ON OBSERVABILITY OF SIGNAL OVER BACKGROUND

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Abstract
Several criteria used by physicists to quantify the ratio of signal to background in planned experiments are compared. An equal probabilities test is proposed for the evaluation of the uncertainty in planned search experiments. This estimation is used for the determination of the exclusion limits in prospective studies of searches. We also consider a probability of discovery as a quantity for comparison of proposals for future search experiments.

1. INTRODUCTION

The aim of a search experiment is to detect an expected new phenomenon. Usually, the theoretical estimations of expected mean number of signal events of a new phenomenon N_s and that of background events N_b are known, and we can define some value of "significance" as a characteristic of the observability of the phenomenon. Some function of the observed number of events x (a statistic) is used to draw a conclusion on observation or non-observation of the phenomenon. The value of this statistic allows one to find the degree of confidence of the conclusion. There exist two types of mistake: to state that a phenomenon does not exist while in fact it exists (Type I error), or to state that a phenomenon exists while it does not (Type II error).

In this paper we compare three "signal significances" S which are suitable to describe the discovery potential of a future experiment:

– "significance" $S_1 = \frac{N_s}{\sqrt{N_b}}$ [1],
– "significance" $S_2 = \frac{N_s}{\sqrt{N_s+N_b}}$ [2, 3],
– "significance" $S_{12} = \sqrt{N_s + N_b} - \sqrt{N_b}$ [4].

For this purpose we apply an equal-tailed test to study the behaviour of Type I and Type II errors as a function of N_s and N_b in planned search experiments with specified values of the "significances" $S_1,\ S_2$ and S_{12}. An equal probabilities test is proposed to estimate the uncertainty in separation of two hypotheses on observability of predicted phenomenon in these experiments. The hypotheses testing results obtained by Monte-Carlo calculations are compared with the result obtained by the direct calculation of probability distributions. The equal probabilities test is used for the determination of exclusion limits in prospective studies of searches.

2. NOTATIONS

Let us assume that the average number of signal events coming from a new phenomenon (N_s) and the average number of background events (N_b) in the experiment are given. We suppose that the events have a Poisson distribution with parameters N_s and N_b, i.e. the random variable $\xi \sim Pois(N_s)$ describes the signal events and the random variable $\eta \sim Pois(N_b)$ describes the background events. Assume that we observed x events – the realization of the process $X = \xi + \eta$ (x is the sum of signal and background events in the experiment). Here N_s, N_b are non-negative real numbers and x is an integer. The classical frequentist methods of testing a precise hypothesis allow one to construct a rejection region and determine associated error probabilities for the following "simple" hypotheses:

$H_0: \ X \sim Pois(N_s+N_b)$ versus $H_1: \ X \sim Pois(N_b)$, where $Pois(N_s+N_b)$ and $Pois(N_b)$ have the probability distributions

$$f_0(x) = \frac{(N_s+N_b)^x}{x!} e^{-(N_s+N_b)}$$

for the case of presence, and

$$f_1(x) = \frac{(N_b)^x}{x!} e^{-(N_b)}$$

for the case of absence of signal events in the whole population.

The probability distributions $f_0(x)$ (a) and $f_1(x)$ (b) for the case of $N_s + N_b = 104$ and $N_b = 53$ ([3], Table.13, cut 6) are shown in Fig. 1. As we see, there is an intersection of these distributions. Let us denote the threshold (critical value) that divides the abscissa in Fig. 1 into the rejection region and the area of accepted hypothesis H_0 by N_{ev}. The incorrect rejection of the null hypothesis H_0, the Type I error (a phenomenon is taken to be absent, while it exists), has the probability $\alpha = \sum_{x=0}^{N_{ev}} f_0(x)$, and the incorrect acceptance of H_0, the Type II error (a phenomenon is taken to be present, while it is absent), has the probability $\beta = \sum_{x=N_{ev}+1}^{\infty} f_1(x)$. The α and β dependences on the value of N_{ev} for the above example are presented in Fig. 2.

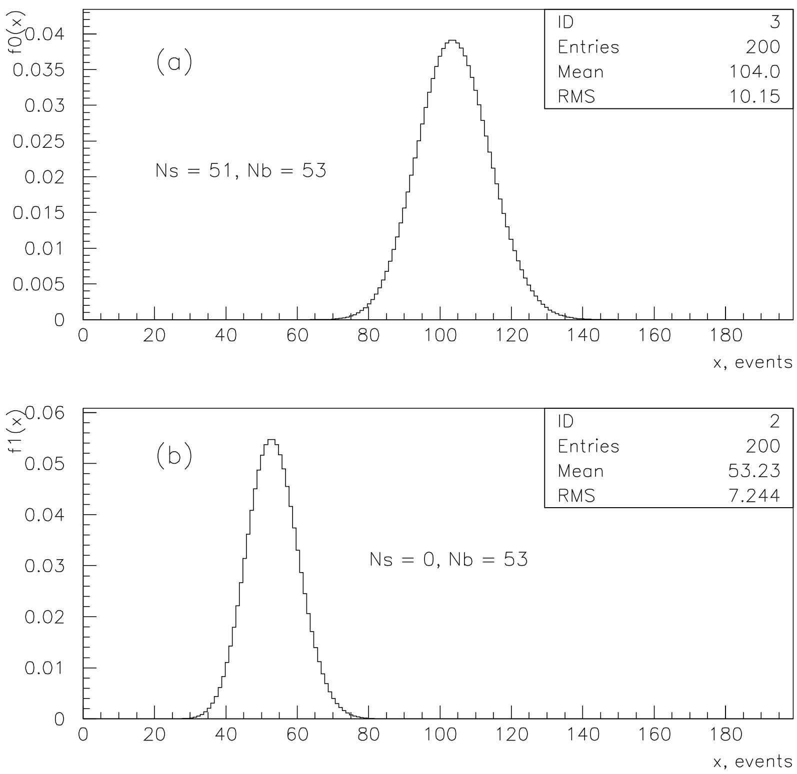

Fig. 1: The probability distributions $f_0(x)$ (a) and $f_1(x)$ (b) for the case of 51 signal events and 53 background events obtained by direct calculations of the probabilities.

