hep-th/0409261
CERN-PH-TH/2004-188
QMUL-PH-04-07

Holographic cubic vertex in the pp-wave

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ABSTRACT

We revisit the cubic interaction of IIB string theory in the maximally supersymmetric pp-wave background. In the supergravity limit, we show that detailed comparison with AdS supergravity determine the vertex completely. Extension of this supergravity vertex to the full string theory leads to a new cubic vertex that combines the previous proposals and contains additional terms. We give an alternative derivation of the holographic duality map in supergravity, first found by Dobashi and Yoneya (hep-th/0406225) and show that our new vertex is consistent with it. We compare some non-BPS amplitudes (including impurity non-preserving ones) with the corresponding field theory correlators, and discuss what they imply for the stringy generalization of the duality map. We also notice that our vertex realizes the $U(1)_Y$ symmetry linearly, and propose a similar modification for the flat space vertex.

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1 Introduction

The BMN duality [1] has drawn a lot of attention for the past two years, largely because it opened up a systematic way to test the AdS/CFT correspondence [2] at the string level. The most striking discovery was that the tree-level string spectrum [3, 4] in the maximally supersymmetric pp-wave background [5, 6] matches exactly (that is, to all orders in the α'-expansion), a particular class of $\mathcal{N} = 4$ super Yang-Mills operators [1]. Since then, much effort has been made to understand how the string interactions (non-zero g_s) fit into the duality. In spite of many important works in the literature[1], the problem has not been fully solved yet. The goal of this paper is to report some progress on this subject.

The simplest type of string interaction is the cubic interaction, in which two strings join to form a single string or vice versa. There are three crucial issues concerning the cubic interaction in the pp-wave duality.

1. Construction of the cubic vertex.

   The string theory in the pp-wave is formulated in terms of the Green-Schwarz superstring in the light-cone gauge. In this set-up, the 3-string vertex is given by the cubic part of the light-cone Hamiltonian. The vertex is usually constructed by imposing the super-symmetry constraints. However, unlike in flat-space, the constraints do not completely fix the pp-wave vertex.

2. Holographic duality map.

   Once the cubic Hamiltonian is known, one can compute its matrix elements and obtain the coupling among three arbitrary string states. On the Yang-Mills side, the natural observable is the coefficient of the (normalized) cubic correlator. To make the comparison between these two observables, one needs a duality map, which must somehow 'know' about the holography underlying the original AdS/CFT correspondence.

3. Choice of basis (Operator mixing)

   It is important to understand how the string and the Yang-Mills Hilbert spaces are mapped to each other. While the matching of the free spectra focuses mainly on the eigenvalues of the physical observables, the duality map for the cubic interaction tests in a much stronger way the dictionary between string and gauge theory states.

In this paper, we will discuss some new findings and considerations on these three points.

Spradlin and Volovich [13, 14] made the first proposal for the cubic vertex, which was further elaborated in [15, 16, 17, 18]. Aside from satisfying the pp-wave super-algebra, the SV vertex has

[1] See the review papers [7, 8, 9, 10, 11, 12] for a detailed bibliography.

two features: (a) it has definite parity under the accidental $\mathbb{Z}_2$ symmetry that exchanges the two manifest $SO(4)$ symmetry groups (the parity is odd in the conventions where the vacuum is $\mathbb{Z}_2$ invariant), (b) it has a smooth 'flat space' limit. Before the question of whether these features are compatible with the putative duality map was answered, another physically different vertex was proposed in [19, 20, 21]. This vertex satisfies the same pp-wave super-algebra, but does not share the above-mentioned features: (a) it has opposite parity under the $\mathbb{Z}_2$, (b) as a consequence of this parity property, it does not have a smooth 'flat space' limit.

Which one of the two vertices is the correct one? In fact, since the constraint from super-algebra essentially gives a set of linear differential equations, the right question would be "Which linear combination of the two is the correct one?" Moreover, there may even exist other independent solutions to the super-algebra equations, ending up with a multi-dimensional space of candidate vertices.

Clearly, to resolve the situation, one has to understand better how holography works in the pp-wave. Among others, Yoneya and collaborators have pursued this line of thought systematically [22, 23]. Recently, in [24], they derived an explicit holographic duality map for the supergravity sector of the pp-wave string theory by taking the semi-classical limit of the GKPW relation [25, 26] in AdS/CFT. This map led them to conclude that the correct vertex is a particular linear combination of the two vertices introduced above which breaks the $\mathbb{Z}_2$ symmetry 'maximally'.

In this paper, we first re-derive the same duality map from a somewhat different perspective, following the idea which first appeared in [27]. Then, we take a closer look at what it implies for the cubic vertex. Among other things, we pay attention to the $U(1)_Y$ symmetry of supergravity as well as the matrix elements of the super-descendants of the chiral primary state. We find that the proposal of [24] should be further modified to include three new terms similar to the second vertex mentioned above, in order for the duality map to hold. Our derivation indicates that this vertex is the unique one compatible with the duality map, although a rigorous proof is not yet available. Finally, we discuss how to extend the duality map to the full string theory. Suggestive as our computation of stringy amplitudes are, the final answer seems to require more work including sub-leading order computations in Yang-Mills.

This paper is organized as follows. Sections 2 and 3 focus on supergravity (or BPS) processes. Section 2 contains the derivation of the holographic duality map. In section 3, we first derive a number of $\text{AdS}_5 \times S^5$ 3-point couplings and study their large J limit. Then we discuss the $U(1)_Y$ symmetry of type IIB supergravity and use it as an additional constraint on the pp-wave cubic vertex. A unique answer for this vertex is obtained by requiring that it reproduce the large J limit of the previously derived $\text{AdS}_5 \times S^5$ 3-point couplings. In section 4, we go beyond the supergravity sector and study the cubic interaction among generic string states. In our construction, we demand that the zero-mode structure of the string vertex reproduce the supergravity results derived in the previous section. By combining the known vertices and also adding some new terms, we present a consistent proposal for the holographic 3-string vertex. In order to test its validity, we compute some stringy amplitudes and compare them against the field theory results by using the simplest generalization to the full string theory of the duality map. Section 5 contains our conclusions along with a discussion of possible future directions.

2 Holography in supergravity

The holographic duality map in the supergravity sector can be derived in two simple steps[2]. The first step is to note that the interaction part of the pp-wave Hamiltonian is equal to that of the original AdS geometry in the Penrose limit. This relation is not restricted to the BPS sector, but should hold even for the full string theory. The second step is to relate the AdS Hamiltonian to the coefficients of the gauge theory correlators via the GKPW relation in supergravity [25, 26]. This is possible since both quantities can be obtained from the same IIB supergravity action on $\mathrm{AdS}_5 \times S^5$.

2.1 From AdS to pp-wave

The first step is a direct consequence of the standard AdS/CFT and pp-wave dictionaries. In the following table, we summarize in the first two columns the two parameters that define each theory and define the dimensionless Hamiltonians in the third column.

	YM-loop / stringy effect	genus / string loop	Hamiltonian
AdS	$\lambda = g_{YM}^2 N = (R_{AdS}/l_s)^4$	$1/N$	$H^{(\mathrm{AdS})} \equiv R_{AdS} P_0$
PP	$\lambda' = g_{YM}^2 N/J^2 = 1/(\mu p^+ \alpha')^2$	$g_2 = J^2/N$	$H^{(\mathrm{PP})} \equiv P_+/\mu$

Since the pp-wave theory describes the dynamics of $\mathrm{AdS}_5 \times S^5$ in the Penrose limit ($N, J \to \infty$, keeping λ' and g_2 fixed), then the two Hamiltonians must be the same in this limit, except for the shift by J, which changes only the *free* part:

$$\boxed{\lim_{\mathrm{Penrose}} \left\{ H^{(\mathrm{AdS})}(\lambda, 1/N) - J \right\} = H^{(\mathrm{PP})}(\lambda', g_2)} \,. \tag{2.1}$$

In passing, we should emphasize that the mass scale μ in the pp-wave has absolutely no physical meaning. The expressions such as '$\mu \to 0$' or '$\mu \to \infty$' often found in the literature should be interpreted as large λ' and small λ', respectively. In particular, the bona-fide flat space ($\mu = 0$) is not related to the '$\mu \to 0$' limit of the pp-wave. If the two were smoothly connected, the

[2]The main ideas of this section were first discussed in a limited setting in the appendix of Ref. [27].

BMN duality would imply a holographic relation between IIB string theory in flat space and a very strongly coupled gauge theory. Of course, some ingredients (for example, the Neumann matrix) of the pp-wave Hamiltonian formally have a smooth $\mu \to 0$ limit. However, as we will see in the next section, the cubic Hamiltonian contains pieces which manifestly break symmetries of flat space. Discontinuity of the '$\mu \to 0$' limit has been recently noticed also in [28], where the causality properties of the pp-wave string theory are studied.

Note also that we are taking the Penrose limit on the AdS Hamiltonian. This is to be contrasted with the approach of [29], where the Hamiltonian is computed directly in the pp-wave geometry.

