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THE REGGEON CALCULUS AND DIFFRACTION SCATTERING

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A B S T R A C T

The Reggeon calculus is developed as a field-theoretic technique for obtaining a general solution of Regge cut discontinuity formulae for the Pomeron (Pomeron unitarity). It is argued that the experimental results of factorization and rising cross-sections, if true asymptotically, lead uniquely to a diffraction peak determined by the critical infra-red behaviour of an effective field theory for the Pomeron. Recent calculations of the diffraction peak are reviewed and compared with ISR data. The remarkable success of the no-parameter prediction of the height of the secondary maximum is emphasized.

Also reviewed, but only briefly, are lattice analogue models of the Pomeron field theory, together with other methods for calculating critical exponents of the infra-red behaviour. Finally the satisfactory s channel status of the critical theory is discussed.

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SECTION 1 INTRODUCTION

In this lecture course*) I shall discuss the solution of the Regge cut discontinuity formulae derived in my previous course [1]. I shall consider the particular case of a Pomeron (or Vacuum) Regge pole with trajectory $\alpha(t)$, where $\alpha(0) = 1$. In this case the Pomeron pole is isolated and is the leading singularity in the angular momentum plane when $t > 0$. However, at $t = 0$ the multi-Pomeron cuts, with trajectory $\alpha_N(t)$, $N = 2,3,\ldots$, which I have shown are required by t-channel unitarity, all coincide with the pole at

$$\alpha_N(0) = N[\alpha(\frac{0}{N^2}) - 1] + 1 = 1 \tag{1.1}$$

To determine the resulting partial-wave amplitude we must solve the complete set of coupled discontinuity formulae for the multi-Pomeron cuts, which we call Pomeron unitarity.

In my previous course (which I shall refer to as I) I showed that if the Pomeron is regarded as a quasi-particle in a non-relativistic two-dimensional space, carrying momentum $\underset{\sim}{k}$ and energy $E = 1-j$, so that the energy-momentum relation is $E = 1 - \alpha(\underset{\sim}{k}^2)$, then Pomeron unitarity is essentially a conventional unitarity condition on the scattering of such quasi-particles. This result suggests that a solution of t-channel unitarity in which the basic Pomeron singularity is not a pole would be analagous to a solution of momentum space unitarity in which the basic state is not a pole. Since no such solutions are known I argue that Regge poles and their associated cuts are the only allowed angular momentum plane singularities. The experimental results of factorisation and non-decreasing asymptotic total cross-sections then lead uniquely to a single Pomeron pole with unit intercept, together with its cuts. Having justified the consideration of this particular case by experiment, I shall then argue, during the course of these lectures, that the corresponding theoretical solution is also unique and determines the asymptotic shape of the diffraction peak.

My approach to the solution of Pomeron unitarity will be to write an effective

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*) The material in this paper is a development of the Reggeon Calculus which is based only on the analytic approach to Regge theory given in Ref. [1]. It was prepared to accompany lectures given at the Les Houches Institute of Theoretical Physics, June (1975) and will be published in the proceedings of this institute.

interacting field theory for the Pomeron treated as a quasi-particle. This is Gribov's Reggeon Calculus [2]. We write a general interaction Lagrangian and use a cut-off in $(E,\underset{\sim}{k})$ space to eliminate all angular momenta and momentum transfers where non-Pomeron interactions should be included. We then have a very general solution of Pomeron unitarity. The behaviour of the elastic diffraction peak (and also general diffractive production) is controlled by the $j \to 1$, $t \to 0$ or $E,k^2 \to 0$ limit, that is the "infra-red" behaviour of the Pomeron field theory.

By using the renormalisation group techniques of Wilson [3], which were originally developed to study critical phenomena in statistical mechanics, we can show that the behaviour of the Pomeron field theory in the infra-red limit is <u>independent</u> of the details of the interaction Lagrangian. A form of universal "critical behaviour" is obtained and the elastic differential cross-section $(\frac{d\sigma}{dt})_{AB}$ for the scattering of particles A and B satisfies

$$(\frac{d\sigma}{dt})_{AB} \quad S \overset{\sim}{\to} \infty \; [g_A g_B \; (\log S)^{\eta} \; \emptyset \; (\frac{\rho}{\rho_o})]^2 \tag{1.2}$$

where $\rho = t \; (\log S)^{\nu}$ and so the total cross-section satisfies

$$\sigma_{AB} \quad S \overset{\sim}{\to} \infty \; g_A g_B \; (\log S)^{\eta} \tag{1.3}$$

These results were first obtained by Migdal [4], Polyakov and Ter Martirosyan and independently by Abarbanel [5] and Bronzan.

The numbers η and ν are two of several "critical exponents" in the theory which, in principle, can be calculated exactly. More importantly the "scaling function" $\emptyset$ can also be calculated exactly. The couplings g_A and g_B and the scale parameter ρ_o can not be calculated, they depend on the details of the underlying strong interaction dynamics which we do not attempt to explain. There are, of course, non-leading terms which should be added to (1.2) and (1.3) and their magnitude determines the experimental applicability of these results.

Two recent results encourage me to believe that it may be possible to use (1.2) and similar results to make extensive predictions about diffraction scattering. The first is the experimental discovery that diffractive production appears to factorise for quite large transverse momentum [6]. The second result is the theoretical calculation [7] of the scaling function $\emptyset(\rho)$ in the ε-expansion of the Pomeron field theory ($\varepsilon = 4-D$, where D is the dimension of the quasi-particle space). As I shall show the result compares very well with

the experimental results from the ISR. (There is some evidence, which I shall briefly discuss in Section 5, that the ϵ-expansion gives the best results in lowest order. In this case we should take $\eta = \frac{1}{6}$ and $\nu = \frac{13}{12}$ in (1.2)).

The factorisation of diffractive production indicates that non-leading terms in (1.2) not associated with the factorising Pomeron propagator are small. The close relation of $\phi(p)$ to the experimental diffraction peak suggests also that the energy at which non-leading terms in the propagator (which should also be added to (1.2)) can be neglected, may not be as high as previous estimates have given. [4], [8], [9]

The major emphasis of this course will be on the theoretical developments leading to (or related to) the theoretical calculation of the diffraction peak. My aim is to show that the results I shall present follow in very general circumstances, from the analyticity and unitarity properties of amplitudes established in previous courses - first in momentum space and then in the angular momentum and helicity-planes. However, if these results also have experimental applicability, as I have suggested above, then the prospects are very exciting. There are many diffractive regions of exclusive and inclusive production processes, where the general Pomeron unitarity equations derived in I can be used to write an effective Pomeron field theory which will give similar asymptotic predictions to (1.2). (The triple Regge region of the one-particle inclusive cross-section has already been discussed from this point of view [7], [10], [11]). While only a small number of diffractive processes are experimentally observable at present machine energies, many more such processes should be observable in the next generation of accelerators and colliding beams.

The layout of these lectures will be as follows. I will begin by writing the elastic partial-wave amplitude as a sum over ("off spin-shell") integrals of Pomeron Greens functions. If the Greens functions are constructed from an interacting Lagrangian field theory then Pomeron unitarity will automatically be satisfied. We introduce the most general interaction Lagrangian by using the (cut-off) functional integral formalism of Calucci [12] and Jengo. By implementing the renormalisation group transformation of Wilson [3], we show that the infra-red behaviour of the general theory is the same as that of the simple theory with a single point triple Pomeron coupling. In Wilson's terminology all other parameters of the Lagrangian are "irrelevant variables."

In Section 3 we consider directly the construction of the triple Pomeron theory. We use the renormalisation group techniques developed by Sugar [13] and myself

which allow the complete Pomeron propagator to be constructed (at least in the ε-expansion). These techniques have been used in Refs. [7] and [11] to calculate both the elastic diffraction peak and the one-particle inclusive cross-section in the triple-Regge region. The results of Ref. [7] are the most extensive and we carry through the calculation of the diffraction peak given there. We then compare the results with the ISR data - the height of the secondary maximum, determined without parameters, is exactly right (note that this number $\sim 10^{-6}$). We also present the inclusive cross-section calculated in Ref [7]. We note that there is no turnover at $t = 0$, of the kind usually associated with a zero of the triple Pomeron vertex.

The calculation of the diffraction peak will be the pinnacle of this course and this is why I shall present it in detail. The subjects covered in the later Sections are treated in much less detail, I shall certainly not have time to give a complete review of all recent work in the Reggeon Calculus. A more comprehensive review of the subject, which includes most work not published within the last six months, can be found in Ref. [8]. Very little of the material covered in these lectures is, however, also covered in Ref. [8]. I shall review just a few of the recent papers that are relevant to the diffraction peak calculation.

In Section 4 I discuss the use of lattice analogue models to study the infra-red behaviour of the Pomeron field theory. If such models can be set-up in a satisfactory manner we expect to be able to apply well-established methods in solid-state physics to calculate the critical exponents (although probably not the scaling function) of (1.2) very accurately. We review the derivation by Cardy [14] and Sugar of an analogue model from the functional integral formulation of the field theory (this is the reverse of the argument for the connection [3] between the Ising model and $\lambda\phi^4$ field theory). We also mention briefly the work by Ellis [15] and Savit on the high temperature expansion of an alternative analogue model [16].

In Section 5 we discuss the intriguing calculations of critical exponents by Dash [17] and Harrington. They work directly with the two-dimensional triple Pomeron theory and demand renormalisation point invariance of "$\beta(g)$" plotted as a function of "$\gamma(g)$". The results are very encouraging, even if their theoretical foundation is not yet clear. They provide the best evidence for the validity of first-order calculations in the ε-expansion.

Section 6 contains a very recent argument* which shows that when the bare Pomeron intercept is placed above the critical value which gives a renormalised Pomeron

*See note added at the end of the end of these lectures.

intercept of one, a renormalised pole with intercept below one reappears. This result is very interesting because it suggests that the Froissart bound can not be saturated in the Pomeron field theory (with just a single Pomeron pole). This also suggests that the black disk of the eikonal model will not persist when Pomeron interactions are added to the model. The result of this Section also closes our uniqueness argument.

Finally Section 7 contains a very brief review of what I classify as subjects associated with the S-channel content of the Pomeron field theory. Some of these subjects will be covered in more detail by DeTar in his lectures. These include the interpretation of the bare Pomeron as being generated by initial short-range multiperipheral production processes. This links up with the very interesting question of the scale of diffraction processes and the approach to asymtopia. It also provides a physical interpretation of the analogy of much of the mathematics of previous Sections with that used to describe critical phenomena. I shall also briefly mention work that has been done on production processes and the general programme of checking the complete consistency of the Pomeron field theory with all S-channel unitarity constraints.

SECTION 2 POMERON GREEN'S FUNCTIONS AND THE WILSON RENORMALISATION GROUP FORMALISM

The final equations of I ((6.33) and (6.34)) state that the elastic partial-wave amplitude $a(j,t) \equiv F(E,\underset{\sim}{k}^2)$ satisfies (near $t = 0$)

$$\text{Im } F(E,\underset{\sim}{k}^2) = \sum_N (-1)^N \int d\rho_N(E,\underset{\sim}{k}^2) \; M_{\underset{\sim}{\alpha}_N}(E,\underset{\sim}{k}^2) \; M^*_{\underset{\sim}{\alpha}_N}(E,\underset{\sim}{k}^2) \qquad (2.1)$$

where $d\rho_N(E,\underset{\sim}{k}^2)$ is the phase-space for N quasi-particle Pomerons with energy E, momentum $\underset{\sim}{k}$ and energy-momentum relation $E = 1 - \alpha(\underset{\sim}{k}^2)$. $M_{\underset{\sim}{\alpha}_N}$ is a multiple fixed-pole residue which we can regard as an amplitude for the scattering of two particles into N Pomerons. Equation (2.1) is illustrated in Fig. 2.1. Note that since $j \sim \alpha(t) \approx 1$ we have dropped the signature factors ξ_{T_N} and $\sin \frac{\pi}{2} j$ which appear in (6.33) of I. They can be harmlessly absorbed into $M_{\underset{\sim}{\alpha}_N}$ and $M^*_{\underset{\sim}{\alpha}_N}$.

From Fig. 2.1 it should be clear that (2.1) can be interpreted as follows. Two particles can act as sources for any number of Pomerons which then propogate satisfying a unitarity condition before being absorbed by two other particles. This analogy suggests the following solution of (2.1). We write

$$F(E,k^2) = \sum_{n,m=1}^{\infty} F_{nm}(E,k^2) \tag{2.2}$$

where $F_{nm}(E,\underset{\sim}{k}^2)$ describes the initial emission of n Pomerons and final absorption of m Pomerons, as illustrated in Fig. 2.2.

$F_{nm}(E,\underset{\sim}{k}^2)$ can therefore be written in the form

$$\begin{aligned} F_{nm}(E,\underset{\sim}{k}) = \int d\underset{\sim}{E}_n d\underset{\sim}{k}_n \; \delta(E - \sum_{i=1}^{n} E_i) \; \delta^2(\underset{\sim}{k} - \sum_{i=1}^{n} \underset{\sim}{k}_i) \; g_n \\ x \int d\underset{\sim}{E}'_m d\underset{\sim}{k}'_m \; \delta(E - \sum_{i=1}^{m} E'_i) \; \delta^2(\underset{\sim}{k} - \sum_{i=1}^{m} \underset{\sim}{k}'_i) \; g_m \\ x \; G_{nm}(\underset{\sim}{E}_n, \underset{\sim}{k}_n, \underset{\sim}{E}'_m, \underset{\sim}{k}'_m) \end{aligned} \tag{2.3}$$

where $G_{nm}(\underset{\sim}{E}_n, \underset{\sim}{k}_n, \underset{\sim}{E}_m, \underset{\sim}{k}_m)$ is a Pomeron Greens function for the scattering of n Pomerons into m Pomerons as shown in Fig. 2.3. g_n and g_m are couplings of the external particles to the initial and final Pomerons respectively, which can also have (regular) dependendance on $\underset{\sim}{E}_n$, $\underset{\sim}{k}_n$ and $\underset{\sim}{E}'_m$, $\underset{\sim}{k}'_m$ respectively. Note that in the case of the propagation of a single Pomeron, all the integrals in (2.3) can be performed by using all the δ-functions. This gives

$$F_{11}(E,\underset{\sim}{k}^2) = g_1^2 x \frac{1}{\Gamma^{(1,1)}(E,\underset{\sim}{k}^2)} \tag{2.4}$$

where

$$G_{1,1}(E_1, \underset{\sim}{k}_1, E_2, \underset{\sim}{k}_2) = \frac{1}{\Gamma^{(1,1)}(E,\underset{\sim}{k}^2)} \tag{2.5}$$

The inverse Pomeron propagator $\Gamma^{(1,1)}(E,\underset{\sim}{k}^2)$ will be the centre of attention for much of this course, since it gives the leading behaviour in (2.2) as $E, \underset{\sim}{k}^2 \to 0$.

It can easily be checked that if the Pomeron Green's functions G_{nm} satisfy Pomeron unitarity, as they will in an interacting Lagrangian field theory, then the unitarity condition (2.1) will be satisfied by (2.2) and (2.3). In fact we can show from (6.33) of I that the "on spin-shell" Pomeron Greens functions ($E_i = 1 - \alpha(\underset{\sim}{k}_i^2)Vi$) is also <u>required</u> to satisfy Pomeron unitarity. So the jump from (2.1) to (2.2) involves no more and no less than the familiar jump from S-Matrix to Quantum Field Theory. We are assuming the existence of <u>some</u> off spin-shell continuation of Pomeron scattering amplitudes which, as we now discuss, in the neighbourhood of $E,\underset{\sim}{k}^2 \sim 0$ ($j \sim 1$, $t \sim 0$), can be described by an effective interaction Lagrangian.

To treat a general interaction Lagrangian we use the functional integral formalism* and follow the treatment of Calucci [12] and Jengo. In our case this formalism can be regarded as a compact way of describing some very complicated manipulations of the Feynman perturbation expansion (which is well-defined to the extent that we have a cut-off in our theory, and so there are no ultra-violet renormalisation problems). We shall drop this formalism once we have obtained the result that we can neglect all but the triple Pomeron interaction. The reader who wishes to avoid functional integral expressions can go straight to the next Section.

We introduce a Pomeron field ψ, which (as a quantum field) in our non-relativistic formalism contains only destruction operators. The free propagator $i\,[E + i\varepsilon - \alpha'_o \underset{\sim}{k}^2 + (\alpha_o - 1)]^{-1}$ represents a bare trajectory function $\alpha_o(t) = \alpha_o + \alpha'_o t$. The higher derivatives can be included in the general interaction which we introduce by writing the action A in the form

$$
\begin{aligned}
A &= \int dt d\underset{\sim}{x}^D \; L(\psi,\psi^*) \\
&= \int dt d\underset{\sim}{x}^D \Big\{ \Big[\frac{1}{2} i\,\psi^* \overleftrightarrow{\frac{\partial}{\partial t}} \psi - \alpha' \underset{\sim}{\nabla}\psi \cdot \underset{\sim}{\nabla}\psi + (\alpha_o - 1)\psi\psi^* \Big]_o \\
&\quad - \Big[+ \alpha''_o \, \nabla^2\psi \, \nabla^2\psi + \alpha'''_o \, \underset{\sim}{\nabla} (\nabla^2\psi^*) \cdot \underset{\sim}{\nabla} (\nabla^2\psi) + \text{---} \Big]_\alpha \\
&\quad - i \Big[r_o (\psi^*)^2\psi + r_o \psi^*\psi^2 + r'_o (\psi^*)^2 (\nabla\psi)^2 + \text{--} \Big) + \text{-} \Big]_r \\
&\quad + \Big[\lambda_{10} (\psi^*)^2\psi^2 + \lambda_{20} (\psi^*\psi^3 + \psi^{*3}\psi) + \text{----} \Big]_\lambda \\
&\quad + \text{---}
\end{aligned}
\tag{2.6}
$$

(Note that we introduce a general space dimension D from the start). In this expression $[\;]_o$ is the bare Lagrangian, $[\;]_\alpha$ contains all higher derivative terms in the trajectory function, $[\;]_r$ contains the triple Pomeron interaction plus derivatives, $[\;]_\lambda$ contains the four-Pomeron interactions plus derivatives etc. Note the i in front of $[\;]_r$ which produces a non-hermitian interaction. This is required by the $(-1)^N$ factor in the Pomeron unitarity equation (2.1).

