A SCALING PROPERTY OF SHRINKING DIFFRACTION PEAKS

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A B S T R A C T

We prove that

$$b(s) \equiv \frac{d}{dt} \ln\left(\frac{d\sigma^A}{dt}(s,t)\right)\Bigg|_{t=0} \underset{s\to\infty}{\leq} \frac{1}{4t_0}\left[\ln\left(\frac{s^2}{\frac{d\sigma^A}{dt}(s,0)}\right)\right]^2,$$

where $(d\sigma/dt)^A(s,t)$ is the absorptive contribution to the elastic unpolarized differential cross-section for particles with arbitrary spins at c. m. energy and momentum transfer $\sqrt{s}$, $\sqrt{-t}$, and t_0 is the right extremity of its Lehmann-Martin ellipse for $s \to \infty$. If this inequality is saturated apart from a constant factor, then there must exist sequences of $s_n \to \infty$ such that

$$\lim_{s_n\to\infty}\left[\frac{d\sigma^A}{dt}\left(s_n, t=-\frac{\tau}{b(s_n)}\right)\Big/\frac{d\sigma^A}{dt}(s_n,0)\right] = f(\tau),$$

where $f(\tau)$ is an entire function of order $\frac{1}{2}$.

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Ref.TH.2210-CERN
10 August 1976

1. - INTRODUCTION

There is considerable interest in the possibility that the elastic diffraction peak possesses a scaling property at high energies [1),2)]. The first rigorous results on this question were obtained by Auberson, Kinoshita and Martin [3)] for a class of scattering amplitudes including those saturating the Froissart bound, and slightly generalized by Cornille and Simao [4)] and Roy [5)]. Substantially new results have been obtained recently by Cornille and Martin [6)].

Here we obtain from unitarity and axiomatic analyticity properties an asymptotic upper bound on

$$b(s) \equiv \frac{d}{dt} \ln \frac{d\sigma^A}{dt}(s,t) \Big|_{t=0} \tag{1}$$

where $(d\sigma/dt)^A(s,t)$ is the absorptive contribution to the elastic differential cross-section for particles of arbitrary spin at c.m. energy and momentum transfer $\sqrt{s}$ and $\sqrt{-t}$, respectively. We show that if this bound is saturated apart from a constant factor, $(d\sigma/dt)^A$ must have a non-trivial scaling property with a scaling variable $\tau \equiv -t\, b(s)$. In the case of dominantly absorptive amplitudes the present results (i) represent a substantial generalization of those of Ref. 3), and (ii) imply a non-trivial scaling property of the differential cross-section provided the diffraction peak width shrinks as $1/(\ln s)^2$ for $s \to \infty$. Such a shrinkage is compatible with, but not implied by, the present high energy data [7)].

2. - BASIC RESULTS

Our starting point is the partial wave expansion, for arbitrary spins,

$$\frac{d\sigma^A}{dt}(s,t) = \sum_{\ell=0}^{\infty} (2\ell+1)\, \sigma_\ell(s)\, P_\ell\left(1 + \frac{t}{2k^2}\right) \tag{2}$$

which converges for physical s for t within the Lehmann-Martin ellipse [8)] of right-extremity $t_0(s)$, with

$$t_0 \equiv \lim_{s\to\infty} t_0(s) \tag{3}$$

(e.g., $t_0 = 4m_\pi^2$ for $\pi\pi$ and πN scattering), k being the c.m. momentum. A fundamental consequence of unitarity proved only recently by Mahoux [9] generalizing an earlier result of Cornille and Martin [6] is that

$$\sigma_\ell(s) \geqslant 0\,, \quad \text{for } \ell = 0,1,2,\ldots \tag{4}$$

It is known in the spinless case that

$$\frac{2}{9}\left[\frac{\sigma_{tot}^2}{4\pi\sigma_{el}} - \frac{1}{k^2}\right] \leq b(s) \underset{s\to\infty}{\leq} \frac{1}{2(t_0-\epsilon)}\left[\ln\left(\frac{s}{\sigma_{tot}}\right)\right]^2, \quad \epsilon > 0, \tag{5}$$

where the left-hand side is due to McDowell and Martin [10] and the right-hand side due to Singh [11] is an improvement of previous results [12]. The left-hand side has been shown by Cornille and Martin [6] to be valid for arbitrary spins provided that the factor 2/9 is replaced by 1/8 ; they also show that if $s\,\sigma_{tot}^2/\sigma_{e\ell} \to \infty$, and if the left-hand side of (5) is saturated apart from a constant factor, $(d\sigma/dt)^A$ must have a "weak scaling" property. Here we generalize to arbitrary spins the right-hand side of the bound (5) and prove that a "strong scaling" property must hold if the resulting bound is saturated apart from a constant factor. Our main results are summarized by the following theorems, valid for elastic scattering of particles with arbitrary spin ; ϵ will denote a positive number which can be chosen arbitrarily small.

