INTRODUCTION TO GAUGE FIELDS

V.N.Popov

Leningrad Branch of V.A.Steklov Mathematical Institute of the Academy of Sciences of the USSR

ABSTRACT

The aim of these lectures is to give an introduction to quantum field theory in the framework of the functional integration method. We outline a functional integral scheme of field quantization and its modification for systems with constaints. A general quantization scheme is developed. This scheme is applied to quantum electrodynamics, Yang-Mills fields, W S model . The role of anomalies in gauge theories is discussed briefly, as well as the problem of extended objects.

LECTURE 1.

Systems with constraints and their quantization

Field theory can be looked upon as an infinite-dimensional analog of a mechanical system. In such an approach the theory of gauge fields is an analog of mechanical systems with constraints [1, 2]

A classical action of the finite-dimensional system with constraints is equal to

$$S = \int \Big(\sum_{i=1}^{n} p_i \dot{q}^i - H(p,q) - \sum_{a=1}^{m} \lambda_a \varphi^a(p,q) \Big) dt \qquad (1.1)$$

It contains besides coordinates q and momenta p the variables λ_a, which come in linearly and play the role of Lagrange multipliers. The coefficients $\varphi^a(p,q)$ have the meaning of constraints. The variables p, q generate the phase space of dimension $2n$. The number of constraints shall be denoted as m. We suppose that $m < n$ and that the constraints φ^a and Hamiltonian H are in involution, i.e. that they fulfill the conditions

$$\{H, \varphi^a\} = \sum_b C^a_b \varphi^b \quad , \quad \{\varphi^a, \varphi^b\} = \sum_d C^{ab}_d \varphi^d \qquad (1.2)$$

Here C^a_b, C^{ab}_d are functions of p, q and $\{f, g\}$ is the Poisson bracket

$$\{f, g\} = \sum_{i=1}^{n} \Big(\frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i} - \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p^i} \Big) \qquad (1.3)$$

The system of equations of motion for action functional (1.1) contains besides canonical equations

$$\dot{q}^i = \frac{\partial H}{\partial p_i} + \sum_{a=1}^{m} \lambda_a \frac{\partial \varphi^a}{\partial p_i} \quad , \quad \dot{p}_i = -\frac{\partial H}{\partial q^i} - \sum_{a=1}^{m} \lambda_a \frac{\partial \varphi^a}{\partial q^i} \tag{1.4}$$

also the constraint equations

$$\varphi^a(p,q) = 0 \quad , \quad a = 1, \ldots, m \tag{1.5}$$

In principle, we can exlude some variables p, q using constraint equations. But in practice the solution of constraint equations often turns out to be rather difficult. So it is desirable to have a formalism where explicit solutions of constraint equations are not required.

Constraint equations (1.5) define the surface M of the 2n-m dimensions in phase space Γ. The involution conditions (1.2) guarantee, for arbitrary functions $\lambda_a(t)$ the fullfillment of constraint equations (1.5), provided those equations are satisfied for initial conditions. In other words, a trajectory which starts on the manifold M does not leave it.

We shall regard as observables on the manifold M the variables which are not influenced by arbitrariness in the choice of $\lambda_a(t)$. This requirement is fulfilled by the functions $f(p,q)$, which obey the conditions

$$\{f, \varphi^a\} = \sum_b d^a_b \varphi^b \tag{1.6}$$

Indeed, in the equations of motion for those functions

$$\dot{f} = \{H, f\} + \sum_a \lambda_a \{\varphi^a, f\} \tag{1.7}$$

the λ_a -depending terms vanish on M.

The function $f(p,q)$, defined on M and satisfying conditions (1.6) does not in fact depends on all variables. Conditions (1.6) can be looked upon as a system of m differential equations of the first order on M for which equations (1.2) are conditions of integrability. The function f is therefore unambiguously defined by its values on a submanifold of the systems initial conditions which has the dimension (2n-m)--m = 2(n-m). It is convenient to take as such a manifold a surface Γ^*,

defined by constraint equations (1.5) and m additional conditions

$$\chi_a(p,q) = 0 \quad , \quad a = 1, \ldots, m. \tag{1.8}$$

The function χ_a must satisfy the condition

$$\det \| \{ \chi_a, \varphi^\beta \} \| \neq 0 \tag{1.9}$$

because only in that case Γ^* can play the role of an initial surface for the equation (1.6). It is convenient to suppose that χ_a mutually commute

$$\{ \chi_a, \chi_\beta \} = 0 \tag{1.10}$$

We mean two functions f, g commute, if their Poisson bracket is equal to zero ($\{f,g\} = 0$). In such a case it is possible to introduce canonical variables onto the manifold Γ^* . Indeed, if condition (1.9) is satisfied, then using canonical transformation in Γ , we can introduce a new set of variables where χ_a take a simple form:

$$\chi_a(p,q) = p_a \quad , a = 1, \ldots, m, \tag{1.11}$$

where p_a ($a = 1, \ldots, m$) is a subset of canonical momenta of the new system of variables. Condition (1.9) can be written in terms of the new variables as

$$\det \| \frac{\partial \varphi^a}{\partial q^\beta} \| \neq 0 \tag{1.12}$$

and the constraint equation (1.6) can therefore be solved with respect to q^a. Finally, the surface Γ^* is given by the equations

$$p_a = 0 \quad , \quad q^a = q^a(p^*, q^*) \tag{1.13}$$

on Γ, so that p^* and q^* are independent variables on Γ^* .

Now let us discuss the quantization procedure for finite-dimensional systems in the functional integral framework. We begin with the one dimensional dynamical system with Hamiltonian $H(p,q)$. The principle of canonical quantization for such a system consists of replacing the coordinate q and momenta p by operators $\hat{q}$ and $\hat{p}$ according to the rule

$$q \to \hat{q} = q \quad , \quad p \to \hat{p} = -i\hbar \frac{\partial}{\partial q} \tag{1.14}$$

where $\hbar$ is the Planck constant. In the following we shall use the system of units with $\hbar = 1$. The operators act on Hilbert space of complex

functions $\psi(q)$.

The time evolution of a state is determined by the Heisenberg equation

$$i\frac{\partial\psi}{\partial t} = \hat{H}\psi, \tag{1.15}$$

where $\hat{H}$ is the energy operator, obtained from the classical Hamiltonian function $H(p,q)$ p and q with operators $\hat{p}$ and $\hat{q}$ according to (1.14). We can write down the formal solution of (1.15) as

$$\psi(t) = \hat{U}(t,t_0)\,\psi(t_0), \tag{1.16}$$

where the evolution operator

$$\hat{U}(t,t_0) = \exp\left(i(t_0 - t)\hat{H}\right) \tag{1.17}$$

is the exponential of the energy operator $\hat{H}$.

The method of functional integration allows us to express the matrix element of the evolution operator as a mean value of the expression

$$\exp i S[t_0, t] \tag{1.18}$$

over trajectories in the phase space where

$$S[t_0,t] = \int_{t_0}^{t}\left(p(\tau)\dot{q}(\tau) - H(p(\tau), q(\tau))\right)d\tau \tag{1.19}$$

is a classical action, corresponding to the trajectories $p(\tau)$, $q(\tau)$ $\left(t_0 \le \tau \le t\,,\ \dot{q}(\tau) = \frac{dq}{d\tau}\right)$.

The mean value over trajectories is called Feynman functional integral. Usially this is defined as a limit of finite dimensional integrals. We shall present here one of the possible definitions.

We devide the interval $[t_0, t]$ with $\tau_1, \ldots, \tau_{N-1}$ points into N equal parts. Let us consider the functions $p(\tau)$ defined on the interval which are constant on the intervals

$$[t_0,\tau_1)\,,\ (\tau_1,\tau_2)\,,\ \ldots,\ (\tau_{N-1}, t] \tag{1.20}$$

and the continious functions $q(\tau)$ linear on the intervals (1.20). We fix the values of the function $q(\tau)$ at the end points of the interval $[t_0, t]$, putting

$$q(t_0) = q_0\,, \quad q(t) = q \tag{1.21}$$

The trajectory $(p(\tau), q(\tau))$ is determined by values of the

piecewise linear function $q(\tau)$ in the points $\tau_1, \ldots, \tau_{N-1}$ (we denote them $q_1, \ldots, q_{N-1}$) and by values of the piecewise constant function $p(\tau)$ on intervals (τ_k, τ_{k+1}). We denote those by $p_1, \ldots p_N$.