Next we write the generating functional

$$
F(J) = \int [d\psi d\psi^*] e^{\,iA[\psi,\psi^*] + i \int J\psi^* + J^*\psi \; dtdx}
\tag{2.7}
$$

*A simple introduction to this formalism can be found in the review article on Gauge Theories by E.S. Abers and B.W. Lee, Physics Reports 9c, 1 (1973).

which is a functional integral over classical fields ψ and ψ^*. We shall discuss the precise definition of this integral further in Section 4. If we define the quantum field theory Green's functions by

$$G_{nm}(E_{\sim n},k_{\sim n},E'_{\sim m},k'_{\sim m})\ \delta(\sum_{i=1}^{n} E_i - \sum_{i=1}^{m} E'_i)\ \delta^D(\sum_{i=1}^{n} k_{\sim i} - \sum_{i=1}^{m} k'_{\sim i})$$

$$= \int \prod_{i=1}^{m} dt_i dx_i^D e^{-iE_i t_i + k_{\sim i}\cdot x_{\sim i}} \prod_{i=1}^{n} dt_i' dx_i'^D e^{-iE_i' t_i' + k'_{\sim i}\cdot x'_{\sim i}}$$

$$\times\ 0|T(\psi^*(t_1,x_{\sim 1})\cdots\ \psi^*(t_n,x_{\sim n})\ \psi(t_1',x_{\sim 1}')\cdots\psi(t_m',x_{\sim m}')\ |0> \qquad (2.8)$$

then this expression can also be written as a functional integral over the classical fields ψ and ψ^* in the form

$$N\ \int d\psi d\psi^* e^{iA[\psi,\psi^*]}\psi^*(E_1,k_{\sim 1})\cdots\ \psi(E'_m,k'_{\sim m}) \qquad (2.9)$$

$$= \frac{\partial^m}{\partial(J^*)^m}\ \frac{\partial^n}{\partial J^n}\ \Big|_{J=J^*=0} F(J) \qquad (2.10)$$

Therefore we can summarise operations on Pomeron Greens functions by operations on the generating functional F(J).

We impose a cut-off in (E,k) space by limiting the functional integration in (2.7) to fields ψ and ψ^* which vanish for $|E| \geq \Lambda^2$, $|k| \geq \Lambda$. We then define a renormalisation group transformation on (2.7) as follows. We write

$$\psi(E,k) = \psi_1(E,k) + \psi_2\ (E,k) \qquad (2.11)$$

where

$$\psi_1(E,k) = \psi(E,k),\ \psi_2 = 0,\ |E| < \frac{\Lambda^2}{a},\ |k| < \frac{\Lambda}{b}$$

$$\psi_2(E,k) = \psi(E,k),\ \psi_1 = 0,\ \frac{\Lambda^2}{a} \leq |E| < \Lambda^2,\ \frac{\Lambda}{b} \leq |k| < \Lambda \qquad (2.12)$$

and so formally

$$A[\psi,\psi^*] = A[\psi_1 + \psi_2,\ \psi_1^* + \psi_2^*] \qquad (2.13)$$

We then factor out the large $|E|$ and $|k|$ integrations in (2.7) by writing

$$e^{iA[\psi_1,\ \psi_1^*]} = \int d\psi_2 d\psi_2^*\ e^{iA[\psi_1 + \psi_2,\ \psi_1^* + \psi_2^*]} \qquad (2.14)$$

The Pomeron Green's functions can now be re-expressed in the form (2.10) with

$\tilde{A}$ replacing A and with the cut-off Λ replaced by $\frac{\Lambda^2}{a}$ for $|E|$ and by Λ/b for $|\underset{\sim}{k}|$. However, if we now make the following change of variables in the functional integration

$$\xi^{1/2}\psi'(E, \underset{\sim}{k}) = \psi(E/a, \underset{\sim}{k}/b) \tag{2.15}$$

and define

$$A'[\psi', \psi'^*] = \tilde{A}[\psi_1, \psi_1^*] \tag{2.16}$$

(The reason for the renormalisation factor ξ in (2.15) will soon be clear). We can return the cut-off to its original value Λ and obtain (suppressing primed momenta and energies for simplicity)

$$G_{nm}(E_i, \underset{\sim}{k}_i)\ \delta(\Sigma E_i)\ \delta^D(\Sigma \underset{\sim}{k}_i)$$

$$= \xi^{\frac{m+n}{2}} N \int d\psi' d\psi'^*\ e^{iA'[\psi', \psi'^*]}\ \psi'^*(aE_1, b\ \underset{\sim}{k}_1)\ \ldots\ldots \tag{2.17}$$

The right-hand side of (2.17) can be interpreted as a Green's function evaluated at $(a\ E_i, b\ \underset{\sim}{k}_i)_{\ i}$ with action A'. Therefore, after eliminating δ-functions on both sides of (2.17) we obtain an extra factor of $a^{-1}b^{-D}$ which gives

$$G^A_{nm}\ (E_i, \underset{\sim}{k}_i) = \xi^{\frac{n+m}{2}}\ a^{-1}b^{-D}\ G^{A'}_{nm}\ (aE_i, b\ \underset{\sim}{k}_i) \tag{2.18}$$

Hence, if we could control the transformation $A \to A'$, we could use (2.18) to determine the behaviour of $G_{nm}\ (E_i, \underset{\sim}{k}_i)$ in the infra-red limit $E_i, \underset{\sim}{k}^2_i \to 0$.

Now it is clear that by exponentiating the right-hand side of (2.14) and Taylor enpanding about the point $\psi = \psi^* = 0$, we can re-express $\tilde{A}$ and hence A' in the general form (2.6). (Actually there are some subtleties connected with the "normal-ordered" form of (2.6) and with single powers of the fields [12], but these can be ignored). Further we can fix ξ by demanding that the kinetic energy term $\psi^* \frac{\overleftarrow{\partial}}{\partial t} \psi$ have the same coefficient in A' as in A. We can also keep α'_o fixed by fixing b in terms of a.

The remaining parameters of A' can not be controlled and clearly (2.18) is only likely to be valuable if it should happen that in the limit a, b $\to \infty$, the parameters of A' approach fixed-values. That is a fixed-point of the renormalization group transformation A $\to$ A', exists. In this case (2.18) gives

$$G^{A}_{nm}\left(\frac{E_i}{a}, \frac{k_i}{b(a)}\right) \underset{a\to\infty}{\sim} [\xi(a)]^{\frac{n+m}{2}} a^{-1}[b(a)]^{-D} G^{A'}_{nm}(E_i, k_i) \tag{2.19}$$

The vital point we make now, having noted the possibility of a fixed-point, is that it follows from dimensional analysis alone that for any parameter λ in A the transformation A $\to$ A' has the form

$$\lambda' = a^{\frac{\nu}{2}} \lambda + R \tag{2.20}$$

$\nu \equiv \nu(\lambda)$ can be determined by taking the dimension of E to be that of k^2 and assigning λ the dimension of k^{ν} by requiring that the action A be dimensionless. R is the change in λ resulting from the presence of the other parameters of A.

(2.20) shows that if $\nu < 0$, the original value of λ in A will be completely suppressed in the transformation A $\to$ A' as a $\to \infty$. Thus we deduce that all parameters with $\nu < 0$ are "irrelevant" parameters in Wilson's sense[3]. For further discussion of this point we refer the reader to Ref. 3. We shall simply accept that the controlling parameters in the transformation are those with $\nu > 0$. It can easily be checked that

$$\begin{aligned} &\nu(\alpha_o) = 1,\ \nu(r_o) = 1 - \frac{D}{4},\ \nu(\lambda_{10}) = \nu(\lambda_{20}) = 1 - \frac{D}{2} \\ &\nu(\alpha_o'') = -1,\ \nu(r_o') = -\frac{D}{2},\ \text{etc.} \end{aligned} \tag{2.21}$$

and so for, $2 < D < 4$, which is the case we shall consider in the next Section, the only relevant parameters are α_o and r_o. The values of the remaining parameters in A' will be the same, in the limit a $\to \infty$, as if we had started with all parameters of A besides α_o, α'_o and r_o equal to zero.

Thus we have reduced the problem to that of determining whether the "scaling behaviour" of (2.18) occurs in the theory with a simple triple Pomeron coupling. That is whether a fixed-point is approached in such a theory. We could pursue this question within the present formalism. However, we shall be able to obtain results going way beyond (2.18) by developing a different formalism to calculate directly the infra-red behaviour of the triple-Pomeron theory. In doing so we shall indirectly verify that (2.18) holds when α_o takes a certain critical value (which puts the renormalized Pomeron intercept at one). At this critical point, as (2.20) and (2.21) suggest, $r_o \to \infty$ as $a \to \infty$. However, a dimensionless coupling constant $\hat{g}_o = \frac{r_o}{\Lambda^{\frac{\epsilon}{4}}}$ is driven to a fixed point and so (2.18) can be used. (Perhaps we should note that in the simple dimensional analysis we have used in this Section $\nu(E) = \nu(k^2) = 2$ and so $\nu(\alpha'_o) = 0$. In general we need not take $\nu(E) = \nu(k^2)$).

SECTION 3 CONTRUCTION OF THE TRIPLE POMERON THEORY AND CALCULATION OF THE DIFFRACTION PEAK IN THE ϵ-EXPANSION

In this Section we consider directly the theory specified by the renormalized Lagrangian $L_R(\psi)$ where

$$L_R = \frac{1}{2} i Z_3 [\psi^+ \overleftrightarrow{\frac{\partial}{\partial t}} \psi] - Z_3 Z_2 \alpha' \underset{\sim}{\nabla}\psi^+ \underset{\sim}{\nabla}\psi$$

$$- \frac{1}{2} i Z_1 r[\psi^+\psi^2 + \psi^{+2}\psi] + Z_3 \Delta \psi^+\psi \qquad (3.1)$$

We can also define a bare Lagrangian L_u which has the same form as (3.1) but with no Z's and containing bare parameters and fields which are denoted by a o suffix (note that $\Delta_o = \alpha_o - 1$). We can identify L_R and L_u by writing

$$\psi_o = Z_3^{\frac{1}{2}} \psi \qquad (3.2)$$

$$\alpha_o' = Z_2 \alpha' \qquad (3.3)$$

$$r_o = Z_1 \, Z_3^{-\frac{3}{2}} \, r \tag{3.4}$$

However, the Greens functions defined by using $L_R(\psi)$ and $L_u(\psi_o)$ in (2.6) and performing the functional integration over fields ψ and ψ_o respectively will differ by a normalization factor. Defining now the (on energy and momentum shell) amputated Greens functions

$$\Gamma^{(N,M)}(E_1, \underset{\sim}{k}_1, \cdots \underset{\sim}{E'}_m, \underset{\sim}{k'}_m) = \prod_{i,j} G^{-1}_{1,1}(E_i, \underset{\sim}{k}_i) \, G^{-1}_{1,1}(E'_j, \underset{\sim}{k'}_j)$$

$$\times \, G^{c}_{NM}(\underset{\sim}{E}_1, \underset{\sim}{k}_1, \cdots, \underset{\sim}{E'}_m, \underset{\sim}{k'}_m) \tag{3.5}$$

where G^c_{NM} is the connected part of G_{NM}, we have

$$\Gamma_R^{(N,M)} = Z_3^{\frac{M+N}{2}} \mu_u^{(N,M)} \tag{3.6}$$

This result is given directly by (2.15) and (2.18) if we take $a = b = 1$.

We fix Z_1, Z_2, Z_3 and Λ in terms of r and α' by requiring that

$$\Gamma_R^{(1,1)}(E, k^2) \Big|_{E = \underset{\sim}{k}^2 = 0} = 0 \tag{3.7}$$

$$\frac{\partial \, i \, \Gamma_R^{(1,1)}}{\partial E}(E, k^2) \Big|_{\substack{E = -E_N \\ \underset{\sim}{k} = \underset{\sim}{k}_N}} = 1 \tag{3.8}$$

$$\frac{\partial \, i \, \Gamma_R^{(1,1)}}{\partial k^2}(E, k^2) \Big|_{\substack{E = -E_N \\ \underset{\sim}{k} = \underset{\sim}{k}_N}} = -\alpha' \tag{3.9}$$

$$\Gamma_R^{(2,1)} \Big|_{\substack{E_1' = 2E_2 = 2E_1 = -E_N \\ \underset{\sim}{k}_1 = \underset{\sim}{k}_i = 0}} = \frac{r}{(2\pi)^{D+1/2}} \tag{3.10}$$

The condition (3.7) will directly constrain Δ (or $\Delta_o = \alpha_o - 1$) in terms of r and α'. The application of the renormalization group to the theory specified by (3.7) - (3.10) was first carried out by Abarbanel and Bronzan [5], using the formalism familiar from relativistic quantum field theory. Here we shall use a slightly different formalism due to Sugar and myself [13], which allows us to explicitly construct Green's functions, at least in the ε-expansion. This formalism is rather complicated but is powerful enough to calculate explicitly the scaling function $\phi(\rho)$ appearing in (1.2). Both the diffraction peak and the triple Regge region of the one-particle inclusive cross-section have been calculated in this way in Refs. 7 and 11. The results of Ref. 7 are the most extensive and since I want to present them in some detail I must develop the more complicated formalism. I shall show that the formalism simplifies enormously if we only wish to prove (1.3).

We shall not use a cut-off to eliminate ultraviolet divergences but rather use the method of dimensional regularization whereby divergent Feynman integrals are analytically continued from dimensions where they are convergent. This is particularly suited to our purpose since we have other reasons (the ε-expansion) for varying the dimension D of $\underset{\sim}{k}$ space in our theory. In fact it follows from simple power counting that for $D < 4$, the theory is super-renormalizable. Z_1, Z_2 and Z_3 are finite (although they are nevertheless important objects in our theory). In this case all the ultra-violet divergences of the theory (that is poles in D) must be absorbed by the intercept counter-term Δ. This can be shown to be the case by use of the following analysis, but we shall not give the full details of this point.

We begin by defining a dimensionless coupling constant (c.f. Section 2 - we now distinguish the dimensions of E and $\underset{\sim}{k}^2$)

$$g_o = \frac{r_o}{(\alpha_o')^{D/4}} (E_N)^{D/4 - 1} = \zeta^{-1} g \tag{3.11}$$

where $$\zeta = Z_3^{3/2}\, Z_1^{-1}\, Z_2^{D/4} \tag{3.12}$$

We also define "renormalization group" functions

$$\gamma_E = \frac{\partial \ln Z_3}{\partial \ln E_N}\Big|_{B,k_N^2}, \qquad \gamma_k = \frac{\partial \ln Z_3}{\partial \ln k_N^2}\Big|_{B,E_N}$$

$$\tau_E = -\frac{\partial \ln Z_2}{\partial \ln E_N}\Big|_{B,k_N^2}, \qquad \tau_k = -\frac{\partial \ln Z_2}{\partial \ln k_N^2}\Big|_{B,E_N} \tag{3.13}$$

$$\beta_E = \frac{\partial g}{\partial \ln E_N}\Big|_{B,k^2_N} = -\frac{\varepsilon g}{4} + g\,\frac{\partial \ln \zeta}{\partial \ln E_N}\Big|_{B,k^2_N}$$

$$\beta_k = \frac{\partial g}{\partial \ln k^2_N}\Big|_{B,E_N} = g\,\frac{\partial \ln \zeta}{\partial \ln k^2_N}\Big|_{B,E_N} \qquad B = \{r_o,\ \alpha_o'\}$$

Z_1, Z_2 and Z_3 are functions of dimensionless parameters only which we take to be g and $\chi = \alpha' k_N^2/E_N$. Chain rule differentiation allows us to write, for example, (using (3.2) - (3.4))

$$\gamma_E = \frac{\beta_E}{g}\,\frac{\partial \ln Z_3}{\partial \ln g} + (\tau_E - 1)\,\frac{\partial \ln Z_3}{\partial \ln \chi}$$

$$\gamma_k = \frac{\beta_k}{g}\,\frac{\partial \ln Z_3}{\partial \ln g} + (\tau_k + 1)\,\frac{\partial \ln Z_3}{\partial \ln \chi} \tag{3.14}$$

which, after elimination of $\dfrac{\partial \ln Z_3}{\partial \ln \chi}$, can be integrated to give

$$Z_3(g,\chi) = \exp \int_o^g \frac{dg'\ \bar{\gamma}(g,'\chi)}{\bar{\beta}(g',\chi)} \tag{3.15}$$

where we have used $Z_3\ (0,\chi) = 1$ and written

$$\bar{\beta} = \beta_E(\ 1 + \tau_k) + \beta_k(1 - \tau_E)$$

$$\bar{\gamma} = \gamma_E(1 + \tau_k) + \gamma_k(\ 1 - \tau_E) \tag{3.16}$$

Similarly

$$Z_2 = e^{-\int_o^g dg' \frac{\bar{\tau}}{\bar{\beta}}} \tag{3.17}$$

and

$$= e^{\int_o^g \frac{dg'}{\bar{\beta}} \left\{ \frac{\bar{\beta}}{g'} + \frac{\varepsilon}{4} [1 + \tau_k] \right\}} \tag{3.18}$$

where

$$\bar{\tau} = \tau_E + \tau_k \tag{3.19}$$

Next we suppose that (as will be the case in the ε-expansion) $\bar{\beta}(g,\chi)$ has a linear zero so that

$$\bar{\beta}(g,\chi) = \bar{\beta}'(\chi)\ (g - g_1(\chi)) + \text{higher terms} \tag{3.20}$$

Then for g near g_1, it follows from (3.15), (3.17) and (3.18) that

$$\ln z_i = \ln(g_1(\chi) - g) + \ln \bar{z}_i(\chi) + \sum_{n=1}^{\infty} a_{n,i}(\chi)(g - g_1(\chi))^n \tag{3.21}$$

where $Z_i = \zeta,\ z_2,\ z_3$ and

$$c = \frac{\varepsilon}{4\bar{\beta}'},\ (1 + \tau_k)_1,\quad c_2 = \left(\frac{-\bar{\tau}}{\bar{\beta}'}\right)_1,\quad c_3 = \left(\frac{\bar{\gamma}}{\bar{\beta}'}\right)_1 \tag{3.22}$$

with $(\ )_1 = (\ )_{g=g_1(\chi)}$. c, c_2 and c_3 must be independent of χ since if this were not the case, substituting (3.21) for $\ln z_3$ in (3.14) would give a logarithmic singularity from the $\frac{\partial \ln z_3}{\partial \ln \chi}$ terms, whereas the other terms can be explicitly shown to be free of logarithmic singularities.

