ELECTROMAGNETIC FORM FACTORS OF NUCLEONS

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I. INTRODUCTION

In these lectures we shall discuss some aspects of the interaction between electrons and nucleons which go beyond quantum electrodynamics. They arise from the fact that the nucleon is not just a Dirac particle coupled to the electromagnetic field but a hadron and, as such, involved in strong interactions. The deviations of the nucleons behaviour in electromagnetic interactions from that of a Dirac particle are usually interpreted as being due to a structure of the nucleon originating in its coupling to pions and other hadrons. Although the effects of strong interactions are already reflected by the static properties of the nucleons, namely by their anomalous magnetic moments, they can be studied in more detail by electron-proton scattering, in particular in processes with large momentum change of the nucleon.

Since it is impossible in three lectures to give both an introduction and a comprehensive survey of the subject, we shall concentrate on those aspects which arise if one evaluates and interprets electron-proton scattering experiments at large momentum transfers[1].

II. GENERAL NOTATION

We consider the process of electron-proton scattering according to

$$e^- + p \rightarrow e^- + p \qquad k + p = k' + p' \ . \tag{1}$$

k, p, k' and p' denote the particle four momenta $(k_0, \vec{k})$ such that

$$k^2 = k_0^2 - \vec{k}^2 = m^2 \qquad p^2 = p_0^2 - \vec{p}^2 = M^2 \ . \tag{2}$$

m and M are the masses of electron and proton, respectively. With the process we can associate a diagram as that of Fig. 1.

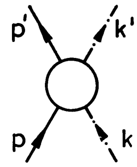

FIG.1

The scattering is described by a scattering amplitude T such that the cross-section can be expressed as :

$$d\sigma = \frac{4}{|\vec{v}_1 - \vec{v}_2|} \frac{(2\pi)^3}{2p_0} \frac{d^3p'}{2p_0'} \frac{d^3k'}{2k_0'} \frac{\delta(p+k-p'-k')}{(2\pi)^4} |T|^2 \quad . \tag{3}$$

$|\vec{v}_1 - \vec{v}_2|$ is the relative velocity of the incoming particles. T is connected with the S-matrix through[2)]

$$< p'k' |\frac{S-1}{2i}| pk > = \delta(p+k-p'-k')\,T \quad . \tag{4}$$

It is more convenient to use variables P, Q, s, t defined as

$$\begin{aligned} P &= (p+k) = (p'+k'); \quad P^2 = s \\ Q &= (p-p') = (k'-k); \quad Q^2 = t \quad . \end{aligned} \tag{5}$$

Frequently $Q^2 = -t$ is also used in the literature.

For electron-proton scattering these variables have the following meaning:

	P :	total energy-momentum four-vector
e-p:	Q :	four-momentum transfer
	s :	square of c.m.s. energy
	t :	invariant momentum transfer.

The process of electron-proton scattering through the substitution law is connected with proton-antiproton annihilation into an electron-positron pair. (Remember that an incoming particle with charge e and momentum p corresponds to an outgoing antiparticle with charge −e and momentum −p.) For p-$\bar{p}$ annihilation P, Q, s, t have a different meaning and assume different values:

		e-p scattering	p-$\bar{p}$ annihilation
	s :	square of c.m.s. energy	inv. momentum transfer
	t :	inv. momentum transfer	square of c.m.s. energy
physical region:		$t \le 0$	$t > 4M^2$

The connection between e-p scattering and p-$\bar{p}$ annihilation has an immediate consequence: the squared matrix elements $\overline{\overline{|T|^2}}$, averaged over spin orientations, entering into the cross-section for unpolarized beam and target and without analysing polarization, is given by the same function, of course for different values of the variables s and t. If p-$\bar{p}$ annihilation proceeds through a finite number of angular momentum states with $l < L$, $\overline{\overline{|T|^2}}$ is of the form

$$\overline{\overline{|T|^2}} = A_0(t) + A_1(t)\cos\vartheta_t + A_2(t)\cos^2\vartheta_t + \dots + A_{2L}\cos^{2L}\vartheta_t \quad . \tag{6}$$

The dependence on s is fully contained in $\cos\vartheta_t$, where ϑ_t is the angle between p and e^+ in the c.m.s. of p-$\bar{p}$ annihilation:

$$\cos\vartheta_t = \frac{2(s - M^2 - m^2) + t}{2(t/4 - M^2)^{1/2}\,(t/4 - m^2)^{1/2}} \quad . \tag{7}$$

This implies for e-p scattering a particular dependence on $ctg^2\,\Theta/2$, where Θ is the scattering angle of the electron in the laboratory system:

$$\cos\vartheta_t = \left\{1 + \frac{1}{1+\tau}\,ctg^2\frac{\Theta}{2}\right\}^{1/2};\quad \tau = \frac{-t}{4M^2} = \frac{q^2}{4M^2} \tag{8}$$

$$\overline{|T|^2} = A_0(t) + \left\{1 + \frac{1}{1+\tau}\,ctg^2\frac{\Theta}{2}\right\}^{1/2} A_1(t) + \ldots + \left\{1 + \frac{1}{1+\tau}\,ctg^2\frac{\Theta}{2}\right\}^{2} A_{2L}(t)\,. \tag{9}$$

An interaction proceeding via angular momentum 1 in $p\text{-}\bar{p}$ annihilation corresponds to the exchange of spin 1 in e-p scattering. Odd powers of $\cos\vartheta_t$ appear only if there is interference between contributions of different parity.