2.2 Hamiltonian vs. Correlator

Now we move on to the second step of the derivation. Suppose we have primary operators $O_i(x)$ in a CFT$_d$ and the corresponding scalar fields φ_i living in AdS$_{(d+1)}$. Assume that the bulk action takes the standard form,

$$S = -\int d^{d+1}x\sqrt{-g}\left[\frac{1}{2}(\nabla\varphi^i)^2 + \frac{1}{2}m_i^2(\varphi^i)^2 + \frac{1}{6}G^c_{ijk}\varphi^i\varphi^j\varphi^k\right], \tag{2.2}$$

where the AdS mass of φ_i and the scaling dimension of O_i are related by $m^2 = \Delta(\Delta - d/2)$. The superscript in G^c_{ijk} is to stress that in this section we are working with canonically normalized fields.

There are two things we can do with this action. First, we can compute in supergravity the normalized 3-point correlators following [26, 30],

$$\langle O_1(x_1)O_2(x_2)O_3(x_3)\rangle = \frac{C_{123}}{|x_1-x_2|^{2\beta_3}|x_2-x_3|^{2\beta_1}|x_3-x_1|^{2\beta_2}}\,, \tag{2.3}$$

$$C_{123} = \frac{G^c_{123}}{2^{5/2}\pi^{d/4}} \times \prod_{r=1}^{3}\left(\frac{\Gamma(\beta_r)}{\{\Gamma(\Delta_r - d/2+1)\Gamma(\Delta_r)\}^{1/2}}\right) \times \Gamma(\sigma - d/2)\,, \tag{2.4}$$

where $\sigma = (\Delta_1+\Delta_2+\Delta_3)/2$, $\beta_r = \sigma - \Delta_r$. Second, we can canonically quantize the free part of the action and read off the matrix elements of the cubic Hamiltonian. As usual, canonical quantization associates a harmonic oscillator to each normalizable solution to the free field equation of motion. For a real scalar in AdS, the expansion takes the following form:

$$\varphi(t,x) = \sum_i \frac{f_i(x)}{\sqrt{2(\Delta+n_i)}}\left(a_i e^{-i(\Delta+n_i)t} + a_i^\dagger e^{i(\Delta+n_i)t}\right)\,, \tag{2.5}$$

where t is the global time and x denotes the d spatial coordinates in the metric,

$$ds^2 = \frac{1}{\cos^2\theta}\left(-dt^2 + d\theta^2 + \sin^2\theta d\Omega^2_{d-1}\right). \tag{2.6}$$

In (2.5), the index i in runs over all solutions, and $f_i(x)$ are the spatial part of the solutions. The excitation number n_i is zero for the ground state and is a positive integer for excited states. The matrix elements of the cubic Hamiltonian can be read off simply by inserting (2.5) into the cubic term of the Hamiltonian. For the ground state wave functions of the scalars,

$$f_0 = \sqrt{\frac{\Gamma(\Delta+1)}{\pi^{d/2}\Gamma(\Delta-d/2+1)}}(\cos\theta)^{\Delta} \,, \tag{2.7}$$

the matrix elements turn out to be[3],

$$H_{123} = \frac{G^c_{123}}{2^{3/2}\pi^{d/4}} \times \prod_{r=1}^{3}\left(\frac{\Gamma(\Delta_r)}{\Gamma(\Delta_r-d/2+1)}\right)^{1/2} \times \frac{\Gamma(\sigma-d/2)}{\Gamma(\sigma)} \,. \tag{2.8}$$

Comparing (2.4) and (2.8), one finds that

$$H_{123} = \frac{2\Gamma(\Delta_1)\Gamma(\Delta_2)\Gamma(\Delta_3)}{\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\beta_3)\Gamma(\sigma)}C_{123}, \tag{2.9}$$

In the pp-wave limit, we take $\Delta_r \to \infty$ and use the relation (2.1) to obtain the holographic duality map as advertised,

$$\boxed{H^{(\mathrm{PP})}_{123} = \lim_{\mathrm{Penrose}} H^{(\mathrm{AdS})}_{123} = \frac{\Delta_{123}}{(\Delta_{123}/2)!}\left(\frac{J_1 J_2}{J_3}\right)^{\frac{\Delta_{123}}{2}} C_{123}} \,, \tag{2.10}$$

where $\Delta_{123} \equiv \Delta_1 + \Delta_2 - \Delta_3$ is kept finite. From here on, for any physical quantity X assigned to each of the three states participating in the cubic interaction, we will use the notation $X_{123} \equiv X_1 + X_2 - X_3$.

2.3 Intuitive picture by Yoneya et al.

Holography in the pp-wave duality has remained a puzzle because the boundary of the original AdS is completely lost in the process of taking the pp-wave limit. Then, how can one derive a relation like (2.10) from the pp-wave string theory (or supergravity) without tracing back to the original AdS? Perhaps one cannot. We did trace back to the original AdS to derive (2.10). In the next section, we will use it as a dynamical *input* in constructing the cubic vertex in the pp-wave. In other words, among all candidate vertices satisfying the (super-)symmetry constraints, we will pick out the one respecting the duality map. This point of view was pursued systematically by Yoneya and collaborators [22, 23, 24]. We briefly review their work here from a slightly different perspective. It will provide an intuitive understanding of what (2.10) means.

[3] Excited states give different values of H_{123} through the overlap integral of wave-functions. However, note that all the wave-functions of a same field share the coupling constant G^c_{123}. This fact will be important in section 3.

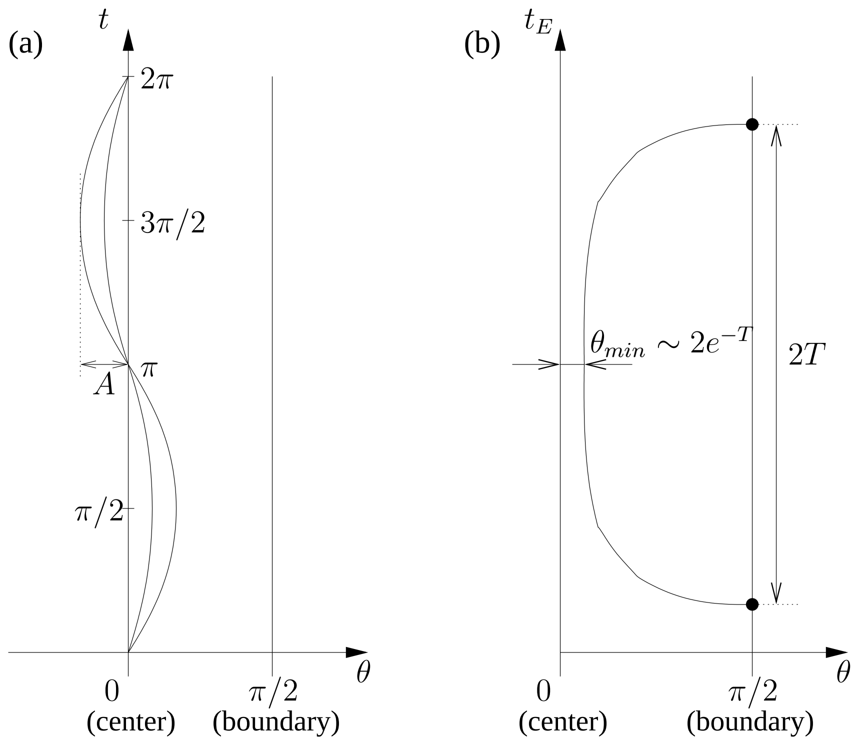

Figure 1: The geodesics in (a) Lorentzian and (b) Euclidean AdS in global coordinates.

One starts with the GKPW relation for the correlators. As emphasized in [22, 23, 24], the bulk to boundary propagator should be understood as a Euclidean path integral. The reason is that in the Lorentzian signature, a massive particle can never reach the boundary. In global coordinates with the metric (2.6), the geodesic equation can be easily solved. For example, the solutions describing a radial motion look like (See Figure 1),

$$\text{Lorentzian}: \quad \sin\theta = \sin A \sin t \ , \qquad \text{Euclidean}: \quad \sin\theta = \frac{\cosh t}{\cosh T} \ . \tag{2.11}$$

The second thing to notice is that in the semiclassical limit ($\Delta \gg 1$), the saddle point approximation to the 'propagator' along the geodesic becomes reliable. For a large value of the distance $2T$ in time direction between the two boundary points, the Euclidean geodesic starting from a boundary point runs exponentially toward the center of the AdS and stays there until it curves back to the other boundary point. This is consistent with the fact that the pp-wave limit magnifies the small region around the center. In [24], the duality map (2.10) was derived by systematically performing the saddle point approximation and constructing an effective action for a particle along the geodesic.