Let us consider the finite dimensional integral

$$(2\pi)^{-N}\int dp_1 dq_1 \ldots dq_{N-1} dp_N \exp(iS[t_0,t]) \equiv J_N(q_0, q, t_0, t), \quad (1.22)$$

where $S[t_0,t]$ is the action (1.13) for the described trajectory $(p(\tau), q(\tau))$ defined by the parameters $p_1, \ldots, p_N, q_1, \ldots, q_{N-1}$. The basis assertion says that the limit of the integral (1.22) for $N \to \infty$ is equal to the matrix element of the evolution operator

$$\lim_{N\to\infty} J_N(q_0, q; t_0, t) = \langle q | \exp(i(t_0 - t)\hat{H}) | q_0 \rangle \quad (1.23)$$

Here we de not dwell on the proof of this statement. It is not hard to check it in the case, when the Hamiltonian H is a function of the coordinate or the momentum only. For the Schrödinger equation the proof is known only if the energy function H is a sum of a function of momentum and a function of coordinate

$$H = H_1(p) + H_2(q) \quad (1.24)$$

Namely the Hamiltonians of the (1.24) type are used in nonrelativistic quantum mechanics.

We denote the functional integral, defined as the $N \to \infty$ limit of the expression (1.24) by the symbol

$$\int_{q(t_0)}^{q(t)} \exp iS[t_0,t] \prod_\tau \frac{dp(\tau)\,dq(\tau)}{2\pi} \quad (1.25)$$

This form is convenient but it does not reflect the fact that in the prelimit expression (1.22) the number of integrations over momenta is higher by one order than that over the coordinates.

The generalization of the functional integral formalism to a system with an arbitrary finite number of degrees of freedom is straightforward.

The action of a mechanical system with n degrees of freedom has the form

$$S[t_0,t] = \int_{t_0}^{t} \left(\sum_{i=1}^{n} p_i \dot{q}^i - H(p,q)\right) d\tau \quad (1.26)$$

Here q^i is the i-th canonical coordinate, p_i is the canonically conjugated momentum, $H(p,q) = H(p_1, \ldots, p_n, q^1, \ldots, q^n)$ is the Hamiltonian.

By definition the functional integral for the evolution operator matrix element is a limit of the finite-dimensional integral obtained from (1.22) by the replacement

$$(2\pi)^{-N} \to (2\pi)^{-Nn}, \quad dq_K \to \prod_{i=1}^{n} dq_K^i, \quad dp_K \to \prod_{i=1}^{n} dp_{i,K}, \tag{1.27}$$

where q_K^i are the values of the i-th coordinate at the point τ_K ($K = 1, \ldots, N-1$), and $p_{i,K}$ are the values of the i-th momentum on (τ_{K-1}, τ_K) interval. It is necessary to keep all the coordinates $q^1, \ldots, q^N$ simultaneously fixed at both ends of the time interval $[t_0, t]$.

We will denote the functional integral defined in such a way by the symbol

$$\int_{q(t')=q'}^{q(t'')=q''} \exp iS \prod_{t} \prod_{i=1}^{n} \frac{dp_i(t)\,dq^i(t)}{2\pi} \tag{1.28}$$

Let us study now what the functional integral for the finite dimensional mechanical system with constraints looks like.

We shall introduce additional conditions $\chi_a(p,q)$ so that relations (1.9) and (1.10) are satisfied.

The basis assertion is that the evolution operator matrix element is given by the functional integral

$$\int \exp\left\{ i \int_{t_0}^{t} \left(\sum_{i=1}^{n} p_i \dot{q}^i - H(p,q) \right) d\tau \right\} \prod_{\tau} d\mu(p(\tau), q(\tau)) \tag{1.29}$$

where the integration measure is given by the formula

$$d\mu(\tau) = (2\pi)^{m-n} \det \| \{\chi_a, \varphi^b\} \| \prod_{a} \delta(\chi_a)\, \delta(\varphi^a) \prod_{i=1}^{n} dp_i(\tau)\, dq^i(\tau) \tag{1.30}$$

To proove the assertion we transform integral (1.29) with measure (1.30) to the integral (1.28), where integration is taken along the trajectories in the physical phase space Γ^*. Using the abovementioned coordinates q^a, q^*, p_a, p^* we may transform (1.29) into the integral with the measure

$$d\tilde{\mu} = (2\pi)^{m-n} \det \left\| \frac{\partial \varphi^a}{\partial q^b} \right\| \prod_{a} \delta(p_a)\, \delta(\varphi^a) \prod_{i=1}^{n} dp_i\, dq^i \tag{1.31}$$

which can be rewritten as

$$\prod_a \delta(p_a)\,\delta(q^a - q^a(p^*, q^*))\,dp_a\,dq^a \prod_{j=1}^{n-m} \frac{dp_j^*\,dq_j^*}{2\pi} \tag{1.32}$$

One need not integrate over p_a and q^a thanks to the δ -functions. As a result the integral (1.29) takes the following form

$$\int \exp i \int_{t_0}^{t} \Big(\sum_{j=1}^{n-m} p_j^* \dot{q}_j^* - H^*(p^*, q^*) \Big) \prod_{\tau} \prod_{j=1}^{n-m} \frac{dp_j^*\,dq_j^*}{2\pi} \tag{1.33}$$

which coincides with (1.28).

Let us note, that integral (1.29) can be rewritten as

$$\int \exp i \int_{t_0}^{t} \Big(\sum_{i=1}^{n} p_i \dot{q}^i - H - \sum_a \lambda_a \varphi^a \Big) d\tau \;\times$$

$$\times \prod_{\tau} \det \| \{\chi_a, \varphi^\beta\} \| \, (2\pi)^{m-n} \prod_a \delta(\chi_a) \prod_{i=1}^{n} dp_i\,dq^i \prod_\beta \frac{\Delta\tau\, d\lambda_\beta}{2\pi} \tag{1.34}$$

The symbol $\prod_\beta (\Delta\tau)(d\lambda_\beta/2\pi)$ shows that in the prelimit expression there are integrals over $\lambda_\beta(\tau_i)$ (τ_i are dividing points of the intervals $[t_0, t]$) of the type

$$\int \exp\Big(-i \sum_{i,a} \lambda_a(\tau_i)\, \varphi^a(p(\tau_i), q(\tau_i))\Delta\tau \Big) \prod_{i,\beta} \Delta\tau \frac{d\lambda_\beta}{2\pi} \tag{1.35}$$

Expression (1.35) is equal to the product of δ -functions

$$\prod_{i,a} \delta(\varphi^a(p(\tau_i), q(\tau_i))) \tag{1.36}$$

It means that in integral (1.36) we can carry out the integration over λ_β and return again to integral (1.29).

It is not difficult to proove, that functional integral (1.29) does not depend on the choice of additional conditions (1.11).

LECTURE 2

Gauge Field Quantization

Field theory can be regarded as the theory of a mechanical system with an infinite number of degrees of freedom. The functional integral in

field theory can be constructed using various methods. First, it is possible to start with field action written in the Hamiltonian form and construct the functional integral over the phase space of a system with an infinite number of degrees of freedom. Second it is possible to start with the action not written explicitly in Hamiltonian form and study the functional integral over all fields. This approach enables us to construct explicitly relativistic theory. In the Hamiltonian approach relativistic invariance is often not explicite and requires special proof.

Let us consider an example the theory of scalar field with action

$$S = \int d^4x \left(\frac{1}{2} g_0^{\mu\nu} \frac{\partial\varphi}{\partial x^\mu} \frac{\partial\varphi}{\partial x^\nu} - \frac{m^2}{2} \varphi^2 - \frac{g}{3!} \varphi^3 \right) \tag{2.1}$$

Here $\varphi(x)$ are field functions depending on a point $x = (x^0, x^1, x^2, x^3)$ of pseudoeuclidean space V_4 and $g_0^{\mu\nu}$ is the diagonal Minkovski tensor $(1,-1,-1,-1)$. The action is the sum of the functional S_0 - the action of the free field theory, quadratic in φ and the integral over $(-g/3!)\,\varphi^3$ that describes the selfinteraction with the coupling constant g.

To define the functional integral over all fields the finite-dimensional approximation is frequently used.

In the space V_4 we take a big cubic volume V, divided into N^4 equal small cubes V_i $(i = 1, \ldots, N^4)$. We approximate function $\varphi(x)$ in the volume V by a function constant in the volumes V_i and the first derivatives $\partial\varphi/\partial x^\mu$ by the finite differences

$$\frac{1}{\Delta\ell} \left[\varphi(x^\nu + \delta^{\mu\nu}\Delta\ell) - \varphi(x^\nu) \right], \tag{2.2}$$

where $\Delta\ell$ is the length of the edge of the cube V_i. Approximating piecewise constant function $\varphi(x)$ is defined through its values in volumes V_i.

Let us consider the finite-dimensional integral

$$\int \exp iS \prod_{i=1}^{N^4} d\varphi(x) \tag{2.3}$$

over the values of function $\varphi(x)$ is volumes V_i. Here S is the action integral for the approximating function $\varphi(x)$ (with (2.2) as an app-

roximation for its first derivatives).

