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UPPER BOUNDS ON THE FINE STRUCTURE CONSTANT AND ON NUCLEON MAGNETIC MOMENTS IN TERMS OF COMPTON SCATTERING CROSS-SECTIONS

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ABSTRACT

We derive and compare with experimental data the bound

$$\alpha \leq -\lambda m_p - m_p \frac{\nu_1^2}{2\pi^2} \int_{\nu_0}^{\infty} \frac{d\nu' \, \sigma_{tot}(\nu')}{(\nu'^2 + \nu_1^2)} + 2\pi m_p \left[\int_{\nu_0}^{\infty} \frac{\nu'^2 \, d\nu' \, \sigma_{tot}(\nu')}{(\nu'^2 + \nu_1^2)\left\{ \nu'^2 \left(\frac{d\sigma}{dt}\right)_0 + \pi\lambda^2 + 2\nu'|\lambda|\sqrt{\pi}\sqrt{\left(\frac{d\sigma}{dt}\right)_0 - \frac{\sigma_{tot}^2}{16\pi}} \right\}} \right]^{-1}$$

where α is the fine structure constant, m_p the proton mass, ν_0 the photo-pion production threshold, σ_{tot} and $(d\sigma/dt)_0$ are unpolarized total hadronic photo-absorption cross-section on protons and unpolarized forward differential cross-section for proton Compton scattering at photon-lab. energy ν', and λ and ν_1 are any real numbers. We derive similar bounds on proton and neutron magnetic moments.

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1. - INTRODUCTION

We assume that the forward nucleon Compton amplitudes obey to lowest order in α the dispersion relations written by Gell-Mann, Goldberger and Thirring [1]. We then deduce rigorously bounds on α and on the nucleon magnetic moments in terms of the pion mass m_π, the nucleon mass m_N, the total hadronic photo-absorption cross-sections on nucleons and the forward differential cross-sections for Compton scattering. These bounds remain valid if in the integrals over the cross-sections only part of the physical region is retained, and thus provide tests of the dispersion relations involving only energy regions where data exist.

Previously, Drell, Hearn [2] and Gerasimov [3] obtained a sum rule for the nucleon magnetic moment by assuming, in addition to the above, an <u>unsubtracted</u> dispersion relation for the amplitude $f_2(\nu)/\nu$, and Truong [4] obtained a bound on α by assuming <u>unsubtracted</u> dispersion relation for the amplitude $[1/f_1(\nu)]$, where ν is the lab. energy of the photon and $f_{1,2}$ are $\gamma N \to \gamma N$ forward amplitudes defined in terms of the helicity amplitudes by

$$f_{1,2} \equiv \frac{f_a \pm f_p}{2} \,, \quad f_a \equiv f_{1\frac{1}{2};1\frac{1}{2}} \,, \quad f_p \equiv f_{1-\frac{1}{2};1-\frac{1}{2}} \,, \tag{1}$$

$$\mathcal{I}m\, f_1(\nu + i0) = \frac{\nu}{8\pi}\left[\sigma_a(\nu) + \sigma_p(\nu)\right] = \frac{\nu}{4\pi}\sigma_{tot}(\nu) \,, \tag{2}$$

$$\mathcal{I}m\, f_2(\nu + i0) = \frac{\nu}{8\pi}\left[\sigma_a(\nu) - \sigma_p(\nu)\right] \,, \tag{3}$$

$$\left(\frac{d\sigma}{dt}\right)_0 = \pi\left[\,\left|\frac{f_1(\nu)}{\nu}\right|^2 + \left|\frac{f_2(\nu)}{\nu}\right|^2\,\right] \,. \tag{4}$$

We are able to avoid their extra assumptions due to a technique of simultaneous use of the positivity properties of forward amplitudes and their inverses introduced recently by Roy and Singh [5].

We state our results in terms of the amplitude

$$f(z) \equiv f_1(\nu) + c\,\frac{f_2(\nu)}{\nu} \,, \quad z \equiv \nu^2 \,, \tag{5}$$

where c is a real number obeying

$$|c| \leq \nu_0 \equiv m_\pi + \frac{m_\pi^2}{2m_N} \tag{6}$$

2. - BOUNDS ON THE FINE STRUCTURE CONSTANT α AND ON NUCLEON ANOMALOUS MAGNETIC MOMENT κ

We prove the following results.

