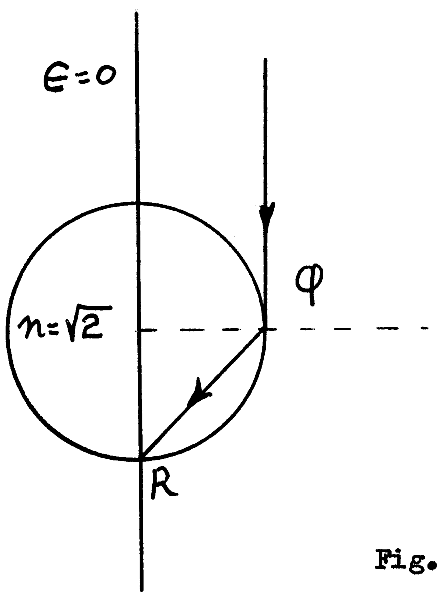
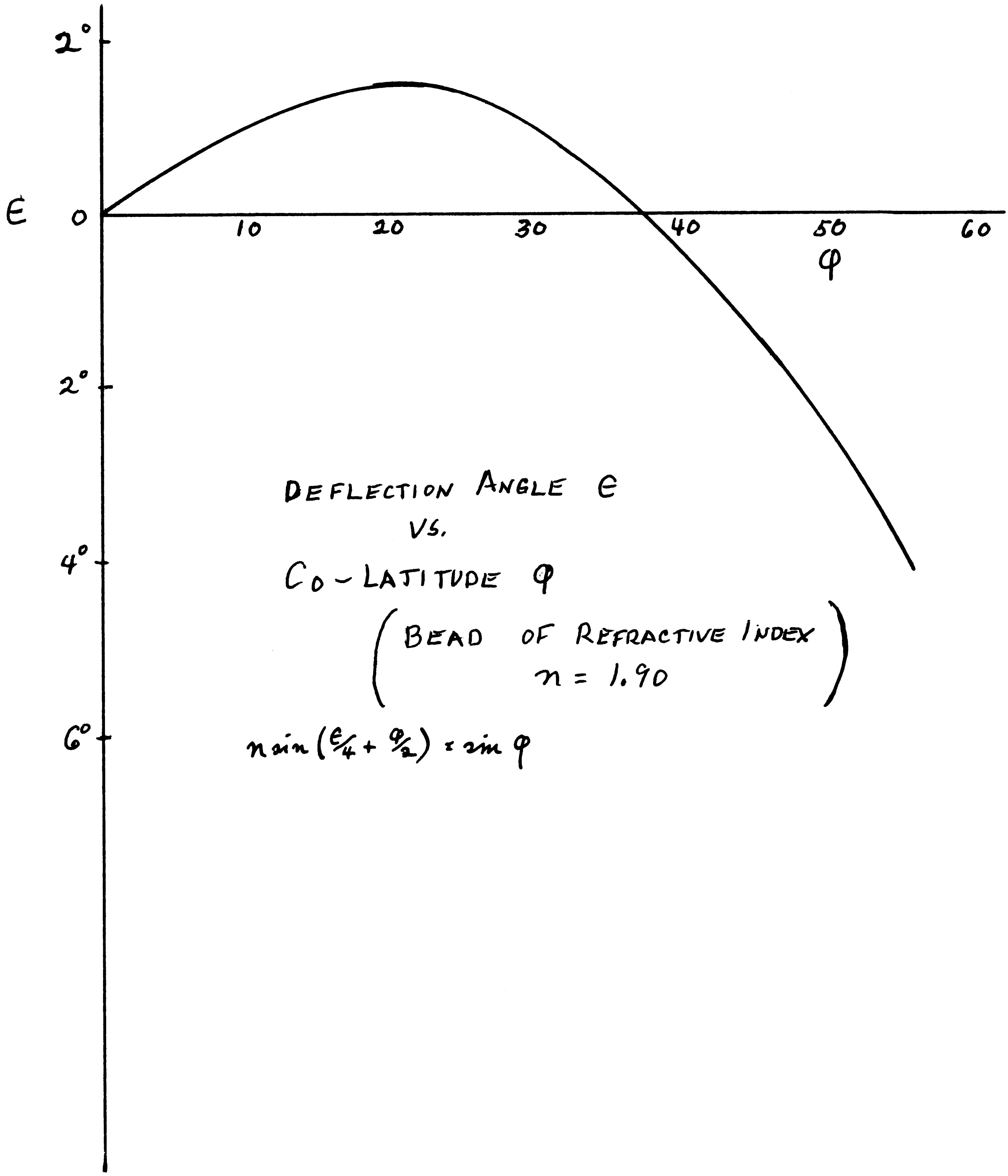