Theorem 1 Upper bound on $(d\sigma/dt)^A(s,t)$ for complex t

For $|t| < t_0$,

$$\left|\frac{\frac{d\sigma^A}{dt}(s,t)}{\frac{d\sigma^A}{dt}(s,0)}\right| \underset{s\to\infty}{\leq} I_0\left(\sqrt{\frac{|t|}{t_0-\epsilon}}\;\omega(s)\right), \tag{6}$$

where I_0 is the modified Bessel function of order zero, and

$$\omega(s) \equiv \ln\left[\frac{s^2}{\frac{d\sigma^A}{dt}(s,0)}\right]. \tag{7}$$

Theorem 2 Upper bound on b(s)

$$b(s) \underset{s\to\infty}{\leq} \frac{[\omega(s)]^2}{4(t_0-\epsilon)} \equiv b_{Max}(s) \tag{8}$$

Theorem 3 Bound on curvature of diffraction peak

$$\left.\frac{d^2}{dt^2}\ln\left[\frac{d\sigma^A}{dt}(s,t)\right]\right|_{t=0} \underset{s\to\infty}{\leq} \frac{b(s)\,[\omega_1(s)]^2}{8(t_0-\epsilon)} \tag{9}$$

where

$$\omega_1(s) \equiv \ln\left[\frac{s^2}{\left.\frac{d}{dt}\left(\frac{d\sigma^A}{dt}(s,t)\right)\right|_{t=0}}\right]. \tag{10}$$

Remark

Theorems 1 to 3 are generalizations to arbitrary spin of Singh's results in the spinless case [11),13)]. Note that $\omega(s) \sim$ const. $\ln s$, for $s \to \infty$, because the lower bound of Jin, Martin and Cornille [14)] which readily generalizes to arbitrary spins using the amplitudes of Mahoux and Martin [8)], gives

$$\frac{d\sigma^A}{dt}(s,0) \geqslant \frac{\sigma_{tot}^2}{16\pi} \underset{s\to\infty}{\geqslant} \text{const.}\ s^{-12} \tag{11}$$

and hence

$$(2-\epsilon)\ln s \underset{s\to\infty}{\leq} \omega(s) \underset{s\to\infty}{\leq} 14\ln s \tag{12}$$

Theorem 4   Bounds on physical region cross-sections

For $-4k^2 \leq t \leq 0$ we have

$$1 + t b(s) \leq \frac{\frac{d\sigma^A}{dt}(s,t)}{\frac{d\sigma^A}{dt}(s,0)} \underset{s\to\infty}{\leq} 1 + t b(s) + \frac{t^2 b(s)[\omega_1(s)]^2}{16(t_0-\epsilon)} \tag{13}$$

Remark

The left-hand side of this inequality is due to Cornille and Martin [6] ; the right-hand side is presumably new.

Theorem 5   Strong scaling theorem

Let

$$f(s,\tau) \equiv \frac{d\sigma^A}{dt}\left(s, t = -\frac{\tau}{b(s)}\right) \Big/ \frac{d\sigma^A}{dt}(s,0) \tag{14}$$

If

$$b(s)/b_{Max}(s) \underset{s\to\infty}{\geq} b_0 \neq 0 \,, \tag{15}$$

where $b_{max}(s)$ is defined by Eq. (8), then every sequence $s'_n \to \infty$ must contain a subsequence $s_n \to \infty$ such that

$$\lim_{s_n\to\infty} f(s_n, \tau) = f(\tau) \tag{16}$$

where (i) the limit is uniform in any bounded set of the complex τ plane, (ii) $f(\tau)$ is an entire function of order half obeying $f(0) = 1$, $f'(0) = -1$ and the representation

$$f(\tau) = \int_{\lambda=0}^{2/\sqrt{b_0}} d\mu(\lambda)\, J_0(\lambda\sqrt{\tau}) \,, \tag{17}$$

where $d\mu(\lambda)$ is a positive measure obeying

$$\int_{\lambda=0}^{2/\sqrt{t_0}} d\mu(\lambda) = 1, \qquad \int_{\lambda=0}^{2/\sqrt{t_0}} d\mu(\lambda)\,\lambda^2 = 4 . \tag{18}$$