Writing (2.10) as $C_{123} = L_{123} \cdot H_{123}$, one could heuristically argue that H_{123} originates from the body of the geodesic passing through the center of AdS captured by the pp-wave, while L_{123}

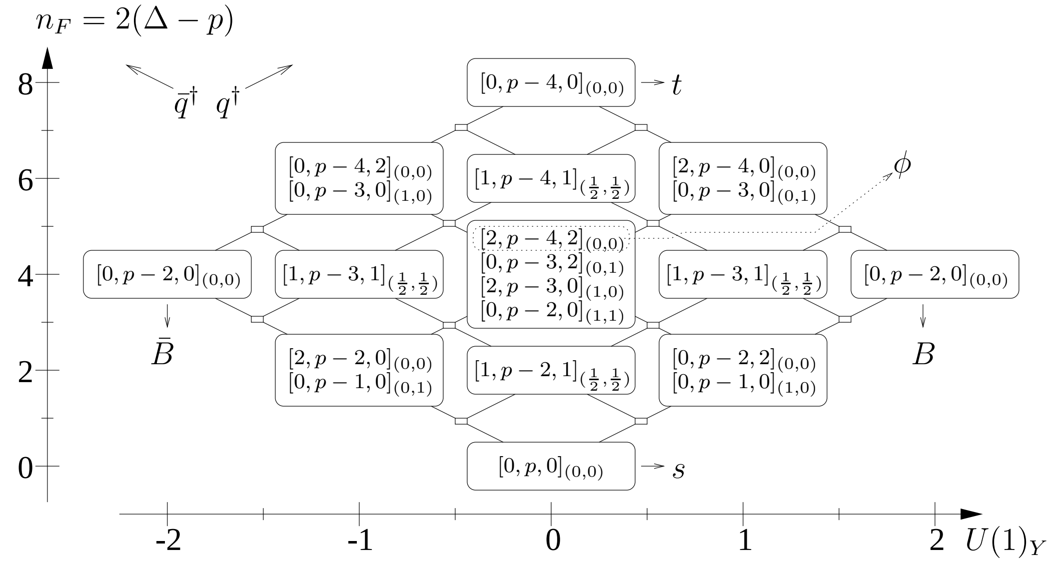

Figure 2: Supergravity multiplet. Fermions are hidden in the small boxes

comes from the 'legs' connecting the center and the boundary. It would be very interesting to generalize this semi-classical picture to the full string theory and derive a similar duality map.

3 Supergravity vertex

In this section, we derive the supergravity vertex consistent with the duality map (2.10), leaving the full string theory vertex to the next one. In the first subsection, we compute several examples of $H_{123}^{(\mathrm{AdS})}$ as described in section 2. In the second one, we determine the form of $H_{123}^{(\mathrm{AdS})}$ by demanding that it satisfy the super-algebra, respect the $U(1)_Y$ symmetry of IIB supergravity, and match the data of the first subsection according to (2.10).

3.1 'Experimental' data

We begin by reviewing the structure of the IIB supergravity multiplet in $\mathrm{AdS}_5 \times S^5$ [31, 32]. After Kaluza-Klein reduction, the supergravity modes form a series of half-BPS multiplets of the $su(2,2|4)$ super-algebra. Each multiplet is labeled by an integer p. Figure 2 shows how such a multiplet splits into several representations of the bosonic $so(2,4) \oplus so(6)$ bosonic algebra. The notation $[a,b,c]_{(i,j)}$ denotes the Dynkin label under $SO(6)$ and the $SU(2) \times SU(2) \approx SO(4) \subset SO(2,4)$ quantum number. The AdS energy Δ of the ground state of a given supergravity mode is the integer p plus half of the number n_F of supercharges needed to reach the given state from the ground state.

In addition to the super-algebra quantum numbers, each mode is assigned the so-called $U(1)_Y$ charge. This $U(1)$ is the subgroup of the $SL(2,\mathbb{R})$ of the IIB supergravity, preserved by the $\mathrm{AdS}_5 \times S^5$ background. The dilaton-axion scalars form a complex scalar field with charge ± 2 and combinations of NSNS and RR two-form fields have charges ± 1, while the graviton and RR 4-form fields are neutral.

Note that this $U(1)_Y$ is an exact symmetry of the AdS and pp-wave supergravity. In particular, the $U(1)_Y$ charge should be conserved in a cubic interaction involving three supergravity states. Even in the full string theory in which the $U(1)_Y$ is broken, the selection rule will continue to hold when all three external states are supergravity states [33, 34]. This selection rule will play an important role in constructing the holographic cubic vertex in the pp-wave.

In the following, we present some explicit examples of the matrix element H_{123}^{AdS} in (2.10). They will impose severe constraints on H_{123}^{PP} through (2.10). For simplicity, we consider only scalar fields in AdS_5. There are four scalar fields that are also scalar on the S^5, as shown in Figure 2. The s and t fields are particular combinations of some components of the graviton and RR 4-form field. The complex B field is the dilaton-axion pair which is related to the standard form $\tau = \chi + ie^{-\phi}$ by the conformal mapping,

$$B = \frac{\tau - \tau_0}{\tau + \tau_0} \, , \tag{3.1}$$

so that, for any constant background value τ_0, the $U(1)_Y$ symmetry acts *linearly* on B. The selection rule becomes manifest in this variable. Finally, we will also consider the field ϕ which is basically the graviton with both indices along the S^5 direction. So, it is a scalar in AdS_5, but a symmetric, traceless tensor on the S^5.

The S^5 scalars s, t, B transforms in $[0, k, 0]$ representation of $SO(6)$. This k is identified with the quadratic Casimir for spherical harmonics on S^5: $\nabla^2 Y = -k(k+4)Y$. As shown in Figure 2, k is related to Δ and p as $k = p - n_F/2 = \Delta - n_F$. More precise definition of the fields and their cubic couplings are summarized in the appendix A and references therein.

Bosonic impurities

The first class of amplitudes we consider involve three s-states. In the pp-wave set-up where a $U(1)$ R-charge is singled out, an $SO(6)$ representation splits into different $SO(4)$ representations. They correspond to the following operators in Yang-Mills, usually called the 'scalar-impurity' operators in the pp-wave literature.

$$O_0 = \mathrm{Tr}(Z^J), \qquad O_1 = \mathrm{Tr}(\phi Z^J), \qquad O_2 = \sum_{l=0}^{J} \mathrm{Tr}(\phi Z^l \psi Z^{J-l}) \, . \tag{3.2}$$

The number of impurities n_B satisfies the relation $J + n_B = k = \Delta$. Since J is conserved, $\Delta_{123} = k_{123} = (n_B)_{123}$ holds. The following table summarizes several amplitudes. The numbers on the first column denote the number of impurities of each operator. The second column contains the value of LHS of (2.10) normalized by $C^{(0)}_{123} \equiv \sqrt{J_1 J_2 J_3}/N$. The third and fourth columns contain the two factors on the RHS of (2.10). For later convenience, we define $q_i \equiv \sqrt{J_i/J_3}$ $(i = 1, 2)$.

$(s^1 s^2 \vert s^3)$	$H_{123}/C^{(0)}_{123} \equiv V_s \Delta_{123}$	$\frac{\Delta_{123}}{(\Delta_{123}/2)!}\left(\frac{J_1 J_2}{J_3}\right)^{\frac{\Delta_{123}}{2}}$	$C_{123}/C^{(0)}_{123}$
$(00\vert 0)$	$1 \cdot k_{123}$	Δ_{123}	1
$(01\vert 1)$	$q_2 k_{123}$	Δ_{123}	q_2
$(02\vert 2)$	$q_2^2 k_{123}$	Δ_{123}	q_2^2
$(11\vert 2)$	$q_1 q_2 k_{123}$	Δ_{123}	$q_1 q_2$
$(11\vert 0)$	$q_1 q_2 k_{123}$	$\frac{J_1 J_2}{J_3}\Delta_{123}$	$\frac{1}{\sqrt{J_1 J_2}}$
$(12\vert 1)$	$q_1 q_2^2 k_{123}$	$\frac{J_1 J_2}{J_3}\Delta_{123}$	$\frac{1}{\sqrt{J_1 J_3}}$
$(22\vert 0)$	$q_1^2 q_2^2 k_{123}$	$\left(\frac{J_1 J_2}{J_3}\right)^2 \frac{\Delta_{123}}{2}$	$\frac{2}{J_1 J_2}$

(3.3)

The variable k_{123} counts the impurity number violation, so it has definite integer values. However, we formally write it as if it is an undetermined variable, even when it vanishes, to facilitate comparison with the pp-wave vertex.

From the point of view of Kaluza-Klein reduction, operators with different scalar impurity configurations correspond to different spherical harmonics wave-functions on the S^5. The factors V_s in the first column come from the spherical harmonics overlap integrals. As such, they are common for all supergravity fields that are scalar on the S^5. So, we will not separately discuss the effects of scalar impurities when we discuss other scalar fields than s below.