(3.21) implies that

$$(g,\chi) \underset{g \to g_1}{\sim} \bar{\zeta}(\chi)(g_1(\chi) - g)^c \tag{3.23}$$

which from (3.11) implies that

$$g_o \sim \frac{g}{\bar{\zeta}} (g_1 - g)^{-c} \tag{3.24}$$

or

$$g \sim g_1 - \left(\frac{\bar{\zeta} g_o}{g_1}\right)^{-\frac{1}{c}} \tag{3.25}$$

and so if $c > o$ $g \to g_1$, $\to g_o \to \infty$, which from (3.11) is equivalent to $E_N \to 0$ with α'_o and r_o fixed. We shall exploit this result.

From (3.21) we similarly obtain

$$Z_2 \underset{g_1 \to g}{\sim} (g_1 - g)^{c_2} \tag{3.26}$$

$$\underset{g_o \to \infty}{\sim} (\frac{\bar{\zeta} g_o}{g_1})^{-\frac{c_2}{c}} \tag{3.27}$$

and also

$$Z_3 \underset{g_o \to \infty}{\sim} (\frac{\bar{\zeta} g_o}{g_1})^{-\frac{c_3}{c}} \tag{3.28}$$

To exploit these relations we need to know both $\bar{\zeta}$ and g_1 as functions of χ which in turn we need to know as a function $\chi(g_o, \chi_o)$ of g_o and χ_o for $g_o \to \infty$, ($\chi_o = \frac{\alpha_o' k_N^2}{E_N}$). However, we can use (3.3) and (3.27) to write

$$\rho = \frac{\alpha_o' k_N^2}{E_N} \frac{1}{1 + \frac{\varepsilon c_2}{4c}} = \frac{\alpha' Z_2 k_N^2}{E_N} \; g_o^{\frac{c_2}{c}} \left(\frac{r_o}{\alpha_o' \frac{D}{4}}\right)^{-\frac{c_2}{c}} \tag{3.29}$$

$$g_o \underset{\chi \to \infty}{\sim} Z \left(\frac{g_1(\chi) \alpha_o' \frac{D}{4}}{\bar{\zeta}(\chi) r_o}\right)^{\frac{c_2}{c}} \tag{3.30}$$

Hence if we keep the "scaling variable" ρ fixed, together with α_o' and r_o, then we know from (3.30) that x remains finite as $g_o \to \infty$ (or $E_N \to 0$) and hence (3.26) and (3.28) give us real information.

Now if we combine (3.8) and (3.9) with (3.6) and use (3.26) and (3.28) we obtain

$$\left.\frac{\partial i \Gamma_u^{(1,1)}}{\partial E}\right|_{\substack{E=-E_N \\ \underset{\sim}{k}=\underset{\sim}{k}_N}} = Z_3^{-1} \tag{3.31}$$

$$\underset{\substack{E_N \to 0 \\ \alpha_o', r_o, \chi \text{ fixed}}}{\sim} E_N^{-\frac{\varepsilon c_3}{4c}} \; Z_3^{-1} \left[\frac{\alpha_o' \frac{D}{4} g_1}{r_o \bar{\zeta}}\right]^{-\frac{c_3}{c}} \tag{3.32}$$

and

$$\left.\frac{\partial i \Gamma_u^{(1,1)}}{\partial k^2}\right|_{\substack{E=-E_N \\ \underset{\sim}{k}=\underset{\sim}{k}_N}} = -\alpha_o' Z_2^{-1} Z_3^{-1} \tag{3.33}$$

$$\underset{E_N \to 0}{\sim} E_N^{-\frac{\varepsilon}{4c}(c_2+c_3)} \alpha_o' Z_3^{-1} Z_2^{-1} \left[\frac{\alpha_o' \frac{D}{4} g_1}{r_o \bar{\zeta}}\right]^{-\frac{(c_2+c_3)}{c}} \tag{3.34}$$

α_o', r_o χ fixed

As we remarked above all the Z's are finite for r_o, α_o fixed and E_N and $\underset{\sim}{k}^2_N$ fixed. Therefore, we are quite content to calculate $\Gamma_u^{(1,1)}$, which is actually independent of E_N and k^2_N. The above manipulations are a bit confusing but some thought will show that all quantities (Z's, Γ_R's, E_N, k^2_N, etc.) associated with renormalisation have simply been used as tools to determine the derivatives of $\Gamma_u^{(1,1)}$ which are given by (3.32) and (3.34) if we simply put $-E_N = E$, $\underset{\sim}{k}_N = \underset{\sim}{k}$, and write $X \equiv X(r_o, \alpha'_o, -E_N, \underset{\sim}{k}_N) = X(r_o, \alpha'_o, E, \underset{\sim}{k}^2)$.

We can write the complete propagator with the zero intercept condition (3.7) imposed by integrating the derivatives given by (3.31) - (3.34) at fixed X to obtain

$$\Gamma_u^{(1,1)}(E,\underset{\sim}{k}^2) = \int_o^E dE' \left[\frac{\partial\Gamma^{(1,1)}}{\partial E'}\bigg|_{k^2} + \frac{\partial\Gamma^{(1,1)}}{\partial k^2}\bigg|_E \frac{\partial k^2}{\partial E}\bigg|_X\right] \tag{3.35}$$

It is shown in Ref. [13] that the perturbation expansion of $\Gamma_u^{(1,1)}(E,\underset{\sim}{k}^2)$ is given by replacing $\int_o^E$ by $-\int_E^\infty$ in (3.35). This implies that the intercept counter-term Δ_o is given by

$$\Delta_o = \int_o^\infty dE' \left[\frac{\partial\Gamma^{(1,1)}}{\partial E'}\bigg|_{k^2} + \frac{\partial\Gamma^{(1,1)}}{\partial k^2}\bigg|_E \frac{\partial k^2}{\partial E}\bigg|_X\right] \tag{3.36}$$

(which is independent of X). This illustrates the point that the ultra-violet divergences coming from $E' = \infty$ are absorbed into Δ_o. $X(E,k^2)$ is still determined implicitly by (3.30). However, it follows directly from (3.35) together with (3.32) and (3.34) that (calculating $\frac{\partial k}{\partial E}\Big|_X$ from (3.30))

$$\Gamma_u^{(1,1)}(E,k^2) \underset{\substack{E\to 0\\ X \text{ fixed}}}{\sim} \frac{i}{1-\frac{\varepsilon c_3}{4c}} (-E)^{1-\frac{\varepsilon c_3}{4c}} \bar{Z}_3^{-1}(X)\left[1 + X\left(1+\frac{\varepsilon c_2}{4c}\right)\right] X \left[\frac{\alpha'^{\frac{D}{4}}_o}{r_o} \frac{g(X)}{\bar{\zeta}(X)}\right]^{-\frac{c_3}{c}} \tag{3.37}$$

This is the original scaling result of Abarbanel and Bronzan rederived using the approach of Ref. [13]. If the Sommerfeld-Watson integral is written with $a(j,t) = g_1^2/\Gamma^{(1,1)}(E,\underset{\sim}{k}^2)$ then the general form of (1.2) follows immediately, with

$$\eta = -\frac{\varepsilon c_3}{4c} \tag{3.38}$$

and (after noting that $X \equiv X(\rho)$ with ρ defined by (3.29)

$$\nu = -1 - \frac{\varepsilon}{4}\frac{c_2}{c} \tag{3.39}$$

Having developed the above formalism we point out that it can be simplified enormously if we only ask for the form of the propagator at $\underset{\sim}{k}^2 = 0$. In this case we can set $\underset{\sim}{k}_N = 0$ in (3.8) and (3.9), so that $\gamma_k = \tau_k = \beta_k = 0$. We then have

$$\bar{\beta} = \beta_E = \beta(g) \tag{3.40}$$

$$\bar{\gamma} = \gamma_E = \gamma(g) \tag{3.41}$$

and so

$$Z_3 = Z_3(g) = \exp \int_0^g dg' \, \frac{\gamma(g')}{\beta(g')} \tag{3.42}$$

From (3.32)

$$\frac{\partial \Gamma^{(1,1)}(E_N,0)}{\partial E_N} = Z_3^{-1} \tag{3.43}$$

$$\underset{E_N \to 0}{\sim} E_N^{-\frac{\epsilon}{4}\frac{c_3}{c}} \tag{3.44}$$

$$= E_N^{-\gamma(g_1)} \tag{3.45}$$

and so

$$\Gamma^{(1,1)}(E,0) \underset{E \to 0}{\sim} E_N^{\,1-\gamma(g_1)} \tag{3.46}$$

where g_1 is now a simple zero of $\beta(g)$. Note that (1.3) follows immediately from this result with

$$\eta = -\gamma(g_1) \tag{3.47}$$

Therefore, if we only wish to calculate η, we need only calculate the simple functions $\beta(g)$ and $\gamma(g)$ (in the ϵ-expansion say). This simple technique can be extended to calculate ν also but if we also ask for the scaling function $\emptyset(p)$ we must use the more complicated formalism.

Note that (3.37) does have the general form of (2.19). Fixing $\underset{\sim}{k}^2$ in terms of E by (3.30) is equivalent to fixing b in terms of a in (2.19). More discussion of the relation between the formalism of this Section and the last Section can be found in the paper by Calucci [12] and Jengo. Here we note only

that imposing (3.7) fixes $(\alpha_o - 1)$ in terms of r_o by the relation (3.36). Our assumption of a zero in $\bar{\beta}(g,X)$ with positive slope is then equivalent to a fixed-point for $\hat{g}_o$.

Our next task is to combine (3.37) with the ε-expansion of the theory to obtain the scaling function $\emptyset(\rho,\varepsilon) \equiv \emptyset(X,\varepsilon)$ which appears in (1.2). The ε-expansion is an expansion of all quantities of the theory in powers of $\varepsilon = 4 - D$. When $\varepsilon = 0$, the theory has a (broken) scale invariance and there is a zero (with positive slope) of $\bar{\beta}(g,X)$ at $g = 0$ (an infra-red stable zero!). This zero survives for small ε as shown in Fig. 3.1. Actually when $\varepsilon \to 0$ the theory changes from a super-renormalisable theory to a just renormalisable theory. That the theory is renormalisable at $\varepsilon = 0$ is equivalent to the property that the limit $\varepsilon \to 0$ is smooth. This in turn requires that a zero with positive slope occur when $\varepsilon \sim 0$.

The above discussion can be rephrased if we give the form of the various functions in (3.13) to lowest order in ε. They are calculated from the Feynman graphs for $\Gamma^{(1,1)}$ and $\Gamma^{(1,2)}$ shown in Fig. 3.2. We shall not give the detailed Feynman rules for calculating these graphs. They can be found in any of Refs. [5], [8], and [13]. The results are

$$\gamma_E = 2\tau_E = -\frac{g^2}{2(8\pi)^2(1+X/2)}$$

$$\gamma_k = 2\tau_k = -\frac{g^2(X/2)}{2(8\pi)^2(1+X/2)}$$

$$\beta_E = -\frac{\varepsilon}{4}g + \frac{2g^3}{(8\pi)^2} - \frac{g^3}{2(8\pi)^2(1+X/2)}$$

$$\beta_k = -\frac{g^3(X/2)}{2(8\pi)^2(1+X/2)} \tag{3.48}$$

which gives

$$\bar{\beta} = -\frac{\varepsilon}{4}g + \frac{3}{2}\frac{g^3}{(8\pi)^2}$$

$$\bar{\tau} = -\frac{g^2}{4(8\pi)^2}, \quad \bar{\gamma} = -\frac{2g^2}{(8\pi)^2} \tag{3.49}$$

Thus we see that to this order in perturbation theory there is a zero of $\bar{\beta}$ at $g = 8\pi\left(\frac{\varepsilon}{6}\right)^{1/2}$. This result will hold if higher terms in the perturbation expansion for $\bar{\beta}$ do not have coefficients which are singular as $\varepsilon \to 0$. However,

this is essentially the requirement that the theory be renormalisable at $\varepsilon = 0$. This, perhaps, helps to explain why the introduction of renormalised quantities is a powerful tool, at least in the ε-expansion.

Inserting (3.48) and (3.49) in the above equations we obtain

$$\zeta = [1 - \frac{6g^2}{(8\pi)^2\varepsilon}]^{1/2}, \quad Z_2 = [1 - \frac{6g^2}{(8\pi)^2\varepsilon}]^{1/12}, \quad Z_3 = [1 - \frac{6g^2}{(8\pi)^2\varepsilon}]^{-1/6} \tag{3.50}$$

which gives

$$c = \frac{1}{2} + O(\varepsilon), \quad c_2 = \frac{1}{12} + O(\varepsilon), \quad c_3 = -\frac{1}{6} + O(\varepsilon) \tag{3.51}$$

and so we can write

$$g_1 = 8\pi \frac{\varepsilon}{6} \; a_1(X,\varepsilon) \qquad \bar{\zeta} = (\frac{8\pi^2\varepsilon}{3})^{-1/4} \; a(X,\varepsilon)$$
$$\bar{Z}_2 = (\frac{8\pi^2\varepsilon}{3})^{-1/24} \; a_2(X,\varepsilon) \qquad \bar{Z}_3 = (\frac{8\pi^2\varepsilon}{3})^{1/12} \; a_3(X,\varepsilon) \tag{3.52}$$

So far all the results I have presented are equivalent to those obtained in Ref. [13]. The important step made in Ref. [7] is to note that the functions in (3.52) are the vital ones for (3.37) and we retain very little of the structure of the scaling function if we simply set $a_1 = a_2 = a_3 = a = 1$ in (3.52), even though this is the lowest order ε-expansion result for these quantities. We do much better if we first calculate general formulae for their derivatives and then use the ε-expansion.