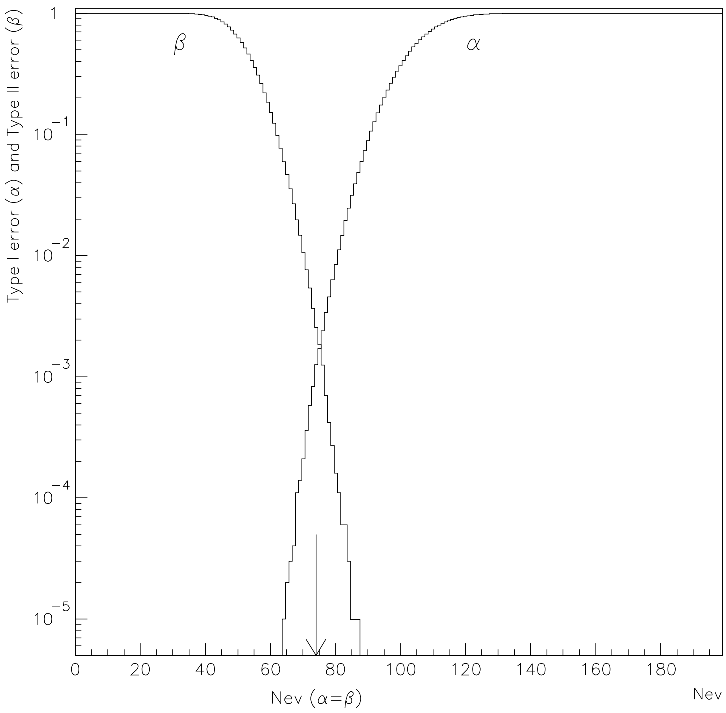

Fig. 2: The dependence of Type I α and Type II β errors on critical value N_{ev} for the case of 51 signal events and 53 background events.

3. HYPOTHESES TESTING

In this Section the construction of a rejection region for the statistic x, the number of observed events, is described. The decision to either reject or accept H_0 will depend on the observed value of x, where small values of x correspond to the rejection of H_0, i.e.

if $x \leq N_{ev}$, reject H_0,

if $x > N_{ev}$, accept H_0.

In compliance with this test, the frequentist reports the Type I and Type II error probabilities as $\alpha = P_0(X \leq N_{ev}) \equiv F_0(N_{ev})$ and $\beta = P_1(X > N_{ev}) \equiv 1 - F_1(N_{ev})$, where F_0 and F_1 are cumulative distribution functions of X under H_0 and H_1, respectively.

The Type I error α is also called a significance level of the test. The value of β is meaningful only when it is related to the alternative hypothesis H_1. The dependence $1 - \beta$ is referred to as a power function that allows one to choose a favoured statistic for the hypothesis testing. It means that for the specified significance level we can determine the critical value N_{ev} and find the power $1 - \beta$ of this criterion. The larger the value of $1 - \beta$, the better the statistic separates hypotheses for a specified value of α.

For a conventional equal-tailed test [1] with $\alpha = \beta$, the critical value N_{ev} satisfies the relation $F_0(N_{ev}) \equiv 1 - F_1(N_{ev})$.

In a similar way we can construct the rejection region, finding the critical values c_1, c_2 and c_{12}, for the statistics $s_1 = \frac{x-N_b}{\sqrt{N_b}}$ ("significance" S_1), $s_2 = \frac{x-N_b}{\sqrt{x}}$ ("significance" S_2) and $s_{12} = \sqrt{x} - \sqrt{N_b}$ ("significance" S_{12}).

[1] See e.g. [5].

The probability distributions of statistics under consideration can be obtained in analytical form or by a Monte-Carlo simulation of a large number of experiments (see as an example [6]) for the given values N_s and N_b. Both approaches were used in our study. The probability distributions for the case of $N_s + N_b = 104$ and $N_b = 53$ events obtained as a result of 10^5 simulations with random variables ξ and η are shown in Fig. 3. There is no significant difference between these distributions compared with the distributions resulting from direct calculation of the probabilities (Fig. 1).

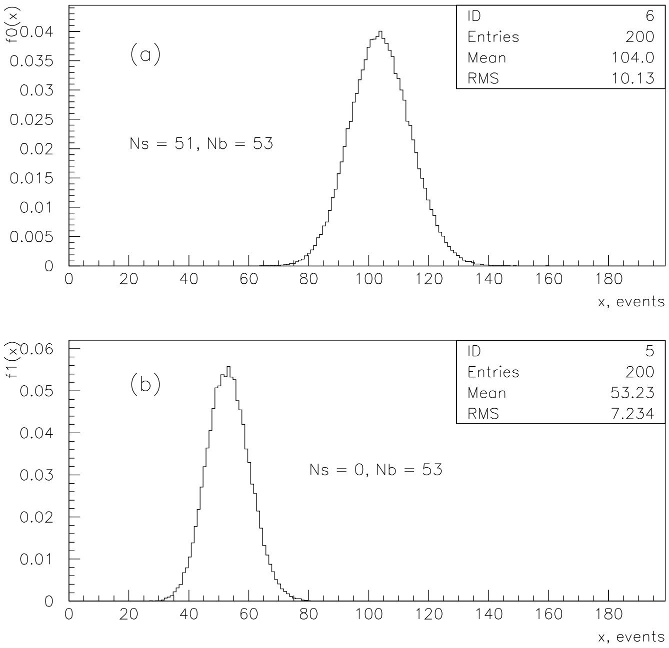

Fig. 3: The probability distributions $f_0(x)$ (a) and $f_1(x)$ (b) for the case of 51 signal events and 53 background events obtained by simulation (10^5 Monte-Carlo trials).