Next, we discuss the effect of the 'vector impurities.' In the supergravity sector, the vector impurities are simply total derivatives acting on a given primary operator,

$$O = \mathrm{Tr}(Z, \phi, \mathrm{etc.}), \quad O^{(1)}_\mu = \partial_\mu O, \quad O^{(2)}_{\mu\nu} = \partial_\mu \partial_\nu O \, . \tag{3.4}$$

The resulting operators are descendants of the primary operator. In any CFT, correlators of descendants are completely determined by those of primaries. This fact is reflected in the supergravity computation. Primary state and descendant states are different wave functions of a same supergravity field. So, they share the same coupling constant. The only difference in H_{123} then comes from the overlap integral of the three wave functions. The following table summarizes an explicit example of the s field.

$$\begin{array}{c|c} (s^1 s^2|s^3) & H_{123}/C^{(0)}_{123} \equiv V_v V_s k_{123} \\ \hline (00|0) & V_s \cdot k_{123} \\ (01|1) & q_2 V_s k_{123} \\ (11|0) & q_1 q_2 V_s k_{123} \end{array} \tag{3.5}$$

The first row contains implicitly the entire table (3.3) with no vector impurities. As one adds vector impurities, the wave function effect shows up as written in the last two rows. Note that the scalar impurities and vector impurities commute with each other. Note also that the 'dynamic' part, $k_{123} = (n_B)_{123}$, counts only the scalar impurities but not the vector impurities. We see that the accidental $\mathbb{Z}_2$ symmetry of the pp-wave string theory is broken. This will be crucial in determining the cubic vertex in the next subsection.

Fermionic impurities

So far, we considered only the s field which lies at the bottom of Figure 2. As we will see in the next subsection, it turns out that the amplitudes listed above are already sufficient to determine the supergravity vertex. Still, by comparing some amplitudes involving other states in Figure 2, we could verify in more detail, the validity of the vertex and as well as the duality map (2.10).

For later convenience, we list all amplitudes involving s, t and B together. In the following table, it is understood that $V_v V_s$ is multiplied to each amplitude when bosonic impurities are added, and that k_{123} counts only the scalar impurities.

$$\begin{array}{cc} \text{process} & H_{123}/C^{(0)}_{123} \\ (s^1 s^2|s^3) & k_{123} \\ (t^1 s^2|s^3) & q_2^8(k_{123}+4) \\ (s^1 s^2|t^3) & \mathcal{O}(1/J^4) \\ (s^1 t^2|t^3) & q_2^8 k_{123} \\ (t^1 t^2|s^3) & \mathcal{O}(1/J^4) \\ (t^1 t^2|t^3) & (k_{123}+4) \end{array} \qquad \begin{array}{cc} \text{process} & H_{123}/C^{(0)}_{123} \\ (s^1 B^2|\bar{B}^3) & q_2^4 k_{123} \\ (B^1 \bar{B}^2|s^3) & q_1^4 q_2^4 (k_{123}+4) \\ (t^1 B^2|\bar{B}^3) & q_2^4(k_{123}+4) \\ (B^1 \bar{B}^2|t^3) & q_1^4 q_2^4 k_{123} \end{array} \tag{3.6}$$

The amplitudes are proportional to either k_{123} or $(k_{123}+4)$. Note that for each process $(12|3)$ listed in (3.6), the constant shift to k_{123} is always equal to $(n_F/2)_{123}$. In other words, including the shift, the coupling is proportional to $(n_B + n_F/2)_{123}$. This expression is most suitable for comparison with the pp-wave vertex. Alternatively, one can use $\Delta = k + n_F$ to write the couplings as $(\Delta - n_F/2)_{123}$. Note that $(\Delta - n_F/2)$ is nothing but the scaling dimension of the chiral primary in the super-multiplet containing the given field. This is natural since the coupling for a chiral primary and those for its super-descendants are expected to be proportional to each other.

Finally, we compute amplitudes involving one ϕ field and two s fields. Unlike the examples discussed above, this amplitude does not contain an explicit factor of k_{123}, because ϕ is not a scalar on the S^5.

$$
\begin{array}{cc}
(\phi^1 s^2|s^3) & H_{123}/C^{(0)}_{123} \\
(\cdot 1|1) & 0 \\
(\cdot 2|0) & q_1^2 q_2^4 \\
(\cdot 0|2) & 0
\end{array}
\qquad\qquad
\begin{array}{cc}
(s^1 s^2|\phi^3) & H_{123}/C^{(0)}_{123} \\
(11|\cdot) & \mathcal{O}(1/J^3) \\
(02|\cdot) & \mathcal{O}(1/J^2)
\end{array}
\qquad (3.7)
$$

3.2 Construction of the vertex

We will closely follow the standard process of constructing the vertex, and our result will share many features with the previous proposals. However, compatibility with the duality map (2.10) and the supergravity data listed in the previous subsection will lead to a final result different from all of the previous proposals.

Let us briefly sketch the standard process.(See, for example, [35, 7]). Quantization of the string theory in the pp-wave is performed in the light-cone gauge. In the light-cone gauge, space-time symmetries are implemented in the interacting theory in two different ways. All the generators that leave the light-cone gauge fixing invariant are called kinematical. They do not receive correction from the interactions and can be promoted to local symmetries on the world-sheet. The remaining generators are called dynamical and do receive corrections when the interactions are turned on. For the string theory in the pp-wave, the light-Hamiltonian and a half of the 32 supercharges are the only dynamical generators.

In principle, the cubic interaction part of the dynamical generators (Hamiltonian H_3 and supercharges Q_3) can be written as an operators in the string Fock space which change the number of strings. In practice, it is more convenient to translate H_3, Q_3 into states $|H_3\rangle, |Q_3\rangle$ in the three string Hilbert space. Construction of $|H_3\rangle, |Q_3\rangle$ takes two steps. First, one builds a kinematical vertex $|V\rangle$ which manifestly respects all the kinematical symmetries. Then, the dynamical generators take the form $|H_3\rangle = \hat{h}_3|V\rangle$, $|Q_3\rangle = \hat{q}_3|V\rangle$ The prefactors $\hat{h}_3, \hat{q}_3$ chosen such that the kinematical constraints are not spoiled and at the same time the commutation relation among the dynamical generators are also satisfied.

Free theory: Review

The IIB supergravity in the pp-wave has manifest $SO(4)\times SO(4)$ rotation symmetry inherited from the $SO(2,4)\times SO(6)$ symmetry of $AdS_5\times S^5$. Following [17], we use the vector index i and bi-spinor indices $\alpha_1,\dot\alpha_1$ for the first $SO(4)\subset SO(2,4)$, and ($i';\alpha_2,\dot\alpha_2$) for the second $SO(4)\subset SO(6)$. The Hilbert space of the free supergravity in the pp-wave is described by 8 bosonic oscillators $\{a^i,(a^i)^\dagger;a^{i'},(a^{i'})^\dagger\}$ and 8 fermionic oscillators $\{b_{\alpha_1\alpha_2},(b^\dagger)^{\alpha_1\alpha_2};b_{\dot\alpha_1\dot\alpha_2},(b^\dagger)^{\dot\alpha_1\dot\alpha_2}\}$. The bosonic oscillators build up $(\Delta;i,j)$ representation of $SO(2,4)$ and $[a,b,c]$ representation of $SO(6)$ in Fig. 1.

The fermionic oscillators are identified with the kinematical super-charges up to a light-cone-momentum dependent factor (we assume $\alpha\equiv\alpha'p^+>0$ throughout this subsection),

$$q_{\alpha_1\alpha_2}=\sqrt{\alpha}b_{\alpha_1\alpha_2},\qquad (q^\dagger)^{\alpha_1\alpha_2}=\sqrt{\alpha}(b^\dagger)^{\alpha_1\alpha_2},$$
$$\bar q_{\dot\alpha_1\dot\alpha_2}=\sqrt{\alpha}b_{\dot\alpha_1\dot\alpha_2},\qquad (\bar q^\dagger)^{\dot\alpha_1\dot\alpha_2}=\sqrt{\alpha}(b^\dagger)^{\dot\alpha_1\dot\alpha_2}\ . \tag{3.8}$$

They form a super-multiplet of the same diamond shape as in Figure 1. The other 16 super-charges are dynamical. Explicitly, they are given by

$$Q^{\dot\alpha_1}{}_{\alpha_2}=(a^\dagger)^{\dot\alpha_1\alpha_1}b_{\alpha_1\alpha_2}-a_{\alpha_2\dot\alpha_2}(b^\dagger)^{\dot\alpha_1\dot\alpha_2},\qquad (Q^\dagger)_{\dot\alpha_1}{}^{\alpha_2}=a_{\alpha_1\dot\alpha_1}(b^\dagger)^{\alpha_1\alpha_2}-(a^\dagger)^{\dot\alpha_2\alpha_2}b_{\dot\alpha_1\dot\alpha_2},$$
$$\bar Q^{\alpha_1}{}_{\dot\alpha_2}=(a^\dagger)^{\dot\alpha_1\alpha_1}b_{\dot\alpha_1\dot\alpha_2}+a_{\alpha_2\dot\alpha_2}(b^\dagger)^{\alpha_1\alpha_2},\qquad (\bar Q^\dagger)_{\alpha_1}{}^{\dot\alpha_2}=a_{\alpha_1\dot\alpha_1}(b^\dagger)^{\dot\alpha_1\dot\alpha_2}+(a^\dagger)^{\dot\alpha_2\alpha_2}b_{\alpha_1\alpha_2}\ . \tag{3.9}$$