For this purpose we first note that from (3.21)

$$\frac{d\ln\bar{Z}_i}{d\ln X} = [\frac{dg_1}{d\ln X} \frac{\partial \ln Z_i}{\partial g} + \frac{\partial \ln Z_i}{\partial \ln X}]_{g = g_1} \tag{3.53}$$

$$= \left[\frac{(\frac{-\partial \ln Z_i}{\partial \ln X})_{g = g_1}}{(\frac{\partial \ln Z_i}{\partial g})_{g = g_1}} \frac{\partial \ln Z_i}{\partial g} + \frac{\partial \ln Z_i}{\partial \ln X} \right]_{g = g_1} \tag{3.54}$$

where this second relation is also a particular consequence of the special form of (3.21). Using (3.14) gives

$$\left.\frac{\partial \ln Z_3}{\partial \ln X}\right|_g = \frac{\gamma_k \beta_E - \gamma_E \beta_k}{\bar{\beta}} \tag{3.55}$$

and when this is combined with (3.15) and substituted in (3.54) we obtain

$$\frac{d\ln \bar{Z}_i}{d\ln X} = \left(\frac{\bar{\gamma}}{\bar{\beta}'}\right)_1 \frac{\partial}{\partial g}\left[\frac{\gamma_k \beta_E - \gamma_E \beta_k}{\bar{\gamma}}\right]_{g = g_1} \tag{3.56}$$

When this is evaluated using (3.48) and (3.49) we obtain for a_3

$$\frac{d\ln a_3(X)}{d\ln X} = -\frac{\varepsilon}{8}\,\frac{X/2}{1+X/2} \tag{3.57}$$

Integrating this and similar equations for a, a_1 and a_2 gives finally

$$\begin{aligned}
g_1(X) &= 8\pi\,\frac{\varepsilon}{6}\,(1+\tfrac{X}{2})^{-\frac{\varepsilon}{12}} \\
\bar{\zeta}(X) &= \left(\frac{8\pi^2\varepsilon}{3}\right)^{-\frac{1}{4}}\,(1+\tfrac{X}{2})^{\frac{\varepsilon}{24}} \\
\bar{Z}_3(X) &= \left(\frac{8\pi^2\varepsilon}{3}\right)^{\frac{1}{12}}\,(1+\tfrac{X}{2})^{-\frac{\varepsilon}{8}} \\
\bar{Z}_2(X) &= \left(\frac{8\pi^2\varepsilon}{3}\right)^{-\frac{1}{24}}\,(1+\tfrac{X}{2})^{\frac{\varepsilon}{16}}
\end{aligned} \tag{3.58}$$

Now, at last, we can substitute (3.58) into (3.37) and obtain the complete form of the inverse Pomeron propagator $\Gamma_u^{(1,1)}(E,\underline{k}^2)$, that is

$$\begin{aligned}
\Gamma_u^{(1,1)}(E,\underline{k}^2) = \frac{i}{1+\varepsilon/12}\,(-E)^{1+\frac{\varepsilon}{12}}\,\left(\frac{8\pi^2\varepsilon}{3}\right)^{\frac{1}{6}}\,\left(\frac{2(\alpha_0')^{\frac{D}{4}}}{r_o}\right)^{\frac{1}{3}} \\
X[1+X(1+\tfrac{\varepsilon}{24})]\,(1+\tfrac{X}{2})^{\frac{\varepsilon}{12}}
\end{aligned} \tag{3.59}$$

where X is still implicitly defined by

$$\frac{\alpha_0' k^2}{(-E)^{1+\frac{\varepsilon}{24}}} = \left(\frac{8\pi^2\varepsilon}{3}\right)^{\frac{1}{12}}\,\left(\frac{2(\alpha_0')^{\frac{D}{4}}}{r_o}\right)^{\frac{1}{6}}\,X(1+\tfrac{X}{2})^{\frac{\varepsilon}{24}} \tag{3.60}$$

An important point to note here is that (3.59) holds uniformly in X. The limits $X \to 0$ and $X \to \infty$ can be smoothly taken in both (3.59) and (3.60) and the results correspond to the forms of $\Gamma^{(1,1)}$ obtained by calculating directly with $k^2 = 0$ or $E = 0$. The simpler procedure of (3.40) - (3.47) can be used for

these cases, and it is clear that (3.59) provides a very nice interpolation between the two cases. A second very important point is that (3.59) has no fixed cuts in E or $\underline{k}^2$, despite the fractional powers appearing in both (3.59) and (3.60) - it is trivial to take E or $\underline{k}^2 \to 0$, with $\underline{k}^2$ or E fixed respectively, and obtain a finite result. We can also see that the square bracket in (3.59) gives a single Pomeron pole for $\underline{k}^2 < 0$ ($t > 0$), while the factor $(1 + \frac{x}{2})^{\varepsilon/12}$ gives a two-Pomeron cut. These singularities become complex conjugate pairs in $t < 0$.

The final stage is to substitute (3.59) into the Sommerfeld-Watson integral to obtain

$$\frac{d\sigma}{dt} = \frac{g_1^4(t)}{16\pi} (\ln S)^{\frac{\varepsilon}{6}} \frac{(1+\frac{\varepsilon}{12})^2}{K^4} F^2\left(-\frac{\alpha_0' t}{K} (\ln S)^{1+\frac{\varepsilon}{24}}\right) \tag{3.61}$$

where

$$F(x) = x^{-\varepsilon/12/1+\frac{\varepsilon}{24}} \int_{-i\infty}^{i\infty} \frac{dw\, e^{-wx^{\frac{1}{1+\varepsilon/24}}}}{(-w)^{1+\frac{\varepsilon}{12}} [1+\bar{x}(1+\frac{\varepsilon}{24})][1+\frac{\bar{x}}{2}]^{\frac{\varepsilon}{12}}} \tag{3.62}$$

$\bar{x}(w)$ is defined by

$$\bar{x}\,(1+\frac{\bar{x}}{2})^{\frac{\varepsilon}{24}} = (-w)^{-1-\frac{\varepsilon}{24}} \tag{3.63}$$

and

$$K = \left[\frac{(8\pi)^2 \xi (\alpha_0')^{\frac{D}{2}}}{6 r_0^2}\right]^{\frac{1}{12}} \tag{3.64}$$

Only the imaginary part of the amplitude is included in (3.61). Other sub-dominant terms are larger than the real part and so it would be inconsistent to keep it. We can finally eliminate the implicit function in (3.63) by changing variables to $\mu = 1 + \frac{1}{2\bar{x}}$. We extract the contribution of the pole at $\bar{x} = -\frac{1}{1+\varepsilon/24}$ leaving the integral over the cut from $\mu = 0$ to 1. We can then set $\varepsilon = 2$ and absorb irrelevant factors into g_1 to obtain

$$\frac{d\sigma}{dt} = \frac{g_1^4}{16\pi} (\ln S) F^2\left(-\frac{\alpha_0' t}{K} (\ln S)^{\frac{13}{12}}\right) \tag{3.65}$$

where now

$$F(x) = \frac{x^{-\frac{2}{13}}}{\frac{13}{12}\cdot 4^{\frac{12}{13}}} \cos\phi\left(\frac{7}{13},x\right) \exp\left\{-\left[\frac{13x}{12\left(\frac{7}{13}\right)^{\frac{1}{12}}}\right]^{\frac{12}{13}} \cos\frac{\pi}{13}\right\}$$

$$\frac{}{\left(\frac{7}{13}\right)^{\frac{15}{13}}\left(\frac{6}{13}\right)^{\frac{11}{13}}}$$

$$-\frac{x^{-\frac{2}{13}}}{2^{\frac{11}{13}}} \frac{1}{\pi} p\int_0^1 \frac{d\upsilon \sin\phi(\upsilon,x)\left[\frac{13\upsilon}{12}^{\frac{1}{12}} - \frac{1}{12}\right]}{\upsilon^{\frac{15}{13}}(1-\upsilon)^{\frac{11}{13}}\left[\frac{13}{12}\upsilon - \frac{7}{12}\right]}$$

$$x \exp\left\{-\left[\frac{x}{2\upsilon^{\frac{1}{12}}(1-\upsilon)}\right]^{\frac{12}{13}} \cos\frac{\pi}{13}\right. \tag{3.66}$$

where $p\int_0^1$ denotes a principal value integral and the auxiliary function $\phi(\upsilon,x)$ is defined by

$$\phi(\upsilon,x) = \frac{\pi}{13} - \left[\frac{x}{2\upsilon^{\frac{1}{12}}(1-\upsilon)}\right]^{\frac{12}{13}} \sin\frac{\pi}{13} \tag{3.67}$$

The diffraction peak, plotted against x, given by (3.65) (calculated on a comp ter) is shown in Fig. 3.3. In the same figure I have plotted the ISR data at $s \simeq 2{,}500$ GeV^{-2}, taking $\alpha_o' = 0.5$ and taking 1 GeV^{-2} as the basic unit of s. Note that $K \simeq 1$ because of the fractional power of $\frac{1}{12}$ in (3.64). In fact the measured value of r_o gives $K \sim 1.25$ and this is the value we used in Fig. 3.3. I hope you will agree that the close fit of (3.66) to the shape of the experimental diffraction peak is remarkable. Note that the height of the secondary maximum relative to the optical point is independent of the parameter α_o', which I used to place the dip in the right position. I think it is very impressive that this number $\sim 10^{-6}$ comes out <u>without parameters</u>.

I shall return to the significance of this calculation in Section 7. Here I note that a similar (but even more complicated) calculation has been carried out in Ref. [7] for the case of the one-particle inclusive cross-section. There is a general scaling law which holds for all large M^2 - missing mass, and $\frac{s}{M^2}$

$$\frac{d\sigma}{dtdM^2} \underset{\ln M^2, \ln\frac{s}{M^2} \to \infty}{\sim} [g_1(t)]^2 \, g_1(0) \, \frac{(\ln\frac{s}{M^2})^{\alpha}}{(\ln M^2)^{\beta}}$$

$$\times \, G\left(t\,(\ln s)^{\nu}, \; t\,(\ln\tfrac{s}{M^2})^{\nu}, \; \frac{\ln s}{\ln M^2}\right) \tag{3.68}$$

This is the contribution of the triple Pomeron graph shown in Fig. 2.4, which is the leading asymptotic contribution. It has not yet been possible to calculate the complete scaling function G except in the limits $\ln M^2 \ll \ln\frac{s}{M^2}$. In these limits we have

$$G \underset{\ln\frac{s}{M^2} >> \ln M^2}{\sim} F^2 \, (t\,(\ln\tfrac{s}{M^2})^{\nu}) \tag{3.69}$$

$$G \underset{\ln M^2 >> \ln\frac{s}{M^2}}{\sim} \hat{F}^2 \, (t\,(\ln M^2)^{\nu}) \tag{3.70}$$

In (3.69) F is the same function, given (apart from some normalisation factor) by (3.62), which appears in the elastic diffraction peak. The function $\hat{F}$ is, however, completely new. It has also been calculated in Ref. [7]. The functional form is similar to (3.62) but we shall not give it here. It is shown graphically, however, in Fig. 3.5. Note the secondary maximum is nearly two-orders of magnitude higher.

Note also that in no part of the triple Regge region does the inclusive cross-section plotted as a function of t, for fixed s and M^2, show a dip in the forward direction (that is $t \to 0$). Thus for all practical purposes there is no triple Pomeron zero. The triple Pomeron vertex of the Pomeron field theory does vanish at $t_i = 0$, $J_i = 1, \forall i$, but the inclusive cross-section does not turn over because Pomeron cuts are present as well as the pole and in this case $J_i \neq 1$ also contributes significantly to the Sommerfeld-Watson transform.

SECTION 4 THE LATTICE ANALOGUE-HIGH TEMPERATURE EXPANSIONS AND RENORMALISATION GROUP TRANSFORMATION

Clearly we would like to escape the restrictions of the ε-expansion which we had to use to calculate explicitly in the last section. We would like to be able to calculate directly in physical space-time, that is D = 2. In this Section I shall describe attempts to do this which exploit the analogy of the mathematics developed in Section 2 with that used to describe critical phenomena. I shall comment further on the physical significance of the analogy of diffraction scattering with a critical phenomenom in Section 7.

The first step we make is to develop an analogue model of interacting spins distributed on a lattice, which, hopefully, has a phase transition or at least critical behaviour with the same critical exponents as the Pomeron field theory. It has been shown by Wilson [3] and Kogut that in this sense $\lambda\phi^4$ field theory is analogous to the Ising model and hence $\lambda\phi^4$ can be used to determine the critical exponents of this model.

Our procedure will be to reverse the argument of Kogut and Wilson. Starting with the functional integral formalism of the Pomeron field theory, we shall be led to a lattice analogue model by evaluating the integral using the saddle-point method. This result is due to Cardy [14] and Sugar and we shall follow their treatment very closely.

A lattice analogue of the Pomeron field theory has also been formulated by Brower [16], Ellis, Savit and Zakrzewski. Their procedure is to choose the degrees of freedom for the lattice field and the interaction so that the theory resembles the continuum theory as closely as possible and in particular has a smooth continuum limit. There is considerable freedom in formulating an analogue model this way and the model studied in Ref. [16] is very complicated compared with that obtained from the functional integral approach. In particular it is not yet clear whether the more complicated model has the same critical behaviour as the simple model. Here we shall concentrate on the derivation of the simple model. We shall discuss the applications of both models only briefly, since these have, so far, been rather limited.

For the reasons discussed in Section 2 we now keep only the triple Pomeron interaction in the Lagrangian, we also change to the real and imaginary part of the field $\Psi = \phi + iX$ as variables. Dropping a time derivative we can write

$$L = 2\,\emptyset\,\frac{\partial X}{\partial t} - \alpha'(\nabla\emptyset)^2 \quad \alpha'(\nabla X)^2 + V \tag{4.1}$$

where

$$V = (\Delta - ir\emptyset)\quad(\emptyset^2 + X^2) \tag{4.2}$$

We drop the o suffices on our parameters in this Section since there will be no renormalisation, nevertheless Δ must be chosen so that $\Gamma^{(1,1)}(\alpha(0),0) = 0$ and so could still be written as $\alpha_o(0) - 1$ where $\alpha_o(0)$ is interpreted as the "bare" Pomeron intercept. The cut-off will be replaced by a finite lattice distance. The generating functional of (2.7) now becomes

$$F(J) = \int d\emptyset dX\; e^{-\int dt d^D x\; [L + J^+\Psi + J\Psi^+]} \tag{4.3}$$

The first step is to replace the continuum (t,x) space by a cubic lattice with spacing a and b in the time and space directions respectively. The generating functional then becomes for J = 0,

$$F(0) = \int \pi_i d\emptyset_i dX_i\; e^{-a\,b^D \sum_i L\,(\emptyset_i,\, X_i)} \tag{4.4}$$

where i labels the lattice points, and the derivatives $\frac{\partial}{\partial t}$ and ∇ are replaced by finite difference operators on the lattice (divided by a and b respectively). The integrations $\int d\emptyset_i dX_i$ are now simple integrals over the real variables $\emptyset_i$ and X_i from $-\infty$ to $+\infty$. If $\Delta < 0$, then these integrals are certainly convergent with L given by (4.1). However, it can be shown from (3.36) that if $\alpha(0) = 1$, then $\Delta = \Delta_c > 0$, that is the <u>bare Pomeron intercept is > 1</u>! (Strictly we need a cut-off in (3.36) if Δ_c is to be finite at D = 2).

(4.4) must therefore be defined in $\Delta > 0$, by analytic continuation from $\Delta < 0$. This can be done by first distorting the $\emptyset$ integration as in Fig. 4.1. The new contour intersects the imaginary axis at $\emptyset = ic$ $(c > 0)$ and is asymptotic to lines making angles $\alpha < \frac{\pi}{6}$ with the real axis. We can now continue in Δ up to any value less than cr, for then $\mathrm{Re}(\Delta + ir\emptyset) < 0$ everywhere on the contour, and $\mathrm{Re}\; ir\emptyset^3 \to -\infty$ as $|\emptyset| \to \infty$. The X integral is kept along the real axis.

Now as we discussed in previous Sections the infra-red limit $E, \underline{k}^2 \to 0, r, \alpha'$ fixed, of our theory is equivalent to the limit $r \to \infty$ $\underline{E}, \underline{k}^2$ fixed. This means we can take r very large in (4.4) in order to explore the $E, \underline{k}^2 \to 0$ limit of the theory. It can also be shown from (3.36) that in two dimensions

$$\Delta_c \underset{r\to\infty}{\sim} \frac{r^2}{\alpha'} \ln\left(\frac{r^2}{\alpha'\Lambda}\right) \tag{4.5}$$

(the factor $\frac{r^2}{\alpha'}$ follows from the dimension of Δ_c alone, the logarithm comes from the simple two Pomeron loop in Fig. (3.2)). From (4.5) we see that $\Delta_c \to \infty$ when $r \to \infty$ and so we can consistently take Δ and r large in (4.4).

In this case the integral (4.4) can be evaluated by the method of steepest descent. Temporarily ignoring the kinetic energy term in L, we look for saddle points in the ϕ and X integrations. $\frac{\partial V}{\partial \phi} = \frac{\partial V}{\partial X} = 0$ gives

$$\text{(a)}\quad \phi = X = 0 \quad \text{(b)}\ \phi = 2i\frac{\Delta}{3r},\ X = 0 \quad \text{(c,d)}\ \phi = \frac{i\Delta}{r};\ X = \pm\frac{\Delta}{r} \tag{4.6}$$

When $\Delta > 0$, (a) and (b) are outside the domain of convergence. (c) and (d), however, are accessible and the ϕ - contour can be distorted (dashed line in Fig. 4.1), so that the contribution from all parts of the contour away from (c) and (d) decreases exponentially when $r,\Delta \to \infty$

Therefore we write

$$\phi_i = \frac{i\Delta}{r} + \phi_i'$$

$$X_i = \frac{\Delta}{r} s_i + X_i' \tag{4.7}$$

where $s_i = \pm 1$. Since the second derivatives of V at the saddle points are $O(\Delta)$ we restrict ϕ_i', $X_i' \lesssim O(\Delta^{-1/2})$. Substituting (4.7) into (4.4) and keeping only second-order terms in ϕ_i' and X_i' gives a multiple Gaussian integral which can be evaluated explicitly. In the limit of interest we then get

$$F = \sum_{s_i = \pm 1} \exp^{-ab^D} \sum_i \left\{ \frac{\Delta^2 \alpha'}{r^2} (\nabla s_i)^2 + \frac{2\Delta\alpha'}{r^2} (s_i \nabla^2 s_i) \frac{\partial s_i}{\partial t} \right\} \tag{4.8}$$

We wish to interpret s_i as a spin at each site of the lattice, and F as the partition function. Denoting each lattice point by co-ordinates t,x (which take integer values) and setting D = 1 for clarity we can write

$$(\nabla s)^2 = a^{-2} (s_{t,x} - s_{t,x+1})^2 = \text{const} - 2a^{-2} s_{t,x} s_{t,x+1} \tag{4.9}$$

$$(s\nabla^2 s)\left(\frac{\partial s}{\partial t}\right) = \frac{1}{2} a^{-1} b^{-2} s_{t,x} (s_{t,x+1} - 2s_{t,x} + s_{t,x-1})(s_{t+1,x} - s_{t-1,x}) \tag{4.10}$$

$$\simeq \frac{1}{2} a^{-1} b^{-2} S_{t,x}(s_{t,x+1} + S_{t,x-1})(s_{t+1,x} - s_{t-1,x}) \qquad (4.11)$$

if we drop $s^2_{t,x}$ $(s_{t+1,x} - s_{t-1,x}) \equiv \frac{\partial}{\partial t} (\frac{1}{3} s^3)$, which sums to zero on the lattice (if periodic boundary conditions are imposed).

