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MATRIX MODELS AND $\mathcal{N}=2$ GAUGE THEORY*

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We describe how the ingredients and results of the Seiberg-Witten solution to $\mathcal{N}=2$ supersymmetric U(N) gauge theory may be obtained from a matrix model.

Dijkgraaf and Vafa discovered that the non-perturbative effective superpotential for certain $d=4$ $\mathcal{N}=1$ supersymmetric gauge theories can be obtained by calculating planar diagrams in a related gauged matrix model[1,2,3] (for a more complete list of references, see Ref. 4). In this talk, we will show that matrix models can also be used to obtain all the ingredients and results of the Seiberg-Witten solution[5] of certain $\mathcal{N}=2$ supersymmetric gauge theories, specifically U(N) theories without matter, or with matter in fundamental, symmetric, or antisymmetric representations[1,2,6,7,8,9].

1. $\mathcal{N}=2$ supersymmetric gauge theory

We will focus on the $\mathcal{N}=2$ U(N) gauge theory with N_f hypermultiplets in the fundamental representation to illustrate the matrix model approach,

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indicating where differences occur in theories with symmetric or antisymmetric hypermultiplets. To apply the insights of Dijkgraaf-Vafa to an $\mathcal{N}=2$ gauge theory, one begins by expressing its field content in terms of $\mathcal{N}=1$ superfields. Let ϕ denote the adjoint $\mathcal{N}=1$ chiral superfield belonging to the $\mathcal{N}=2$ vector multiplet, and q^I and $\tilde{q}_I$ the $\mathcal{N}=1$ chiral superfields that comprise the $\mathcal{N}=2$ hypermultiplets transforming in the fundamental representation. The $\mathcal{N}=2$ theory has the superpotential

$$W_{\mathcal{N}=2}(\phi,q,\tilde{q}) = \sum_{I=1}^{N_f} \left[\tilde{q}_I \phi\, q^I + m_I \tilde{q}_I q^I\right] \tag{1.1}$$

where m_I are the masses of the fundamentals. The Coulomb branch of the moduli space of vacua is characterized by an arbitrary diagonal vev for the scalar field in ϕ, but we may select a specific (but generic) point $\phi = \mathrm{diag}(e_1,\cdots,e_N)$, at which the $\mathrm{U}(N)$ gauge group is broken to $\mathrm{U}(1)^N$, by adding a perturbation to the superpotential

$$W(\phi,q,\tilde{q}) = W_{\mathcal{N}=2}(\phi,q,\tilde{q}) + W_0(\phi) \tag{1.2}$$

where $W_0'(x) = \alpha \prod_{i=1}^N (x-e_i)$. The perturbation breaks the supersymmetry to $\mathcal{N}=1$, but the full $\mathcal{N}=2$ supersymmetry will be restored by sending $\alpha \to 0$ at the end of the matrix model calculation.

2. The perturbative matrix model

Each $\mathcal{N}=1$ chiral superfield described in the previous section has an analog (denoted by a capital letter) in the corresponding matrix model; specifically, an $M\times M$ hermitian matrix Φ, and M-dimensional vectors Q^I and $\tilde{Q}_I$. (The analog of a symmetric or antisymmetric matter hypermultiplet would be an $M\times M$ symmetric or antisymmetric matrix.) The superpotential (1.2) is reinterpreted[1] as the potential of the matrix model, whose partition function is thus

$$Z = \frac{1}{\mathrm{vol}(G)} \int \mathrm{d}\Phi\, \mathrm{d}Q^I \mathrm{d}\tilde{Q}_I \exp\left(-\frac{W(\Phi,Q,\tilde{Q})}{g_s}\right) \tag{2.1}$$

where G is the unbroken matrix model gauge group, and g_s is a parameter that will be taken to zero in the planar limit $M\to\infty$. The matrix integral (2.1) can be evaluated perturbatively about an extremum

$$\Phi_0 = \begin{pmatrix} e_1 \mathbb{1}_{M_1} & 0 & \cdots & 0 \\ 0 & e_2 \mathbb{1}_{M_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e_N \mathbb{1}_{M_N} \end{pmatrix}, \qquad (Q^I)_0 = 0, \qquad (\tilde{Q}_I)_0 = 0 \tag{2.2}$$

where the M_i are arbitrary, subject to $\sum_{i=1}^{N} M_i = M$. The U(M) symmetry of the matrix model is broken by Φ_0 to $G = \prod_{i=1}^{N} \mathrm{U}(M_i)$. The residual gauge symmetry must be gauge-fixed, and ghosts introduced; for details, see Refs. 2, 6. For the U(N) gauge theory with an antisymmetric representation[7,8,9] of mass m, Φ_0 must include an additional diagonal block $m\mathbb{1}_{M_0}$ and the antisymmetric matrix a corresponding block J (the symplectic unit), breaking the symmetry of the matrix model to $\mathrm{Sp}(M_0) \times \prod_{i=1}^{N} \mathrm{U}(M_i)$. The inclusion of the extra block for the antisymmetric case has been put in a broader context in Ref. 10.