Note that all of them annihilate the oscillator vacuum, and that they are eigenstates of the $U(1)_Y$ symmetry. In what follows, the anti-commutators among the dynamical supercharges are important. The only nonvanishing terms are

$$\{Q^{\dot\alpha_1}{}_{\alpha_2},(Q^\dagger)_{\dot\beta_1}{}^{\beta_2}\}\ =\ 2\delta^{\dot\alpha_1}_{\dot\beta_1}\delta^{\beta_2}_{\alpha_2}H+\text{rotations},$$
$$\{\bar Q^{\alpha_1}{}_{\dot\alpha_2},(\bar Q^\dagger)_{\beta_1}{}^{\dot\beta_2}\}\ =\ 2\delta^{\alpha_1}_{\beta_1}\delta^{\dot\beta_2}_{\dot\alpha_2}H+\text{rotations}\ . \tag{3.10}$$

Cubic vertex

As a starting point, we use the kinematical vertex proposed in [19],

$$|V\rangle=\exp\left\{\frac{1}{2}\sum_{r,s=1}^{3}a^\dagger_{(r)}M^{rs}a^\dagger_{(s)}\right\}\exp\left\{-\sum_{i=1}^{2}b^\dagger_{(i)}q_ib^\dagger_{(3)}\right\}|v\rangle_{123}, \tag{3.11}$$

where M^{rs} are the supergravity Neumann coefficients ($q_i=\sqrt{|\alpha_i/\alpha_3|}$, $i=1,2$, as before),

$$M=\begin{pmatrix} q_2^2 & -q_1q_2 & -q_1\\ -q_1q_2 & q_1^2 & -q_2\\ -q_1 & -q_2 & 0\end{pmatrix}. \tag{3.12}$$

In order for the prefactor not to spoil the kinematical constraint, it should consist of the following combinations of oscillators.

$$\begin{aligned} K^i &= q_1(a_2^\dagger)^i - q_2(a_1^\dagger)^i, \quad Y^{\alpha_1\alpha_2} = q_1(b_2^\dagger)^{\alpha_1\alpha_2} - q_2(b_1^\dagger)^{\alpha_1\alpha_2}, \\ L^{i'} &= q_1(a_2^\dagger)^{i'} - q_2(a_1^\dagger)^{i'}, \quad Z^{\dot\alpha_1\dot\alpha_2} = q_1(b_2^\dagger)^{\dot\alpha_1\dot\alpha_2} - q_2(b_1^\dagger)^{\dot\alpha_1\dot\alpha_2}. \end{aligned} \tag{3.13}$$

Following the literature on the construction of the vertex, we assume that the prefactor has at most two powers of bosonic oscillators. It will be justified by matching the amplitudes via the duality map. The $U(1)_Y$ symmetry demands that $|H_3\rangle$ contains only terms with the same number of Y and Z, and that the supercharges have terms like $Y^nZ^{n\pm1}$ depending on their $U(1)_Y$ charges. It is straightforward to enumerate all possible terms allowed to appear in the supercharges. Schematically,

$$\begin{aligned} |Q_3\rangle &= (c_1LZ + c_2KYZ^2 + c_3LY^2Z^3 + c_4KY^3Z^4)|V\rangle, \\ |Q_3^\dagger\rangle &= (d_1KY + d_2LY^2Z + d_3KY^3Z^2 + d_4LY^4Z^3)|V\rangle, \\ |\bar Q_3\rangle &= (\bar c_1LY + \bar c_2KY^2Z + \bar c_3LY^3Z^2 + \bar c_4KY^4Z^3)|V\rangle, \\ |\bar Q_3^\dagger\rangle &= (\bar d_1KZ + \bar d_2LYZ^2 + \bar d_3KY^2Z^3 + \bar d_4LY^3Z^4)|V\rangle\ . \end{aligned} \tag{3.14}$$

Define the sum of super-charges in the free theory,

$$Q = \sum_{r=1}^{3} Q_r^{(2)}\ , \tag{3.15}$$

for the four kinds of dynamical supercharges. The super-algebra at the cubic level demands that

$$Q|Q_3^\dagger\rangle + Q^\dagger|Q_3\rangle = |H_3\rangle, \quad \bar Q|\bar Q_3^\dagger\rangle + \bar Q^\dagger|\bar Q_3\rangle = |H_3\rangle\ , \tag{3.16}$$

and that similar equations with $\mathrm{RHS} = 0$ hold for all the other combinations of super-charges.

A straightforward but tedious calculation gives a seemingly over-constrained set of linear equations among the coefficients in (3.14). However, it turns out that many of the relations are linearly dependent, and there are three independent solutions. The solution for the cubic Hamiltonian is given by,

$$\begin{aligned} |H_3\rangle &= h_0\Big((L^2-K^2)(1+Y^4Z^4) + 2KL(YZ+Y^3Z^3) - K^2(YZ)^2 + L^2(YZ)^2\Big)|V\rangle \\ &\quad + h_-(K^2+L^2)|V\rangle + h_+(K^2+L^2+8)(YZ)^4|V\rangle, \end{aligned} \tag{3.17}$$

where h_0, h_-, h_+ are so far undetermined constants. As expected, the super-algebra alone does not fix the vertex completely. It is now time to use our knowledge on holography discussed above. Using the conservation laws for the kinematical symmetries and $q_1^2 + q_2^2 = 1$, one can show that

$$(L^{i'})^2|V\rangle = (n_B)_{123}|V\rangle, \tag{3.18}$$

that is, $(L^{i'})^2$ counts the change in the number of scalar impurities. Similarly, one can show that the $(K^i)^2$ term counts vector impurities. However, we saw that vector impurities contribute only the 'wave-function factor' V_v and do not affect the coupling constant. This fact demands via the duality map (2.10) that $|H_3\rangle$ should not contain a factor of $(K^i)^2$ when all external states are $SO(4) \times SO(4)$ scalars. This implies that $h_- = h_0 = h_+$. The overall normalization can be fixed by matching any one of the non-zero amplitudes listed in (3.3). All in all, the final answer is[4],

$$\begin{aligned} |H_3\rangle &= C_{123}^{(0)}\Big((L^{i'})^2 + \{(L^{i'})^2+4\}Y^4Z^4 + K^{\dot\alpha_1\alpha_1}L^{\dot\alpha_2\alpha_2}(Y_{\alpha_1\alpha_2}Z_{\dot\alpha_1\dot\alpha_2} + Y^3_{\alpha_1\alpha_2}Z^3_{\dot\alpha_1\dot\alpha_2})\Big)|V\rangle \\ &\quad + \frac{C_{123}^{(0)}}{2}\Big(L^{\dot\alpha_2\alpha_2}L^{\dot\beta_2\beta_2}Y^2_{\alpha_2\beta_2}Z^2_{\dot\alpha_2\dot\beta_2} - K^{\dot\alpha_1\alpha_1}K^{\dot\beta_1\beta_1}Y^2_{\alpha_1\beta_1}Z^2_{\dot\alpha_1\dot\beta_1}\Big)|V\rangle\ . \end{aligned} \tag{3.19}$$

Note that we used only the 'bosonic impurity' part of the previous subsection to determine the vertex. The $(L^{i'})^2$ factor gives k_{123} part of (3.3) and (3.5). The wave-function factor V_s and V_v match exactly elements of the bosonic Neumann matrix.

Now, all the amplitudes containing 'fermionic impurities' provide further checks on the vertex. First, note that the factor $((L^{i'})^2 + 4)$ multiplying Y^4Z^4 matches $(k_{123}+4)$ in (3.6). In fact, the $(YZ)^0$ term and $(YZ)^4$ terms can be combined into $(n_B + n_F/2)(1+Y^4Z^4)|V\rangle$. The factors of q_1, q_2 in (3.6) come from both the fermionic Neumann matrix, and for those proportional to $(k_{123}+4)$, also from Y^4Z^4. Finally, one can check that the $L^2(YZ)^2$ term gives the non-vanishing entry in the table (3.7) for the (ϕss) amplitudes.

4 String theory vertex

We now turn to the task of generalizing the supergravity vertex (3.19) to the full string theory. We first need to enlarge the Fock space to include also the states created by the stringy oscillators $a_n^\dagger$ and $b_n^\dagger$. Then we need to find a 3-string vertex satisfying two main constraints: it should realize the pp-wave super-algebra at cubic level and it should reduce to the supergravity expression (3.19) in the $\mu\alpha_i \to 0$ limit. In principle one could proceed in a systematic way as done in the previous section for the BPS sector, but this exhaustive approach is rather complicated at the string level. It is easier and also more instructive to derive the cubic vertex by combining the results derived in previous works.

We start by choosing a coherent state that realizes the kinematical part of the algebra $|V\rangle = E_a\, E_b\, |v\rangle_{123}$, where the two terms contain the bosonic and the fermionic contributions respectively.