Remarks

(i) In Ref. 3), an analogous scaling property has been proved under a condition, which for purely absorptive amplitudes reads $\sigma_{tot} > \text{const}(\ln s)^2$; then the McDowell-Martin bound implies that the condition (15) for validity of Theorem 5 also holds. On the other hand, the condition (15) places no restriction on the behaviour of σ_{tot} allowing $\sigma_{tot} \sim s^{\gamma}$, $(\gamma > 0)$, as well as $\sigma_{tot} \sim (\ln s)^2$. Thus, for purely absorptive amplitudes, Theorem 5 is of more general applicability.

(ii) The scaling variable $\tau = -t\, b(s)$ is not necessarily a constant multiple of $t\,(d\sigma/dt)^A(s,0)/\sigma_{e\ell}$ because the condition (15) allows $b(s)\sigma_{e\ell}/(d\sigma/dt)^A(s,0) \to \infty$ for $s \to \infty$. Correspondingly, the asymptotic behaviour of our scaling function can be quite different from that in Refs. 3), 6), as discussed later.

(iii) As in Ref. 3), uniqueness of the scaling function is not proved.

Theorem 6 Upper bound on $d\sigma/dt^A(s,t)$ at finite energies in terms of $\sigma_{e\ell}$ and $d\sigma/dt^A(s,0)$

For any physical s and for $-1 \leq \cos\theta \equiv 1 + t/2k^2 \leq 1$,

$$\frac{d\sigma^A}{dt}(s,t) \leq \frac{\sigma_{el}}{4k^2}\left[\sum_{\ell=0}^{L-1}(2\ell+1)\left(1+\ell(\ell+1)\sin^2\theta\right)^{-1/4} + (2L+1)\epsilon_L\left(1+L(L+1)\sin^2\theta\right)^{-1/4}\right] \tag{19}$$

where the integer L and the fraction ϵ_L are given by

$$\frac{d\sigma^A}{dt}(s,0) = \frac{\sigma_{el}}{4k^2}\left[\sum_{0}^{L-1}(2\ell+1) + (2L+1)\epsilon_L\right], \quad 0 \leq \epsilon_L < 1 . \tag{20}$$

Further, if $s(d\sigma/dt)^A(s,0)/\sigma_{e\ell} \underset{s\to\infty}{\to} \infty$, then we have the asymptotic bound

$$\frac{\frac{d\sigma^A}{dt}(s,t)}{\frac{d\sigma^A}{dt}(s,0)} \underset{\substack{s\to\infty \\ \tau'\text{-fixed}}}{\leq} \frac{(1+4\tau')^{3/4}-1}{3\tau'}, \quad \text{for } \tau' \geq 0, \tag{21}$$

where

$$\tau' \equiv (-t)\frac{d\sigma^A}{dt}(s,0)\Big/\sigma_{e\ell} . \tag{22}$$

3. - PROOF OF THEOREMS 1 TO 4

To prove Theorem 1, we pose the problem of finding an upper bound on $(d\sigma/dt)^A(s,t)$ for t within the Lehmann-Martin ellipse (in particular $|t| < t_o - \epsilon$), given $(d\sigma/dt)^A(s,0)$, and the information that $\sigma_\ell \geq 0$ and

$$\frac{d\sigma^A}{dt}(s, t_o-\epsilon) \underset{s\to\infty}{<} \text{Const.}\, s^2 . \tag{23}$$

Using the facts that $|P_\ell(1+(t/2k^2))| < P_\ell(1+(|t|/2k^2))$, $P_\ell(1+(|t|/2k^2))$ increases with ℓ, and $P_\ell(1+(|t|/2k^2))/P_\ell(1+((t_o-\epsilon)/2k^2))$ decreases with increasing ℓ for $|t| < t_o - \epsilon$

$$\left|\frac{d\sigma^A}{dt}(s,t)\right| \leq \sum_{\ell=0}^{L(s)}(2\ell+1)\sigma_\ell(s)P_\ell\left(1+\frac{|t|}{2k^2}\right) + \sum_{\ell=L(s)+1}^{\infty}(2\ell+1)\sigma_\ell(s)P_\ell\left(1+\frac{|t|}{2k^2}\right)$$