2.1. *Topological expansion*

The connected diagrams of the perturbative expansion of (2.1) may be organized, using standard 't Hooft double-line notation, in a topological expansion characterized by the Euler characteristic χ of the surface in which the diagram is embedded

$$\log Z = \sum_{\chi \le 2} g_s^{-\chi} F_\chi(S), \qquad S_i = g_s M_i, \qquad \chi = 2 - 2g - h - q \tag{2.3}$$

with g the number of handles, h the number of boundary components, and q the number of crosscaps. We now take the large M limit, letting $M_i \to \infty$, $g_s \to 0$ with S_i held fixed. In this limit, the dominant contribution $F_{\mathrm{s}}(S) \equiv g_s^2 \log Z\big|_{\mathrm{sphere}}$ arises from planar diagrams that can be drawn on a sphere. Theories with fundamental representations contain surfaces with boundaries; the dominant such contribution $F_{\mathrm{d}}(S) \equiv g_s \log Z\big|_{\mathrm{disk}}$ comes from planar diagrams on a disk. Theories with symmetric or antisymmetric representations contain nonorientable surfaces; the dominant nonorientable contribution $F_{\mathrm{rp}}(S) \equiv g_s \log Z\big|_{\mathbb{RP}^2}$ comes from planar diagrams on $\mathbb{RP}^2$, a sphere with one crosscap.

2.2. *The effective superpotential*

The values of S_i in the matrix model, hitherto arbitrary, are determined by the extremization of the effective superpotential, given by[1,11,12]

$$W_{\mathrm{eff}}(S) = -\left[\sum_{i=1}^{N} \frac{\partial}{\partial S_i} F_{\mathrm{s}}(S) + F_{\mathrm{d}}(S) + 4F_{\mathrm{rp}}(S) \right] \tag{2.4}$$

in the case where the gauge group U(N) of the gauge theory is broken to $\mathrm{U}(1)^N$. The resulting vevs $\langle S_i \rangle$ may be computed in an expansion in Λ, the scale in the matrix model. For the U(N) theory with N_f fundamentals, the leading term is[6]

$$\langle S_i \rangle = \alpha \frac{\prod_{I=1}^{N_f}(e_i + m_I)}{\prod_{j\neq i}(e_i - e_j)} \Lambda^{2N-N_f} + \mathcal{O}(\Lambda^{4N-2N_f}) \tag{2.5}$$

and the Λ^{4N-2N_f} term is also computed in Ref. 6.

2.3. *Tadpole diagrams*

The Seiberg-Witten solution of the $\mathcal{N}=2$ gauge theory is expressed, not in terms of the parameters e_i, but in terms of the renormalized order parameters a_i, defined as the periods of the Seiberg-Witten differential[5]. The matrix model prescription for computing a_i was presented and motivated in Ref. 6:

$$a_i = e_i + \left[\sum_{j=1}^{N} \frac{\partial}{\partial S_j} g_s \langle \mathrm{tr}\,\Psi_{ii} \rangle_{\mathrm{sphere}} + \langle \mathrm{tr}\,\Psi_{ii} \rangle_{\mathrm{disk}} + 4 \langle \mathrm{tr}\,\Psi_{ii} \rangle_{\mathrm{rp}} \right] \Bigg|_{\langle S \rangle} \tag{2.6}$$

where Ψ_{ii} is the ith diagonal block of $\Phi - \Phi_0$, and the vevs in Eq. (2.6) represent tadpole diagrams with the specified topology. Equation (2.6) may be computed in an expansion in Λ; the Λ^{2N-N_f} contribution agrees[6] with the one-instanton relation between a_i and e_i computed in SW theory[13].