Finite-dimensional integrals of type (1.3) are present in the prelimit expressions used for the definition of functional integrals encountered in field theory. We shall define the Green function as

$$G(x,y) = -i \langle \varphi(x)\varphi(y) \rangle =$$

$$= -i \lim_{\substack{V\to\infty \\ v_i \to 0}} \frac{\int (\exp iS)\, \varphi(x)\varphi(y) \prod_{i=1}^{N^4} d\varphi(x)}{\int (\exp iS) \prod_{i=1}^{N^4} d\varphi(x)} \tag{2.4}$$

We denote the limit on the r.h.s. of (2.4) by the symbol

$$\frac{\int (\exp iS)\, \varphi(x)\varphi(y) \prod_x d\varphi(x)}{\int (\exp iS) \prod_x d\varphi(x)} \tag{2.5}$$

The method of functional integration over all fields can be explained and justified if it is possible to transform the functional integral obtained here into integrals of a Hamiltonian form which represent a field theory generalization of the integrals obtained above, in the quantization of the finite-dimensional mechanical systems.

Continuing the examination of the scalar field example, we shall write down in Hamiltonian form the functional integral

$$\int \exp iS \prod_x d\varphi(x) \tag{2.6}$$

To proceed we consider the integral

$$\int \exp iS[\varphi,\pi] \prod_x d\varphi(x)\, d\pi(x) \tag{2.7}$$

where the expression

$$S[\varphi,\pi] = \int \left(\pi \partial_0 \varphi - \frac{\pi^2}{2} - \frac{1}{2}(\vec{\nabla}\varphi)^2 - \frac{m^2}{2}\varphi^2 - \frac{g}{3!}\varphi^3 \right) \tag{2.8}$$

coincides with action (2.1) provided $\partial_0 \varphi(x)$ is substituted for $\pi(x)$.

Action (2.8) is of Hamiltonian form and the corresponding Hamiltonian function is

$$H = \int d^3x \left(\frac{1}{2}\pi^2 + \frac{1}{2}(\vec{\nabla}\varphi)^2 + \frac{m^2}{2}\varphi^2 + \frac{g}{3!}\varphi^3 \right) \tag{2.9}$$

where the functions $\varphi(x)$, $\pi(x)$ have the meaning of coordinate and conjugated momentum densities respectively. We show that integral (2.7) over variables φ and π results in integral (2.6) over all fields. To achieve this, it is sufficient to notice that the integral over π in formula (2.5) can be expressed explicitly if the shift

$$\pi(x) \to \pi(x) + \partial_0 \varphi(x) \tag{2.10}$$

is performed which couses the integral to transform into the product of integral (2.6) over φ and the integral over π

$$\int \exp\left(-\frac{i}{2}\int \pi^2 d^4x \right) \prod_x d\pi(x) \tag{2.11}$$

leading to the product of normalization factors.

In such a way we have succeeded in expression the functional integral of scalar field theory through the Hamiltonian form, artifically introducing an integral over a new variable - the canonical momentum. Such an approach turns out to be useful for the proof of the Hamiltonicity of given systems of quantum field theory and statistical physics.

The scheme of functional integration over all fields produces a method of quantization of Bose fields.

Quantization of Fermi fields can be performed using the functional integral over anticommuting variables. The following basic facts are necessary here.

The integral over Fermi fields (over an infinite Grassmann algebra) is defined as a limit of the integral on an algebra with a unit element and a finite number of generators X_i, X_i^* $(i = 1, \ldots, n)$ obeying commutation relations

$$X_i X_j + X_j X_i = 0 \quad , \quad X_i^* X_j^* + X_j^* X_i^* = 0 \quad , \quad X_i X_j^* + X_j^* X_i = 0 \tag{2.12}$$

Any of the elements of the algebra $f(x, x^*)$ is a polynomial of the form

$$f(x,x^*) = \sum_{a_i,\beta_i=0,1} c_{a_1,\ldots,a_n,\beta_1,\ldots,\beta_n}\, x_1^{a_1}\ldots x_n^{a_n}(x_1^*)^{\beta_1}\ldots(x_n^*)^{\beta_n} \qquad (2.13)$$

On the algebra we can introduce the integral

$$\int f(x,x^*)dx^*dx = \int f(x_1,\ldots,x_n,x_1^*,\ldots,x_n^*)\,dx_1^*dx_1\ldots dx_n^*dx_n \qquad (2.14)$$

This integral is defined through the relations

$$\int dx_i = 0\,,\quad \int dx_i^* = 0\,,\quad \int x_i dx_i = 1\,,\quad \int x_i^* dx_i^* = 1 \qquad (2.15)$$

We demand also that the symbols dx_i, dx_i^* anticommute with each other and with the generators and we impose the natural condition of linearity

$$\int (c_1 f_1 + c_2 f_2)\,dx^*dx = c_1\int f_1 dx^*dx + c_2\int f_2\,dx^*dx \qquad (2.16)$$

When we integrate the sum (2.13), only the contribution of the term with $a_i=\beta_i=1$ for all $i=1,\ldots,n$ is different from zero.

The following two formulae will be used later on

$$\int \exp(-x^*Ax)\,dx^*dx = \det A\,, \qquad (2.17)$$

$$\frac{\int \exp(-x^*Ax+\eta^*x+x^*\eta)\,dx^*dx}{\int \exp(-x^*Ax)\,dx^*dx} = \exp(\eta^*A^{-1}\eta) \qquad (2.18)$$

where

$$x^*Ax = \sum_{i,k} a_{ik}x_i^*x_k \qquad (2.19)$$

is a quadratic form of the generators x_i, x_i^*, corresponding to the matrix A.

The expressions

$$\eta^*x = \sum_i \eta_i^* x_i\,,\qquad x^*\eta = \sum_i x_i^*\eta_i \qquad (2.20)$$

are linear of the generators x_i, x_i^*, whose coefficients η_i, η_i^*

anticommute with each other and with the generators. The elements η_i, η_i^* together with the generators χ_i, χ_i^* can be regarded as generators of a larger algebra. The expression $\eta^* A^{-1} \eta$ in (2.20) is a quadratic form of the matrix A^{-1} inverse to A.

Now we are ready to discuss the gauge field quantization. Gauge fields can be regarded as connections on a fibre bundle, its base being the space time V_4 and its fiber a finite-dimensional space carrying a representation of a group.

The geometrical nature of gauge fields also demonstrates itself when constructing the corresponding quantum theory. The most suitable way of gauge field quantization is to use the functional integration approach.

The method of quantization is based on the following idea. The fields obtained from other fields through gauge transformation (e.g. A_μ and $A_\mu + \partial_\mu \lambda$ in electrodynamics) describe the same physical (geometrical) situation and are therefore physically indistinguishable. This leads to an idea that the classes of those fields which can be obtained from other fields through gauge transformations should be the basic objects of the theory. In such a way all fields of the type $A_\mu + \partial_\mu \lambda$ are unified into one class.

The action in the gauge field theory is the same for all fields obtained through gauge transformations. In order words the action is a functional defined on classes.

In the functional integral formalism it is possible to obtain a theory whose basic objects are classes if we can write down the functional integral as an integral over all classes. It can be accomplished, e.g. if the integration is taken over the surface in the manifold of all fields whose elements intersect each of the classes once. Then each class will have exactly one representative on that surface. The integration measure arising on such surfaces changes with variation of the surface, but all physical results must be independent of the choice ofthe surface.

We now formulate the quantization scheme of gauge fields in the formalism of functional integration over all fields.

We shall denote the gauge fields by A, its components A_μ^a, where $\mu = 0, 1, 2, 3$ is a space-time index, and a is an isotopic index.

The gauge group is a direct product of the groups G_c , operating at every point x of space-time

$$G = \prod_x G_c(x) \tag{2.21}$$

Let Ω be an element of the gauge group which is a function on V_4 with functional values in G_c . We denote A^Ω the result of action of the element Ω on the field A . The set of fields A^Ω with A fixed, Ω ranning through gauge group G is called the gauge group orbit.

We have seen that quantization of a field with action S leads to averaging $\exp iS$ over all fields. In the theory of gauge fields the action $S[A]$ is gauge invariant, i.e.

$$S[A^\Omega] = S[A] \tag{2.22}$$

The measure in the functional integral exhibit the property of gauge invariance

$$d_\mu[A^\Omega] = d_\mu[A] \tag{2.23}$$

as well as the action $S[A]$. The invariance of action $S[A]$ and the measure $d_\mu[A]$ with respect to gauge transformations $A \to A^\Omega$ implies that the corresponding functional integral

$$\int \exp iS[A]\, d_\mu[A] \tag{2.24}$$

becomes propertional to the "orbit volume" i.e. to the functional integral

$$\int \prod_x d\Omega(x) \tag{2.25}$$

over gauge group G . Here $\prod_x d\Omega(x)$ is an invariant measure on group G which is equal to the product of measures on G_c , operating at every point of space-time V_4 .