III. STRUCTURE OF THE SCATTERING AMPLITUDE

The electromagnetic interaction between electrons and protons can be described by the exchange of photons, i.e. of quanta of the electromagnetic field. We decompose the scattering amplitude into terms corresponding to different numbers of photons being exchanged (cf. Fig. 2).

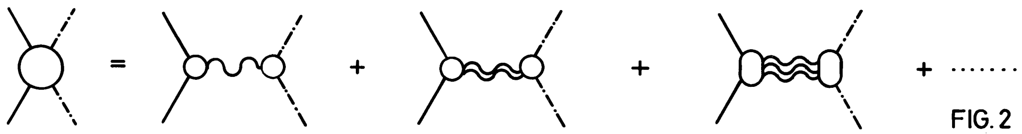

FIG. 2

Such a decomposition arises if one uses perturbation theory for the electromagnetic interaction. We shall restrict our discussion mainly to the one-photon exchange contribution, which seems to describe most experimental data very well. This agreement might be correlated with the fact that the multiple-photon exchange term contains higher powers of the fine-structure constant α, which is small ($\approx 1/137$). But we shall discuss explicitly methods to test the validity of the one-photon exchange approximation.

The one-photon contribution, according to the Feynman rules, turns out to be

$$T = \frac{1}{2i}\,(-i)(2\pi)^4\; e\,\frac{\bar{U}(k')\,\gamma_\mu U(k)}{(2\pi)^3}\,\frac{(-i)}{(2\pi)^4}\,\frac{1}{Q^2}\,(-i)(2\pi)^4\,\frac{\langle J_\mu\rangle}{(2\pi)^3}$$

$$T = \frac{1}{2}\,\frac{e}{(2\pi)^2}\,\bar{U}(k')\,\gamma_\mu U(k)\,\frac{1}{t}\,\langle J_\mu\rangle\,. \tag{10}$$

$\langle J_\mu\rangle$ is the matrix element of the electromagnetic current between the states of the particle by which the electron is scattered:

$$\langle J_\mu\rangle = (2\pi)^3\,\langle\vec{p}'s'|J_\mu(0)|\vec{p}\,s\rangle\,. \tag{11}$$

Here, s and s' are the spin quantum numbers of the proton. The electric charge e appearing as coupling constant in Eq. (10) is normalized such that $e^2/4\pi = \alpha$.

Usually one works with unpolarized electron beams and does not analyse the polarization of the outgoing electrons. The appropriate spin average of $|T|^2$ can be written as:

$$\overline{|T|^2} = \frac{1}{2}\,\alpha\,\frac{1}{(2\pi)^3}\,\{(k'+k)_\mu\,(k'+k)_\nu + (tg_{\mu\nu} - Q_\mu Q_\nu)\} < J_\mu > < J_\nu >^* \,. \qquad (12)$$

If also the target is unpolarized and if the polarization of the outgoing target particles is not observed, the corresponding spin average entering into the cross-section can be written as:

$$\frac{1}{2S+1} \sum_{\text{spins}} < J_\mu > < J_\nu >^* = e^2 \{a(t)(p'+p)_\mu\,(p'+p)_\nu + b(t)(tg_{\mu\nu} - Q_\mu Q_\nu)\}\,. \qquad (13)$$

This general form is independent of the magnitude S of the spin of the target particle. For a spin 0 and for a spin ½ particle, coupled only to the electromagnetic field, a(t) and b(t) according to the Feynman rules turn out to be:

$$\begin{aligned} &\text{spin } 0 \;:\; < J_\mu > = e(p'+p)_\mu \to a(t) = 1 \qquad b(t) = 0, \\ &\text{spin } 1/2 \;:\; < J_\mu > = e\,\bar{U}(p')\,\gamma_\mu U(p) \to a(t) = 1 \quad b(t) = 1. \end{aligned} \qquad (14)$$

Here, of course, it has been assumed that the spin ½ particle has no anomalous magnetic moment. Particles with couplings to the electromagnetic field, as those indicated in Eq. (14), we shall call particles with point-like electric charges.

From Eq. (13) we obtain for $\overline{\overline{|T|^2}}$:

$$\overline{\overline{|T|^2}} = \frac{\alpha^2}{4\pi^2}\,\frac{2}{t^2}\,\{2[t(S-m^2) + (S-M^2-m^2)]\;a(t) + t(t+2m^2)\,b(t)\}\,. \qquad (15)$$

This gives for the cross-section[3]:

$$d\sigma = (d\sigma)_{NS} \left\{a(t) + \frac{t(t+2m^2)}{t(S-m^2) + (S-M^2-m^2)}\,\frac{b(t)}{2}\right\}. \qquad (16)$$

Here $(d\sigma)_{NS}$ is the cross-section for the scattering of electrons by a target with no spin and electric point charge. The differential cross-section in the laboratory system can be expressed as:

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_{NS} \left(a(t) + 2\tau\;tg^2\,\frac{\Theta}{2}\,b(t)\right), \qquad (17)$$

with

$$\left(\frac{d\sigma}{d\Omega}\right)_{NS} = \left(\frac{d\sigma}{d\Omega}\right)_{Mott} \frac{1}{1+\frac{2E}{M}\sin^2\frac{\Theta}{2}}\;;\quad \left(\frac{d\sigma}{d\Omega}\right)_{Mott} = \frac{\alpha^2}{4E^2}\,\frac{\cos^2\frac{\Theta}{2}}{\sin^4\frac{\Theta}{2}}\,. \qquad (18)$$

E is the laboratory energy of the incoming electron. $E \gg m$ has been assumed. The functions $a(t)$ and $b(t)$ can be determined as follows: $(d\sigma/d\Omega)/(d\sigma/d\Omega)_{NS}$ for fixed t plotted versus $tg^2\,\Theta/2$ gives a straight line. Slope and intercept give $a(t)$ and $b(t)$.

If in Eq. (15) we express the dependence on s in terms of $\cos\vartheta_t$ we obtain

$$\overline{\overline{|T|^2}} = \frac{\alpha^2}{\pi}\left\{\left[4M^2 t\,(1+\tau)\,a(t) + t(t+2m^2)\,\frac{b(t)}{2}\right] + \left(\frac{t}{4} - M^2\right)\left(\frac{t}{4} - m^2\right) a(t)\cos^2\vartheta_t\right\} . \tag{19}$$

This is just the form expected for an interaction through angular momentum 1 with definite parity. This reflects that the photon has spin 1 and parity minus.

IV. STRUCTURE OF THE VERTEX FUNCTION, FORM FACTORS

We now turn to the discussion of $< J_\mu >$. The current $J_\mu(x)$ plays the role of a source of the electromagnetic vector potential $A_\mu(x)$ as can be seen from the familiar equation

$$\Box A_\mu(x) = - J_\mu(x) .$$

If we compute the amplitude for the scattering of electrons by an external field, $< J_\mu >$ entering into the expression corresponding to Eq. (10) is the Fourier transform of the charge-current distribution producing the external field. If we have a static charge distribution only $< J_0 > \neq 0$.

For a point source $< J_0 > = e$, while for a charge distribution

$$< J_0 > = e \int \rho(\vec{x})\, e^{i\vec{Q}\cdot\vec{x}}\, d^3x = e\,f(t) . \tag{20}$$

The function $f(t)$, i.e. the Fourier transform of the static charge distribution, is called "form factor". The scattering cross-section can be written as

$$d\sigma = (d\sigma)_p\,|f(t)|^2 , \tag{21}$$

where $(d\sigma)_p$ is the cross-section for the scattering by a fixed point charge.

The transition to the scattering by a particle with finite mass will produce modifications due to recoil and associated acceleration. Even in classical physics we expect that:

a) the movement of the charge will in general make $< \vec{J} > \neq 0$,

b) the acceleration will make the charge distribution vary during the scattering process.

The effect a) can be eliminated by going to an appropriate co-ordinate system such that the contributions to $< \vec{J} >$ cancel for a particle without magnetic moment.

Consider a spin zero particle with point charge. We have $< j_\mu > = e(p' + p)_\mu$. We require $< \vec{J} > = 0$. This corresponds to a frame of reference where $\vec{p}' = - \vec{p}$. Such a system is well known in scattering theory. It is called the "Breit system" or "brickwall system". The momenta in the Breit system can be represented as in Fig. 3 (see next page).

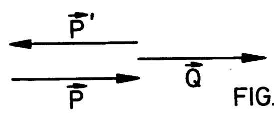

FIG. 3

Quantities in the Breit system we shall mark with a subscript B. We have

$$< j_0 >_B = e(p_0' + p_0); \quad < \vec{j} >_B = 0 \tag{22}$$

for spinless point charge.

The general structure of $< j_\mu >$ in the Breit system for a spinless particle follows from invariance considerations [which the limited space does not permit us to give in detail, c.f. reference 4)]:

$$< j_0 >_B = e(p_0' + p_0)\, G_E(t), \quad < \vec{j} >_B = 0 \quad \text{spin } 0 \,. \tag{23}$$

The function $G_E(t)$, which reflects the deviation from a point charge, is called the electric form factor of the particle.

Although the transition from Eq. (22) to Eq. (21) looks similar to the transition from a fixed point charge to a fixed extended charge, some caution is necessary when interpreting $G_E(t)$ as the Fourier transform of a charge distribution. The charge-current distribution $\rho_\mu(x)$ for a state φ is the expectation value, i.e. the diagonal matrix element of $j_\mu(x)$:

$$\rho_\mu(x) = <\varphi | J_\mu(x) | \varphi> \quad , \tag{24}$$

whereas into the scattering amplitude there enters the Fourier transform of

$$< j_\mu(x) > = <\varphi' | j_\mu(x) | \varphi > \; , \tag{25}$$

where φ' moves relative to φ. Therefore in general $< J_\mu(x) > \neq \rho_\mu(x)$; only for $\varphi' = \varphi$, i.e. in forward scattering, we have $< J_\mu(x) > = \rho_\mu(x)$. This quantum mechanical argument can be supplemented by a classical consideration: during scattering the charge distribution is accelerated and therefore - unless the charge distribution is rigid, which relativistically is impossible - $G_E(t)$ reflects the influence of a varying charge distribution. We therefore shall refrain from taking the Fourier transform of $G_E(t)$ literally to be "the" charge distribution of the particle.