2.4. *Period matrix and prepotential*

In Seiberg-Witten theory, the matrix τ_{ij} of gauge couplings of the unbroken $\mathrm{U}(1)^N$ gauge theory is given by[5,13]

$$\tau_{ij}(a) = \frac{\partial^2 \mathcal{F}(a)}{\partial a_i \partial a_j} \tag{2.7}$$

$$\mathcal{F}(a) = \mathcal{F}_{\mathrm{pert}}(a) + \frac{\Lambda^{2N-N_f}}{2\pi i} \sum_i \frac{\prod_{I=1}^{N_f}(a_i + m_I)}{\prod_{j\neq i}(a_i - a_j)^2} + \mathcal{O}(\Lambda^{4N-2N_f})$$

where $\mathcal{F}_{\mathrm{pert}}(a)$ is the perturbative prepotential and the one-instanton prepotential is shown explicitly.

The matrix model prescription for the gauge coupling matrix is[1]

$$\tau_{ij} = \frac{1}{2\pi i} \frac{\partial^2 F_{\mathrm{s}}(S)}{\partial S_i \partial S_j} \bigg|_{S=\langle S \rangle} . \tag{2.8}$$

In Ref. 6, this quantity was computed and re-expressed in terms of a_i using Eq. (2.6), and was shown to agree with Eq. (2.7) to one-instanton accuracy.

For theories with symmetric or antisymmetric hypermultiplets, the prescription (2.8) must be modified by including relative signs among the various contributions to $F_{\rm s}(S)$. The justification for these signs, together with a prescription for computing τ_{ij}, was given in Refs. 8, 9. With this modification, and also the inclusion of the extra block for the case with antisymmetric matter discussed above, the matrix model calculation of τ_{ij} agrees with the SW calculation to one-instanton accuracy[9].

3. SW curve and differential from the matrix model

In this section, we will indicate how the usual ingredients of the Seiberg-Witten approach, the SW curve and SW differential, may be obtained from the matrix model using saddle point methods. In this approach, one introduces the trace of the resolvent

$$\omega(z) = g_s \left\langle \mathrm{tr} \left(\frac{1}{z - \Phi} \right) \right\rangle \tag{3.1}$$

which, like the free energy (2.3), may be expressed in terms of a topological expansion, with $\omega_{\rm s}(z)$ the leading term in the large M limit. The saddle-point approximation to (2.1) implies[1]

$$\omega_{\rm s}^2(z) - W_0'(z)\,\omega_{\rm s}(z) + \tfrac{1}{4} f(z) = 0 \tag{3.2}$$

where $f(z)$ is a polynomial, given by

$$f(z) = 4 g_s \left\langle \mathrm{tr} \left(\frac{W_0'(z) - W_0'(\Phi)}{z - \Phi} \right) \right\rangle . \tag{3.3}$$

The polynomial $f(z)$ is determined by extremizing the effective superpotential (2.4); this may be done exactly[6] using Abel's theorem, or perturbatively[9] using (2.5), as follows:

$$\begin{aligned} f(z) &= 4 \sum_i \frac{W_0'(z)}{z - e_i} \langle S_i \rangle + \mathcal{O}(S^2) \qquad (3.4) \\ &= 4\,\alpha^2 \Lambda^{2N-N_f} \left[\prod_{I=1}^{N_f} (z + m_I) - \tilde{T}(z) \prod_{i=1}^{N} (z - e_i) \right] + \mathcal{O}(\Lambda^{4N-2N_f}) \end{aligned}$$

where $\tilde{T}(z)$ is the polynomial part of $\prod_{I=1}^{N_f}(z+m_I)/\prod_{i=1}^{N}(z-e_i)$. Defining $y(z) = -2\omega_{\rm s}(z) + W_0'(z)$, one obtains

$$y^2 = W_0'(z)^2 - f(z) \tag{3.5}$$

precisely the Seiberg-Witten curve for this theory[13] for the choice of moduli e_i consistent with Eq. (2.6). See Refs. 6, 9 for details.

The Seiberg-Witten differential may also be obtained in the matrix model approach as[14,6,3,8]

$$\lambda_{SW} = z \left[\sum_{i=1}^{N} \frac{\partial}{\partial S_i} \omega_{\rm s} + \omega_{\rm d} + 4\,\omega_{\rm rp} \right] {\rm d}z. \qquad (3.6)$$

The cubic Seiberg-Witten curve (and associated SW differential) for the gauge theory with symmetric or antisymmetric matter hypermultiplets may also be obtained[7,8,9] from the matrix model approach, using saddle-point methods, together with extremization of the effective superpotential.

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