$$\leq P_{L(s)}\left(1+\frac{|t|}{2k^2}\right)\frac{d\sigma^A}{dt}(s,0) + P_{L(s)+1}\left(1+\frac{|t|}{2k^2}\right)\frac{d\sigma^A}{dt}(s,t_o-\epsilon)\Big/P_{L(s)+1}\left(1+\frac{t_o-\epsilon}{2k^2}\right),$$

and, since

$$P_\ell(z) \leq I_o\left((2\ell+1)\sqrt{\frac{z-1}{2}}\right)$$

for $z \geq 1$, $\ell = 0,1,2,\ldots$ [Ref. 5), p. 193]

$$\left| \frac{d\sigma^A}{dt}(s,t) \right| \underset{s\to\infty}{\leq} I_0\left((2L(s)+1)\sqrt{\frac{|t|}{4k^2}}\right)\frac{d\sigma^A}{dt}(s,0)\left[1+o(1)\right], \quad L(s) \equiv \frac{\sqrt{s}\,\omega(s)}{2\sqrt{t_0-2\epsilon}} \tag{24}$$

which is equivalent to Theorem 1. Theorems 2 and 3 follow exactly similarly, and we omit their proof. For theorem 4, we use the inequality, valid for $-1 \leq \cos\theta \leq 1$, $\ell = 0, 1, 2, \ldots$,

$$1 + (\cos\theta - 1)\left[P_\ell'(\cos\theta)\right]_{\cos\theta=1} \leq P_\ell(\cos\theta) \leq 1 + (\cos\theta - 1) P_\ell'(\cos\theta = 1) + \frac{(\cos\theta-1)^2}{2} P_\ell''(\cos\theta = 1) \tag{25}$$

(whose left-hand side is due to Singh [11]) and right-hand side to Cornille [6]), and obtain, after inserting Theorem 3, the desired result.

4. - PROOF OF THEOREM 5

From Theorem 1 and assumption (15), we see that for S large enough, $\{f(s,\tau) | s > S\}$ is a family of analytic functions of τ in the disc $|\tau| < b_0(\ln S)^2$, uniformly bounded in this disc by

$$|f(s,\tau)| \underset{s>S}{<} I_0\left(2\sqrt{\frac{|\tau|}{b_0}}\right) \tag{26}$$

We may thus repeat the arguments of Ref. 3) to conclude that every sequence $s_n' \to \infty$ must contain a subsequence $s_n \to \infty$ such that $f(s_n, \tau)$ converges (uniformly in any bounded region of the τ plane) to an entire function $f(\tau)$ of order $\leq \frac{1}{2}$. We know from the uniformity of the convergence, and from Theorem 4 that $f(0) = 1$, and

$$1 - \tau \leq f(\tau) \leq 1 - \tau + \frac{\tau^2}{4b_0}, \quad \text{for } \tau \geq 0, \tag{27}$$

and hence $f(\tau)$ cannot be identically equal to one. Further, from analyticity inside a circle C of radius R, around $\tau = 0$,

$$\left| \frac{df(\tau)}{d\tau} - \frac{df(s_n,\tau)}{d\tau} \right|_{\tau=0} = \left| \frac{1}{2i\pi} \oint_C d\tau' \frac{f(\tau') - f(s_n,\tau')}{\tau'^2} \right|$$

$$\leq \frac{1}{R} \max_{\tau' \in C} \left| f(\tau') - f(s_n,\tau') \right| \xrightarrow[s_n \to \infty]{} 0 \tag{28}$$

Hence $f'(\tau=0) = -1$; further, since $|P_\ell(\cos\theta)| \leq 1$ for $-1 \leq \cos\theta \leq 1$

$$f(s,\tau) \leq 1 \text{ , and } f(\tau) \leq 1 \text{, for all } \tau \geqslant 0 . \tag{29}$$

If the order of $f(\tau)$ were less than half, the Phragmén-Lindelöf theorem [15] and Eq. (29) would imply that $f(\tau)$ is bounded everywhere and hence a constant ; this is not the case. Hence $f(\tau)$ must be of order half.