Next, we turn to particles with spin ½. We remember that j_μ corresponds to a source of photons, i.e. of particles with spin 1. For real photons ($Q^2 = 0$) only two spin orientations are possible with spin projections onto $\vec{Q}$ $S_{ph} = \pm 1$, corresponding to right- and left-handed circular polarization. Virtual photons ($Q^2 \neq 0$) also permit $S_{ph} = 0$. Now let $\vec{Q}$ be parallel to the 3-axis. Then circularly polarized photons carry a 3-component of angular momentum. Therefore a term corresponding to the interaction of circularly polarized photons will also exhibit a change of the spin projection of the particle (i.e. spin flip). The interaction with $S_{ph} = 0$ on the other hand will occur only without spin flip.

The four components $< j_\mu >$ are not independent as can be seen from current conservation:

$$\partial_\mu < J_\mu(x) > = 0 \rightarrow Q_\mu < J_\mu > = 0 \rightarrow Q_0 < J_0 > - \vec{Q} \cdot < \vec{J} > = 0 \quad . \tag{26}$$

With our choice of 3-axis this means that

$$< J_3 >_B = 0 \quad . \tag{27}$$

Of the remaining terms $< J_0 >_B$ corresponds to the interaction of photons with $S_{ph} = 0$. $< J_{1,2} >$ represent the source of transversely polarized photons. It is more convenient to work with circular polarization, i.e. with $< J^\pm >_B$:

$$< J^\pm >_B = \frac{1}{\sqrt{2}} \{ \mp < J_1 >_B - i < J_2 >_B \} \quad . \tag{28}$$

We now have

$$\begin{aligned} < J_0 >_B &= e\, \delta_{s's}\, \tilde{G}_E(t) \\ < J^\pm >_B &= e\, \delta_{s's\pm 1}\, \tilde{G}_M(t) \quad . \end{aligned} \tag{29}$$

The term with $\tilde{G}_M$ corresponds to the source of the field produced by the magnetic moment. We return to $< \vec{j} >$ and express the result in terms of matrix elements between Pauli spinors χ'_s, χ_s:

$$\begin{aligned} < J_0 >_B &= e\, \chi^+_{s'}\, \chi_s\, \tilde{G}_E(t) \\ < \vec{J} >_B &= \frac{ie}{2M}\, \chi^+_{s'}\, \vec{Q} \times \vec{\sigma} \chi_s\, \tilde{G}_M(t) \quad . \end{aligned} \tag{30}$$

Now we normalize the functions $\tilde{G}_E$ and $\tilde{G}_M$ appearing in Eq. (30) such that for a particle without structure they become equal to one. We have for such a particle:

$$\begin{aligned} < J_\mu > = e\, \bar{U}(p',s')\, \gamma_\mu U(p,s) \rightarrow < J_0 >_B &= e\, 2M\, \chi^+_{s'} \chi_s \\ < \vec{J} >_B &= \frac{ie}{2M}\, 2M\, \chi^+_{s'}\, \vec{Q} \times \vec{\sigma} \chi_s \quad . \end{aligned} \tag{31}$$

Therefore our general result for spin ½ becomes:

$$\begin{aligned} < J_0 >_B &= e\, 2M\, \chi^+_{s'}\, \chi_s\, G_E(t) \\ < \vec{J} >_B &= \frac{ie}{2M}\, 2M\, \chi^+_{s'}\, \vec{Q} \times \vec{\sigma} \chi_s\, G_M(t) \quad . \end{aligned} \tag{32}$$

The normalization is such that $G_E(0)$ = static charge in units of e, $G_M(0)$ = static magnetic moment in units of e/2M, i.e. with anomalous magnetic moment κ: $G_M(0) = 1 + \kappa$.

The general expression for $< j_\mu >$ can now be written in terms of a covariant matrix between Dirac spinors:

$$\begin{aligned} < J_\mu > &= e\, \bar{U}(p',s') \left\{ \frac{(p'+p)_\mu}{2M} \frac{G_E(\tau)}{1+\tau} + \frac{i}{2M}\, r_\mu \frac{G_M(t)}{1+\tau} \right\} U(p,s) \\ r_\mu &= \frac{i}{2} \left\{ Q \cdot \gamma\, \gamma_\mu (p'+p) \cdot \gamma - (p'+p) \cdot \gamma\, \gamma_\mu\, Q \cdot \gamma \right\} \quad . \end{aligned} \tag{33}$$