The approach to the integration over classes consists of the explicit factorization of that factor from the functional integral. Such a factorization can be realized by several methods.

One of them consists of the transition from integral (2.24) over all fields to the integral over the surface in the manifold of all fields, whose elements intersect with each of the gauge-group orbits just once.

Let the equation of the surface be

$$f(A) = 0 \tag{2.26}$$

The equation $f(A^\Omega) = 0$ should have a unique solution with respect to $\Omega(x)$ for any $A(x)$.

We introduce the functional $\Delta_f[A]$ defined by the condition

$$\Delta_f[A] \int \prod_x \delta\left(f\left(A^\Omega(x)\right)\right) d\Omega(x) = 1 \tag{2.27}$$

Here the integration is carried out over gauge group G of the infinite dimensional δ-function $\prod_x \delta\left(f\left(A^\Omega(x)\right)\right)$. Such a δ-function is a functional defined by the specification of the rules for its integration with other functionals. In the following we shall demonstrate several specific examples of the evaluation of integrals of type (2.27). Let us notice that the functional $\Delta_f[A]$ is gauge invariant, i.e.

$$\Delta_f[A^\Omega] = \Delta_f[A] \tag{2.28}$$

To factorize factor (2.25) from functional integral (2.20) we insert the left hand side of (2.27) (which is equal to one) into the integral and make the substitution $A^\Omega \to A$. The measure $d_\mu[A]$ and functionals $S[A]$, $\Delta_f[A]$ are invariant under such a substitution. Integral (2.24) leads to the multiplication of the group's volume $\int \prod_x d\Omega(x)$ by the integral

$$\int \exp(iS[A])\, \Delta_f[A] \prod_x \delta(f(A))\, d_\mu[A] \tag{2.29}$$

Just this integral provides a starting point for the quantum theory of gauge fields.

It is easy to demonstrate that integral (2.29) formally depending on the choice of the surface $f(A) = 0$, is in fact invariant with respect to the choice of the surface. To prove it we insert into the integrand (2.29) "another unit element"

$$1 = \Delta_g[A] \int \prod_x \delta\left(g\left(A^\Omega(x)\right)\right) d\Omega(x) \tag{2.30}$$

where $g(A) = 0$ is an equation of another surface which, like the surface $f(A) = 0$, intersects each of the orbits of group G only once.

Interchanging the integration over A and Ω , performing then the

shift $A^{\Omega} \to A$ and finally interchanging again the integration over A and Ω we can express integral (2.29) as

$$\int \exp(iS[A])\, \Delta_g[A] \prod_x \delta(g(A))\, d\mu[A] \qquad (2.31)$$

The method described allows one to pass in the functional integral from one surface to another or, we can say, from one gauge to another. Especially, such a method is suitable for the transition from the Hamiltonian form of the functional integral to the integral in the relativistic gauge. There exists a method for the factorization of the volume of a gauge group from the functional integral which is more general than the method just described. Let us take the functional $F[A]$ which is not gauge invariant. We define a gauge invariant functional $\phi[A]$ by the equation

$$\phi[A] \int F[A^{\Omega}] \prod_x d\Omega(x) = 1 \qquad (2.32)$$

It is necessary, however, to require that the functional on the l.h.s. of (2.32) really exists. Inserting the l.h.s. of (2.32) into integral (2.34) and then performing the shift $A^{\Omega} \to A$ we obtain the product of the group volume (2.35) with the integral

$$\int \exp(iS[A])\, \phi[A]\, F[A]\, d\mu[A] \qquad (2.33)$$

Integral (2.29) is a special case of (2.33). The independence of integral (2.33) on the choice of functional $F[A]$ can be proved in the some way as the independence of integral (2.29) on the choice of the surface $f(A) = 0$.

In the theory of gauge fields Green's function is defined as an expectation value of the product of field functions at different points of space-time V_4. The generating functional of Green's functions has the form

$$Z[\eta] = \frac{\int \exp\{iS[A] + i\int \eta A\, d^4x\}\, F[A]\, \phi[A]\, d\mu[A]}{\int \exp(iS[A])\, F[A]\, \phi[A]\, d\mu[A]}, \qquad (2.34)$$

where $S[A]$ is the action of the field A, $d\mu[A]$ is the local gauge invariant measure, the functionals F and ϕ are defined above. The linear functional

$$\int \left(\sum_{\mu,a} \eta_a^\mu(x) A_\mu^a(x) \right) d^4x \tag{2.35}$$

is denoted as $\int \eta A \, d^4x$, where $\eta_a^\mu(x)$ are arbitrary test functions. Green's functions - functional derivatives of function (2.34) - depend on the choice of gauge i.e., on the choice of the functional $F[A]$. Physical results, obtained by averaging gauge-invariant functionals, however, do not depend on the choice of gauge.

LECTURE 3

Examples of gauge theories. Electrodynamics, Yang-Mills field.

The simplest example of gauge field is electromagnetic field. The action of a free electromagnetic field

$$S = -\frac{1}{4} \int (\partial_\mu A_\nu - \partial_\nu A_\mu)^2 d^4x \tag{3.1}$$

is invariant under the Abelian group of gauge transformations

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \lambda(x). \tag{3.2}$$

We have seen that the quantization of gauge fields is realized using the functional integral of the functional $\phi F \exp iS$ where S is the action, F is an arbitrary nongauge invariant functional, and ϕ^{-1} is the expectation value of F averaged over the gauge group. The local integration measure

$$d\mu[A] = \prod_x \prod_{\mu=0}^{3} dA_\mu(x) \tag{3.3}$$

is evidently gauge invariant. Functionals of the type

$$\begin{aligned} F_1[A] &= \prod_x \delta(\partial_\mu A_\mu(x)), \\ F_2[A] &= \prod_x \delta(\mathrm{div}\, \vec{A}(x)), \\ F_3[A] &= \exp\left(-\frac{i}{2\alpha} \int (\partial_\mu A_\mu)^2 d^4x\right) \end{aligned} \tag{3.4}$$

turn out to be most convenient for the construction of the perturbation theory. The functionals F_1 and F_3 lead to an explicitly relativistic

quantization and the use of F_2 is convenient when passing to the Hamilton theory. The corresponding gauge invariant functionals are given by the formulae

$$\Phi_1^{-1}[A] = \int \prod_x \delta\left(\partial_\mu (A_\mu + \partial_\mu \lambda)\right) d\lambda(x) ,$$

$$\Phi_2^{-1}[A] = \int \prod_x \delta\left(\mathrm{div}\,(\vec{A} + \vec{\nabla}\lambda)\right) d\lambda(x) , \tag{3.5}$$

$$\Phi_3^{-1}[A] = \int \exp\left(-\frac{i}{2\alpha}\int \left(\partial_\mu (A_\mu + \partial_\mu \lambda)\right)^2 d^4x\right) \prod_x d\lambda(x)$$

All these functionals do not, in fact, depend on the field $A_\mu(x)$, as c can be seen if we perform the shift $\lambda \to \lambda - \Box^{-1}\partial_\mu A_\mu$ in the first and third functionals and $\lambda \to \lambda - \Delta^{-1}\,\mathrm{div}\,\vec{A}$ in the second one. Thus, with precision up to an (infinite) constant factor we can take

$$\Phi_1 = \Phi_2 = \Phi_3 = 1 \tag{3.6}$$

Now the form of functional integral is defined in all 3 cases. The use of the functional F_2 means the integration over the fields satisfying the equation

$$\mathrm{div}\,\vec{A} = 0 \tag{3.7}$$

This is a well-known Coulomb-gauge condition. We shall show how the integral with the functional F_2 can be transformed into the integral of an explicitly Hamilton type. Such a transformation is possible if the integral over auxiliary fields is introduced. In our case we are led to the functional integral of the form

$$\int \exp i S[A_\mu, F_{\mu\nu}] \prod_x \delta(\mathrm{div}\vec{A}(x)) \prod_\mu dA_\mu(x) \prod_{\mu<\nu} dF_{\mu\nu}(x) \tag{3.8}$$

with the action

$$S[A_\mu, F_{\mu\nu}] = \int \left(\frac{1}{4} F_{\mu\nu}F_{\mu\nu} - \frac{1}{2} F_{\mu\nu}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\right) d^4x \tag{3.9}$$

depending not only on the vector $A_\mu(x)$, but also on the antisymmetric tensor $F_{\mu\nu}(x)$. In the classical theory $F_{\mu\nu}$ are the electromagnetic field strengths:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{3.10}$$

Here we regard A_μ, $F_{\mu\nu}$ as independent variables and integrate over them as over independent variables. The integral over $F_{\mu\nu}$ in (3.9) can be evaluated exactly. To accomplish this it is sufficient to perform a shift

$$F_{\mu\nu} \to F_{\mu\nu} + \partial_\mu A_\nu - \partial_\nu A_\mu \tag{3.11}$$

which transforms integral over A_μ into the product of the integral over A_μ and the integral over $F_{\mu\nu}$

$$\int \exp\left(\frac{i}{4}\int F_{\mu\nu}F_{\mu\nu}\,d^4x\right)\prod_x\prod_{\mu<\nu} dF_{\mu\nu} \tag{3.12}$$

which is just a normalization constant.