The probability distributions of statistic s_2 for the case of $N_s = 51$, $N_b = 53$ (a) and the case of $N_s = 0$, $N_b = 53$ (b) are shown in Fig. 4. The behaviour of probabilities α and β as a function of the critical value c_2 for the statistic s_2 is also presented in Fig. 4(c).

We stress that the second approach allows one to construct the probability distributions and, correspondingly, the acceptance and the rejection regions for complicated statistics, taking into account the systematic errors and the uncertainties in the estimations of N_b and N_s.

4. EQUAL-TAILED TEST

What is the exact meaning of the statement that

$S_1 = \frac{N_s}{\sqrt{N_b}} = 5$ or $S_2 = \frac{N_s}{\sqrt{N_s+N_b}} = 5$?

Tables 1 and 2 give the answer to this question. Here the values α and β have been determined by applying equal-tailed test (in this study we use the conditions $min(\beta - \alpha)$ and $\alpha \leq \beta$). One can see the dependence of α (or β) on the value of N_s and N_b. The case of $N_s = 5$ and $N_b = 1$ for S_1 (Fig. 5) is perhaps the most dramatic example. Having 5σ deviation and rejecting the hypothesis H_0, we are mistaken in 6.2% of the cases; if we accept the hypothesis H_0, we are mistaken in 8.0% of the cases.

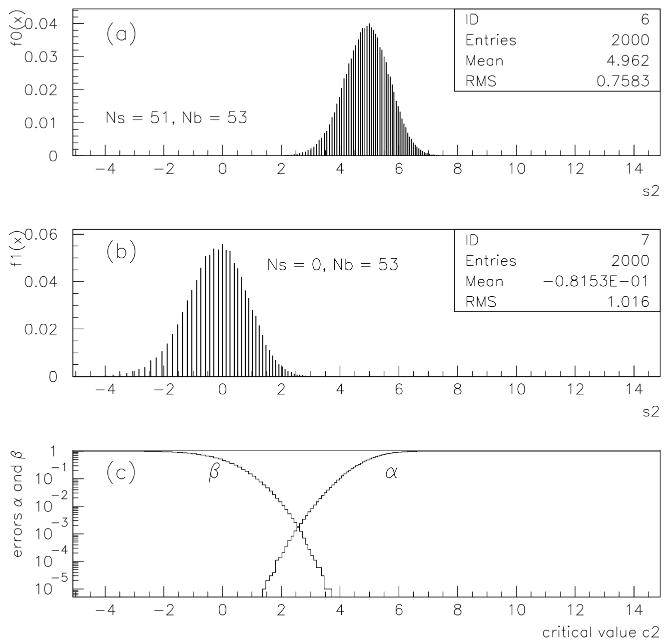

Fig. 4: The probability distributions $f_0(x)$ (a) and $f_1(x)$ (b) for statistic s_2. The dependence of Type I and Type II errors on the critical value c_2 (c) for the case of 51 signal events and 53 background events.

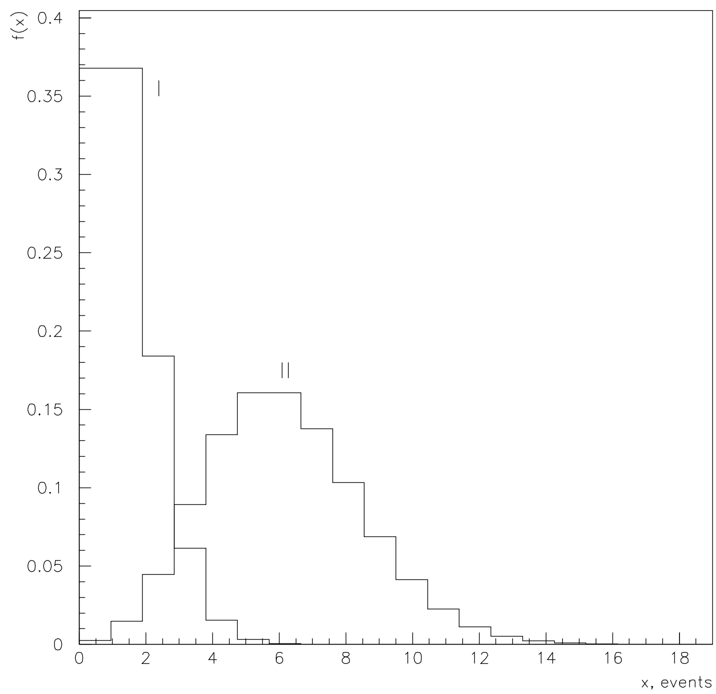

Fig. 5: The probability distributions $f_0(x)$ (II) and $f_1(x)$ (I) of statistic s_1 for the case of 5 signal events and 1 background events.

One can point out that the values of α and β for S_1 and S_2 converge when we increase the number of events. It means that, for a sufficiently large value of N_b, the values of α and β obtained by equal-tailed tests have a constant value close to 0.0062 for both S_1 and S_2. The standard deviation tends to be unity both for the distribution of s_1 (Fig. 6) and for the distribution of s_2, i.e. these distributions in case of large N_b and N_s can be approximated by a standard Gaussian function $\mathcal{N}(0,1)$ [2] for a pure background and by a Gaussian function $\mathcal{N}(5,1)$ for a signal mixed with a background. Therefore, the equal-tailed test for normal distributions gives the critical value $c_1 = 2.5$ and $\alpha = \beta = 0.0062$. These are the limiting values of α and β for the requirement $S_1 = 5$, or $S_2 = 5$, or $S_{12} = 2.5$.