If we use that

$$\bar{U}\, r_\mu\, U = -i\, 2M\tau\, \bar{U}\, \gamma_\mu U - \bar{U}\, \sigma_{\mu\nu} Q_\nu U \; ,$$
$$(p' + p)_\mu \bar{U}U = 2M\, \bar{U}\, \gamma_\mu U + i\, \bar{U}\, \sigma_{\mu\nu} Q_\nu U \; , \tag{34}$$

the expression (33) can be rewritten as

$$< J_\mu > = e\, \bar{U}(p',s') \left\{ \gamma_\mu F_1(t) - \frac{i}{2M} \sigma_{\mu\nu} Q_\nu F_2(t) \right\} U(p,s)$$
$$= e\, \bar{U}(p',s') \left\{ \gamma_\mu G_M(t) - \frac{(p'+p)}{2M} \mu\, F_2(t) \right\} U(p,s) \; . \tag{35}$$

Here,

$$F_1 = \frac{G_E + \tau G_M}{1+\tau} \; , \qquad F_2 = \frac{G_M - G_E}{1+\tau}$$
$$G_E = F_1 - \tau F_2 \; , \qquad G_M = F_1 + F_2 \; ; \tag{36}$$

$F_1(t)$ and $F_2(t)$ are called Dirac and Pauli form factors, respectively. The general expression for $< j_\mu >$ now gives for the functions $a(t)$ and $b(t)$ appearing in the cross-section:

$$a(t) = (1+\tau)\, F_2^2 - 2F_2\, G_M + G_M^2 = \frac{G_E^2 + \tau\, G_M^2}{1+\tau}$$
$$b(t) = G_M^2 \; . \tag{37}$$

The cross-section may then be written as:

$$d\sigma = (d\sigma)_{NS} \left\{ G_E^2 + \tau \left(1 + 2\, tg^2 \frac{\Theta}{2}\right) G_M^2 \right\} \cdot \frac{1}{1+\tau} \; . \tag{38}$$

This formula is called the Rosenbluth formula. The fact that no interference terms between G_M and G_E appear is easily understood: G_E and G_M are the factors of the non-spin flip and spin-flip terms in the Breit system. When writing Eq. (37) use has been made of the fact that both G_E and G_M are real for electron-proton scattering. This property can be derived from time reversal invariance[4)].

V. TEST OF VALIDITY OF ONE-PHOTON APPROXIMATION

The one-photon approximation has two characteristic features:

a) a particular form of the dependence of $\overline{|T|^2}$ on s and t,

b) the form factors are real.

The particular dependence on s and t reflects the fact that spin 1 is exchanged. If the Rosenbluth formula is verified by experiment this means that the interaction goes via spin 1; but multiple photon exchange with exchanged total angular momentum equal to one and parity minus is not excluded. Furthermore, an additional interaction through $J^P = 0^-$ would not modify the structure of the Rosenbluth formula since it would only give an additional contribution to A_0 of Eq. (6) but no interference term. In general, a modification through exchange of states with J^P other than 1^- and 0^- will produce additional terms in the cross-section formula with powers of

$$\left\{1 + \frac{1}{1+\tau}\,\mathrm{ctg}^2\,\frac{\Theta}{2}\right\}^{1/2} = \cos\vartheta_t \;.$$

Any deviations from the Rosenbluth formula can be expected to become more pronounced for small angles Θ. For instance, an additional 1^+ interaction will make the cross-section become:

$$d\sigma = (d\sigma)_{NS}\left\{a(t) + 2\tau\, b(t)\,\mathrm{tg}^2\,\frac{\Theta}{2} + c(t)\,\mathrm{tg}^2\,\frac{\Theta}{2}\left(1 + \frac{1}{1+\tau}\,\mathrm{ctg}^2\,\frac{\Theta}{2}\right)^{1/2}\right\}. \quad (39)$$

The deviation of the Θ dependence of the interference term from the form "constant + $\mathrm{tg}^2\,\Theta/2$ constant" increases for small Θ. Our result is that a test of the Rosenbluth formula basically is the test for the exchange of $J^P = 0^-, 1^-$.

A more powerful test uses the measurement of the polarization of the outgoing nucleons and is based on the fact that the form factors are real. Since one works with an unpolarized electron beam and without analysing the polarization of the outgoing electrons for simplicity we neglect the electron spin; its inclusion would not change the result. The scattering amplitude can be written as matrix between Pauli spinors

$$T = \chi^+_{s'}\,(f + i\vec{\sigma}\,\vec{g})\,\chi_s \;. \quad (40)$$

The polarization can be written as:

$$P = \frac{\sigma_+ - \sigma_-}{\sigma_+ + \sigma_-} = \frac{2\,\mathrm{Im}\, f\,\vec{g}^* \cdot \vec{n}}{|f|^2 + |\vec{g}|^2} \;. \quad (41)$$

σ_+ and σ_- are the cross-sections for the outgoing protons, spin being parallel or antiparallel to the unit vector $\vec{n}$. Now, in the one-photon exchange approximation both f and $\vec{g}$ are real. There should be no polarization if the approximation is valid. If there is an additional two-photon exchange we write

$$f = e\,f_1 + e^2 f_2 \;, \qquad \vec{g} = e\,\vec{g}_1 + e^2 \vec{g}_2 \;. \quad (42)$$

The factors e indicate the different powers of the coupling constant for the electron. The indices 1 and 2 refer to one and two-photon exchange. Then

$$P \sim (e\,f_1 + e^2\,\mathrm{Re}\, f_2)\,\mathrm{Im}\, e^2\,\vec{g}_2^* \cdot \vec{n} + e^2\,\mathrm{Im}\, f_2\,(e\,\vec{g}_1 \cdot \vec{n} + e^2\,\mathrm{Re}\,\vec{g}_2 \cdot \vec{n}) \;. \quad (43)$$

Even small two-photon contributions can give rise to polarization since P contains interference terms between one-photon and two-photon amplitudes. Basically polarization measurement is a test for the scattering amplitude being complex.