Integral representation

As in proof of Theorem 1, we show easily that, uniformly for $-T \leq \tau \leq 0$, for $s \to \infty$,

$$f(s,\tau) = o(1) + \sum_{\ell=0}^{L(s)} (2\ell+1)\sigma_\ell(s) P_\ell\left(1 - \frac{\tau}{2k^2 b(s)}\right) \Big/ \sum_{\ell'=0}^{L(s)} (2\ell'+1)\sigma_{\ell'}(s) , \tag{30}$$

with $L(s)$ given by Eq. (24) ; further for $\ell \leq L(s)$, $z = 1-(\tau/2k^2b(s))$, [see Ref. 5), p. 19]

$$0 \leq I_0\left((2\ell+1)\sqrt{\frac{z-1}{2}}\right) - P_\ell(z) \leq I_0\left((2\ell+1)\sqrt{\frac{z-1}{2}}\right) - I_0\left(\ell \frac{\sqrt{z^2-1}}{z+\sqrt{z^2-1}}\right)$$

$$\leq \left[(2L(s)+1)\sqrt{\frac{z-1}{2}} - \frac{L(s)\sqrt{z^2-1}}{z+\sqrt{z^2-1}} \right] I_0'\left((2L(s)+1)\sqrt{\frac{z-1}{2}} \right) = o(1),$$

uniformly for $-T \leq \tau \leq 0$. Hence we may approximate $P_\ell(z)$ by $I_0((2\ell+1)\sqrt{(z-1)/2})$ to obtain

$$f(s,\tau) = \int_{\lambda=0}^{2/\sqrt{b_0}} d\mu_s(\lambda)\, I_0(\lambda\sqrt{-\tau}) + o(1), \; -T \leq \tau \leq 0,\; s \to \infty, \tag{31}$$

$$d\mu_s(\lambda) = \frac{d\lambda}{\sum_{\ell'=0}^{L(s)} (2\ell'+1)\sigma_{\ell'}(s)} \sum_{\ell=0}^{L(s)} (2\ell+1)\sigma_\ell(s)\, \delta\left(\lambda - \frac{2\ell+1}{\sqrt{4k^2 b(s)}}\right), \tag{32}$$

$$\|\mu_s\| \equiv \int_{\lambda=0}^{2/\sqrt{b_0}} d\mu_s(\lambda) = 1 . \tag{33}$$

Consider a sequence $s'_n \to \infty$ such that $f(s'_n,\tau) \to f(\tau)$. It is known that[16] for every sequence of positive measures $d\mu_{s'_n}(\lambda)$ of unit norm on $\lambda = [0, 2/\sqrt{b_0}]$, there exists a subsequence s_n and a positive measure $d\mu(\lambda)$ of unit norm such that, for every continuous function $g(\lambda)$,

$$\lim_{s_n \to \infty} \int_0^{2/\sqrt{b_0}} d\mu_{s_n}(\lambda)\, g(\lambda) = \int_0^{2/\sqrt{b_0}} d\mu(\lambda)\, g(\lambda). \tag{34}$$

Choosing $g(\lambda) = I_0(\lambda\sqrt{-\tau})$, we have

$$f(\tau) = \lim_{s_n \to \infty} \int_0^{2/\sqrt{b_0}} d\mu_{s_n}(\lambda)\, I_0(\lambda\sqrt{-\tau}) = \int_0^{2/\sqrt{b_0}} d\mu(\lambda)\, I_0(\lambda\sqrt{-\tau}), \tag{35}$$

first for $-T \leq \tau \leq 0$, and by analytic continuation, for all complex τ. Finally, $f'(0) = -1$ yields Eq. (18).

5. - PROOF OF THEOREM 6

From

$$\sigma_\ell(s) = \frac{1}{2} \int_{-1}^{1} d(\cos\theta)\, P_\ell(\cos\theta)\, \frac{d\sigma^A}{dt}(s, t = -2k^2(1-\cos\theta)), \tag{36}$$

the positivity of $(d\sigma/dt)^A(s,t)$, and $|P_\ell(\cos\theta)| \leq 1$, we have

$$\sigma_\ell(s) \leq \sigma_0(s) = \frac{\sigma_{e\ell}^A}{4k^2} \leq \frac{\sigma_{e\ell}}{4k^2}, \tag{37}$$

where $\sigma_{e\ell}^A$ denotes the absorptive contribution to $\sigma_{e\ell}$. Further from [17]

$$|P_\ell(\cos\theta)| \leq [1+\ell(\ell+1)\sin^2\theta]^{-1/4}, \quad -1 \leq \cos\theta \leq 1, \tag{38}$$

we have

$$\frac{d\sigma}{dt}^A(s,t) \leq \sum_{\ell=0}^{\infty}(2\ell+1)\sigma_\ell(s)[1+\ell(\ell+1)\sin^2\theta]^{-1/4}. \tag{39}$$

We seek then an upper bound on the right-hand side of this equation given $(d\sigma/dt)^A(s,0)$, and the constraints $0 \leq \sigma_\ell(s) \leq \sigma_{e\ell}/(4k^2)$, and readily derive Theorem 6.