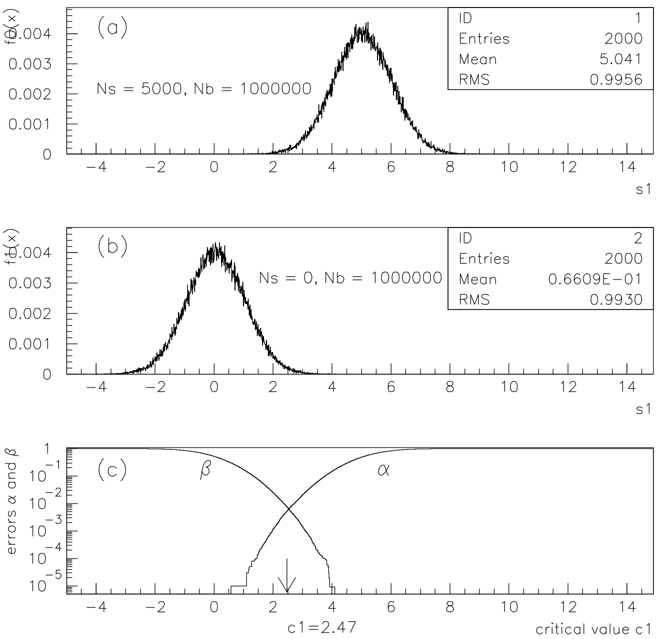

Fig. 6: The probability distributions $f_0(x)$ (a) and $f_1(x)$ (b) of statistic s_1. The dependence of Type I and Type II errors on the critical value c_1 (c) for the case of 5000 signal events and 10^6 background events.

In a similar way we can determine the Type I and Type II errors for small values N_s and N_b and predict the limiting values of α and β for a large number of events for other statements about "significance" S_1 (Table 3) or any other estimator.

5. EQUAL PROBABILITIES TEST

The last columns in Tables 1, 2 and 3 contain the values of probability κ [4] which is a characteristic of the observability of a phenomenon in future experiments with given N_s and N_b. In particular, it is the fraction of probability distribution $f_0(x)$ for a statistic x that can be described by the fluctuation of the background. The value of κ is equal to the area of the overlapping probability distributions $f_0(x)$ and $f_1(x)$ (Fig. 1). If we superimpose the distributions $f_0(x)$ and $f_1(x)$ and choose the intersection point ($N_{ev} = [\frac{N_s}{ln(1+\frac{N_s}{N_b})}]$) as a critical value for the hypotheses testing, we obtain $\kappa \equiv \alpha + \beta$. In this point $f_0(N_{ev}) = f_1(N_{ev})$ (in our case conditions $min(f_0(N_{ev}) - f_1(N_{ev}))$ and $f_1(N_{ev}) \leq f_0(N_{ev})$ are used). Hence this kind of check can be called an equal probabilities test. If κ equals to 1 a phenomenon will never be found in the experiment, if κ equals to 0 the first measurement with probability one has to

[2] $\mathcal{N}$(mean,variance) is a traditional notation for normal distribution.

answer the question about presence or absence of new phenomenon (this case is not realized for Poisson distribution). The dependences of κ on the number of signal events for the criteria $S_1 = 5,\ S_2 = 5$ and $S_{12} = 2.5$ are shown in Fig. 7. Correspondingly, the dependences of N_b versus N_s for these criteria are presented in Fig. 8.

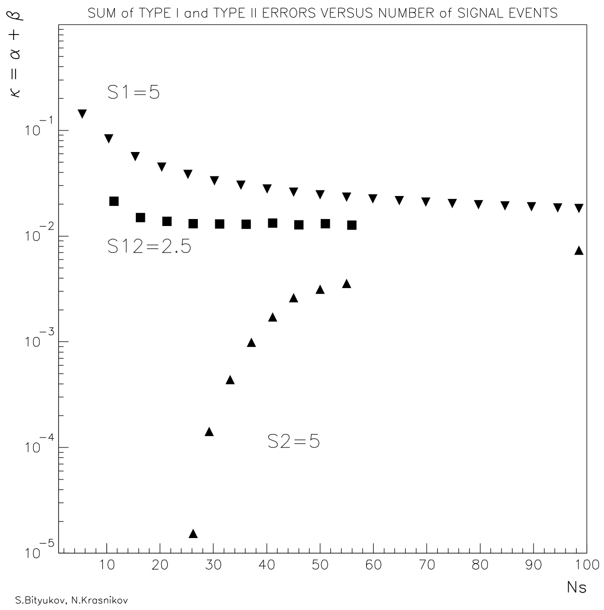

Fig. 7: The dependences of κ on the number of signal events for "significances" $S_1 = 5,\ S_2 = 5$ and $S_{12} = 2.5$.

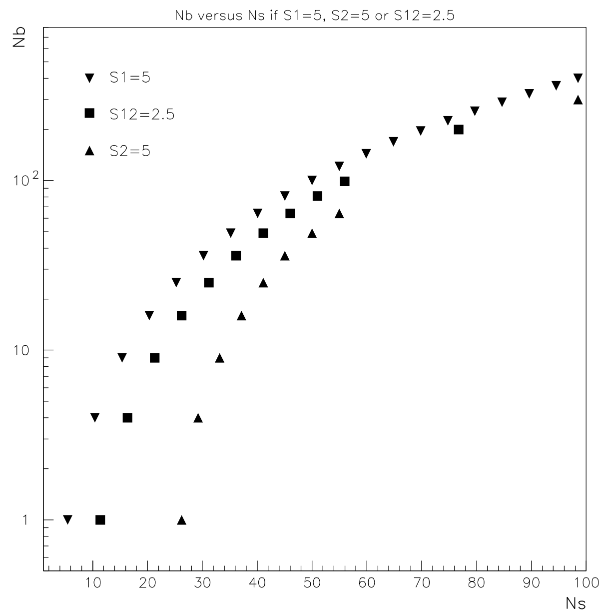

Fig. 8: The dependences of the number of background events on the number of signal events for "significances" $S_1 = 5,\ S_2 = 5$ and $S_{12} = 2.5$.

Note that the equal probabilities test can be applied for probability distributions with several points of intersection (Fig. 9). The relative uncertainty of the observability of a new phenomenon in a future experiment $\tilde{\kappa}$ is equal to $\frac{\kappa}{2-\kappa}$.