[4]Our definitions for the products of Y and Z are slightly different from those of [17]: $Y^2 = Y_P^2/2, Y^3 = Y_P^3/3, Y^4 = Y_P^4/12$ and similarly for Z. Accordingly, the polynomials v and s appearing in (4.3) should be understood as $s(Y) = Y + iY^3$, $v^{ij} = \delta^{ij}(1+Y^4)(1+Z^4) - i[(Y^2)^{ij}(1+Z^4) - (Z^2)^{ij}(1+Y^4)] + (Y^2Z^2)^{ij}$ and $v^{i'j'} = \delta^{i'j'}(1-Y^4)(1-Z^4) - i[(Y^2)^{i'j'}(1-Z^4) - (Z^2)^{i'j'}(1-Y^4)] + (Y^2Z^2)^{i'j'}$.

Let us recall their explicit expressions[5]. The bosonic exponential reads [13]

$$E_a = \exp\left\{\frac{1}{2}\sum_{r,s=1}^{3} a^{\dagger}_{n(r)} N^{rs}_{nm} a^{\dagger}_{m(s)}\right\} . \tag{4.1}$$

The string Neumann coefficients are usually written in terms of products of infinite matrices. From this formal definition many properties can be derived [36, 15], however it is difficult to obtain an explicit value of the N^{rs}_{nm} in terms of n, m and the α_i's, since the original product expression contains the inverse of an infinite matrix. A detailed study of the Neumann coefficients for $\mu \neq 0$ can be found in [37, 38, 39]. In the fermionic sector we will use the coherent state introduced in [20], which can be written in the $SO(4) \times SO(4)$ notation as done in [17]

$$E_b = \exp\left[\sum_{r,s=1}^{3}\sum_{m,n\geq 0}\left(b^{\alpha_1\alpha_2\,\dagger}_{-m(r)} b^{\dagger}_{n(s)\,\alpha_1\alpha_2} + b^{\dot\alpha_1\dot\alpha_2\,\dagger}_{m(r)} b^{\dagger}_{-n(s)\,\dot\alpha_1\dot\alpha_2}\right) Q^{rs}_{mn}\right] . \tag{4.2}$$

As we reviewed in the introduction, this kinematical part can be completed into a fully supersymmetric interacting Hamiltonian in (at least) two physically different ways. One possible completion was first proposed by [13, 14, 16] in the $SO(8)$ formalism. Subsequently the same vertex was recast in the $SO(4)\times SO(4)$ language [17, 18] and here we will stick to the $SO(4)\times SO(4)$ notation

$$\begin{aligned} |H_3\rangle_I \;=\; & - \Big[\Big(K_i\tilde K_j + \frac{1}{2}\,\delta_{ij}\Big) v^{ij}(Y,Z) - \Big(L_{i'}\tilde L_{j'} + \frac{1}{2}\,\delta_{i'j'}\Big) v^{i'j'}(Y,Z) \\ & - K^{\dot\alpha_1\alpha_1}\tilde L^{\dot\alpha_2\alpha_2} s_{\alpha_1\alpha_2}(Y) s^{*}_{\dot\alpha_1\dot\alpha_2}(Z) - \tilde K^{\dot\alpha_1\alpha_1} L^{\dot\alpha_2\alpha_2} s^{*}_{\alpha_1\alpha_2}(Y) s_{\dot\alpha_1\dot\alpha_2}(Z)\Big]|V\rangle , \end{aligned} \tag{4.3}$$

where again we follow the conventions and notation of [17], except for the normalization the bosonic constituents which is slightly different,

$$K^i = \sqrt{\frac{\alpha'}{2\mu|\alpha_1\alpha_2\alpha_3|}}\; K^i_P \;\; , \quad L^{i'} = \sqrt{\frac{\alpha'}{2\mu|\alpha_1\alpha_2\alpha_3|}}\; K^{i'}_P . \tag{4.4}$$

Another possibility for writing a supersymmetric vertex is discussed in [21], where it was proposed to use simply the free Hamiltonian as prefactor of the coherent state

$$|H_3\rangle_D = \sum_{r=1}^{3} H_r |V\rangle . \tag{4.5}$$

However, it was first noticed in [24] that neither of the two vertices (4.3) and (4.5) have the expected behaviour from the holographic point of view. We can rephrase this observation in a somehow

[5]For the conventions on the string oscillators and the explicit definition of the Neumann matrices the reader is referred to [15] and references therein.

different way by using the results of the previous section: the supergravity limit of the vertices (4.3) and (4.5) breaks the relation (2.1) because they contain some K_0^2 term in the prefactor which is absent on the AdS side. So it was proposed [24] that the holographic cubic vertex for the pp-wave background is proportional to $|H\rangle_I + |H\rangle_D$. It is interesting to notice that this combination reproduces, when restricted to the scalar bosonic oscillators, the 'phenomenological' prefactor introduced in [40] to explain the field theory results from a string theory point of view.

However, a closer comparison between the proposed vertex $|H\rangle_I + |H\rangle_D$ and the large J limit of the AdS couplings shows that relation (2.1) is not yet satisfied. In fact, when we restrict the combination $|H\rangle_I + |H\rangle_D$ to the supergravity sector, the only term that perfectly matches the AdS expectation is the one without fermionic insertions (of Y_0 and Z_0). However, it is not difficult to see how to modify the vertex (4.5) in such a way that its combination with (4.3) gives the expected supergravity answer. First we should add two pieces quartic in the fermions (Y^4 and Z^4) so that at the supergravity level the $U(1)_Y$ violating terms of (4.3) are canceled. Then a contribution with eight fermionic insertions ($Y^4 Z^4$) should be added to match the second term in (3.19). Thus our final proposal for the holographic cubic vertex is

$$|H\rangle = \frac{C_{123}^{(0)}}{2} \Big(|H\rangle_I + |H\rangle_{II} \Big) \,, \tag{4.6}$$

where

$$|H_3\rangle_{II} = \left(\sum_{r=1}^{3} H_r \right) \Big(1 + Y^4\Big)\Big(1 + Z^4\Big) |V\rangle \,. \tag{4.7}$$

Clearly this contribution to the vertex is a natural generalization of the (4.5) and it satisfies by itself the supersymmetry constraints. In fact the combination $(1 + Y^4 + Z^4 + Y^4 Z^4)|V\rangle$ satisfies all the requirement related to the kinematical part of the pp-wave algebra. Thus it can be 'dressed' with the free supercharges or Hamiltonian as done in [21] in order to produce a consistent system of interacting correction to the free generators. Notice also that the commutation of the kinematical constraints with $\sum_r H_r$ is again a combination of the kinematical constraints and thus does not spoil the properties of the coherent state $|V\rangle$.

4.1 Some checks on the string vertex

From the gauge theory point of view the holographic vertex contains a great deal of information on non-BPS quantities, since the dependence of the Neumann matrices on $\mu\alpha_i$ in (4.6) translates, in the SYM theory, into the exact dependence on the 't Hooft coupling. Moreover in the non-supersymmetric sector, the comparison with the gauge theory is the only way at our disposal to check the correctness of the proposal (4.6). However, it is still not entirely clear how to relate in general string and gauge theory results, since the dictionary (2.10) between 3-point correlators in the two descriptions has been derived only in the supergravity approximation. The authors of [24] proposed a small modification of (2.10)

$$\Delta_{123}\left(\frac{J_1 J_2}{J_3}\right)^{\frac{\Delta_{123}}{2}} C_{123} = (f)^{-\frac{\Delta_{123}}{2}}\,\Gamma\left(\frac{\Delta_{123}}{2}+1\right) H_{123}^{(\mathrm{PP})}\,, \tag{4.8}$$

where f is a combination that appears in various places of the string computations (see, for instance, [15]): $f=(1-4\mu\alpha K)$. With this prescription the 3-point functions among BPS states are independent of $\mu\alpha_i$, even if the full string vertex (4.6) is used to compute the correlator. This can be checked by using the relation between the stringy Neumann coefficients and the supergravity ones: $N^{ij}_{00} = f M^{ij}$ for $1 \leq i,j \leq 2$ and $N^{i3}_{00} = M^{i3}$. The requirement to have constant 3-point functions among BPS states is in accordance with the expected non-renormalization theorem [41] of the SYM correlators among three BPS operators. Of course it would be very interesting to *derive* (4.8) in order to check the non-renormalization theorem, instead of imposing it. Moreover it is quite likely that other α'-dependent modifications will appear in the exact dictionary between C_{123} and $H_{123}^{(\mathrm{PP})}$. However, if we focus on the first order in the λ' expansion, the simple Eq. (4.8) is able to capture completely the relation between gauge theory and string theory. Let us briefly summarize the evidence collected so far supporting this proposal.