We rewrite action (3.9) using 3-dimensional notations

$$\int\left(\vec{E}\,\partial_0\vec{A} - \frac{1}{2}E^2 + \frac{1}{2}H^2 - (\vec{H}, \mathrm{rot}\,\vec{A}) + A_0\,\mathrm{div}\,\vec{E}\right)d^4x, \tag{3.13}$$

where

$$E_i = F_{0i}\,, \quad H_1 = F_{23}\,, \quad H_2 = F_{31}\,, \quad H_3 = F_{12} \tag{3.14}$$

We shall integrate over $\vec{H}$ in (3.8). This leads to the substitution $\vec{H} \to \mathrm{rot}\,\vec{A}$ in action (3.13). Then we integrate over A_0 , which yields the functional

$$\prod_x \delta(\mathrm{div}\,\vec{E}(x)) \tag{3.15}$$

We obtain the integral

$$\int\exp(iS[\vec{A},\vec{E}])\prod_x \delta(\mathrm{div}\,\vec{A}(x))\,\delta(\mathrm{div}\,\vec{E}(x))\prod_{i=1}^{3} dA_i(x)\,dE_i(x) \tag{3.16}$$

with a Hamiltonian-type-action

$$\int\left(\vec{E}\,\partial_0\vec{A} - \frac{1}{2}E^2 - \frac{1}{2}(\mathrm{rot}\,\vec{A})^2\right)d^4x \tag{3.17}$$

Integral (3.16) is an analog of integrals considered in Lecture 1 when we quantized finite-dimensional systems with constraints. Here the role of the constraint is played by $\mathrm{div}\vec{E}$, the role of the additional condition by the Coulomb gauge equation (3.7). It is possible to use the transverse (in the 3-dimensional sense) components of vectors $\vec{A}$ and $\vec{E}$ as independent variables.

The Lagrangian of spinorial quantum electrodynamics

$$\bar{\psi}(x)\left(i\gamma_\mu\left(\partial_\mu - ieA_\mu(x)\right) - m\right)\psi(x) - \frac{1}{4}\left(\partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)\right)^2, \quad (3.18)$$

where γ_μ are Dirac matrices, also contains besides the electromagnetic potential $A_\mu(x)$, the four-component spinors $\psi(x)$, $\bar{\psi}(x)$, describing the Fermi electron-positron field. Lagrangian (3.18) is invariant under the Abelian group of gauge transformations

$$A_\mu(x) \to A_\mu(x) + \partial_\mu\lambda(x), \quad \psi(x) \to e^{ie\lambda(x)}\psi(x), \quad \bar{\psi}(x) \to \bar{\psi}(x)e^{-ie\lambda(x)} \quad (3.19)$$

In the functional integration scheme we shall consider the components of spinors $\psi_\alpha(x)$, $\bar{\psi}_\alpha(x)$ to be anticommuting elements of Grassmann algebra and we shall integrate $\exp iS$ with the measure

$$F[A]\prod_x\prod_\mu dA_\mu(x)\prod_\alpha d\bar{\psi}_\alpha(x)\,d\psi_\alpha(x) \equiv F\prod_x dA\,d\bar{\psi}\,d\psi \quad (3.20)$$

Here F is equal to F_1 or F_2 or F_3.

The form of Green's functions depends on the choice of functional $F[A]$, but all the physical results do not depend on the gauge condition.

The theory of Yang-Mills fields is the simplest example of the theory with a nonabelian gauge group [3]

The vector Yang-Mills field, connected with simple compact Lie group G can be described by matrices $B_\mu(x)$ acquiring values in the Lie algebra of that group

$$B_\mu(x) = \sum_{a=1}^{n} b_\mu^a(x)\,\tau_a. \quad (3.21)$$

Here τ_a are linear independent matrices in the adjoint representation of

the Lie algebra, normalized by the conditions

$$tr\,\tau_a \tau_b = -2\delta_{ab}, \tag{3.22}$$

n is the number of group parameters, $b_\mu^a(x)$ is a C-number function with the vector index μ and the "isotopic" index a. As is well known, in the adjoint representation it is possible to use the latter index for the enumeration of matrix elements, so that

$$(B_\mu)_{ab} = (\tau_c)_{ab}\, b_\mu^c = t_{abc}\, b_\mu^c, \tag{3.23}$$

where t_{abc} are group-structure constants, antisymmetric in all three indices.

The Lagrangian of the Yang-Mills field

$$\frac{1}{8} tr\, F_{\mu\nu} F_{\mu\nu}, \tag{3.24}$$

where

$$F_{\mu\nu} = \partial_\nu B_\mu - \partial_\mu B_\nu + \varepsilon [B_\mu, B_\nu], \tag{3.25}$$

is invariant under the gauge transformations

$$B_\mu \to \Omega B_\mu \Omega^{-1} + \varepsilon^{-1} \partial_\mu \Omega \cdot \Omega^{-1} \tag{3.26}$$

with the matrix Ω acting in the adjoint representation of the group.

The analogues of the functionals $F[A]$, used above for quantum electrodynamics, are suitable also for quantization of Yang-Mills field. Now they have the form

$$F_1[B] = \prod_x \delta(\partial_\mu B_\mu(x)) = \prod_{x,a} \delta(\partial_\mu b_\mu^a(x)),$$

$$F_2[B] = \prod_x \delta(div\, \vec{B}(x)) = \prod_{x,a} \delta(\partial_i b_i^a(x)), \tag{3.27}$$

$$F_3[B] = exp\left(\frac{i}{4\alpha}\int tr(\partial_\mu B_\mu)^2 d^4x\right) =$$

$$= exp\left(-\frac{i}{2\alpha}\int \sum_a (\partial_\mu b_\mu^a)^2 d^4x\right)$$

Here, the functionals F_1 and F_2 select among all fields, those satisfying the conditions

$$f_L[B] = \partial_\mu B_\mu = 0 \quad \text{for } F_1 \; ; \; f_R[B] = \operatorname{div} \vec{B} = 0 \quad \text{for } F_2 \tag{3.28}$$

Each of these equations is a matrix equation and in fact it represents additional conditions (according to the number of parameters of the group G).

The factor Φ_1, corresponding to the functional F_1, will be denoted by Δ_L. In the functional integral, this factor stands before the δ-function of $\partial_\mu B_\mu$ and it is therefore sufficient to know its values only for the transverse fields ($\partial_\mu B_\mu = 0$). In such a case, the whole contribution to the integral

$$\Delta_L^{-1}[B] = \int \prod_x \delta(\partial_\mu B_\mu^{\Omega}(x))\, d\Omega(x) \tag{3.29}$$

comes from a neighbourhood at the unit element. Here the substitution

$$\Omega(x) = 1 + \varepsilon u(x) \tag{3.30}$$

can be performed ($u(x)$ is an element of the Lie algebra) and only the terms linear in u remain in

$$\begin{aligned} \partial_\mu B_\mu^{\Omega} &= \partial_\mu \left(B_\mu + \varepsilon [u, B_\mu] + \partial_\mu u \right) = \\ &= \Box u - \varepsilon [B_\mu, \partial_\mu u], \end{aligned} \tag{3.31}$$

where $\Box = \hat{A}_0$ is the d'Alambert operator. Instead of the matrices $u(x)$ we introduce the column

$$u(x) = \sum_{a=1}^{n} \tau_a u_a(x) \tag{3.32}$$

which the operator $\hat{A}$ acts on according to the rule

$$\begin{aligned} (\hat{A}u)_a &= (\Box u - \varepsilon [B_\mu, \partial_\mu u])_a = (\Box \delta_{ac} - \varepsilon (B_\mu)_{ac} \partial_\mu) u_c = \\ &= \Box u_a - \varepsilon t_{abc} B_\mu^b \partial_\mu u_c \end{aligned} \tag{3.33}$$

Integral (3.29) can be written as

$$\Delta_L^{-1}[B] = \int \prod_{x,a} \delta((\hat{A}u)_a)\, du_a(x) \tag{3.34}$$

Formally $\Delta_L[B]$ is the determinant of the operator $\hat{A}$. Putting out the trivial (infinite) factor $\det \Box$ it is possible to expand the logarithm Δ_L into series in ε :

$$\ln \Delta_L[B] = \ln \det \hat{A}/\hat{A}_0 = Tr \ln(1 - \varepsilon \Box^{-1} B_\mu \partial_\mu) =$$

$$= -\sum_{n=2}^{\infty} \frac{\varepsilon^n}{n} \int d^4x_1 \ldots d^4x_n \; tr\left(B_{\mu_1}(x_1) \ldots B_{\mu_n}(x_n)\right) \times \tag{3.35}$$

$$\times \partial_{\mu_1} D(x_1 - x_2) \ldots \partial_{\mu_n} D(x_n - x_1)$$

$D(x)$ is the Green's function of the d'Alambert operator $\Box$. Tr in (3.35) means the trace in an operator sence in contradistinction to tr - the trace of a matrix.