Fig. 6



2°
ε
0
10
20
30
40
50
60
φ
2°
4°
6°
DEFLECTION ANGLE ε
VS.
CO-LATITUDE φ
(BEAD OF REFRACTIVE INDEX n = 1.90)
$n \sin(\epsilon/4 + \varphi/2) = \sin\varphi$

Fig. 7

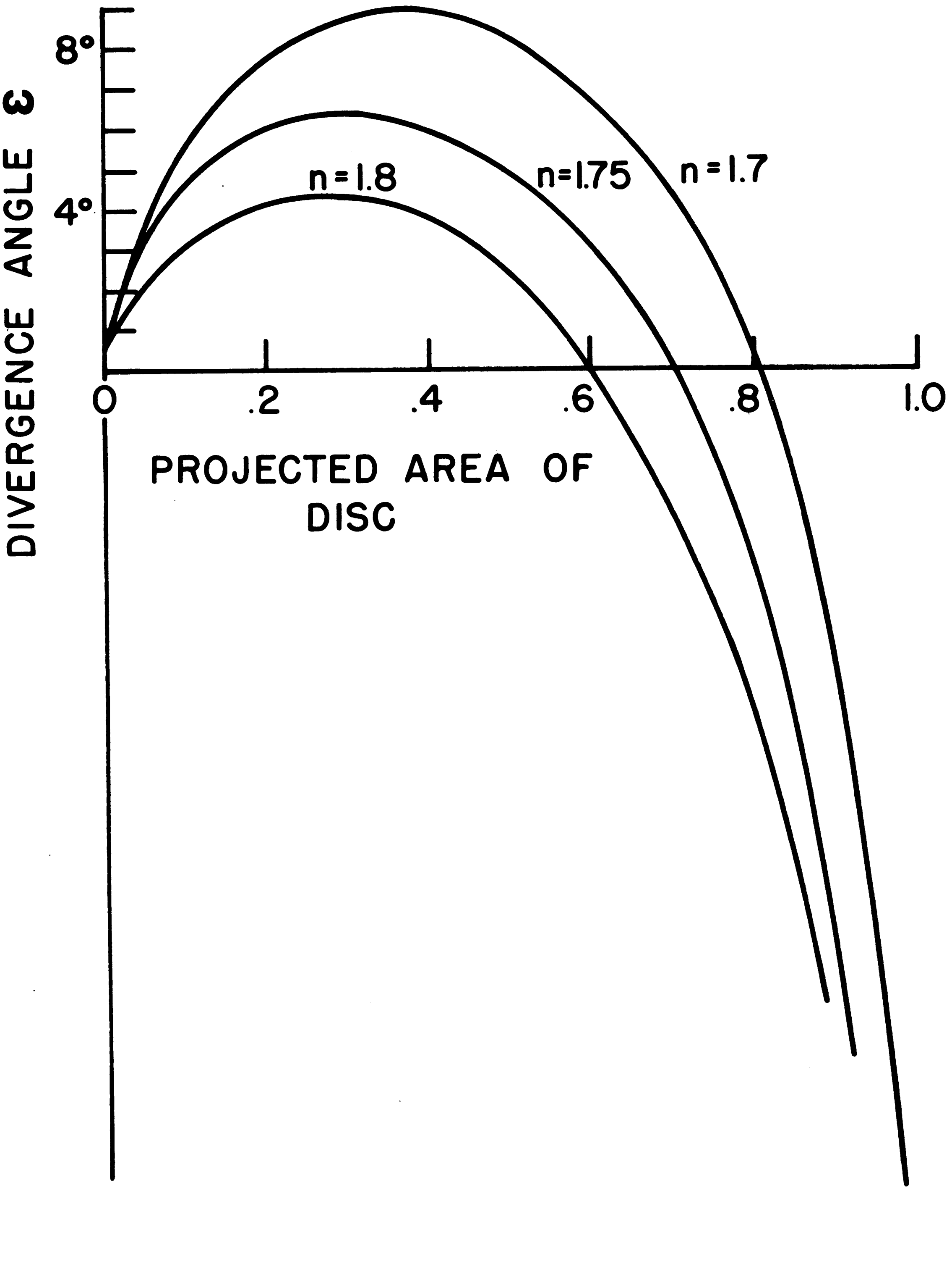

8°
4°