– The first thing we want to verify is that the new terms introduced in (4.6) do not spoil the agreement between string and gauge theory correlators found in previous works. It is clear that for purely bosonic amplitudes the new terms present in (4.7) are irrelevant and so all the checks already done in this subsector[6] supports our proposal (4.6). On the contrary, the amplitudes with four or more fermionic impurities are sensitive to the novelties contained in (4.6). However, in the situation studied in [24], the four fermions are divided in an impurity preserving way, that is two of them act on the ingoing state (the one with negative α_i) and the others act on the outgoing states (those with $\alpha_i > 0$). In this case, the new contributions in (4.7) appear only at the next-to-leading order in λ', In fact the Y^4 and Z^4 terms appearing in $|H\rangle_{II}$ are multiplied by $\sum_r H_r$ and in the impurity preserving processes $\sum_r H_r \sim O(\lambda')$. Similar terms quartic in the fermions are present also in $|H\rangle_I$, but they do not have the energy difference as additional factor and so their contribution survives also at the first order in the λ' expansion. Thus the $O(\lambda')$ result for these amplitudes is again in agreement also with the vertex (4.6). This situation is very similar to that encountered in the study of the processes where the number of impurities is preserved, but their flavor changes (like the process considered in [40]). Also in this case only $|H\rangle_I$ contributes to the leading order result of the string amplitude.

– In the truly non-impurity preserving processes, where also the number of impurities changes from the operator O_3 to the operators O_1 and O_2, the full vertex (4.6) enter. We have already seen

[6]This applies also to the recent papers [42], as well as to previous works [40, 43].

in section 3 that the new terms in (4.7) are necessary to have agreement with the large J limit of supergravity results. In the BPS sector this ensures that the string amplitudes do agree also with the gauge theory answer, thanks to the standard AdS/CFT duality. Let us see how this works by focusing for instance on the sixth case in the table (3.3). The relevant operators are

$$O_1 = \mathrm{Tr}(\phi Z^{J_1}) \ , \qquad O_2 = \frac{1}{\sqrt{J_2}} \sum_{l=0}^{J_2} \mathrm{Tr}(\bar{\phi} Z^l \psi Z^{J_2-l}) \ , \qquad O_3 = \mathrm{Tr}(\psi Z^{J_3}) \ , \tag{4.9}$$

and it is straightforward to see that the gauge theory combinatorics reproduces in the large J limit the third column of (3.3)

$$\langle \bar{O}_3(x_3)\, O_2(x_2)\, O_1(x_1)\rangle = \frac{1}{\sqrt{J_1 J_3}} \frac{C_{123}^{(0)}}{|x_1-x_2|^{2\beta_3}|x_2-x_3|^{2\beta_1}|x_3-x_1|^{2\beta_2}} \ . \tag{4.10}$$

On the string side one obtains

$$_{123}\langle v|\; a_{0(3)}^{\bar{\psi}}\, a_{0(2)}^{\bar{\phi}} a_{0(2)}^{\phi}\, a_{0(1)}^{\phi} |H\rangle = 2 N_{00}^{12} N_{00}^{23} = 2 f M_{00}^{12} M_{00}^{23} \ . \tag{4.11}$$

By using (3.12) and, in this case, $\Delta_{123} = 2$, we see that this is equal to the first column of the table (3.3) multiplied by the factor f which is the difference between the supergravity and the full Neumann matrices for the elements N_{00}^{ij} with $1 \leq i, j \leq 2$. However, the dictionary (4.8) was engineered to cancel the factors of f and in fact we get the same μ–independent answer obtained in section 3. It is also easy to study the same amplitude in the string case, where the second operator is replaced by

$$O_2 = \frac{1}{\sqrt{J_2}} \sum_{l=0}^{J_2} \mathrm{Tr}(\bar{\phi} Z^l \psi Z^{J_2-l})\; \mathrm{e}^{2\pi i \frac{nl}{J_2+1}} \ . \tag{4.12}$$

In this case the tree-level result on the gauge theory side is zero, because the phase forces the final sum over l to vanish. On the string side, the only difference with the BPS case is that now the result is proportional to the Neumann matrices $N_{0n}^{12} N_{n0}^{23}$, while before we had $n = 0$. By using the results of [37], we find that in this case the first non-trivial contribution to the RHS of (4.8) starts at order λ', in agreement with the gauge theory results which fixes the tree-level contribution to be zero.

– The last case of table (3.3) presents the prototypical case of impurity non-preserving processes. In this case both ‘outgoing’ operators contain two impurities. On the gauge theory side the large J limit of this amplitude does not change when we pass from BPS operators to stringy ones with the BMN phase (like that of Eq. (4.12)). This is because only particular terms in the sum defining the operators contribute to the amplitude in the *planar* approximation and $\mathrm{e}^{2\pi i n/J} \to 0$ for any $n \neq 0$ in the BMN limit. On the string side this observation implies that the elements N_{nm}^{ij}

with $1 \leq i, j \leq 2$ of the Neumann matrices are, *at leading order in* λ', basically the same as the zero-mode elements. Again by using the results of [37] one can check that this is indeed the case. Thus we can use the agreement between string and supergravity/CFT results at the BPS level in order to claim that impurity non-preserving amplitudes agree at leading order in λ' also for generic non-BPS states.

5 Discussion

In the usual approach to the BMN duality, one first tries to build the pp-wave string Hamiltonian by using only the internal consistency of the theory and then looks for a string/SYM dictionary compatible with the string vertex. Since the two vertices (4.3) and (4.5) are rather different, they motivated two different ways to relate string theory interactions with the dual gauge theory results. Inspired by the string bit proposal [44, 45], various authors [46, 47, 48, 49] studied the relation between the string vertex (4.3) and the mixing between single and double trace operators on the field theory side (see also [50, 51] for further checks in this direction). In particular, they proposed to identify the 3-string couplings derived from (4.3) with the matrix elements of the gauge theory dilatation generator in a particular basis in the space of the single and double trace operators. On the other hand the vertex (4.5) was motivated by realizing in string theory the proposal of [52] that relates the 3-string couplings with the correlators among the BMN operators on the gauge theory side. Notice that also this point of view is consistent with the string bit picture, since it identifies, in the $\mu\alpha_i \to \infty$ limit, the world-sheet dynamics with the free contractions among the constituents of the three operators (see for instance the figures for the 3-point function in [52] and [12]). Even though these two proposals were checked in various different cases, the situation was not completely satisfactory. First the agreement between string and field theory results was checked only at leading order in λ'. Then, on the conceptual ground, it was rather unclear the role played in the duality by the gauge theory operators that are exact eigenstates of the dilatation generator. At leading order in g_2 these eigenvectors are a particular combination of single and double trace operators. However, on the one hand the comparison between the string vertex (4.5) and the gauge theory results gave agreement only by using the original BMN operators [27, 53, 21] and ignoring the multi-trace corrections of the dilation eigenvectors. On the other hand the string/gauge theory comparison with the vertex (4.3) required a mixing between single and double traces that was *different* from the one necessary to define the dilatation eigenvectors. In fact the field theory computations of [40, 43], that are done with the dilatation eigenvectors, represented for long time a puzzle from the string point of view, since they seemed to be not related to either of the two vertices (4.3)-(4.5).

In order to overcome these problems, in this paper we reversed the approach commonly adopted so far and constructed a 3-string interaction in the PP-wave background by taking into consideration all possible information from different descriptions from the very beginning. In particular, we study systematically the constraints on the string dynamics coming from the large J limit of $\mathrm{AdS}_5 \times S^5$ supergravity. Our results confirm the physical picture of Dobashi and Yoneya [24] and show how the string vertex has to be generalized in order to describe correctly also impurity non-preserving processes. Moreover, as explained in section 5 of [24], this approach is able to explain also the partial success, for impurity preserving processes, of the previous string/gauge theory comparisons (see the discussion above). For these processes, it is possible to separate the contributions coming from the free field theory combinatorics from those responsible of the operator mixing and map them into the $|H_3\rangle_D$ and $|H_3\rangle_{SV}$ parts of the full string vertex $|H_3\rangle$.

Let us conclude by summarizing here the main results derived in this paper and focusing on the properties of the vertex (4.6). A first unexpected feature is that the string interaction must break the $\mathbb{Z}_2$ symmetry of the pp-wave background, which, on the contrary, was preserved by the free spectrum. It was first noticed in [19] that SV vertex [13] had a definite parity under this discrete symmetry. It was further proposed that one should build a different 3-string vertex, with opposite parity, in order to make a direct comparison with gauge theory correlators possible. This idea was in striking contrast with the belief that there was a unique possible interacting Hamiltonian realizing the relevant supersymmetry algebra. However an explicit realization of this proposal [21] showed the necessity of further constraints in order to fix completely the string cubic Hamiltonian. However, it turns out that the behaviour under the $\mathbb{Z}_2$ symmetry is not a reliable input for fixing the form of the string vertex. A first signal that this $\mathbb{Z}_2$ was not a good symmetry at the interacting level came from the study [54] of field theory correlators among dilatation eigenstates containing vector impurities. Here instead we used the insights coming from supergravity and we showed that the interacting Hamiltonian must contain both odd and even terms under $\mathbb{Z}_2$. Moreover the vertex (4.6) singled out by our analysis contain new $SO(4) \times SO(4)$ preserving combinations of the various building blocks [15, 21], realizing once more a situation quite common in physics (i.e. everything that is not forbidden is compulsory). It is natural at this point to ask whether it is necessary to add further corrections to Eq. (4.6) that are not captured by our supergravity analysis. Although this seems unlikely we can not rule out such corrections. For instance we still use as an additional input the requirement that the prefactor is at most quadratic in the bosonic oscillators. In order to clarify completely this point it would be necessary to derive the cubic Hamiltonian from first principles, for instance by applying a standard path integral approach also in the derivation of the prefactor (and not only for the exponential part, as it was done in [12]).