Theorem 1

For $\nu_0^2 > z_1 \geq z$ and λ real,

$$\frac{\alpha}{m_N}\left(Q^2 + c\,\frac{\kappa^2}{2m_N}\right) \leq \frac{I(z,z_1)}{2}\left[\sqrt{1 + \frac{4}{I(z,z_1)\,J(z,z_1,\lambda)}} - 1\right] - \lambda - I(z_1,0), \tag{7}$$

where Q is the nucleon charge ($Q = 1$ for proton, 0 for neutron), and

$$\left\{I(z,z_1), J(z,z_1,\lambda)\right\} \equiv \frac{z_1 - z}{\pi}\int_{\nu_0^2}^{\infty} \frac{dz'\,\mathrm{Im}\,f(z'+i0)}{(z'-z)(z'-z_1)}\left\{1, \frac{1}{|f(z'+i0)-\lambda|^2}\right\}. \tag{8}$$

Further if $\lambda > f(\nu_0^2)$ and if $|f(z'+i0) - \lambda|$ does not vanish for any $z' > \nu_0^2$ then the bound (7) must be an equality for all z, z_1.

Remarks

i) The case $c = 0$, $Q = 1$, leads to an upper bound on α. The cases $\nu_0 \geq c > 0$ and $-\nu_0 \leq c < 0$ lead to upper and lower bounds on κ^2 respectively, considering α as given.

ii) Note that $I(z,z_1) \geq 0$, $I(z_1,0) \geq 0$, and $J(z,z_1,\lambda) \geq 0$, because the integrands are non-negative. The parameters λ, z and z_1 are to be chosen to make the bound optimal and sensitive to the energy region required.

iii) The right-hand side of (7) increases when J is decreased ; hence an upper bound on $(\alpha/m_N)(Q^2+c\varkappa^2/2m_N)$, in terms of ν_o and the cross-sections σ_p, σ_a and $(d\sigma/dt)_o$ is obtained when we replace in Eq. (8) $|f(z'+i0)-\lambda|^2$ by the upper bound

$$|f(z'+i0)-\lambda|^2 \leq \frac{\nu^2+c^2}{\pi}\left(\frac{d\sigma}{dt}\right)_0 + \lambda^2 - \left[\frac{2c\,\sigma_{tot}-\nu(\sigma_a-\sigma_p)}{8\pi}\right]^2 + 2|\lambda|\left[\frac{\nu^2+c^2}{\pi}\left\{\left(\frac{d\sigma}{dt}\right)_0 - \frac{\sigma_{tot}^2}{16\pi} - \frac{(\sigma_a-\sigma_p)^2}{64\pi}\right\}\right]^{1/2}. \qquad (9)$$

Theorem 2

For $\nu_o^2 > z_1$ and λ real,

$$\frac{\alpha}{m_N}\left(Q^2 + c\frac{\varkappa^2}{2m_N}\right) \leq \frac{1}{J(z_1,\lambda)} - \lambda - I(z_1,0) , \qquad (10)$$

where

$$J(z_1,\lambda) \equiv \frac{1}{\pi}\int_{\nu_0^2}^{\infty}\frac{dz'}{z'-z_1}\;\frac{\mathrm{Im}\, f(z'+i0)}{|f(z'+i0)-\lambda|^2} \qquad (11)$$

Remarks

i) In the proof $J(z_1,\lambda)$ is not assumed to be finite. But (10) shows assuming $\alpha > 0$, that for $\lambda \geq 0$, $c \geq 0$ and $z_1 \leq 0$, $J(z_1,\lambda)$ must be finite !

ii) The bound remains valid if we retain in the integrals $I(z_1,0)$ and $J(z_1,\lambda)$ only part of the energy region where data exist. Further, we may replace $|f(z'+i0)-\lambda|^2$ in Eq. (11) by its upper bound given by Eq. (9).

3. - PROOF

i) Due to the optical theorem

$$\operatorname{Im} f(z'+i0) \geq 0 \quad \text{for } z' \geq \nu_0^2 \tag{12}$$

ii) From the forward dispersion relations [1] for f_1 and f_2, we deduce

$$f(z) = f(z_1) - I(z, z_1), \tag{13}$$

where z_1 is a subtraction point and $I(z,z_1)$ is given by Eq. (8). Following Jin and Martin [6] we prove easily, using the positivity property (12), that $\operatorname{Im} f(z) > 0$ for $\operatorname{Im} z > 0$, and hence that $f(z)$ and $-[f(z)]^{-1}$ are Herglotz functions [7].

iii) For any $f(z)$ obeying Eqs. (12) and (13), Roy and Singh [5] have used the Herglotz property to prove the following result.