A third test of the one-photon approximation is the comparison of electron-proton scattering. Both electron and positron scattering are described by the same spin average $\overline{\overline{|T|^2}}$ of the square of the amplitude except that the electron coupling constant e is replaced by $-e$. If - neglecting spins - we indicate the dependence on the coupling constant explicitely:

$$T = e\,T_1 + e^2\,T_2 \;, \quad (44)$$

where the indices 1 and 2 refer to single and double photon exchange, we have

$$|T|^2 = e^2 |T_1|^2 + e^4 |T_2|^2 + 2 e^3 T_1 \, \mathrm{Re}\, T_2 \ . \tag{45}$$

The interference term changes sign when going from electron to positron scattering:

$$\sigma_{e^-} - \sigma_{e^+} = \sim T_1 \, \mathrm{Re}\, T_2 \ . \tag{46}$$

VI. GENERAL STRUCTURE OF FORM FACTORS

We have introduced electric and magnetic form factors G_E and G_M (for the proton: G_E^p, G_M^p). A similar analysis can be done for the neutron giving G_E^n and G_M^n. Since the static charge of the neutron is zero: $G_E^n(0) = 0$ and $G_M^n = \kappa_n$. But from the scattering of slow neutrons it is known that $d G_E^n(t)/dt\big/_{t=0} \neq 0$. In practice the form factors of the neutron can not be determined as easily as those for the proton. Since free neutrons are not available as target one has to use bound neutrons, e.g. deuterium, which causes difficulties due to the strong interaction between the neutron and the proton.

For treating the influence of strong interactions it is more convenient to introduce isoscalar and isovector form factors G^S and G^V:

$$G^S = \frac{G^p + G^n}{2} \ , \qquad G^V = \frac{G^p - G^n}{2} \ . \tag{47}$$

In nucleon-antinucleon annihilation G^S corresponds to the isosinglet contribution (I = 0) and G^V to the isotriplet contribution (I = 1) of the one-photon channel. We can write the inverse of Eq. (47) as matrix element between nucleon isospinors η:

$$G^{\substack{p\\n}} = \eta^+_{\substack{p\\n}} (G^S + \tau^3 G^V) \, \eta_{\substack{p\\n}} \ . \tag{48}$$

The normalization is such that:

$$\begin{aligned} G_E^S(0) &= \frac{1}{2} & G_E^V(0) &= \frac{1}{2} \\ G_M^S(0) &= \frac{1}{2} + \frac{\kappa_p + \kappa_n}{2} \ , & G_M^V(0) &= \frac{1}{2} + \frac{\kappa_p - \kappa_n}{2} \ . \end{aligned} \tag{49}$$

The influence of strong interactions on the electromagnetic structure of the nucleons is most conveniently discussed within the frame of dispersion relations. It has been shown in perturbation theory that the form factors are functions analytic in the complex t-plane cut along the real axis from some value t_0 to ∞. The corresponding representation by a Cauchy integral gives the dispersion relation:

$$G(t) = G(0) + \frac{t}{\pi} \int_{t_0}^{\infty} \frac{\mathrm{Im}\, G(t')}{(t' - t) t'} \, dt' \ . \tag{50}$$

With this representation there is associated a decomposition into contributions of different intermediate states indicated in Fig. 4.

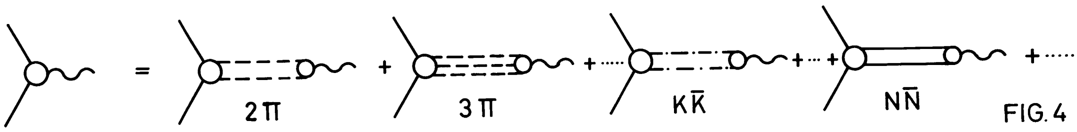

FIG. 4

To each diagram corresponds a cut of the function G for values of t such that the particles in the intermediate state are on the mass shell, i.e. for $t \geq (\Sigma m_i)^2$, where m_i are the masses of the particles. The value of t_0 in Eq. (50) is given by the intermediate state of lowest mass. The intermediate states must satisfy the selection rules, i.e. they must have

angular momentum J = 1 parity minus
isospin I = 1 for isovector form factor
I = 0 for isoscalar form factor .