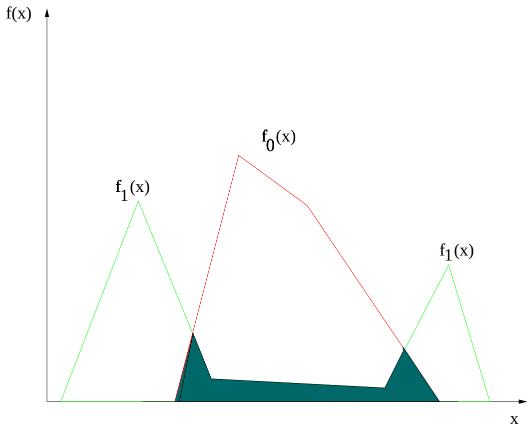

Fig. 9: The estimation of uncertainty in hypotheses testing for arbitrary distributions by using of equal probabilities test.

As is seen from Tables 1, 2 and 3, the value of κ is also close to the sum of $\alpha + \beta$ determined by using the equal-tailed test. Clearly, the accuracy of the determination of κ by Monte-Carlo calculations depends on the number of trials made. Fig. 10 shows the distribution of 40 estimations of the $\alpha + \beta$ for the case $N_s = 100$, $N_b = 500$ and for the 10^5 Monte-Carlo trials in each estimation. The result obtained by the direct calculation of the probability distributions is also given in the Fig. 10.

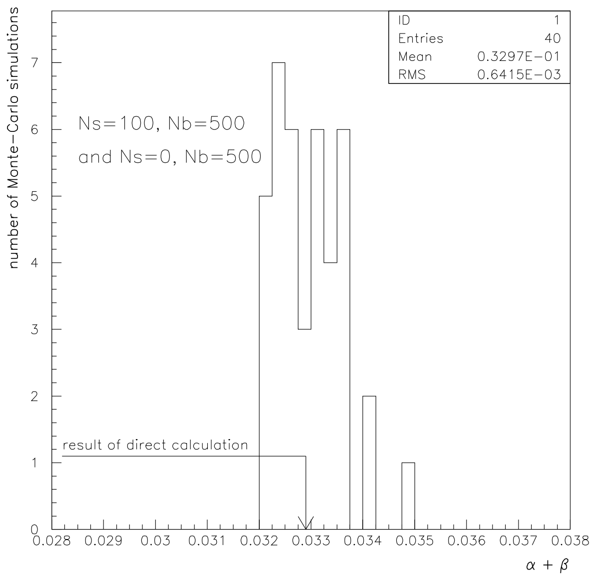

Fig. 10: The variation of $\alpha + \beta$ in the equal-tailed hypotheses testing ($N_s = 100$, $N_b = 500$ versus $N_s = 0$, $N_b = 500$ in 40 Monte-Carlo simulations of probability distributions).

6. ESTIMATION OF EXCLUSION LIMITS ON NEW PHYSICS

Suppose we know the background cross section σ_b and we want to obtain bound on signal cross section σ_s which depends on some parameters (masses of new particles, coupling constants, ...) and describes some new physics beyond standard model. We have to compare two Poisson distributions with and without new physics. The results of Section 5 are trivially generalized to the case of the estimation of exclusion limits on signal cross section and, hence, on parameters (masses, coupling constants, ...) of new physics.

Consider at first the case when the Gaussian distributions approach the Poisson distributions ($N_b \gg 1$). As it has been mentioned in Section 5 the common area of probability distributions with background events and with background plus signal events is the probability that "new physics" can not be described by the "standard physics". For instance, when we require the probability that "new physics" can be described by the "standard physics" is more or equal 10% (i.e. S_{12} is larger than 1.64) it means that the formula

$$\sqrt{N_b + N_s} - \sqrt{N_b} \leq 1.64 \tag{1}$$

gives us 90% exclusion limit on the average number of signal events N_s. In general case when we require the probability that "new physics" can be described by the "standard physics" is more or equal to ϵ the formula

$$\sqrt{N_b + N_s} - \sqrt{N_b} \leq S(\epsilon) \tag{2}$$

allows us to obtain $1 - \epsilon$ exclusion limit on signal cross section. Here $S(\epsilon)$ is determined by the κ [3], i.e. we suppose that $\epsilon = \kappa$. It should be stressed that in fact the requirement that "new physics" with the probability more or equal to ϵ can be described by the "standard physics" is our definition of the exclusion limit as $(1 - \epsilon)$ probability for signal cross section. From the last formula we find that

$$\sigma_s \leq \frac{S^2(\epsilon)}{L} + 2S(\epsilon)\sqrt{\frac{\sigma_b}{L}}. \tag{3}$$

Here $N_b = \sigma_b L,\ N_s = \sigma_s L$, where L is integrated luminosity.

For the case of not large values of N_b and N_s we have to compare the Poisson distributions directly and the corresponding method has been formulated in Section 5.

In refs.[7, 8] different methods to derive exclusion limits in future experiments have been suggested. As is seen from Fig. 11 the essential differences in values of the exclusion limits take place. Let us compare these methods by the use of the equal probabilities test. In order to estimate the various approaches of the exclusion limit determination we suppose that new physics exists, i.e. the value N_s equals to one of the exclusion limits from Fig. 11 and the value N_b equals to the corresponding value of expected background. Then we apply the equal probability test to find critical value N_{ev} for hypotheses testing in future measurements. Here a zero hypothesis is the statement that new physics exists and an alternative hypothesis is the statement that new physics is absent. After calculation of the Type I error α (the probability that the number of observed events will be equal or less than the critical value N_{ev}) and the Type II error β (the probability that the number of observed events will be more than the critical value N_{ev} in the case of the absence of new physics) we can compare the methods. In Table 4 the result of the comparison is shown. As is seen from this Table the "Typical experiment" approach [8] gives too small values of exclusion limit. The difference in the 90% CL definition is the main reason of the difference between our result and the exclusion limit from ref. [7]. We require that $\epsilon = \kappa$. In ref [7] the criterion for determination exclusion limits: $\beta < \Delta$ and $\frac{\alpha}{1-\beta} < \epsilon$ is used, i.e. the experiment will observe with

[3] Note that $S(1\%) = 2.57$, $S(2\%) = 2.33$, $S(5\%) = 1.96$ and $S(10\%) = 1.64$

probability at least $1-\Delta$ at most a number of events such that the limit obtained at the $1-\epsilon$ confidence level excludes the corresponding signal [4].