For $\nu_0^2 > z_1 \geq z$ and for z_2 and λ real,

$$\operatorname{Re} f(z_2+i0) \geq \lambda + \operatorname{Re} I(z_1, z_2+i0) + \\ + \frac{I(z,z_1)}{2}\left[1 - \sqrt{1 + \frac{4}{I(z,z_1)J(z,z_1,\lambda)}}\right]. \tag{14}$$

Further if $\lambda > f(\nu_0^2)$ and if $|f(z'+i0) - \lambda| \neq 0$ for all $z' > \nu_0^2$ then the bound (14) must be an equality for all z, z_1.

iv) The low energy theorem of Low [8] and Gell-Mann and Goldberger [9] for nucleon Compton scattering yields

$$f(0) = -\frac{\alpha}{m_N}\left[Q^2 + c\,\frac{\kappa^2}{2m_N}\right]. \tag{15}$$

Combining this with (14) for $z_2 = 0$, we obtain Theorem 1.

v) To obtain Theorem 2, note first that for $\nu_0^2 > z_1 \geq z$ and λ real,

$$\frac{I(z,z_1)}{2}\left[\sqrt{1+\frac{4}{I(z,z_1)\,J(z,z_1,\lambda)}}-1\right] \leq \frac{1}{J(z,z_1,\lambda)} . \tag{16}$$

Further $J(z,z_1,\lambda) \geq J^\Lambda(z,z_1,\lambda)$, where J^Λ denotes the corresponding integral cut-off at the upper limit Λ. In this finite range integral take the limit $z \to -\infty$, and finally set $\Lambda = \infty$ to deduce Theorem 2.

4. - COMPARISON WITH PROTON COMPTON SCATTERING DATA

We test Theorem 2 for the case $c = 0$ which involves only unpolarized cross-sections after the replacement

$$\left|f(z'+i0)-\lambda\right|^2 \longrightarrow \frac{\nu^2}{\pi}\left(\frac{d\sigma}{dt}\right)_0 + \lambda^2 + 2|\lambda|\sqrt{\frac{\nu^2}{\pi}\left[\left(\frac{d\sigma}{dt}\right)_0 - \frac{\sigma_{tot}^2}{16\pi}\right]} . \tag{17}$$

We obtain from the theorem an upper bound on $J(z_1,\lambda)$ in terms of α, m_N, m_π and σ_{tot}, and compare it with the rather poor data on $J(z_1,\lambda)$. The cases $c \neq 0$ must await future direct experiments on σ_p, σ_a ; their derivation from photo-production data is not sufficiently reliable [10].

Previously dispersion relation calculations of Damashek and Gilman [11),12)] and Armstrong et al. [13)] have yielded $[\mathrm{Re}\, f_1/\mathrm{Im}\, f_1]^2 < 1/4$ for $\nu > 2$ GeV ; data on $(d\sigma/dt)_0$ from 2.45 to 17 GeV give a qualitative confirmation [14)] assuming $|f_2|^2 < 0.1\,|f_1|^2$. The present method has the advantage of not involving principal value integrations and of making rigorous statements based on incomplete data, e.g., in the absence of low and high energy data.

i) Choice of Compton scattering data. The input data (see Table I) on $(d\sigma/dt)_0$ from 2.45 to 17 GeV from Ref. 14), and from 0.65 to 2.45 GeV from Ref. 15), are chosen to be those extrapolations to $t = 0$ which do not use dispersion relations ; when this value is smaller than $\sigma^2_{tot}/(16\pi)$ and thus violates unitarity, we replace it by $\sigma^2_{tot}/(16\pi)$ in calculating $J(z_1,\lambda)$.

ii) Choice of σ_{tot} data. We use, from threshold up to 2.678 GeV values listed in Ref. 13), and from 2.678 to 38.8 GeV the fit of all data in this range [16),]

$$\sigma_{tot}^{\gamma p \to hadrons} = (99.8 \pm 1.6) + (57.0 \pm 3.0)/\sqrt{\nu} \quad mb, \tag{18}$$

which matches with the Ref. 13) fit at 2.678 GeV.

iii) Choice of λ, z_1. The contribution to $J(z_1,\lambda)$ of the region 0.15 to 0.64 GeV where no data exist can be suppressed by choosing λ and $(-z_1)$ positive and sufficiently large ; on the other hand, to have non-trivial tests of dispersion relations, λ and $(-z_1)$ must be sufficiently small such that the upper bound $J^{optical}_{max}$ obtained by neglecting $[\mathrm{Re}\, f(z'+i0) - \lambda]$ in Eq. (11) is significantly larger than the upper bound given by Theorem 2. Optimum results are quoted in Table II and the Figure.