Ignoring a multiplicative constant, the generating functional now takes the standard form of a partition function in a thermodynamic model

$$F = \sum_{s_i} e^{-H} \qquad (4.12)$$

with

$$H = - k \sum_{<i,j>} s_i s_j \pm L \sum_{<ijk>} s_i s_j s_k \qquad (4.13)$$

where (in D = 2)

$$k = \frac{2\Delta^2 \alpha'}{r^2} \quad a \qquad L = \frac{\Delta \alpha'}{r^2} \qquad (4.14)$$

The k term in (4.13) represents a sum over nearest neighbours in space directions and the L term a sum over groups of three neighbouring spins in any of the configurations shown in Fig. 4.2.

An important point to note now is that H in (4.13) still has all the symmetry properties of the original Lagrangian with respect to ($t \to -t$, $x \to -x$) $\equiv$ (T = time reversal; $s \to -s$ spin flip) and ($x \to -x \equiv s$ = space reversal). Also the only time dependant term in changes sign under T alone. This preservation of symmetry properties by the lattice analogue should ensure that, if the system has a phase transition then the critical exponents will be that of the Pomeron field theory.

Next we consider the Greens functions of the theory. Consider the two-point function $<0|\ T\ [\Psi(t,x),\ \Psi^+(0,0)]\ |0>$, which gives $\Gamma^{(1,1)}$. This is difficult to calculate directly because of the problems associated with normal ordering and a non-zero expectation value of $\emptyset$ at the saddle-point. These problems can be avoided, however, by observing that for $t \geq 0$

$$<0|\chi(t,x)\ \chi(0,0)|0> = -\frac{1}{4} <0|[\Psi^+(t,x) - \Psi(t,x)][\Psi^+(0,0) - \Psi(0,0)]|0>$$

$$= \frac{1}{4} <0|\Psi(t,x)\ \Psi^+(0,0)|0> \qquad (4.15)$$

since Ψ annihilates the vacuum. The left-hand side of this equation is

$$F^{-1}\int\delta\phi\delta X\ X(t,x)\ X(0,0)\ e^{-\int dtd^{D}x\ \ (\phi,X)} \tag{4.16}$$

and so we can carry out the saddle-point integration, keeping only the leading terms in X(t,x) and X(0,0) to obtain

$$\langle 0|T[\Psi(t,x)\ \Psi(0,0)]|0\rangle \propto \langle s_{t,x}\ s_{0,0}\rangle \tag{4.17}$$

$$\equiv G(t,\underset{\sim}{x},\Delta)$$

This is a particular example of the relation between the Green's functions of the Pomeron field theory and the spin correlation functions of the analogue model. It shows that a system with a doubly infinite number of degrees of freedom at each lattice site has been reduced to a system with just two.

Having arrived at a lattice analogue model which is (or can be) set up in two dimensions, what are its uses? Firstly we would like a positive answer to the question, is there a phase transition in two dimensions? The connection may not be transparent but if there is a phase transition then this would show (or could be used to show) that there is a fixed-point of the β function, defined in the last Section, in two dimensions. If there is a phase-transition then we expect scaling behaviour of the spin correlation functions $G(t,x,\Delta)$ appearing in (4.17) of the form

$$G(t,\underset{\sim}{x},\Delta) \underset{\Delta\to\Delta_c}{\sim} (\Delta-\Delta_c)^{\alpha}\ f\left(\frac{t}{(\Delta-\Delta_c)^{\beta}}\ ,\ \frac{\underset{\sim}{x}^2}{(\Delta-\Delta_c)^{\alpha}}\right) \tag{4.18}$$

The requirement that the limit $\Delta \to \Delta_c$ exists for G also leads to scaling laws at $\Delta = \Delta_c$, for example,

$$G(t,\underset{\sim}{x},\Delta_c) \underset{t,\underset{\sim}{x}^2\to\infty}{\sim} t^{\frac{\alpha}{\beta}}\ f\left(\frac{t}{(\underset{\sim}{x}^2)^{\frac{\beta}{\alpha}}}\right) \tag{4.19}$$

When combined with (4.17) this result would imply a scaling law for the Pomeron Greens functions in co-ordinate space which would then give a corresponding scaling law in E and $\underset{\sim}{k}$ space.

While it seems unlikely that the scaling function calculated in the previous Section could ever be calculated like this, we might hope that the critical exponents a,b,c could be calculated accurately. Unfortunately neither of Refs. [14], [15] were able to prove the existence of a phase transition in their respective models, although this was largely due to the complexity of

the calculations. However, in both cases applications of the analogue models were discussed. Ellis [15] and Savit discussed the high temperature expansion of their lattice analogue model. In general this method has been very successful for the determination of critical exponents in thermodynamic lattice models. To exploit an analogue model in this way we write

$$H = \frac{A}{T} \tag{4.20}$$

and identify T with the thermodynamic temperature. Clearly from (4.13) and (4.14) Δ^{-1} is closely analogous to T. (In fact if we write $a' = \Lambda a$ in (4.14) we can directly identify T with Δ^{-1}.)

The high temperature expansion is an expansion of the theory in inverse powers of T (or equivalently powers of H). The scaling law (4.18) can be rephrased in terms of $T \to T_c$ where T_c is the "critical temperature". For example,

$$G(0,0,T) \underset{T \to T_c}{\sim} (T - T_c)^{\alpha}$$

$$= T^{\alpha} \sum_{n=0}^{\infty} \frac{(\alpha+1)}{(n)\ (\alpha+1-n)} \left(\frac{T_c}{T}\right)^n \tag{4.21}$$

$$\underset{T \to T_c}{\sim} \sum_{\ell=u}^{\infty} a_\ell\, T^{-\ell} \tag{4.22}$$

Taking ratios of successive terms gives

$$L(\ell) \quad \frac{a_\ell}{a_{\ell-1}} = T_c \left(\frac{\alpha+1}{\ell} - 1\right) \tag{4.23}$$

and so $L(\ell)$ should be a linear function of $\frac{1}{\ell}$ with intercept $-T_c$ and slope $T_c(\alpha+1)$. Hence, by calculating $L(\ell)$ for sufficiently large ℓ, we would hope to calculate both T_c and α. In principle it would also be possible to determine β and γ in (4.18) by high-temperature expansions of suitable quantities.

Values for the critical exponents of the Pomeron field theory were obtained by Ellis and Savit in their model using this approach (assuming the existence of a phase transition). These values were slightly higher than the results given by the ε-expansion although since they were unable to calculate more than three terms in the expansion (4.22), they were unable to place much confidence on the exactness of their results. Their main conclusion was that their results confirmed the approximate validity of the ε-expansion results.

Cardy [14] and Sugar discussed a rather different application of their analogue model. They considered what is a relatively old idea in critical phenomena,

but which has only recently been applied. An explicit renormalisation group transformation is carried out on the lattice. The lattice sites are grouped into cells, which consist of a finite number of sites and have the same periodic structure as the original lattice. With each cell is associated a spin s' which takes the value of ± 1, according to whether the sum of all the site spins in the cell is positive or negative. The configurations when the sum is zero are assigned either to $s' = +1$ or $s' = -1$, in such a way that changing all site spins transforms a configuration with $s' = +1$ into one with $s' = -1$. The site spins can now be relabelled by the cell spin s' plus an internal cell variable σ. A new Hamiltonian H' is determined by summing over σ

$$e^{H'(s') - N_g} = \sum_{\sigma i} e^{-H(s)} \qquad (4.24)$$

N_g depends on the number of cells and can be thought of as the self-energy of a cell. This transformations defines new couplings k and L in (4.13) and once again the point of the exercise is to look for a fixed-point of the transformation. This would correspond to scaling behaviour as discussed above.

Cardy and Sugar were not able to prove the existence of a fixed-point although there is some hope that further numerical work might lead to this result. However, they were able to prove that <u>if a fixed point exists</u>, then the Froissart bound is satisfied. That is if a solution of the form (1.2) does exist in two dimensions then <u>$\eta \leq 2$</u>. They were also able to show that the elastic cross-section is less than the total cross-section.

SECTION 5 THE LOOP EXPANSION AND THE CALLAN-SYMANZIK EQUATION

In this Section we briefly discuss an alternative route to calculating directly at $D = 2$, which has been employed by Dash [17] and Harrington. The method used is again analogous to a technique used for critical phenomena except that Dash and Harrington have extended the idea in what seems at the moment to be a very promising direction.

The basic idea is simply to calculate the theory described in Section 3, in two dimensions, in perturbation theory, and see if the results are sensible. Now the infra-red problems associated with imposing the intercept one condition (3.7) are very serious in two-dimensions, and standard perturbation theory certainly can not be used [13]. The constructional method of Section 3 can be used in principle by first calculating $\bar{\gamma}$ and $\bar{\beta}$ from lowest order perturbation theory (they will be convergent), then calculating z_3, z_2 and ζ from (3.15), (3.17) and (3.18) and then using (3.31), (3.33) and (3.35) to calculate the complete propagator. If this propagator is used in the next order of perturbation theory no divergence problems will be encountered and the process can be repeated. In this way the complete theory can in principle be calculated in two dimensions [13] (provided a zero of $\bar{\beta}$ exists at each stage). Unfortunately this process is very complicated and while it would be very nice if somebody had the strength to carry it through, at present this looks unlikely.

Alternatively it is possible to not impose the intercept one condition, and calculate in the "massive" theory with intercept less than one. In this case it is possible to calculate using the Callan-Symanzik equation. This also involves β and γ functions but these are now defined as derivatives with respect to the intercept (that is "mass") rather than the renormalisation point, as in Section 3. It can then be argued that ultra-violet scaling behaviour observed in the massive theory at a critical value of the coupling constant must be identical to that found in the massless theory, since the mass (or intercept) is unimportant in the ultra-violet region. (Precisely the same argument is used to justify the calculation of the critical exponents of massless $\lambda\phi^4$ using the massive theory [19]). Hence one calculates the β function of the Callan-Symanzik equation, looks for a zero and calculates the associated critical exponents.

This procedure was carried out in Ref. [17] for the Pomeron field theory. However, although the results were not too discouraging - the critical exponents were again of the same order of magnitude as in the ε-expansion - the precise

results were very sensitive to the normalisation point chosen to renormalise the ultra-violet divergences of the theory. To all orders of perturbation theory the critical exponents should be independent of the renormalisation procedure. However, too finite order in perturbation theory, i.e., the loop expansion, this will not necessarily be the case.

Consequently Dash and Harrington were led to demand invariance under changes of the renormalisation point. Conventionally we ask for a zero of the β function regarded as a function of the dimensionless coupling constant g and than calculate $\gamma(g)$, say, to determine a critical exponent (compare (3.40) - (3.47)). Instead we can plot β as a function of γ for various values of the renormalisation point E_N. (For the purpose of illustration the reader can associate β and γ directly with the functions defined in (3.40) and (3.41), although strictly they are Callan-Symanzik functions - in two dimensions). The results are shown in Figs. 5.1 and 5.2 in the one and two-loop approximation.

In the one-loop approximation a zero of the β-function, of course, exists for all E_N. However, note that the intersection point is much better defined than the zero as E_N is varied. In the two loop approximation there is no zero of the β-function. (Although the zero will, of course, return in the three-loop approximation simply because $\beta(g) \underset{g\to\infty}{\sim} + g^7$ and $\beta'(0) < 0$. In the two-loop approximation $\beta(g) \underset{g\to\infty}{\sim} - g^5$ and there need not be a zero). However, there is a point γ_c where $\beta'(\gamma_c)$ is independent of E_N. To all orders of perturbation theory we expect the zero of β to be the only point with this property. Further γ_c changes very little as we move from the one loop to the two loop approximations. Hence, should we accept γ_c as the best estimate of η in (1.2) and (1.3)? I suspect so and I believe that further theoretical investigation will help us understand why.

Dash and Harrington are currently applying this method to the critical exponents of solid state physics, where, of course, most of the exponents have been accurately determined both experimentally and theoretically. Apparently the initial results are encouraging. Here we note only that the results for $\eta = \gamma_c$ given by Figs. 5.1 and 5.2 lie very close to the first order ε expansion result, that is

$$\eta \lesssim 0.11 < \frac{1}{6} = 0.166 < 0.18 \qquad (5.1)$$

The one and two loop results for ν are equally encouraging, they give 1.05 and 1.08 respectively and so

$$\nu \gtrsim 1.05 < \frac{13}{12} = 1.08 \leq 1.08 \qquad (5.2)$$

That this calculation procedure gives (apparently) convergent numbers which lie very close to the first order ε-results is very encouraging and perhaps helps to explain why the corresponding calculation of the scaling function in Section 3 gave such good results. It is well-known that the second-order ε-expansion results [20] for η and ν discourage the belief that the second-order is any better (or worse!) than the first order.

SECTION 6 UNIQUENESS OF THE SOLUTION - WHAT HAPPENS WHEN $\Delta > \Delta_c$

In this Section we close the argument for uniqueness of our solution that we referred to in the introduction by presenting a recent argument developed by Sugar.

We argued in the introduction that unitarity in the angular momentum plane looks very like unitarity in momentum space and stated that we know of no solutions to this sort of condition besides basic poles plus cuts, (apart from degenerate massless theories). Note that the Pomeron trajectory must have a t-dependence and in particular must have a singularity at the two-pion threshold. This means that a degenerate Schwartz trajectory with the Pomeron and all multi-Pomeron trajectories co-inciding is not possible, that is we can not solve (1.1) with $\alpha_N(t) = \alpha(t)$ when $\alpha(t)$ is singular at the two-pion threshold). It might also be possible to argue that Regge poles plus their associated cuts are the only allowed angular-momentum plane singularities as follows. Mandelstam [21] has argued that only Regge poles would be allowed in the angular momentum plane if elastic (t-channel) unitarity were exact, and as we know the cuts are associated with the multiparticle unitarity condition. It is possible that the multiparticle equations could be introduced successively and at each stage it could be proved that the only new angular momentum singularities allowed were those generated by the singularities present at the previous level. Certainly a maximal analyticity hypothesis could be formulated this way, which would imply the result we require.

In the introduction we next used the experimental facts of factorisation plus rising cross-sections to argue that the Pomeron must be a single isolated pole in $t > 0$ and must have $\alpha(0) = 1$. In Section 2 we argued that the general class

of solutions of Pomeron unitarity given by a field theory of "quasi-Pomerons" with arbitrary interaction Lagrangian had the same $E,\underline{k}^2 \to 0$ behaviour as the simple theory with a point triple Pomeron interaction studied in Section 3. (It is possible to avoid this result by arranging a cut-off dependent cancellation of terms in the Renormalisation group transformation [12]. Such a cancellation is very artificial and it seems extremely unlikely that this form of solution would avoid the famous decoupling problems. As I discuss briefly in Section 7, the triple Pomeron theory does avoid these problems).

Nevertheless there is further possibility for a different solution which we have not yet eliminated. In Section 3 we imposed the condition (3.7), that is $\Gamma^{(1,1)}(0,0) = 0$ and that implied $\Delta = \Delta_c = \alpha_o(0) - 1 \quad (>0)$. Now we can ask what happens if $\Delta > \Delta_c$? If $\Delta < \Delta_c$ then the situation is relatively simple, we simply get a leading (renormalised) pole with intercept less than one (this can easily be shown from perturbation theory, which is convergent in this case). However, if $\Delta > \Delta_c$ we might perhaps expect the contributions of other Pomeron Greens functions, besides the propagator, in (2.2) to become important. Model calculations have led some people to believe that the eikonal result of saturation of the Froissart bound would occur when the sum over m and n in (2.2) was carried out [23], [24]. Personally I have never understood how a satisfactory angular momentum plane structure could arise in this case. The following argument (due to Sugar [18] and as yet unpublished) suggests very strongly that if $\Delta > \Delta_c$ then we simply recover a leading Pomeron pole with intercept less than one.