6. - ZEROS AND ASYMPTOTIC BEHAVIOUR

A) - Exactly as in Ref. 3), we deduce that $f(\tau)$ has infinitely many zeros in a small neighbourhood of the positive τ axis (i.e., negative t axis).

B) - Unlike Ref. 3), our assumption (15) allows the left-hand side of the equation

$$b(s)\,\sigma_{e\ell}^A \Big/ \frac{d\sigma}{dt}^A(s,0) = \int_0^{4k^2 b(s)} d\tau\, f(s,\tau) \tag{40}$$

to be unbounded for $s \to \infty$, and hence allows $f(\tau)$ to be non-integrable in $\tau = [0,\infty]$. This is most easily seen from the following example in the spinless case, with $a_\ell(s)$ denoting partial waves of the absorptive part : $a_\ell(s) = 1$, $\ell = (0, L_1)$ and (L_2, L_3) ; $a_\ell(s) = 0$ otherwise.

$$L_1 \equiv \sqrt{s}\sqrt{\frac{\sigma}{16\pi}\left(1-\frac{b}{2c^2}\right)}, \quad L_2 \equiv c\sqrt{s}\,\ln s\left[1-\frac{\sigma b}{64\pi c^4}\frac{1}{(\ln s)^2}\right],$$
$$L_3 \equiv c\sqrt{s}\,\ln s, \quad b < 2c^2, \quad c < 1/(2\sqrt{t_0}) \tag{41}$$

Then, for $s \to \infty$,

$$\sigma_{el}^{A}(s) = \sigma_{tot}(s) \to \sigma, \quad \frac{b(s)}{(\ln s)^2} \to b, \quad \frac{b(s)\,\sigma_{el}^{A}}{(\ln s\ \sigma_{tot})^2} \to \frac{b}{\sigma},$$

$$f(s,\tau) \underset{s\to\infty}{\longrightarrow} f(\tau) = \left[1 - \frac{b}{2c^2} + \frac{b}{2c^2} J_0\left(\frac{2c}{\sqrt{b}}\sqrt{\tau}\right)\right]^2 \underset{\tau\to\infty}{\longrightarrow} \left(1 - \frac{b}{2c^2}\right)^2. \tag{42}$$

Thus $f(\tau)$ can approach a constant for $\tau \to \infty$.

For comparison, note that in the (spinless) strong scaling case of Ref. 3), $f(\tau)$ is not only integrable on $\tau = [0,\infty]$ but obeys the local bound $|f(\tau)| < c/\sqrt{\tau}$ for $\tau \to \infty$.

In the weak scaling case of Ref. 6), it was shown that

$$\int_0^\infty d\tau' f(\tau') < \text{Const}, \quad \tau' \equiv -t \, \frac{d\sigma^A}{dt}(s,0)/\sigma_{el} \tag{43}$$

From unitarity,

$$\int_0^\infty d\tau' f^2(\tau') < \int_0^\infty d\tau' f(\tau') < \text{Const}. \tag{44}$$

and hence we have the Plancherel formula [3)]

$$f(\tau') = \frac{1}{2} \int_0^\infty du\, h(u) J_0(\sqrt{\tau' u}); \quad \int_0^\infty du\, h^2(u) = \int_0^\infty d\tau' f^2(\tau') < \text{Const}. \tag{45}$$

Further, from Theorem 6, we deduce that

$$f(\tau') < \frac{(1+4\tau')^{3/4} - 1}{3\tau'}, \quad \tau' \geqslant 0 \tag{46}$$

Here

$$f(\tau') < \text{const.}(\tau')^{-1/4}, \quad \text{for } \tau' \to \infty \tag{47}$$

Thus the Hankel transform representation of our scaling function $f(\tau)$ and its asymptotic behaviour are quite different from the previously known cases of strong and weak scaling.

ACKNOWLEDGEMENT

We wish to thank A. Martin for useful discussions.

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