Another interesting aspect of our string proposal is to see how the $U(1)_Y$ symmetry is realized at the level of BPS (or supergravity) interactions. Actually this is a general observation, not restricted to the particular pp-wave background we are focusing on. In fact a similar pattern appears also in the construction of the flat space IIB string field theory: in [55] it was noticed that the $U(1)_Y$ symmetry forces the supergravity prefactor to be quartic in the fermionic fields. However, the full string construction [56] requires the presence of other terms that survive also when the amplitudes are restricted to the supergravity sector. The original observation in [56] was that these new terms are proportional (at the supergravity level) to the difference of the free Hamiltonians ($\sum_r H_r$) and thus are zero on-shell. In order to have a conserved $U(1)_Y$ symmetry also off-shell, [56] proposed that the $U(1)_Y$ generator should get corrections in the interacting theory. Here we show that there is a simpler way out: one can define the off-shell cubic Hamiltonian for the flat space to be a simple combination of the Brink, Green and Schwarz vertex and of the following vertex

$$|H_3\rangle = |H_3\rangle_{BGS} - \left(\sum_r H_r\right)\left(1 + \prod_{a=1}^{8} Y^a_{BGS}\right)|V\rangle_{BGS}\ , \qquad (5.1)$$

where we are now using the conventions of [56]. Notice that the additional piece is irrelevant if we just want to compute on-shell scattering amplitudes because in flat space the energy is conserved. Thus previous checks on S-matrix elements like those in [57] are not affected by the modification proposed here. However, the inclusion of the new terms in (5.1) yields a $U(1)_Y$ preserving (supergravity) vertex also off-shell. In the pp-wave case this feature is necessary since we clearly do not want any conservation law on H_r in the physical observables and so the terms proportional to $\sum_r H_r$ can not be disregarded. However, the modification proposed in (5.1) is important also in flat space every time one needs to go off-shell. Problems of this type are constructing a 4-string vertex by sewing two 3-string vertices or computing the energy of an arbitrary string configuration including the cubic contributions H_3. It is known that the vertex $|H_3\rangle_{BGS}$ is incomplete and can not be used to deal consistently with these questions. Because of these problems it has been proposed that the light-cone string field theory contains also quartic terms [58, 59, 60, 61]. It would be very interesting to reconsider these issues by using the 3-string vertex (5.1) to see whether it can provide a different completion of $|H_3\rangle_{BGS}$ that does not require quartic corrections.

Also on the field theory side the string vertex (4.6) together with the duality map (4.8) yields some interesting and counterintuitive consequences. For instance, it is common to write the BMN operators by focusing only on the leading term in the $J \to \infty$ limit, even if in principle they are combinations of various contributions with the same quantum numbers[7]. This is the so-called 'dilute gas approximation' where the impurities are always thought to be far apart from each other. However in the impurity non-preserving processes this approximation breaks down even in the simplest situations, since in the holographic dictionary (2.10) between gauge and string theory correlators there is a compensating J-dependent factor. This term plays an important rôle in the correlators with *different* barred and unbarred operators (i.e. with different 'ingoing' and 'outgoing' states). In the dilute gas approximation this kind of amplitudes is trivially vanishing, while

[7] The importance of certain compensating terms, subleading in the $J \to \infty$ limit, was already stressed in [62, 54].

on the string side the corresponding processes are non-zero, since they get a non-zero contribution from the various term in the prefactor containing the fermionic insertions. The presence of the compensating factor in (2.10) enhances the contributions coming from the subleading (in J) terms in the definition of the BMN operators and gives a non-zero answer also on the gauge theory side.

Finally a very important open issue is the full justification of the holographic dictionary. For example, the duality map (4.8), if correct, can provide a resolution to the puzzle of fractional powers of λ' raised in [63]. While the map in the supergravity sector (2.10) has been derived from directly from the rules of the AdS/CFT duality, its string generalization (4.8) has been proposed [24] by imposing the non-renormalization of the 3-point BPS correlators. It is clearly important to test and possibly completely fix this holographic dictionary. Two complementary approaches are possible: either one can work from the bulk point of view and generalize the physical picture sketched in section 2.3 from the particle to the string case, or one starts from the field theory by pushing the computations to the subleading order in λ'.

Acknowledgement

We are grateful to A. Tanzini for useful discussions.

A Coupling constants

We summarize the cubic couplings needed to compute the amplitudes in section 3. The (sss) coupling was first computed in [41]. Couplings for two s and another arbitrary field were worked out in [64, 65]. Other couplings listed below can be derived in a similar way. We follow the notations of [65]. The part of AdS_5 supergravity action relevant to our discussion can be written as

$$S = \frac{N^2}{8\pi^2}\int d^5x\sqrt{-g}\{L_2 + L_3\} \ . \tag{A.1}$$

The quadratic Lagrangian takes the following form

$$L_2 = -\sum_{\varphi=s,t,\phi}\frac{A_\varphi}{2}\{(\nabla\varphi)^2 + m_\varphi^2\varphi^2\} - A_B\{|\nabla B|^2 + m_B^2|B|^2\} \ . \tag{A.2}$$

The mass of each scalar is determined by the usual relation $m^2 = \Delta(\Delta - 4)$ and the relation between Δ and k mentioned in subsection 3.1. The normalization constants are given by

$$A_s = 2^5\tfrac{k(k-1)(k+2)}{k+1}z(k), \quad A_t = 2^5\tfrac{(k+4)(k+5)(k+2)}{k+3}z(k), \quad A_B = z(k), \quad A_\phi = \tfrac{1}{2}z(k) \ . \tag{A.3}$$

The cubic Lagrangian is given by

$$L_3 \;=\; -\frac{1}{6}G_{123}^{(sss)}s^1s^2s^3 - \frac{1}{6}G_{123}^{(ttt)}t^1t^2t^3 - \frac{1}{2}G_{123}^{(tss)}t^1s^2s^3 - \frac{1}{2}G_{123}^{(stt)}s^1t^2t^3$$

$$
-\frac{1}{2}G^{(\phi ss)}_{123}\phi^1 s^2 s^3 - G^{(sB\bar{B})}_{123}s^1 B^2 \bar{B}^3 - G^{(sB\bar{B})}_{123}t^1 B^2 \bar{B}^3, \tag{A.4}
$$

where the coupling constants are given by

$$
\begin{aligned}
(s^1 s^2 s^3) \quad &: \quad 2^9 \frac{\alpha_1\alpha_2\alpha_3}{(k_1+1)(k_2+1)(k_3+1)}\frac{(\sigma+2)!}{(\sigma-3)!}a(k_1,k_2,k_3)\langle C^1 C^2 C^3\rangle, \\
(t^1 t^2 t^3) \quad &: \quad 2^9 \frac{(\alpha_1+2)(\alpha_2+2)(\alpha_3+2)}{(k_1+3)(k_2+3)(k_3+3)}\frac{(\sigma+8)!}{(\sigma+3)!}a(k_1,k_2,k_3)\langle C^1 C^2 C^3\rangle, \\
(t^1 s^2 s^3) \quad &: \quad 2^9 \frac{(\sigma+2)(\alpha_2+2)(\alpha_3+2)}{(k_1+3)(k_2+1)(k_3+1)}\frac{\alpha_1!}{(\alpha_1-5)!}a(k_1,k_2,k_3)\langle C^1 C^2 C^3\rangle, \\
(s^1 t^2 t^3) \quad &: \quad 2^9 \frac{(\sigma+4)\alpha_2\alpha_3}{(k_1+1)(k_2+3)(k_3+3)}\frac{(\alpha_1+6)!}{(\alpha_1+1)!}a(k_1,k_2,k_3)\langle C^1 C^2 C^3\rangle, \\
(s^1 B^2 \bar{B}^3) \quad &: \quad 2^4 \frac{(\sigma+2)(\alpha_1+2)\alpha_2\alpha_3}{k_1+1}a(k_1,k_2,k_3)\langle C^1 C^2 C^3\rangle, \\
(t^1 B^2 \bar{B}^3) \quad &: \quad 2^4 \frac{(\sigma+4)\alpha_1(\alpha_2+2)(\alpha_3+2)}{k_1+3}a(k_1,k_2,k_3)\langle C^1 C^2 C^3\rangle, \\
(\phi^1 s^2 s^3) \quad &: \quad 2^5 \frac{\sigma(\sigma+1)(\alpha_1-1)(\alpha_1-2)}{(k_2+1)(k_3+1)}h(k_1,k_2,k_3)\langle T^1 C^2 C^3\rangle \ .
\end{aligned} \tag{A.5}
$$

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