DIVERGENCE ANGLE ε
n=1.8
n=1.75
n=1.7
0
.2
.4
.6
.8
1.0
PROJECTED AREA OF DISC

Fig. 8

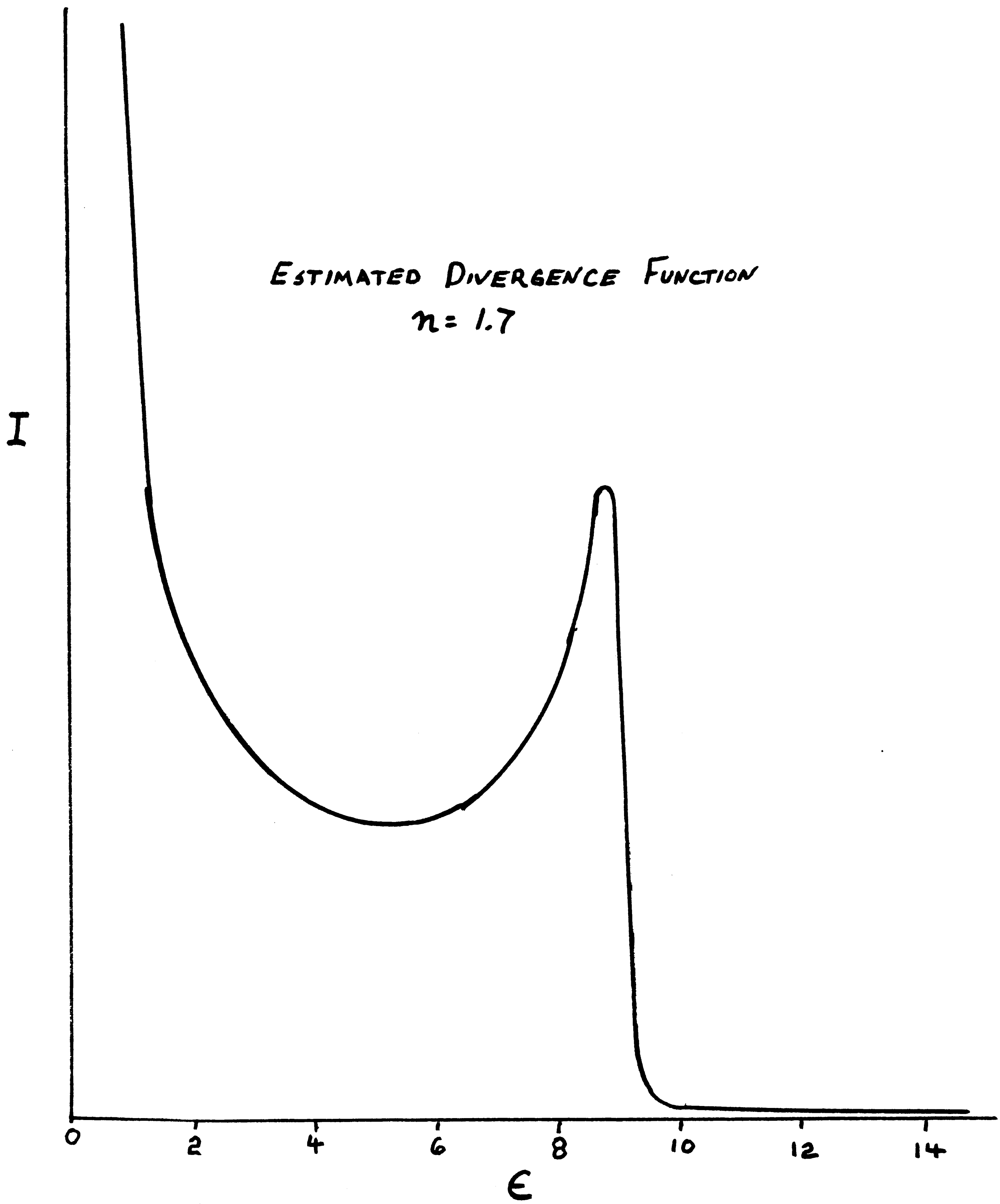


Fig. 9

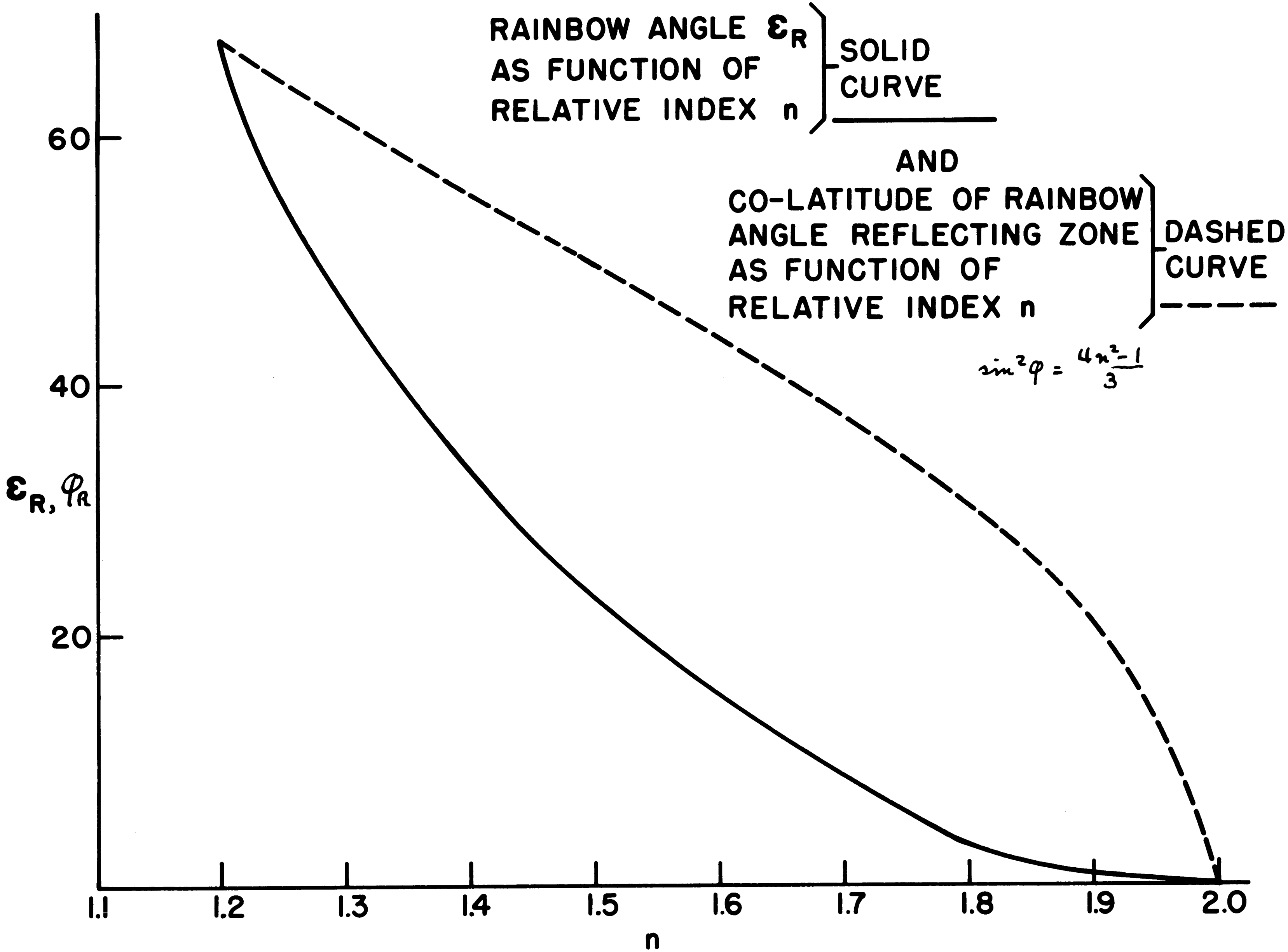

RAINBOW ANGLE ε_R AS FUNCTION OF RELATIVE INDEX n — SOLID CURVE
AND
CO-LATITUDE OF RAINBOW ANGLE REFLECTING ZONE AS FUNCTION OF RELATIVE INDEX n — DASHED CURVE
$\sin^2\varphi = \frac{4n^2-1}{3}$
60
40
20
ε_R, φ_R
1.1
1.2
1.3
1.4
1.5
1.6
1.7
1.8
1.9
2.0
n

Fig. 10