As we discussed in Section 4 when $\Delta >> 0$ the functional integral giving the generating functional of the theory is dominated by the saddle-points (c) and (d) in (4.6). Suppose we pick on one and give the field corresponding vacuum expection values. (As we shall see this will lead to sensible results for $\Delta >> r^2/\alpha'$, but for $\Delta \sim \frac{r^2}{\alpha'}$ which is the case for $\Delta \sim \Delta_c$ as we discussed in Section 4, it would not give sensible results). We could choose either, but suppose we choose (d), where $\psi = 0$ and $\psi^* = \frac{2i\Delta}{r}$, and write

$$\psi = \psi', \qquad \psi^* = \frac{2i\Delta}{r} + \psi'^* \tag{6.1}$$

so that $\mathcal{L} = \mathcal{L}_o + \mathcal{L}_I$ where

$$\mathcal{L}_o = \frac{1}{2} i \psi'^* \overleftrightarrow{\frac{\partial}{\partial t}} \psi' - \alpha' \nabla\psi'^* \cdot \nabla\psi' - \Delta\psi'^*\psi' \tag{6.2}$$

$$L_I = \Delta\psi'^2 + \frac{ir}{2}\ \psi'^*\ \psi'\ (\psi'^* + \psi') \tag{6.3}$$

Note that it is $-\Delta\psi'^*\psi'$ which appears in (6.2), so that L_0 is the free Lagrangian for a bare Pomeron with intercept $\alpha_0(0) - 1 = -\Delta$, that is $\alpha_0(0) < 1$. We can not yet do perturbation theory with L_I given by (6.3), since our Greens functions are still given as vacuum expectation values of ψ and ψ^*, e.g.,

$$G^{(1,1)}(\underset{\sim}{x},t) = \langle 0|\psi(\underset{\sim}{x},t)\ \psi^*(0,0)|0\rangle \tag{6.4}$$

$$= \frac{1}{F(0)}\ \frac{\delta}{\delta J^*}\ \frac{\delta}{\delta J(0,0)}\ F(J)\Big|_{J=J^*=0} \tag{6.5}$$

$$= \langle 0|\psi'(\underset{\sim}{x},t)\ \left[\frac{2i\Delta}{r} + \psi'^*(0,0)\right]|0\rangle \tag{6.6}$$

However, order by order in perturbation theory

$$\langle 0|\psi'(\underset{\sim}{x},t)||0\rangle = 0 \qquad\qquad \langle 0|\psi'(\underset{\sim}{x},t)|0\rangle \neq 0 \tag{6.7}$$

since ψ' commutes with the interaction term $\Delta\psi'^2$, but ψ'^* does not. Therefore

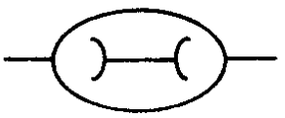
$$G^{(1,1)}(\underset{\sim}{x},t) = \langle 0|\psi'(\underset{\sim}{x},t)\ \psi'^*(0,0)|0\rangle \tag{6.8}$$

and we can simply do standard perturbation theory with ψ' and ψ'^* as our basic fields, and L_I given by (6.3).

Suppose now that we do perturbation theory to a given order η and calculate the self-energy which we write as

$\Sigma_\eta \equiv$  (6.9)

(the reason for this graphical notation will soon become clear). Suppose we add one more line attached to two triple Pomeron vertices

$\Sigma_{\eta'} =$  (6.10)

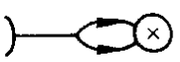
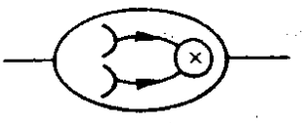
This gives one more logs integration with three extra inverse propagators which are of order Δ if $\Delta >> 0$ and so

$$r^2\int\frac{dEdk^2}{\Gamma^3} \sim \frac{r^2}{\alpha'\Delta} << 1 \tag{6.11}$$

If we denote $(-\Delta)\psi'^2$ by

$-\Delta\psi'^2 \quad \leftrightarrow$  (6.12)

There are two possibilities for adding such vertices to ξ_η

1)  2)  (6.13)

In both cases we have four new propagators, two new r's, one new Δ and one new logs integration giving

$$r^2\Delta\int\frac{dEdk^2}{4} \quad \sim \quad \frac{r^2}{\alpha'\Delta} \tag{6.14}$$

Consequently for $\Delta >> \frac{r^2}{\alpha'}$ we have

$$G^{(1,1)}(E,\underline{k}^2) = \frac{i}{E-\alpha'\underline{k}^2-\Delta}\ [1 + O(\frac{r^2}{\alpha'\Delta})] \tag{6.15}$$

and so we recover an isolated leading pole with intercept less than one. For $\Delta \tilde{\sim} \Delta_c$ the above arguments clearly do not work. However, there seems to be no reason for any change of structure as Δ approaches Δ_c. Therefore, it seems safe to assume that <u>there is only one critical value of Δ at which the Pomeron Field theory gives a rising total cross-section</u> and the scaling form (1.2) and (1.3) results. Saturation of the Froissart bound does not seem to be possible with a single factorising Pomeron pole, as suggested by experiment. Of course, it remains an open question whether such behaviour could be obtained with some finite (or possibly infinite) number of poles.

SECTION 7 S-CHANNEL UNITARITY: THE BARE PERTURBATION EXPANSION, THE ASYMPTOTIC SCALE, AND PRODUCTION PROCESSES

So far we have concentrated almost exclusively on the elastic partial-wave amplitude and in particular the contribution of the Pomeron propagator. I have emphasised that we can make a precise analysis of the infra-red behaviour of the propagator and so predict the asymptotic diffraction peak. As I have stated (but not proved) the propagator does give the leading behaviour of the differential cross-section (sufficiently near t = o) and since this can be calculated exactly my attitude has been to simply calculate this behaviour and compare with experiment afterwards. As I shall discuss shortly I believe this approach can be extended to the asymptotic behaviour of all exclusive and inclusive production processes. As I have also emphasized it is the content of <u>t-channel</u> unitarity that we are exploiting (together with crossing and analyticity). While I believe this is the determining factor, it means that

we can only indirectly check the constraints of s-channel unitarity by such means as checking that inclusive sum rules are satisfied asymptotically.

We can try to develop a more detailed understanding of the s-channel production processes that are producing the Pomeron field theory results by introducing the bare perturbation expansion. The theoretical significance of this will be discussed in a lot more detail by DeTar. Here we note only that there is an expansion of the Greens functions of the triple Pomeron theory of Section 3 in which we write the complete set of Feynman graphs (as illustrated in Fig. 7.1) where the propagator is given by

$$\Gamma_0^{(1,1)} = E - \alpha_0' k^2 + \Delta_0 \qquad (7.1)$$

and the vertex is r_0 (apart from a normalisation factor). Since $\Delta_0 > 0$, we expect the convergence of this expansion to be somewhat delicate. We can now attempt to identify the bare propagator with some elementary object in our underlying strong interaction theory. For example, in a dual model it might correspond (roughly - see DeTar's lectures) to the set of graphs which contain a single Pomeron. In a field theory it would be the basic set of graphs generating a vacuum pole near one. Alternatively we can try to be more physical by taking the imaginary part of our amplitude and identifying the various graphs with production processes through unitarity.

Accepting the experimental evidence that production processes are successfully described by a basic multiperiphal process (uniform spread in rapidity with a sharp momentum transfer cut-off, and short-range correlations in rapidity) we might identify this process with the imaginary part of the bare Pomeron - as in Fig. 7.2. The second graph in Fig. 7.2 is then identified (partly) with diffractive production of large missing masses as in Fig. 7.3. As DeTar will elaborate there are other contributions to the imaginary part of the graph of Fig. 7.3. All Pomeron Greens functions would have similar expansions and their imaginary parts are interpreted as connected with diffractive production or absorptive corrections as in Fig. 7.4.

If we suppose that the bare expansion (or rather the first few terms) are what we see at present energies, then we can try to extract the bare parameters r_0, α_0', Δ_0 etc. from experiment. The present rise of the total cross-section can be attributed to a bare Pomeron pole with intercept above one. The series has an energy dependence which guarantees that it will diverge at asymptotic energies and reproduce the asymptotic results of Section 3. Note that this

interpretation of the experimental results fits nicely with the critical phenomenon analogy. The basic production process is a short-range (in rapidity) interaction which by itself would simply produce a simple Pomeron Regge pole. However, the interaction is so strong that it produces a rising total cross-section, with the absorption balanced at its most critical value so that the cross-section continues to rise asymptotically ($\Delta = \Delta_c$). To achieve this it is essential that long-range interactions set in and critical behaviour develops.

Having developed the short-range interaction or perturbation picture, we can try to estimate the energy at which the asymptotic (or long-range) behaviour sets in. This is discussed in Refs. 7 and 11 as well as in a recent paper by Amati [25] and Jengo. Before discussing this here, let us note that both asymptotically and sub-asymptotically, the contribution of the higher Pomeron Greens functions are determined by the arbitrary couplings g_m, g_n appearing in (2.2). In general these contributions will be associated with non-factorising diffraction processes, which become increasingly important for large t. Since factorisation seems to be so good [6] we assume that we can neglect $g_2 \cdots g_n, \cdots$ and need discuss only the factorising Pomeron propagator.

There is general agreement in Refs. 7, 11, and 25 that the transition between the bare expansion and the asymptotic expansion is determined by a factor

$$k \frac{r_o^2}{\alpha_o'} \ln S \qquad (7.2)$$

where k depends on the normalisation convention. Essentially by using the techniques of Section 3 all three sets of authors construct models which illustrate the transition. There is also agreement that when r_o and α_o' are determined from present experiments the transition region is

$$\ln S \sim 10 \quad \text{or} \quad S \sim 10^4 - 10^5 \ \text{G.eV}^2 \qquad (7.3)$$

Fig. 6.5 shows a plot of recent cosmic ray experimental results for the total cross-section of protons on air [26]. If you look closely you can see that there is a transition from the sharp energy rise around the ISR region of 10^3GeV^{-1} to a slower energy dependence, which occurs around $10^4 - 10^5 \text{GeV}^{-1}$.

How do we reconcile the scale of (7.3) with the close fit of the diffraction peak calculated in Section 3 with the ISR data? Firstly it could well be that since the diffraction pattern is changing only slowly with energy at the ISR

(the dip is moving in approximately like ln S as required by (1.2)) it is already close to its asymptotic shape and will not change much as the energy rises through another order of magnitude. Physically we know that the total cross-section reflects all production processes and so may approach its asymptotic behaviour more slowly.

However, I would add a note of caution to using the bare expansion to extract the parameters r_o and α'_o. Conventionally the sharp exponential t-dependence of differential cross-sections is associated with the residue functions g_A and g_B in (1.2). This t-dependence is then factorised out of the inclusive cross-section before extracting the triple Pomeron coupling which is then also parametrised with an exponential in order to extract r_o. However, as we see from Fig. 3.3 very little of the t-dependence may be due to g_A and g_B. If we fit $\frac{d\sigma}{dt}$ directly with (3.65) and fit $\frac{d\sigma}{dtdM^2}$ with (3.68) we may reach very different conclusions on the parameters r_o and α'_o. (A detailed form of (3.68) can be found in Ref. [7]).

A further advantage that is often claimed for the bare expansion is that the cutting rules of Abramovskii [27], Gribov and Kanchelli (which DeTar will discuss) can be used to calculate the Reggeon Calculus rules for inclusive cross-sections directly from the rules for the elastic amplitude. However, I am optimistic that we will be able to take discontinuities in a well-defined way within the formalism of I. We should then obtain well-defined Pomeron unitarity equations (and hence a Pomeron Field Theory prescription) for all inclusive discontinuities without the "AGK rules" - or rather we would prove these rules.

Given explicit Pomeron Field theory formulae for all exclusive and inclusive amplitudes there will be many s-channel unitarity constraints to check. The one-particle inclusive sum rule has been finally checked in full detail in the triple Pomeron region in Ref. [7]. Using the AGK rules Caneschi [28] and Jengo have verified that the multiplicity moments $\langle n^P \rangle$ have the behaviour

$$\langle n^P \rangle \sim \ln S^{P(1+\eta)} \tag{7.4}$$

where η also appears in (1.2). Further they have shown that

$$C_p = \frac{\langle n^P \rangle}{\langle n \rangle^P} \simeq \left(1 + \frac{c\varepsilon}{p^2} e^{P \ln 2}\right)\left(1 - \frac{\varepsilon}{12} p \ln p\right) \tag{7.5}$$

which not only is in agreement with all positivity requirements but also is in surprisingly good agreement with experiment [28]. Finally Bartels [29] and

Rabinovici have used the Pomeron Field Theory rules for production amplitudes previously extracted from Hybrid Feynman graphs by Bartels [30] to show that the old Finkelstein-Kajantie problem is definitely avoided. They did, however, find a bound on the magnitude of the particle/two Pomeron coupling.

NOTE VERY RECENT DEVELOPMENTS

Since these lectures were prepared a large number of papers have appeared which are relevant to the material of the latter part of the lectures. As a result, I have prepared a list of all the papers that I am aware of, giving the sections of the lectures that they are relevant to together with some short comments. This list appears below.

I would also like to emphasize that the argument of Section 6, which was communicated to me by Sugar, was just one result of an extensive investigation by a large collaboration. This work is more completely reported in the first paper listed under Section 6 below.

Section 2

W.A. Bardeen, J.W. Dash, S.S. Pinsky and V. Rabl, Phys. Rev. D 12, 1820 (1975). This paper discusses explicitly the stability of the triple Pomeron theory against the addition of a four-Pomeron interaction.

Section 4

G. Parisi, Phys. Letters 56B, 470 (1975). This gives a third lattice model, which is quite similar to that of Ref. [16].

A general discussion of the relation between high-energy behaviour and statistical mechanics is given by

J.R. Ellis and R. Savit, Nuclear Phys. B94, 477 (1975).

Further material relevant to this section appears in

R.C. Brower, J.R. Ellis, R. Savit and J. Zinn-Justin, CERN TH 2125 (1976), to appear.

Section 5

J.W. Dash and S.J. Harrington, Phys. Letters 59B, 249 (1975); LBL preprint 3885 (1975). These papers give improved results for the calculations reported in Ref. [17].

J.A. Shapiro and T.F. Wong, Phys. Rev. D 12, 2390 (1975). This paper criticizes the work of Dash and Harrington by comparing it with similar calculations of $\lambda\phi^4$ critical exponents.

Section 6

H.D.I. Abarbanel, J.B. Bronzan, A. Schwimmer and R.L. Sugar, SLAC-PUB-1619 (1975). This paper gives a general argument that the instability of the vacuum shifts a bare Pomeron intercept above the critical value to one below this value.

There has been a lot of controversy surrounding this subject, particularly with respect to how the Green's functions of a shifted Pomeron field should couple to the external particles. Relevant papers are

S.A. Jackson, CERN TH 2058 (1975).
J.R. Ellis and R. Savit, CERN TH 2094 (1975).
D. Amati, L. Caneschi and R. Jengo, CERN TH 2047 (1975).
V. Alessandrini, D. Amati and R. Jengo, CERN TH 2089 (1975).

In the last two papers the one-dimensional Reggeon Field Theory is studied in detail.

W.J. Zakrzewski, Durham preprint (1975). This paper studies the effect of higher order couplings on the intercept shift.

In the light of the controversy it seems to me that the vital question for a theory that starts with a bare Pomeron intercept above the critical value is "can the theory be organized to demonstrate that Pomeron unitarity is satisfied by the complete partial-wave amplitude?" This question has not been answered yet. The perturbation expansion of Section 6 is unsymmetric with respect to ψ and $\dot{\psi}^+$ and so does not satisfy Pomeron unitarity perturbatively. It seems to me that the only possibility to satisfy Pomeron unitarity may be to obtain a renormalized pole (+ cuts) with intercept less than one. In this case (essentially) nothing new has been learned by formally starting with the bare intercept above one.

Section 7

There have also been many new papers relevant to this section. New approaches to the bare perturbation expansion of the Reggeon Calculus are given by

M. Ciafaloni, G. Marchesini and G. Veneziano, WIS 75/22Ph and CERN TH 2048 (1975). These authors develop the "topological expansion" which is motivated by both dual- and field-theoretic models. Not only is the expansion crossing symmetric so that t- and s-channel unitarity can be simultaneously studied, but it also seems to resolve the counting problems inherent in deriving the Reggeon Calculus bare perturbation expansion from an underlying theory. The AGK cutting rules are also derived.

M. Baker and L.D. Mclerran, Seattle preprint (1975). This paper discusses the physical basis of the bare expansion when it is combined with the AGK cutting rules. The AGK cutting rules are also discussed from many different points of view in the following papers. There seems to be universal agreement that these rules are very general.

T. De Grand, Phys. Rev. D 8, 2233 (1975).

T. De Grand and C.E. De Tar, CERN TH 1962 (1975).

J. Koplik and A.H. Mueller, CO-2271-57 (1975).

L.D. Mclerran, J.H. Weis, RLO-1388-673 (1975).

N. Staton, Imperial Coll. preprint (1975).

P. Suryani, Phys. Rev. D 12, 2124 (1975).

J.L. Cardy and P. Suryani, Santa Barbary, Cincinnati preprint (1975).

The phenomenological application of the bare expansion to both elastic and inclusive experiments has been studied by

A. Capella, J. Kaplan and J. Tran Thanh Van, Nuclear Phys. B27, 493 (1975) and CERN TH 2083 (1975). These authors conclude that the bare parameters of the Pomeron are not compatible with a "critical" Pomeron.

The addition of low energy threshold parameters to the bare propagator, analogous to the temperature in statistical mechanics, is advocated by

J. Dash, Oregon preprint OITS-75-37 (1975).

For present energies Gribov himself advocates a perturbative approach based on the smallness of the Pomeron slope.

V.N. Gribov, Leningrad preprint I48 (1975). This gives $\sigma_T \sim (\ln \ln s)^2$ at applicable energies. The approach is somewhat similar to that of Amati et al. above.