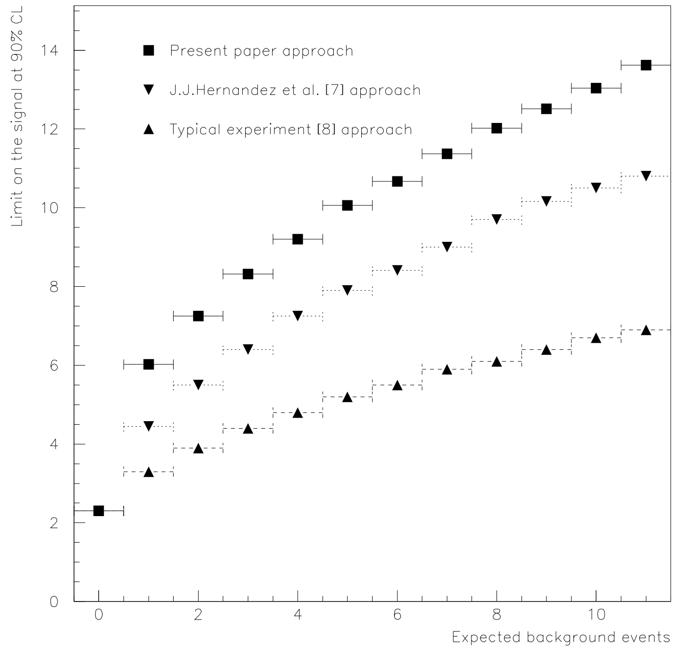

Fig. 11: Estimations of the 90% CL upper limit on the signal in a future experiment as a function of the expected background. The method proposed in ref. [8] gives the values of exclusion limit close to ”Typical experiment” approach.

7. THE PROBABILITY OF NEW PHYSICS DISCOVERY

It is also very important to determine the probability of new physics discovery in future experiment. According to common definition (for example,[9, 10]) the new physics discovery corresponds to the case when the probability that background can imitate signal is less than 5σ or in terms of the probability less than $5.7 \cdot 10^{-7}$ (here of course we neglect any possible systematic errors).

So we require that the probability $\beta(\Delta)$ of the background fluctuations for $n > n_0(\Delta)$ is less than Δ, namely

$$\beta(\Delta) = \sum_{n=n_0(\Delta)+1}^{\infty} P(N_b, n) \leq \Delta \tag{4}$$

The probability $1-\alpha(\Delta)$ that the number of signal events will be bigger than $n_0(\Delta)$ is equal to

$$1-\alpha(\Delta) = \sum_{n=n_0(\Delta)+1}^{\infty} P(N_b + N_s, n) \tag{5}$$

It should be stressed that Δ is a given number and $\alpha(\Delta)$ is a function of Δ. Usually physicists claim the discovery of phenomenon [9, 10] if the probability of the background fluctuation is less than 5σ that corresponds to $\Delta_{dis} = 5.7 \cdot 10^{-7}$ [5]. So from the equation (4) we find $n_0(\Delta)$ and estimate the probability $1-\alpha(\Delta)$ that an experiment will satisfy the discovery criterion.

[4]If we define ϵ as normalized κ ($\epsilon = \tilde{\kappa} = \frac{\kappa}{2-\kappa}$) we have the result close to ref. [7], i.e., for example, $\kappa = 0.17$ corresponds to $\epsilon = 0.0929$.

[5]The approximation of Poisson distribution by Gaussian for tails with area close to or less than Δ_{dis} for values of N_s and N_b under consideration gives strong distinction in determination of $1-\alpha$.

As an example consider the search for standard Higgs boson with a mass $m_h = 110\ GeV$ at the CMS detector. For total luminosity $L = 3 \cdot 10^4 pb^{-1} (2 \cdot 10^4 pb^{-1})$ one can find [10] that $N_b = 2893(1929), N_s = 357(238)$, $S_1 = \frac{N_s}{\sqrt{N_b}} = 6.6(5.4)$. Using the formulae (4, 5) for $\Delta_{dis} = 5.7 \cdot 10^{-7}$ (5σ discovery criterion) we find that $1 - \alpha(\Delta_{dis}) = 0.96(0.73)$. It means that for total luminosity $L = 3 \cdot 10^4 pb^{-1} (2 \cdot 10^4 pb^{-1})$ the CMS experiment will discover at $\geq 5\sigma$ level standard Higgs boson with a mass $m_h = 110\ GeV$ with a probability 96(73) percent.

8. CONCLUSION

In this paper the discussion on the observation of new phenomena is restricted to the testing of simple hypotheses in the case of predicted values N_s and N_b and an observable value x. As is stressed in [5], the precise hypothesis testing should not be done by forming a traditional confidence interval and simply checking whether or not the precise hypothesis is compatible with the confidence interval. A confidence interval is usually of considerable importance in determining where the unknown parameter is likely to be, given that the alternative hypothesis is true, but it is not useful in determining whether or not a precise null hypothesis is true.

To compare several criteria used for the hypotheses testing, we employ both a method that allows one to construct the rejection regions via the determination the probability distributions of these statistics by Monte-Carlo calculations and direct calculations of probabilities distributions. An equal-tailed test was used to compare the criteria. An equal probabilities test is proposed to estimate the uncertainty in separating two hypotheses about observability of predicted phenomenon in a planned experiment. This estimation is used for determination of exclusion limits in prospective studies of searches. The method has been used to draw a conclusion on the observability of some predicted phenomena [4]. We also considered a probability of discovery as a quantity for comparison of proposals for future search experiments.