Since the pions are bosons, two pions with J = 1 necessarily have I = 1. J = 1 and I = 0 can only be obtained with at least three pions; therefore

$$\left.\begin{array}{ll} t_0 = (2\mu)^2 & \text{for isovector} \\ t_0 = (3\mu)^2 & \text{for isoscalar} \end{array}\right\} \text{form factors; } \mu = \text{pion mass .} \tag{51}$$

Each diagram of Fig. 4 is composed of diagrams corresponding to other processes, e.g. for the two-pion intermediate state corresponding to $N+\bar{N} \rightarrow 2\pi$ and to the pion form factor. This is reflected in the rule for the computation of the weight function:

$$\mathrm{Im}\, G = \sum_{\substack{\text{intermediate} \\ \text{states } z}} T^*_{N\bar{N};z}\, \Gamma_{zi\gamma} \; . \tag{52}$$

Here $T_{N\bar{N};z}$ denotes the scattering amplitude for $N+\bar{N} \rightarrow z$, $\Gamma_{zi\gamma}$ denotes the electromagnetic vertex. Each of the functions entering into Im G according to Eq. (52) itself is connected with amplitudes for other processes. For instance, the pion form factor entering into the two-pion contribution is related to the amplitude for pion-pion scattering as indicated in Fig. 5.

FIG. 5

In practice it is impossible to take into account all relations between the functions appearing in Eq. (52) and the amplitudes for the related processes. But it is hoped that in the physical region for electron-nucleon scattering, i.e. $t < 0$, for small values of $|t|$ the first portion of the cut will be more important than the far distant parts, for which the factor $1/(t' - t)$ appearing in the dispersion integral is smaller. Now, the first part of the cut is largely determined by the pion-pion interaction which is known to show strong resonances. Therefore, one frequently interprets the experimental results in terms of resonance models involving the ρ, ω and φ mesons.

Before turning to the approximations we shall mention some general considerations which impose constraints on parameters of the model. If a form factor $G(t)$ for $t \to -\infty$ vanishes sufficiently rapidly in the limit, the dispersion integral must cancel the constant term in Eq. (50). This gives

$$G(0) = \frac{1}{\pi} \int_{t_0}^{\infty} \frac{\operatorname{Im} G(t')}{t'} \, dt \; . \tag{53}$$

If $G(t)$ vanishes faster than $\sim 1/|t|$, $\operatorname{Im} G(t)$ can not be ≥ 0 throughout the interval of integration. In particular if $\lim_{t \to -\infty} tG(t) = 0$, we have

$$\int_{t_0}^{\infty} \operatorname{Im} G(t') \, dt' = 0 \quad . \tag{54}$$

Unfortunately no rigorous statement about the asymptotic behaviour can yet be given. We therefore shall not go into details[5].

Another constraint is imposed by the behaviour at $t = 4M^2$, i.e. the physical threshold for nucleon-antinucleon annihilation. The matrix element of the electromagnetic current in the c.m.s. for nucleon-antinucleon annihilation has the structure

$$\langle J_\mu \rangle \sim \chi^{+}_{-s'} \left(\vec{\sigma} \cdot \frac{\vec{p}\vec{p}}{\vec{p}^{\,2}} (G_M - G_E) + \vec{\sigma} G_M \right) \chi_s \quad . \tag{55}$$

The second term involving only $\vec{\sigma}$ corresponds to a ^3S contribution while the first term contains the ^3D contribution. At threshold there should remain only the S-wave amplitude which requires: $G_M = G_E$ at $t = 4M^2$. Another argument for this relation to hold goes as follows: at threshold there exists no privileged direction in space. Therefore the angular distribution should be isotropic. Now, $\overline{\overline{|T|^2}}$ in general contains a term proportional to $\cos^2 \vartheta_t$ which should vanish at threshold:

$$\left(\frac{t}{4} - M^2\right)\left(\frac{t}{4} - m^2\right) a(t) \cos^2 \vartheta_t = -\left(\frac{t}{4} - m^2\right) M^2\left(G_E^2 - \frac{t}{4M^2} G_M^2\right) \cos^2 \vartheta_t$$

$$\rightarrow 0 \quad \text{for } t \rightarrow 4M^2 \qquad (56)$$

$$\text{i.e.} \quad G_E^2 \rightarrow G_M^2 \quad \text{for } t \rightarrow 4M^2 .$$

The reasoning for obtaining expressions to fit the form factors goes as follows: the pion-pion interaction at small energies is dominated by resonances (e.g. ρ, ω, φ). It is assumed that the contribution to the weight function in the dispersion relation for the nucleon form factors comes mainly from the vicinity of resonances. In the limit of an infinitely narrow resonance one has a δ-function contribution to Im G(t) leading to a form factor contribution:

$$\text{contribution of narrow resonance} \sim \frac{\text{constant}}{t - m_r^2} . \qquad (57)$$

m_r is the mass of the particle associated with the resonance. The constant appearing in Eq. (57) is the product of the coupling constants $N-\bar{N}$ resonance and γ resonance. This approximation corresponds to substituting one particle for a two or more pion intermediate state. For the two-pion contribution a ρ meson is substituted according to Fig. 6.

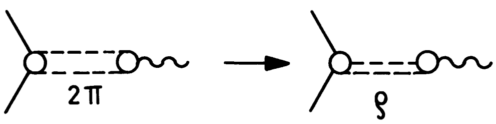

FIG. 6

Candidates for approximating the isoscalar form factors are the ω and φ mesons. It can not be expected that all of the integral can be approximated by resonant contributions of the form (57). There will be a slowly varying background which sometimes is approximated by a constant. Thus one arrives at a formula of the structure:

$$G(t) \approx a + \sum \frac{b_i}{t - m_i^2} . \qquad (58)$$

A formula of this type with one pole term is known as "Clementel-Villi formula"[6].