Thorough discussions of the avoidance of decoupling problems by the Strong Coupling Pomeron can be found in

J.L. Cardy, Phys. Rev. D 12, 3346 (1975).
J. Bartels and E. Rabinovici, FNAL 75/55 (1975).

Finally we note three other general review papers

J.L. Cardy, APS proceedings, Seattle (1975).
R. Savit, CERN TH 2059 (1975).
A.B. Kaidalov and V.A. Khoze, Leningrad preprint I93 (1975).

We also note that the Reggeon unitarity relation, on which these lectures are based, has been challenged by B.M. McCoy and T.T. Wu, FNAL 74/88, 74/89 (1974). Their arguments are based on field theory. However, the challenge has been refuted by I.T. Drummond and I.G. Halliday CERN TH 2086, 2108 (1975).

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[6] R. Webb et al., Phys. Letters 55B, 336 (1975).

[7] H.D.I. Abarbanel, J. Bartels, J.B. Bronzan and D. Sidhu, Phys. Rev. D 12, 2459 and 2798 (1975).

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[12] D. Calucci and R. Jengo, Nuclear Phys. B84, 413 (1975).

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[15] J.R. Ellis and R. Savit, Nuclear Phys. B94, 477 (1975).

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[28] L. Caneschi and R. Jengo, Nuclear Phys. B89, 19 (1975).

[29] J. Bartels and E. Rabinovici, Phys. Letters 58B, 171 (1975).

[30] J. Bartels, Phys. Rev. D 11, 2977 and 2989 (1975).

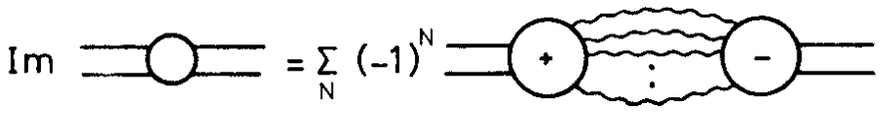

Fig. 2.1 Pomeron unitarity

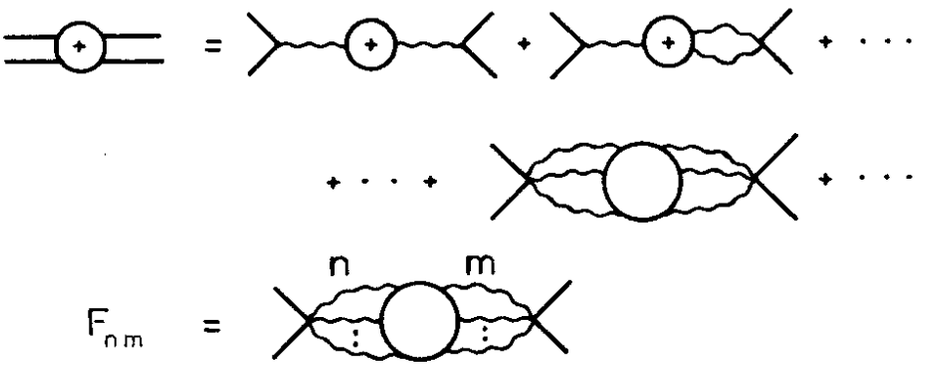

Fig. 2.2 The expansion of the Froissart-Gribov amplitude in terms of Pomeron Green's functions

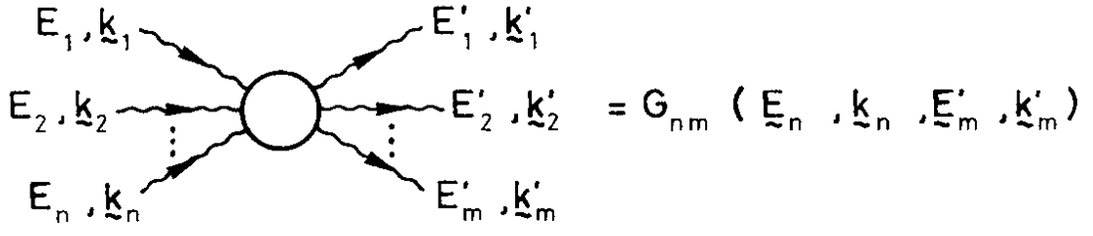

Fig. 2.3 The general Pomeron Green's function

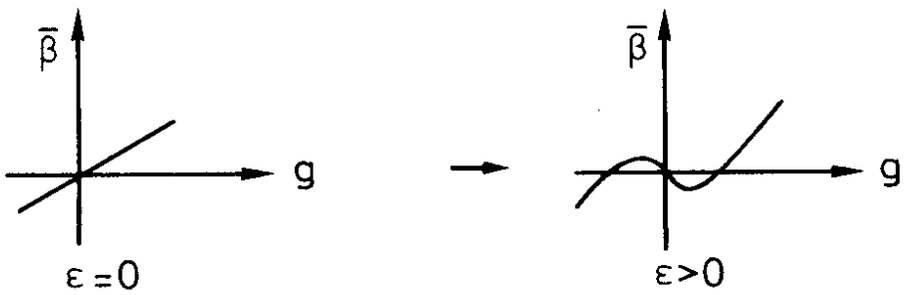

Fig. 3.1 The dependence of $\bar{\beta}(g)$ on ε

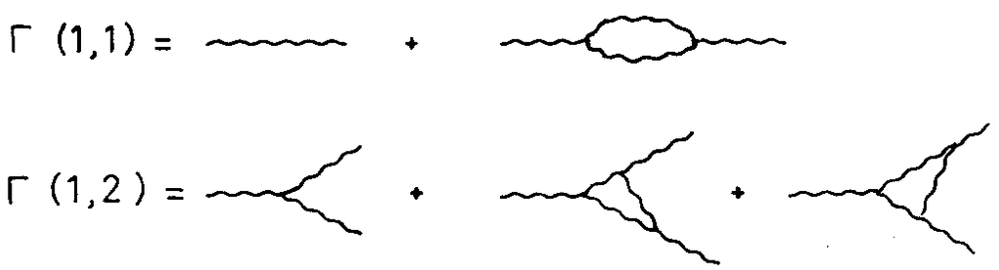

Fig. 3.2 Perturbation theory diagrams for $\Gamma^{(1,1)}$ and $\Gamma^{(1,2)}$

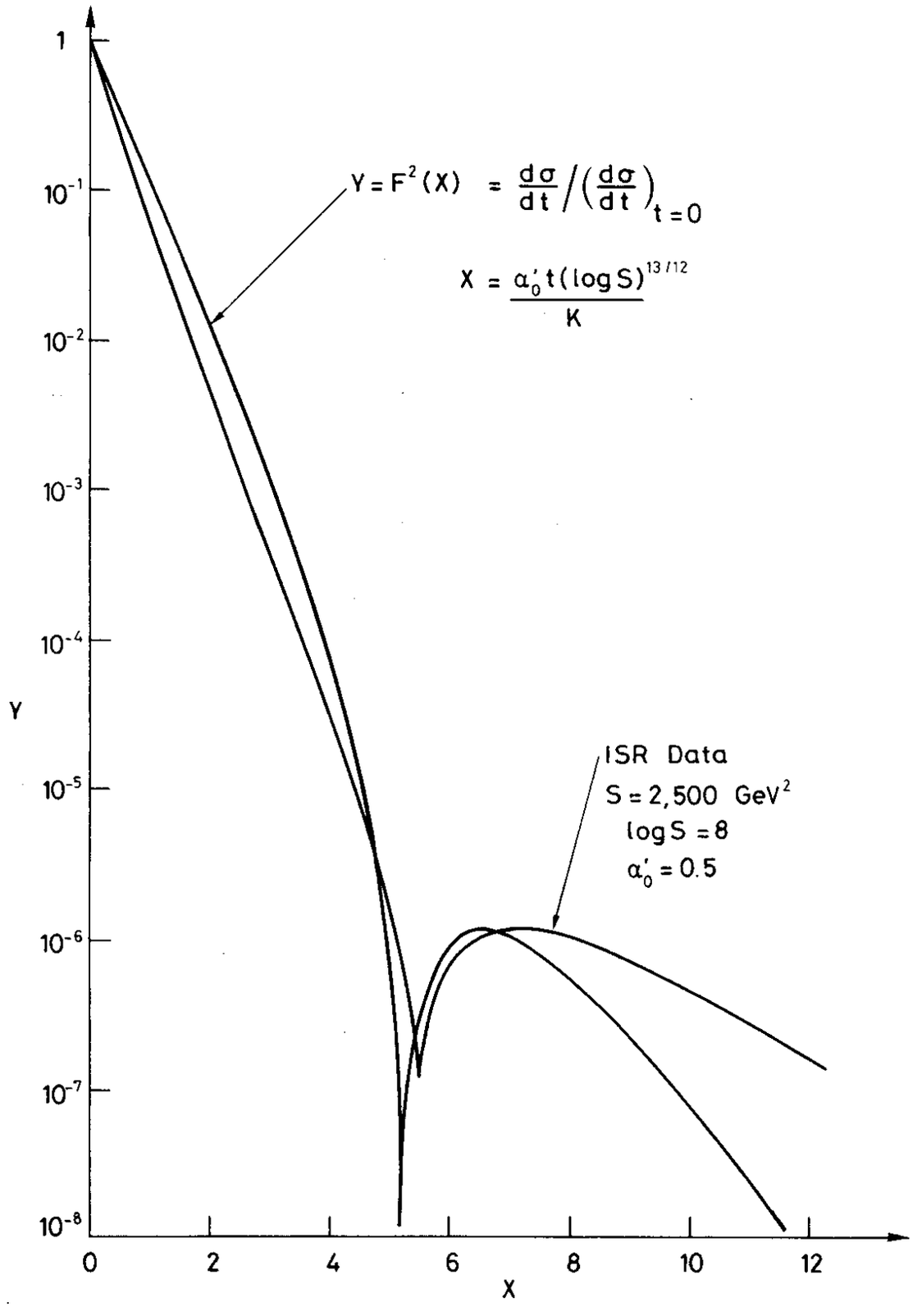

Fig. 3.3 Comparison of the scaling function for the diffraction peak with ISR data

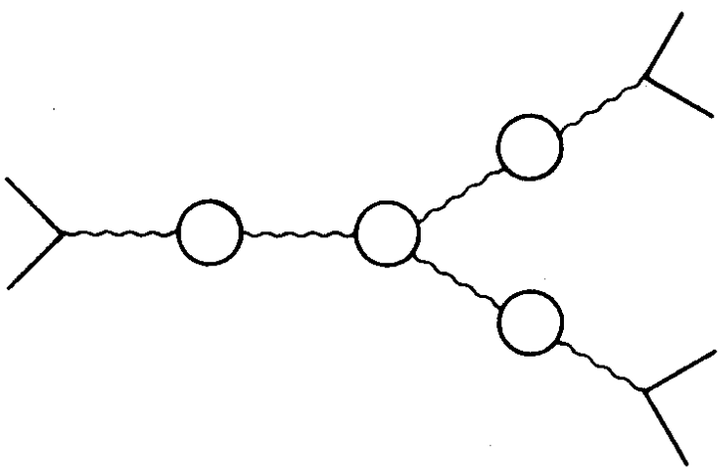

Fig. 3.4 The triple Pomeron contribution to the one-particle inclusive cross-section

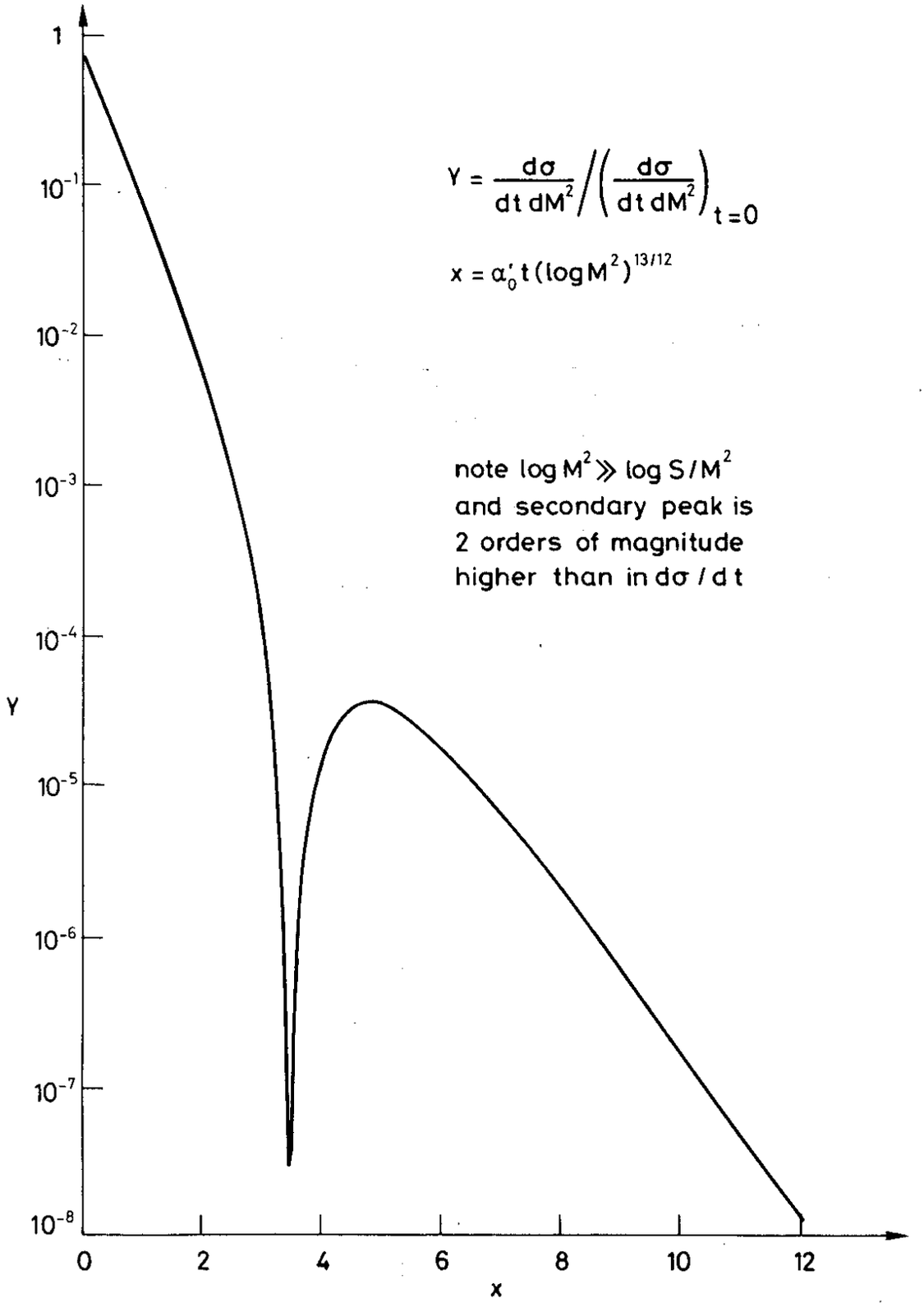

Fig. 3.5 The scaling function in that part of the triple Regge region where $\log M^2 >> \log S/M^2$