The corresponding factor in the Coulomb gauge is denoted by Δ_R. Analogous evaluation leads to the formula

$$\ln \Delta_R[B] = Tr \ln(1 - \varepsilon \Delta^{-1} B_i \partial_i) =$$

$$= -\sum_{n=2}^{\infty} \frac{\varepsilon^n}{n} \int d^4x_1 \ldots d^4x_n \; tr\left(B_{i_1}(x_1) \ldots B_{i_n}(x_n)\right) \times \tag{3.36}$$

$$\times \partial_{i_1} \tilde{D}(x_1 - x_2) \ldots \partial_{i_n} \tilde{D}(x_n - x_1)$$

where

$$\tilde{D}(x) = -\frac{1}{(2\pi)^4} \int \frac{d^4k}{\vec{k}^2} e^{ikx} = -\delta(x_0)(4\pi|\vec{x}|)^{-1} \tag{3.37}$$

The indices $i_1, \ldots, i_n$ in (3.36) acquire the values 1, 2, 3.

It is not difficult to construct the perturbation theory in the Lorentz gauge $\partial_\mu B_\mu = 0$. It arises as a result of the expansion of the

functional

$$\Delta_L[B]\exp\left(iS[B]\right) = \exp\left(iS[B] + \ln\Delta_L[B]\right) \tag{3.38}$$

into series in ε . The expression $\ln\Delta_L$ may be interpreted as an addition to the action S . The term of n-th order in the expansion of $\ln\Delta_L$ into series in ε leads to the diagram vertex with n outgoing lines. The explicit expression for this term, which follows from (3.35), suggests its interpretation as a circle with n outgoing lines along which the ghost scalar particle propagates. This statement can be interpreted exactly if we write down the determinant as an integral over the anticommuting variables

$$\begin{aligned} &\det\left(\Box - \varepsilon B_\mu \partial_\mu\right) = \\ &= \int \exp\left(i\int L(B_\mu, \bar\eta, \eta)\, d^4x\right) \prod_{x,a} d\bar\eta^a(x)\, d\eta^a(x) \,, \end{aligned} \tag{3.40}$$

where

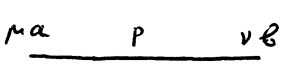
$$\begin{aligned} L[B_\mu, \bar\eta, \eta] &= \frac{1}{2}\,\mathrm{tr}\,\bar\eta\left(\Box - \varepsilon B_\mu \partial_\mu\right)\eta = \\ &= \bar\eta^a \Box \eta - \varepsilon\, t_{abc}\, B_\mu^c\, \bar\eta^a \partial_\mu \eta^b \end{aligned} \tag{3.41}$$

Formula (3.40) is an infinite-dimensional integral of the (2 17) type.

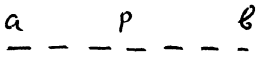
So, out system can be looked upon as a system of Bose fields $B_\mu^a(x)$ interacting with each other and with scalar Fermi fields $\eta^a(x)$, $\bar\eta^a(x)$.

The elements of the diagram technique in the Yang-Mills theory are lines of two types, corresponding to the transverse vector and ghost scalar particles, and also the vertices describing the interaction of vector particles with the scalar ones and with each other.

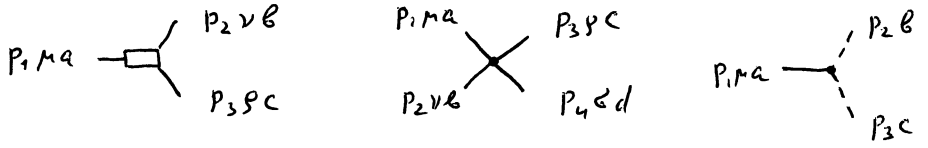
We shall represent the vector particles with solid lines and the ghost particles with dashed lines. The elements of the diagrams are the lines and vertices of the form



$G_{\mu\nu}^{ab}(p)$



$G^{ab}(p)$



(3.42)

$V^{abc}_{\mu,\nu\rho}$ $\qquad$ $V^{abcd}_{\mu\nu\rho\sigma}$ $\qquad$ V^{abc}_{μ}

The expressions of the diagram elements (3.42) are

$$G^{ab}_{\mu\nu}(p) = -\delta_{ab}\,(p^2\delta_{\mu\nu} - p_\mu p_\nu)\,(p^2+i0)^{-2},$$

$$G^{ab}(p) = -\delta_{ab}\,(p^2+i0)^{-1},$$

$$V^{abc}_{\mu,\nu\rho} = i\varepsilon\, t_{abc}\,(p_{1\nu}\delta_{\mu\rho} - p_{1\rho}\delta_{\mu\nu}), \qquad (3.43)$$

$$V^{abcd}_{\mu\nu\rho\sigma} = \varepsilon^2\, t_{abe}\, t_{cde}\,(\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho}),$$

$$V^{abc}_{\mu} = \frac{i\varepsilon}{2}\, t_{abc}\,(p_1 - p_2)_\mu$$

To find the contribution of a given diagram it is necessary to integrate over independent 4-momenta the product of expressions which correspond to all its elements, sum over all independent discrete indices and multiply the result by

$$\mathcal{Z}^{-1}\left(\frac{i}{(2\pi)^4}\right)^{\ell-v-1}(-2)^S \qquad (3.44)$$

where v is the number of diagram vertices, ℓ is the number of its internal lines, S is the number of closed loops of ghost scalar particles and $\mathcal{Z}$ is the order of the symmetry group of the diagram. Let us remark

that $\ell - v - 1 = c$ is the number of independent contours of the diagram.

This perturbation theory is not the only possible one. Another form of perturbation theory and diagram technique emerges in the so-called first order formalism. This formalism can be obtained if Lagrangian (3.24) is written as

$$-\frac{1}{8}\,\mathrm{tr}\,F_{\mu\nu}F_{\mu\nu} + \frac{1}{4}\,\mathrm{tr}\,F_{\mu\nu}\left(\partial_\nu B_\mu - \partial_\mu B_\nu + \varepsilon\,[B_\mu, B_\nu]\right) \tag{3.45}$$

and the integration over B_μ, $F_{\mu\nu}$ as over independent variables is performed. The expression "first order formalism" means that the symbol of the derivative enters Lagrangian (3.45) in an order not higher than the first.

Using the Lorentz gauge we obtain the functional integral of the form

$$\int \exp\left(iS[B,F]\right)\Delta_L[B]\prod_x \delta(\partial_\mu B_\mu)\,dB\,dF\,, \tag{3.46}$$

where the expression

$$dB(x)\,dF(x) = \prod_a \prod_\mu d\,\ell^a_\mu(x) \prod_{\mu<\nu} d\,f^a_{\mu\nu}(x) \tag{3.47}$$

as well as $dB(x)$, is gauge invariant.

The formalism of the first order is convenient for the passage to the canonical quantization. We shall examine such a transition starting from the integral over $B_\mu, F_{\mu\nu}$ in the Coulomb gauge

$$\int \exp\left(iS[B,F]\right)\Delta_R[B]\prod_x \delta(\mathrm{div}\,B)\,dB\,dF \tag{3.48}$$

In 3-dimensional notation Lagrangian (3.45) acquires the form

$$\mathrm{tr}\Big(-\frac{1}{8}F_{ik}F_{ik} + \frac{1}{4}F_{0i}F_{0i} + \frac{1}{4}F_{ik}\left(\partial_k B_i - \partial_i B_k + \varepsilon\,[B_i, B_k]\right)$$

$$-\frac{1}{2}F_{0i}\partial_0 B_i - \frac{1}{2}B_c\left(\partial_i F_{0i} - \varepsilon\,[B_i, F_{0i}]\right)\Big) \tag{3.49}$$

We can integrate over B_c, F_{ik}. Integration over B_c is equivalent to the appearance of δ-functional

$$\prod_x \delta\left(\partial_i F_{0i} - \varepsilon\,[B_i, F_{0i}]\right) \tag{3.50}$$

The integration over F_{ik} reduces to the replacement F_{ik} by

$$H_{ik} = \partial_k B_i - \partial_i B_k + \varepsilon [B_i, B_k] \tag{3.51}$$

in the integral over the remaining variables B_i, F_{0i}.