5. - CONCLUSIONS

The contributions to J from the region 0.64 to 17 GeV are comfortably below the upper bound. To see what to expect when $(d\sigma/dt)_0$ data below 0.64 GeV become available we have also quoted the contribution to J from 0.15 to 17 GeV by adding a dispersion relation estimate of the energy region below 0.64 GeV ; we see then that the bound is saturated within the large errors of the data. Hence, it would be very interesting to have measurements of $(d\sigma/dt)_0$ below 0.64 GeV as well as to have more accurate measurements from 0.64 to 2.5 GeV.

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ν (GeV)	0.65	0.70	0.75	0.80	0.85	0.90	0.95	1.0	1.25	1.5-2.5	2.45
$(d\sigma/dt)_o$ $\mu b/GeV^2$	2.0 ±0.5	3.0 ±0.6	2.4 ±0.4	6.0 ±1.5	2.4 ±0.4	2.0 ±0.5	2.1 ±0.3	1.1 ±0.3	0.7 ±0.25	1.0 ±0.15	1.26±0.13
ν (GeV)	2.95	3.45	3.95	4.6	5.6	6.5	8.0	11.5	16	17	-
$(d\sigma/dt)_o$ $\mu b/GeV^2$	1.14±0.11	1.24±0.11	1.02±0.14	0.92±0.09	0.76±0.06	0.76±0.07	0.82±0.04	0.63±0.06	0.69±0.03	0.55±0.05	-

TABLE I - Extrapolated values of $(d\sigma/dt)_o^{\gamma p \to \gamma p}$ without using dispersion relations from Ref. 14) for 2.45 to 17 GeV and from Ref. 15) for 0.65 to 2.45 GeV.

		$\lambda = 4\ \mu b\ GeV$			$\lambda = 8\ \mu b\ GeV$		
$(-z_1)^{1/4}$ $[GeV]^{\frac{1}{2}}$	$J(0.15, 17)$ Optical $[\mu b\ GeV]^{\frac{1}{2}}$	$J(0.64, 17)$ $[\mu b\ GeV]^{-1}$	$J(0.15, 17)$ $[\mu b\ GeV]^{-1}$	J_{max} $[\mu b\ GeV]^{-1}$	$J(0.64, 17)$ $[\mu b\ GeV]^{-1}$	$J(0.15, 17)$ $[\mu b\ GeV]^{-1}$	J_{max} $[\mu b\ GeV]^{-1}$
0	0.380±0.015	0.062±0.010	0.149±0.014	0.143	0.054±0.009	0.098±0.011	0.091
0.2	0.367±0.015	0.062±0.010	0.147±0.014	0.140	0.054±0.009	0.096±0.011	0.090
0.4	0.256±0.010	0.061±0.010	0.123±0.013	0.117±0.001	0.053±0.009	0.086±0.010	0.080
0.6	0.148±0.006	0.057±0.009	0.090±0.011	0.085±0.001	0.050±0.008	0.068±0.009	0.063±0.001
0.8	0.095±0.004	0.051±0.008	0.066±0.009	0.062±0.001	0.045±0.007	0.054±0.007	0.050±0.001
1.0	0.066±0.003	0.042±0.007	0.050±0.007	0.048±0.001	0.038±0.006	0.042±0.006	0.041±0.001

TABLE II - Contributions from the energy ranges 0.64 to 17 GeV $[J(0.64, 17)]$ and 0.15 to 17 GeV $[J(0.15, 17)]$ to the integral $J(z_1,\lambda)$ defined by Eqs. (11) and (17) are compared with the upper bound on $J(z_1,\lambda)$ $[J_{max}]$ given by Theorem 2 for proton Compton scattering with $c = 0$. To see quantitatively the importance of $\mathrm{Re}\, f$, we also list the upper bound $J_{optical}(0.15, 17)$ obtained by neglecting $\mathrm{Re}\, f - \lambda$ in Eq. (11).