Acknowledgements

We would like to thank the Workshop co-convenors Fred James and Louis Lyons and local organizer Yves Perrin. We are greatly indebted to M.Dittmar for useful discussions which were one of the motivations to perform this study. We are grateful to V.Genchev, V.A.Matveev, V.F.Obraztsov, V.L.Solovianov and Yu.P.Gouz for the interest and valuable comments. This work has been supported by RFFI grants 99-02-16956 and 99-01-00091.

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Table 1: The dependence of α and β determined by using the equal-tailed test on N_s and N_b for $S_1 = 5$; κ is the area of intersection of probability distributions $f_0(x)$ and $f_1(x)$.

N_s	N_b	α	β	κ
5	1	0.0620	0.0803	0.1423
10	4	0.0316	0.0511	0.0828
15	9	0.0198	0.0415	0.0564
20	16	0.0141	0.0367	0.0448
25	25	0.0162	0.0225	0.0383
30	36	0.0125	0.0225	0.0333
35	49	0.0139	0.0164	0.0303
40	64	0.0114	0.0171	0.0278
45	81	0.0124	0.0136	0.0260
50	100	0.0106	0.0143	0.0245
55	121	0.0114	0.0120	0.0234
60	144	0.0100	0.0126	0.0224
65	169	0.0106	0.0109	0.0216
70	196	0.0095	0.0115	0.0209
75	225	0.0101	0.0102	0.0203
80	256	0.0091	0.0107	0.0198
85	289	0.0096	0.0097	0.0193
90	324	0.0088	0.0101	0.0189
95	361	0.0081	0.0106	0.0185
100	400	0.0086	0.0097	0.0182
150	900	0.0078	0.0084	0.0162
500	10^4	0.0068	0.0068	0.0136
5000	10^6	0.0062	0.0065	0.0125

Table 2: The dependence of α and β determined by using the equal-tailed test on N_s and N_b for $S_2 \approx 5$. Here κ is the area of intersection of probability distributions $f_0(x)$ and $f_1(x)$.

N_s	N_b	α	β	κ
26	1	$0.519 \cdot 10^{-5}$	$0.102 \cdot 10^{-4}$	$0.154 \cdot 10^{-4}$
29	4	$0.661 \cdot 10^{-4}$	$0.764 \cdot 10^{-4}$	$0.142 \cdot 10^{-3}$
33	9	$0.127 \cdot 10^{-3}$	$0.439 \cdot 10^{-3}$	$0.440 \cdot 10^{-3}$
37	16	$0.426 \cdot 10^{-3}$	$0.567 \cdot 10^{-3}$	$0.993 \cdot 10^{-3}$
41	25	$0.648 \cdot 10^{-3}$	$0.118 \cdot 10^{-2}$	$0.172 \cdot 10^{-2}$
45	36	$0.929 \cdot 10^{-3}$	$0.193 \cdot 10^{-2}$	$0.262 \cdot 10^{-2}$
50	49	$0.133 \cdot 10^{-2}$	$0.185 \cdot 10^{-2}$	$0.314 \cdot 10^{-2}$
55	64	$0.178 \cdot 10^{-2}$	$0.179 \cdot 10^{-2}$	$0.357 \cdot 10^{-2}$
100	300	$0.317 \cdot 10^{-2}$	$0.428 \cdot 10^{-2}$	$0.735 \cdot 10^{-2}$
150	750	$0.445 \cdot 10^{-2}$	$0.450 \cdot 10^{-2}$	$0.894 \cdot 10^{-2}$

Table 3: The dependence of α and β determined by using equal-tailed test on N_s and N_b for $S_1 = 2$, $S_1 = 3$, $S_1 = 4$, $S_1 = 6$ and $S_1 = 8$. Here κ is the area of intersection of probability distributions $f_0(x)$ and $f_1(x)$.

S_1	N_s	N_b	α	β	κ
2	2	1	0.199	0.265	0.4634
	4	4	0.192	0.216	0.4061
	6	9	0.184	0.199	0.3817
	8	16	0.179	0.188	0.3680
	∞	∞	0.1587	0.1587	0.3174
3	3	1	0.0906	0.263	0.3184
	6	4	0.0687	0.216	0.2408
	9	9	0.0917	0.123	0.2159
	12	16	0.0722	0.131	0.1952
	∞	∞	0.0668	0.0668	0.1336
4	4	1	0.0400	0.263	0.2050
	8	4	0.0459	0.110	0.1406
	12	9	0.0424	0.0735	0.1130
	16	16	0.0407	0.0572	0.0977
	∞	∞	0.0228	0.0228	0.0456
6	6	1	0.0301	0.0806	0.1008
	12	4	0.0217	0.0217	0.0434
	18	9	0.0089	0.0224	0.0271
	24	16	0.00751	0.0132	0.0198
	∞	∞	0.00135	0.00135	0.0027
8	8	1	0.0061	0.0822	0.0402
	16	4	0.0049	0.0081	0.0131
	24	9	0.0016	0.0052	0.00567
	32	16	0.00128	0.00237	0.00331
	∞	∞	0.000032	0.000032	0.000064

Table 4: The comparison of the different approaches to determination of the exclusion limits. The α and β are the Type I and Type II errors for the equal probability test. The κ equals to the sum of α and β.