We shall insert into the integral (3.48) the factor

$$\int \prod_x \delta(\Delta C + \partial_i F_{0i}) \, dC(x) \tag{3.52}$$

which in fact foes not depend on F_{0i}, and then perform the shift $F_{0i} \to F_{0i} - \partial_i C$. The functional $\prod_x \delta(\Delta C + \partial_i F_{0i})$ is transformed into $\prod_x \delta(\partial_i F_{0i})$ and the functional $\prod_x \delta(\partial_i F_{0i} - \varepsilon [B_i, F_{0i}])$ into the expression $\prod_x \delta(\Delta C - \partial_i F_{0i} - \varepsilon [B_i, \partial_i C] + \varepsilon [B_i, F_{0i}])$, which is equal to $\prod_x \delta(\Delta C - \varepsilon [B_i, \partial_i C] + \varepsilon [B_i, F_{0i}])$ due to $\partial_i F_{0i} = 0$.

Let $C_0(x)$ be a solution of the equation

$$\Delta C - \varepsilon [B_i, \partial_i C] = - \varepsilon [B_i, F_{0i}] \tag{3.53}$$

which can be expressed in terms of the Green's function depending on B :

$$C_0(x) = - \varepsilon \int D(x, y, B) [B_i(y), F_{0i}(y)] \, d^3y \tag{3.54}$$

After the shift $C \to C + C_0$ the functional $\prod_x \delta(\Delta C - \varepsilon [B_i, \partial_i C])$ originates and the function $C(x)$ can be put equal to zero everywhere except in the argument of the δ -functional. The integral

$$\int \prod_x \delta(\Delta C - \varepsilon [B_i, \partial_i C]) \tag{3.55}$$

cancels with the factor $\Delta_R[B]$. As a result, functional (3.45) takes the form

$$\int \exp\left(i S[B_i, F_{0i}]\right) \prod_x \delta(\partial_i B_i) \, \delta(\partial_i F_{0i}) \prod_i dB_i \, dF_{0i}, \tag{3.56}$$

where

$$S[B_i, F_{0i}] = \int dx_0 \left(\int f^a_{0i} \, \partial_0 B^a_i \, d^3x - H \right) \tag{3.57}$$

$$H = \int d^3x \left(\frac{1}{4} h^a_{ik} h^a_{ik} + \frac{1}{2} f^a_{oi} f^a_{oi} + \frac{1}{2} \partial_i c^a_o \partial_i c^a_o \right) \tag{3.58}$$

In those formulae $S[B_i, F_{oi}]$ is the action corresponding to the Hamiltonian H, where the transverse fields ℓ_i, f_{oi} have the meaning of canonically conjugated coordinates and momenta.

As we have shown above, the functional integration over canonically conjugated coordinates and momenta is equivalent to the canonical quantization. When the canonical quantization is applied to the system described by Hamiltonian (3.58), it results in the replacement of functions ℓ^a_i, f^ℓ_{oj} through which h^a_{ik}, c^a_o are expressed, by operators $\hat{\ell}^a_i(\vec{x})$, $\hat{f}^\ell_{oj}(\vec{y})$ satisfying commutation relations

$$\left[\hat{\ell}^a_i(\vec{x}), \hat{f}^\ell_{oj}(\vec{y}) \right] = i \delta_{a\ell} \delta^{tr}_{ij}(\vec{x}-\vec{y}) =$$

$$= \frac{i \delta_{a\ell}}{(2\pi)^3} \int d^3k \, e^{i(\vec{k},\vec{x}-\vec{y})} \left(\delta_{ik} - \frac{k_i k_j}{|\vec{k}|^2} \right) \tag{3.59}$$

Hamiltonian (3.58) becomes a self-conjugated and positive definite energy operator. Such a quantization of the Yang-Mills field has been suggested by Schwinger[5] It has been shown how the functional integral formalism leads to Schwinger's canonical quantization. Let us emphasise that the existence of the factor $\Delta_R[B]$ in the original integral (3.48) is important in bringing the integral to an explicitly Hamiltonian form.

LECTURE 4

Electroweak Interactions, QCD, Extended Objects, Anomalies.

Yang-Mills field theory discussed above is a fundament for construction more realistic nonabelian gauge theories which can be applied to the real high energy physics. Historically the first such theory was the Weinberg-Salam (W.S.) model of electroweak interactions.

The main idea of W.S. model is the spontaneous breaking of the original invariance with respect to the gauge transformation of massless vector

fields of the Yang-Mills type. The gauge group of the model is group $U(2)$ This group is isomorphic to the group of 2x2 unitary matrices and equal to the product of the $U(1)$ group of phase transformations and the group of 2x2 unitary matrices with unit determinant.

The connection generated by group $U(2)$ consists of two types of vector fields - the Yang-Mills multiplet A_μ^a ($a = 1, 2, 3$) and the field B_μ. Besides these fields scalar fields exist that induce the spontaneous symmetry breaking of the $U(2)$ gauge invariance. The lepton fields involved in the W.S.model are the electron-type fields

$$L = \frac{1}{2}(1+\gamma_5)\begin{pmatrix}\nu_e\\ \psi_e\end{pmatrix}, \qquad R = \frac{1}{2}(1-\gamma_5)\psi_e, \tag{4.1}$$

where ψ_e is the electron field and ν_e is the electron neutrino field. The scalar fields form the doublet

$$\varphi = \begin{pmatrix}\varphi_0\\ \varphi_-\end{pmatrix} \tag{4.2}$$

The Lagrangian of the model is

$$L = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu + g[A_\mu, A_\nu])^2 - \frac{1}{4}(\partial_\mu B_\nu - \partial_\nu B_\mu)^2 -$$

$$- \bar{R}\gamma^\mu(\partial_\mu - ig'B_\mu)R - \bar{L}\gamma^\mu(\partial_\mu + ig\frac{\tau_a}{2}A_\mu^a - \frac{i}{2}g'B_\mu)L -$$

$$- \frac{1}{2}(\partial_\mu\varphi - ig\frac{\tau_a}{2}A_\mu^a\varphi - \frac{i}{2}g'B_\mu\varphi)^2 -$$

$$- G_e(\bar{L}\varphi R + \bar{R}\bar{\varphi}L) - M_1^2\varphi^+\varphi + h(\varphi^+\varphi)^2 \tag{4.3}$$

where g, g' are the coupling constants of the multiplet A_μ and singlet B_μ respectively.

The mechanism of spontaneous symmetry breaking and mass generation, which was first proposed by Higgs, is based on the appearance of the anomalous average value

$$\lambda = \langle\varphi^0\rangle \tag{4.4}$$

of the zeroth component of the φ field. Such a mechanism is a well-known in the superfluidity theory. Let us proceed from the original fields to the "physical" ones substracting from the φ fields their anomalous average values. For the physical fields we take the φ field and

$$\varphi_1 = (\varphi^0 + \bar{\varphi}^0 - 2\lambda)/\sqrt{2} \quad , \quad \varphi_2 = (\varphi^0 - \bar{\varphi}^0)/i\sqrt{2} \tag{4.5}$$

In the first order of the perturbation theory the quantity λ is determined by the minimum of the expression $-M_1^2(\varphi^+\varphi)^2 + h(\varphi^+\varphi)^2$ supposing that $\varphi^0 = \lambda$, $\varphi^- = 0$. This leads to

$$\lambda^2 = M_1^2/2h \tag{4.6}$$

After these operations the φ_1 fields acquire the mass M_1 and fields φ_2, φ^- remain massless. The appearance of massless excitations in models with spontaneous symmetry breaking was discovered by Goldstone. In this case, however, the excitations have no immediate physical meaning and can be removed by a gauge $U(2)$ transformation.

The mass of the φ_1 meson appears to be too large (compared to the electron mass me) and that is why the coupling of φ to other fields may be neglected.