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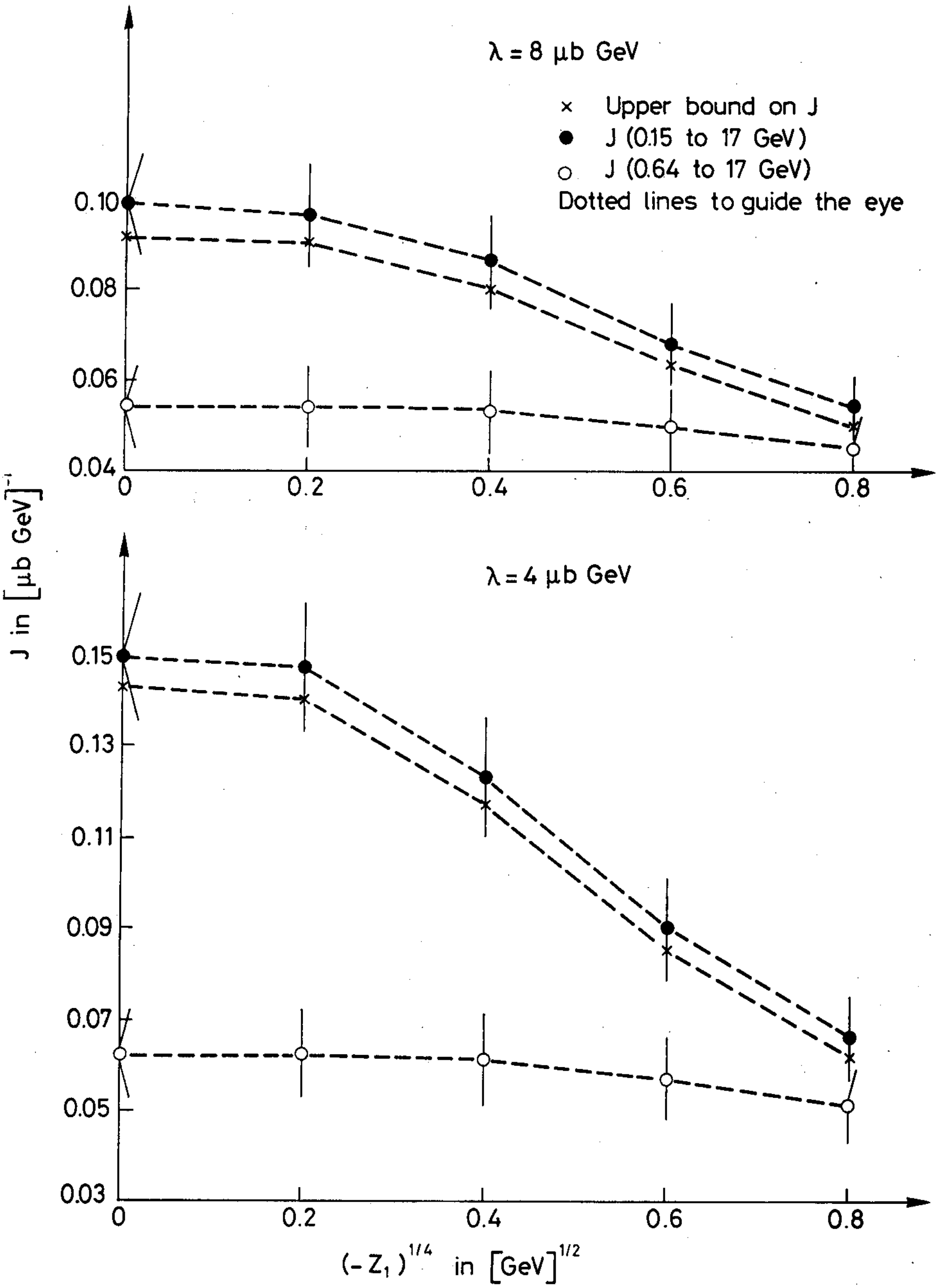
FIGURE CAPTION

Contributions from the energy ranges 0.64 to 17 GeV (ϕ) and 0.15 to 17 GeV ($\blacklozenge$) to the integral $J(z_1,\lambda)$ defined by Eqs. (11) and (17) are compared with the upper bound on $J(z_1,\lambda)$ (×) given by Theorem 2 for proton Compton scattering for the case $c=0$. The points $\blacklozenge$ are obtained by adding to the points ϕ a dispersion relation estimate of the region 0.15 to 0.64 GeV where $(d\sigma/dt)_0$ measurements are not available.



λ = 8 μb GeV
× Upper bound on J
● J (0.15 to 17 GeV)
○ J (0.64 to 17 GeV)
Dotted lines to guide the eye
0.10
0.08
0.06
0.04
0
0.2
0.4
0.6
0.8
J in [μb GeV]⁻¹
λ = 4 μb GeV
0.15
0.13
0.11
0.09
0.07
0.05
0.03
0
0.2
0.4
0.6
0.8
(- Z₁)^1/4 in [GeV]^1/2

FIG.1