	this paper				ref. [7]				ref. [8]			
N_b	N_s	α	β	κ	N_s	α	β	κ	N_s	α	β	κ
1	6.02	0.08	0.02	0.10	4.45	0.09	0.08	0.17	3.30	0.20	0.08	0.28
2	7.25	0.05	0.05	0.10	5.50	0.13	0.05	0.18	3.90	0.16	0.14	0.30
3	8.32	0.07	0.03	0.10	6.40	0.09	0.08	0.18	4.40	0.14	0.18	0.32
4	9.20	0.05	0.05	0.10	7.25	0.13	0.05	0.18	4.80	0.23	0.11	0.34
5	10.06	0.07	0.03	0.10	7.90	0.10	0.07	0.17	5.20	0.20	0.13	0.34
6	10.67	0.06	0.04	0.10	8.41	0.09	0.08	0.18	5.50	0.19	0.15	0.34
7	11.37	0.05	0.05	0.10	9.00	0.08	0.10	0.18	5.90	0.17	0.17	0.34
8	12.02	0.07	0.03	0.10	9.70	0.10	0.06	0.17	6.10	0.17	0.18	0.35
9	12.51	0.06	0.04	0.10	10.16	0.09	0.07	0.17	6.40	0.16	0.20	0.36
10	13.04	0.05	0.05	0.10	10.50	0.09	0.08	0.17	6.70	0.22	0.14	0.36
11	13.62	0.04	0.06	0.10	10.80	0.08	0.09	0.18	6.90	0.21	0.15	0.36

APPENDIX

Let us try to generalize approach of the Section 5 to case when we have measurements.

We want to test the hypotheses: $h_0 : \ X \sim Pois(N_s + N_b)$ versus $h_1 : \ X \sim Pois(N_b)$. Denote N_s via s, N_b via b and the area of overlapping of probability distributions f_0 and f_1 via $\kappa(s|b)$. Assume that the result of experiment is x and we make decision about observation of Phenomenon in the case of two simple hypotheses. Also we may construct "a posteriori" probabilities of hypotheses h_0 and h_1 independent of decision . If likelihood functions are $L_0 = L(x|h_0)$ and $L_1 = L(x|h_1)$ then

$P(h_0|x) = \frac{L_0}{L_0+L_1}$ and $P(h_1|x) = \frac{L_1}{L_0+L_1}$.

It means that we associate for any pair of b and $s > 0$ the probability $P(s|x, b) = P(h_0|x)$.

In case of unpredicted value of s we must consider hypotheses

$H_0 : \ s > 0$ versus $H_1 : \ s = 0$

and we can determine "a posteriori" ("mean") uncertainty of hypothesis H_0

$\kappa(H_0|x, b) = \int_0^{\infty} P(s|x, b)\kappa(s|b)ds$.

Discussion after talk of Serguei Bityukov. Chairman: Wilbur Venus.

L. Lyons

Could you explain the motivation for your test statistic $\sqrt{S+B} - \sqrt{B}$?

S. Bityukov

The reason is that when we approximate the Poisson by Gaussian we analytically calculate the area of overlapping probability density for pure background and probability density for background plus signal. After that we derive this formula.

For example, let us draw two Poisson distributions with parameters $\mu_1 = N_b$ and $\mu_2 = N_s + N_b$. Let N_b be large enough to approximate these distributions by normal distributions $\mathcal{N}(\mu_1, \sigma_1)$ and $\mathcal{N}(\mu_2, \sigma_2)$, where $\sigma_1 = \sqrt{\mu_1}$ and $\sigma_2 = \sqrt{\mu_2}$. The transformation of the distributions to standard normal distribution (see Figure) and exploitation of the equalities

$$x_t = \frac{x_0 - N_b}{\sqrt{N_b}} = -\frac{x_0 - (N_s + N_b)}{\sqrt{N_s + N_b}}$$

allows one to find the points $x_0 = \sqrt{N_s + N_b}\sqrt{N_b}$ and, correspondingly, $x_t = \sqrt{N_s + N_b} - \sqrt{N_b}$. It allows us to use both the language of probability and the language of standard deviations. Note that in this approximation an equal-tailed test coincides with equal probabilities test.

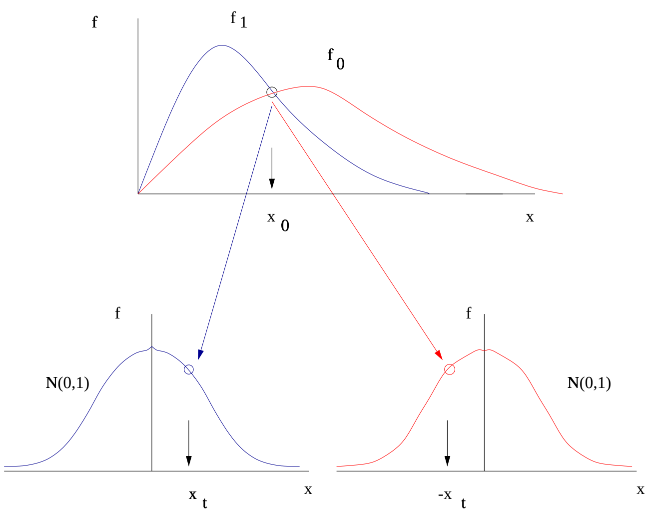

Fig. 12: A sketch of transformation of Poisson probability distributions to standard normal probability density function.

H. Prosper

Just a point of clarification. In your definition of κ which is equal to $\alpha + \beta$, in the $\alpha + \beta$ there is the number of events observed. How do you determine that or how do you get rid of the fact that you do not know the number of events observed. In your method, α and β are the sums of the Poisson distribution, but in the sum you start at some number and you go from N+1 to infinity; what determines the N in those sums?

S. Bityukov

We use an equal probabilities test to determine the uncertainty in future hypothesis testing about observability of the new phenomenon, which the planned experiment has before measurements (in the case of predicted numbers of signal and background events).

R. Cousins

Somebody did tell me about this paper, and the way they explained it to me it sounded very interesting. The idea was: Suppose you have a theory that predicts a certain amount of signal, and from your apparatus you predict how much background you're going to see, so a typical proposal will say: "For this much running we'll get a 3 sigma effect", but you're not taking into account the fact that your signal and background will fluctuate. As I understand it, this formula allows you to tell the program committee what the chance is you'll actually make the discovery of the signal the theory predicts, taking into account the fact that your experiment's going to be chosen from an ensemble of experiments, and you don't know which data you're going to get. So if the formula does that, then that's a really nice formula.

L. Lyons

Yes, some five sigmas are better than other five sigmas.