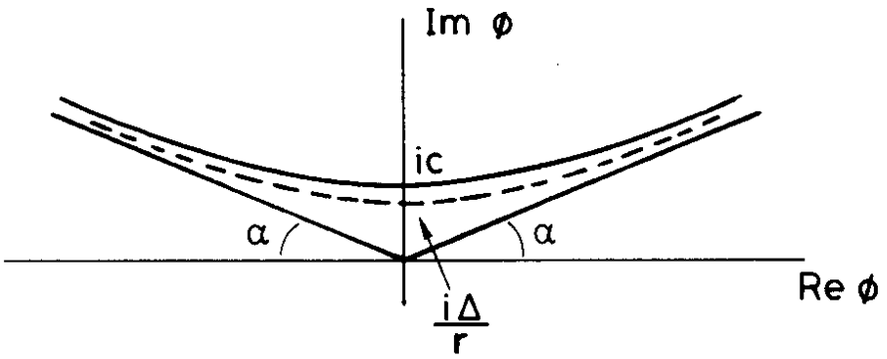

Fig. 4.1 The integration contour for the functional integral

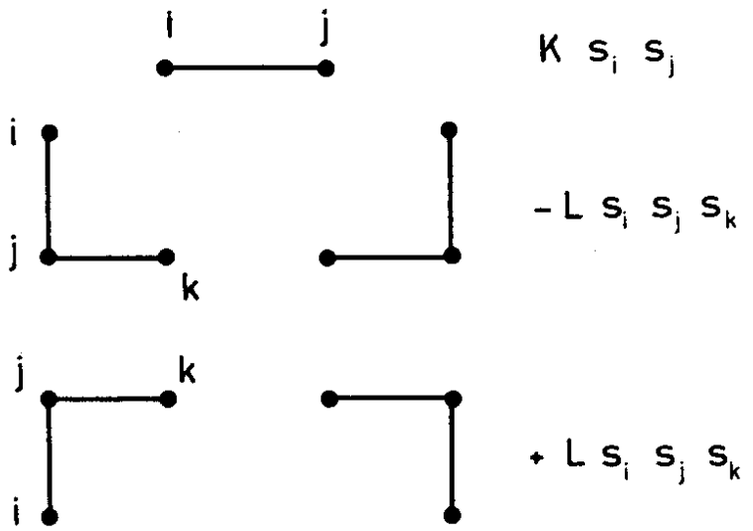

Fig. 4.2 The interactions in the analogue model

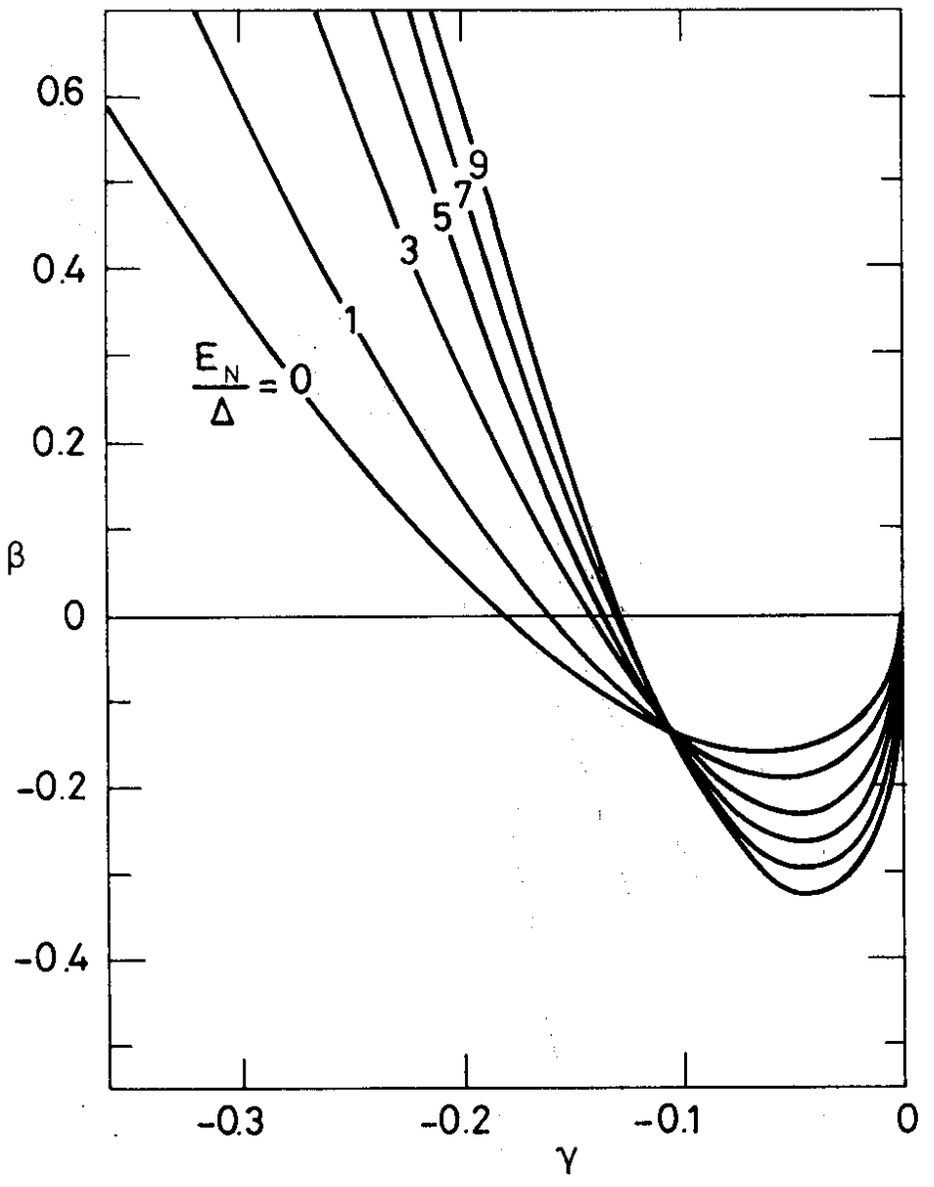

Fig. 5.1 The twisted fan for β(g) in lowest order

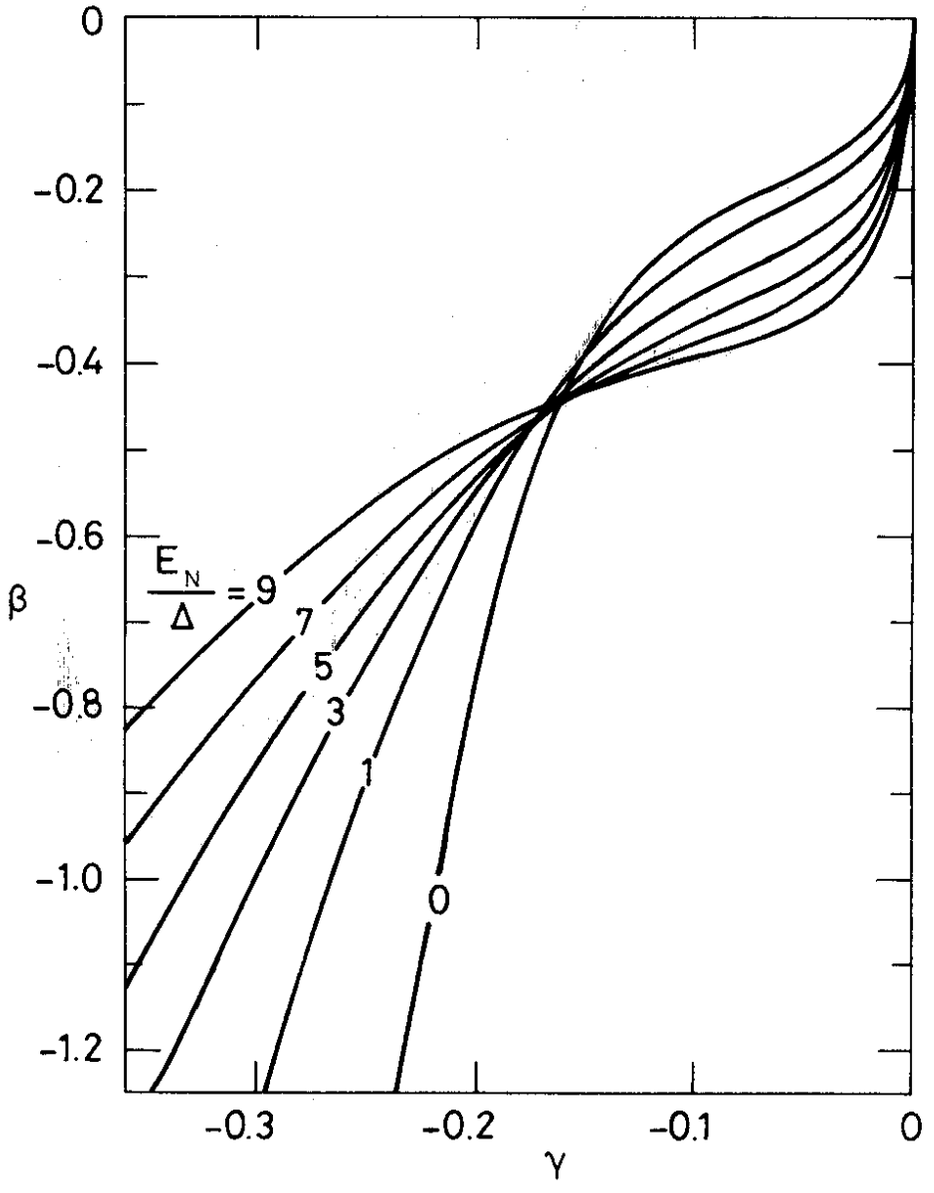

Fig. 5.2 The twisted fan for β(g) to second order

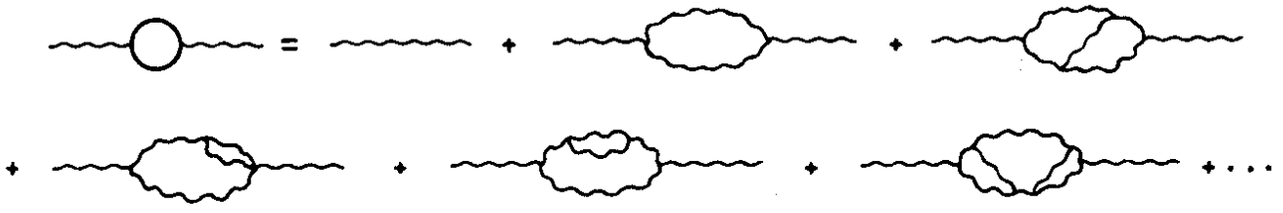

Fig. 7.1 The bare expansion for the Pomeron propagator

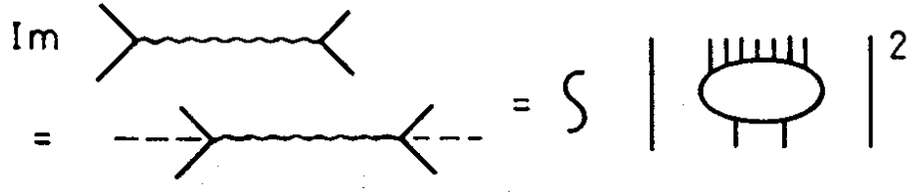

Fig. 7.2 Cutting the bare propagator

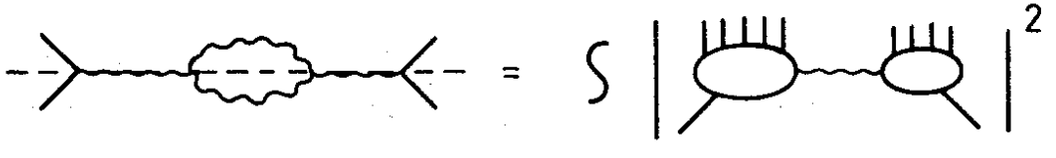

Fig. 7.3 Cutting the first Pomeron interaction diagram

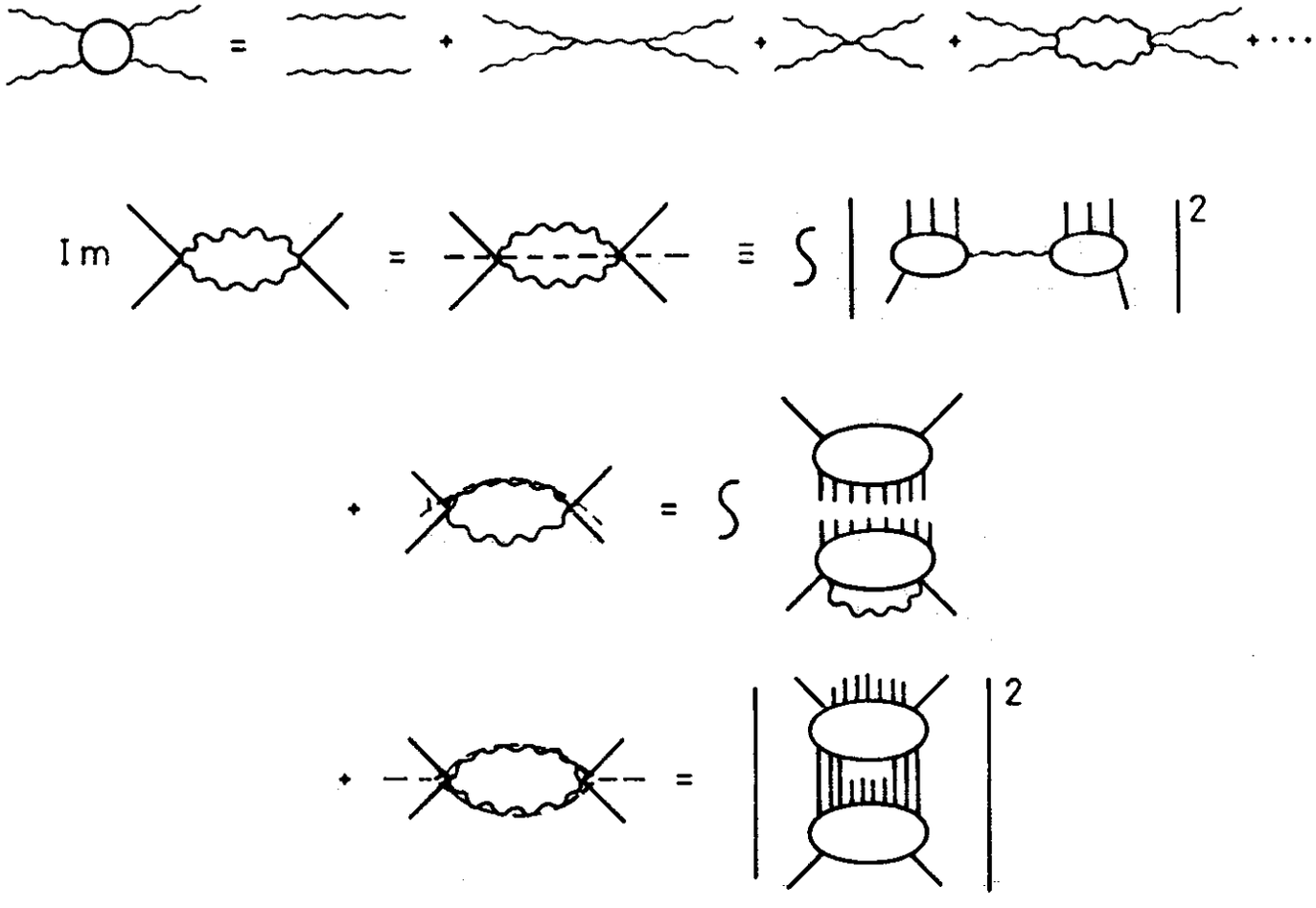

Fig. 7.4 The bare expansion of the four Pomeron Green's function, and the cutting of the first term

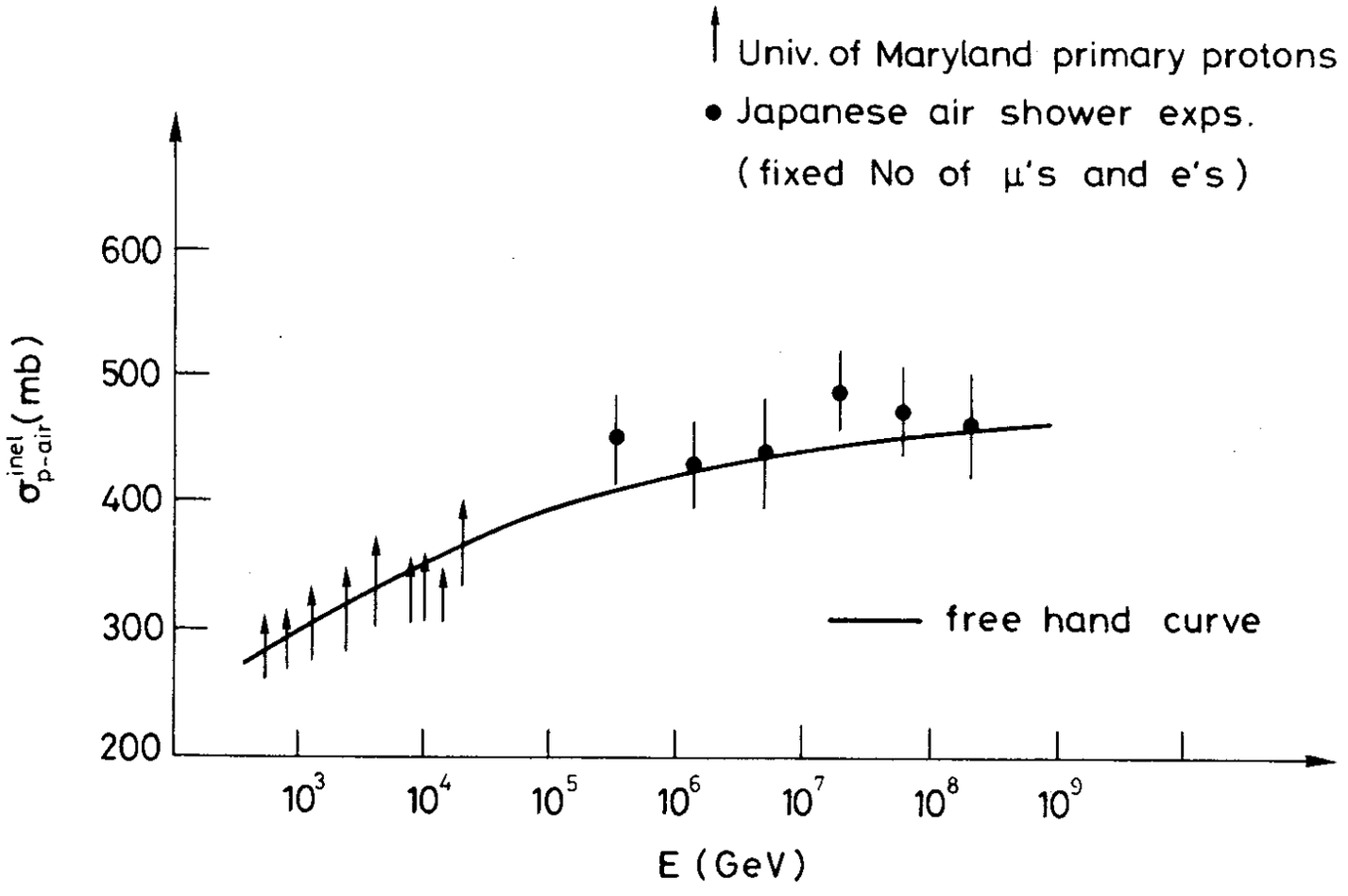

Fig. 7.5 The inelastic cross-section for protons on air