Our considerations leading to the expression (58) have only been a motivation but not a derivation. This should be born in mind when applying it to the experiment.

We have added the constant term a in order to represent a slowly varying background. If the inclusion of such a term gives a good fit to the experiment it does not necessarily mean that the rigorous expression for G(t) contains a constant. An equally good fit can be obtained if instead of the constant a an additional pole term is used such that $-b_i/m_i^2 = a$ and m_i^2 sufficiently large. The constant term only simulates a contribution of a slowly varying Im G(t) for large values of t'. Since the Fourier transform of a constant is a δ-function, the constant a sometimes is interpreted as being due to a hard core. But in the light of the fact that expression (58) is only an approximation this interpretation should not be taken too literally.

The number of pole terms necessary to give a good fit to the experiments is not known a priori. Of course, one expects the known resonances with appropriate quantum numbers to give contributions. There are the ρ meson for the isovector form factors and the ω and φ mesons for the isoscalar form factors. It turns out that a better fit is obtained if one pole more - a ρ' - is included for the isovector form factor. The apparent necessity to include such a pole does not necessarily mean that a corresponding resonance exists. It only says that the behaviour of the dispersion integral is thus better approximated.

The masses entering into (58) can not a priori be expected to be exactly those of the known resonances. The resonances have a finite width which in the case of the ρ is considerable. As pointed out by Ball and Wong, a more detailed calculation of the weight function entering into the isovector form factors shows that the maximum is shifted to lower energies and furthermore the region of small t' is weighted rather heavily. Therefore, in general, shift of the masses m_i relative to the masses of the known resonances is expected.

The constants b_i are restricted by the normalization at $t = 0$. Furthermore, sometimes the condition $G_E = G_M$ at $t = 4M^2$ is used as a constraint. But it should be born in mind that this point lies outside the region where one can expect the approximation to hold.

At present the experiments indicate that there is no substantial deviation from the predictions of the one-photon approximation. The Rosenbluth formula with its particular dependence on $tg^2\,\theta/2$ seems to hold, a measurement of polarization at $t = 16\ F^{-2}$ gives $P = 0.031 \pm 0.025$ which is no significant deviation from $P = 0$, and a comparison of e^+ and e^- scattering also does not indicate a substantial two-photon contribution.

If the scattering data are analysed in terms of form factors, the surprising result is that G_E^p, $G^p_{M/1+\kappa_p}$, and G^n_{M/κ_n} seem to be equal up to $q^2 = 50\ F^{-2}$. This equality in the region $t < 0$ is not expected to hold for all values of t since it contradicts $G_E = G_M$ at $t = 4M^2$. If the experimental results are represented by a Clementel-Villi-type formula, the values for the parameters depend on the number of them left free. If one takes the masses of ρ, ω and φ and introduces a fictitious ρ' with $m\rho' = 940$ MeV, one obtains[7])

$$G_E^s = \frac{1.24}{1 + q^2/15.8} - \frac{0.74}{1 + q^2/26.7}$$

$$G_E^v = \frac{2.01}{1 + q^2/14.4} - \frac{1.51}{1 + q^2/23.0}$$

(57)

$$G_M^s = \frac{1.12}{1 + q^2/15.8} - \frac{0.68}{1 + q^2/26.7}$$

$$G_M^v = \frac{6.23}{1 + q^2/14.5} - \frac{3.87}{1 + q^2/23.0} \quad .$$

This fit does not include a constant term. But it is also possible to represent the data with ρ, ω and φ only together with constant terms. This gives[8])

$$G_E^s = \frac{2.6}{1 + q^2/15.8} - \frac{3.1}{1 + q^2/26.7} + 1$$

$$G_E^v = \frac{0.9}{1 + q^2/14.5} - 0.4$$

(58)

$$G_M^s = \frac{3.4}{1 + q^2/15.8} - \frac{3.8}{1 + q^2/26.7} + 0.8$$

$$G_M^v = \frac{3.2}{1 + q^2/14.5} - 0.8 \quad .$$

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Further references to the original papers can be found in the above-mentioned articles.

2) We normalize our states such that for a spinless particle

$$\langle \vec{p}' | \vec{p} \rangle = 2 p_0 \, \delta(\vec{p} - \vec{p}')$$

while for a particle with spin

$$\langle \vec{p}' s' | \vec{p} \, s \rangle = 2 \, p_0 \, \delta(\vec{p} - \vec{p}_1') \, \delta_{ss'} \, .$$

Furthermore $\bar{U}(p,s') \, U(p,s) = 2M\delta_{ss'}$ is being used.

3) For a discussion of the structure of the cross-section see also

M. Gourdin, Nuovo Cimento 36, 129 (1965).

4) A more detailed discussion of the kinematic structure of the electromagnetic vertex can be found in

L. Durand III, in Lectures in Theoretical Physics, IV, W.E. Brittin, ed., New York 1962 (Interscience).
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6) E. Clementel and C. Villi, Nuovo Cimento 4, 1207 (1956).

7) J.R. Dunning et al., Phys.Rev. Letters 13, 631 (1964).

8) M.W. Kirson, Phys.Rev. 132, 1249 (1963).