Eventually it became obvious that the effect of the appearance of anomalous value (4.4) can be reduced in the first order to the substitution of the φ field by its vacuum expectation value

$$\langle\varphi\rangle = \lambda \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{4.7}$$

By this substitution Lagrangian (4.3) transforms to

$$-\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu + g[A_\mu, A_\nu])^2 - \frac{1}{4}(\partial_\mu B_\nu - \partial_\nu B_\mu)^2 -$$

$$-\bar{R}\gamma^\mu(\partial_\mu - ig'B_\mu)R - \bar{L}\gamma^\mu(\partial_\mu + ig\frac{\tau^a}{2}A_\mu^a - \frac{1}{2}g'B_\mu)L \tag{4.8}$$

$$-\frac{1}{8}\lambda^2 g^2((A_\mu^1)^2 + (A_\mu^2)^2) - \frac{\lambda^2}{8}(gA_\mu^3 + g'B_\mu)^2 - \lambda G_e \bar{\Psi}_e \Psi_e$$

The electron acquires the mass

$$m_e = \lambda G_e \tag{4.9}$$

The charged vector field

$$W_\mu = 2^{-1/2} \left(A_\mu^1 + i A_\mu^2 \right) \tag{4.10}$$

describes an intermediate boson with the mass

$$M_W = \frac{\lambda}{2} g \tag{4.11}$$

From the neutral fields A_μ^3, B_μ the combinations

$$\begin{aligned} Z_\mu &= (g^2 + g'^2)^{-1/2} \left(g A_\mu^3 + g' B_\mu \right), \\ A_\mu &= (g^2 + g'^2)^{-1/2} \left(-g' A_\mu^3 + g B_\mu \right) \end{aligned} \tag{4.12}$$

with the masses

$$M_Z = \frac{1}{2} \lambda (g^2 + g'^2)^{1/2} \quad , \quad M_A = 0 \tag{4.13}$$

can be formed. In this way, one component of the vector field multiplet has zero mass and it is therefore considered to be the photon field.

The interaction term of lepton and vector fields can be written as

$$\begin{aligned} &\frac{ig}{2\sqrt{2}} \bar{\psi}_e (1+\gamma_5) \nu W_\mu + i \frac{g g'}{(g^2+g'^2)^{1/2}} \bar{\psi}_e \gamma_\mu \psi_e A_\mu + \\ &+ \frac{i (g^2+g'^2)^{1/2}}{4} \left[\frac{3(g'^2-g^2)}{g^2+g'^2} \bar{\psi}_e \gamma_\mu \psi_e - \bar{\psi}_e \gamma_\mu \gamma_5 \psi_e + \bar{\nu} \gamma_\mu (1+\gamma_5) \nu \right] Z_\nu \end{aligned} \tag{4.14}$$

The second term in (4.14) implies that the electron charge e is

$$e = g g' (g^2 + g'^2)^{-1/2} \tag{4.15}$$

and is therefore smaller that any of the two original charges g, g'. Supposing that W_μ is, as usual, coupled to hadrons and to a muon we obtain

$$G_w/\sqrt{2} = \frac{g^2}{8M_w^2} = \frac{1}{2\lambda^2} \qquad (4.16)$$

It follows from (4.12) and (4.16) that masses of intermediate bosons are very large

$$M_z > 80 \text{ Gev} \quad , \quad M_w > 40 \text{ Gev} \qquad (4.17)$$

compared not only to the electron mass but also to the hadron masses.

Nevertheless, such intermediate bosons were observed experimentally on the GERN collider, as well as the neutral currents, representing the other prediction of the W.S. model.

Strong interactions also can be included into the scheme of gauge theories. The best candidate for the theory of strong interactions is quantum chromodynamics QCD. The main idea is that any strong interacting fermion (hadron) consists of 3 other fermions (quarks) which a described by the fundamental representation of $SU(3)_c$ "colour" group, and also under the "flevor" group G_f, so the full gauge group is the product $SU(3)_c \times G_f$. The QCD Lagrangian

$$-\frac{1}{8}\,\mathrm{tr}\,F_{\mu\nu}F_{\mu\nu} - \bar{\psi}\left(\gamma_\mu\left(i\partial_\mu - \varepsilon B_\mu\right) - m\right)\psi \qquad (4.18)$$

is invariant under the gauge transformations of the type of (3.26), where ψ is a quark spinor field, which has spinorial indices α and also isotopic indices.

Such a Lagrangian describes satisfactorily high energy strong interaction processes. The main problem is how to deal with the low energy interactions, for example, how to evaluate masses of strong interacting particles and low energy scattering amplitudes. The hypothesis of <u>quark confinement</u> explains why quarks can be observed only in hadrons, but not separately. This hypothesis can be justified in 3-dimensional field theories, but not in the real 4-dimensional case.

Another hypothesis in QCD is that of quark and gluon condensates. It states, that there exsist the "anomalous averages".

$$\begin{aligned} &\langle\bar{\psi}\psi\rangle && \text{quark condensate} \\ &\langle F^a_{\mu\nu}F^a_{\mu\nu}\rangle && \text{gluon condensate} \end{aligned} \qquad (4.19)$$

This supposition allows to move to the quantative description of strong interactions. Here I am not going to dwell on this subject.

At the conlusion let me touch two more aspects of gauge theories. The first one is the presence of extendend objects in gauge theories. These objects are connected with nontrivial classical solutions of the field equations. In the functional integral formalism we have to integrate over fields, which are "close" to some classical solutions. May be the most famous of such solutions is the Polyakov - t'Hooft monopole, which exists in the theory of the Yang-Mills field $\mathcal{B}_\mu^a$, interacting scalar isotopic field φ_a. The action of such a system is

$$-\frac{1}{2}\int\Big[\sum_{\mu,a}(\partial_\mu\varphi_a+\varepsilon\varepsilon_{abc}\mathcal{B}_\mu^b\varphi_c)^2-\lambda\sum_a\varphi_a^2+\frac{g}{2}\Big(\sum_a\varphi_a^2\Big)^2\Big]d^4x$$

$$-\frac{1}{2}\int\sum_{\mu,a}(\partial_\mu\mathcal{B}_\nu^a-\partial_\nu\mathcal{B}_\mu^a+\varepsilon\varepsilon_{abc}\mathcal{B}_\mu^b\mathcal{B}_\nu^c)^2d^4x \qquad (4.20)$$

We can look for a solution of the form

$$\varphi_a(x)=x_a u(r)r^{-1},\quad \mathcal{B}_\mu^a(x)=\varepsilon_{\mu ab}x_c\,(a(r)-(\varepsilon r^2)^{-1}) \qquad (4.21)$$

It can be shown that this solution has a finite energy functional and consequently can be treated as a new particle[5,6]

The other interesting solution is the so called <u>instanton</u>, which is the solution of the Yang-Mills field equations in the euclidean space with a finite action functional. Such field configuration gives a contribution when performing functional integration over Yang-Mills fields.

Attempts to find out other more realistic field theoretical models with vortex-line solutions belong to very actual problems. It is not excluded that the key to the construction of a successive strong interaction theory lies just on that path.

The last thing I should like to touch in these lectures is the "anomaly". Anomalies were invented by Adler, Bardeen, Gross, and Jackiw[7-9]. Recently a new approach to the anomalies was sugfested by L.D.Faddeev and S.L. Shatashvili.[10]

Let us consider Lagrangian, describing left fermions interacting the Yang-Mills field A (A are matrices in the Lie algebra).

$$L = i\bar{\psi}\left(\gamma_\mu\left(\partial_\mu + \tfrac{1}{2}(1-\gamma_5)A_\mu\right)\psi\right) \tag{4.22}$$

This Lagrangian is invariant under gauge transformations

$$A_\mu \to \Omega^{-1} A_\mu \Omega + \Omega^{-1}\partial_\mu \Omega \quad , \quad \psi \to \Omega^{-1}\psi \tag{4.23}$$

But, as it is well-known, the functional integral

$$Z[A] = \int \mathcal{D}\bar{\psi}\,\mathcal{D}\psi \exp\left(i\int L d^4x\right) \tag{4.24}$$

is not gauge invariant, i.e. $Z[A^\Omega] \neq Z[A]$.

In order to charactirize the behaviour of $Z[A]$ more precisely, let us introduce the generators of infinitezimal gauge transformations

$$T^a(x) = -\nabla_\mu \frac{\delta}{\delta A^a_\mu(x)} = -\left(\partial_\mu \frac{\delta}{\delta A^a_\mu} + t^{abc} A_\mu^b \frac{\delta}{\delta A^c_\mu}\right) \tag{4.25}$$

These operators obey the commutation relations

$$\left[T^a(x), T^b(y)\right] = t^{abc}\, T^c(y)\, \delta(x-y) \tag{4.26}$$

The calculations which were performed by Bardeen, Gross and Jackiw show, that

$$T^a(x)\, Z[A] = i\, U^a(x)\, Z[A] \tag{4.27}$$

where $U^a(x)$ are polinomials on $A^a_\mu(x)$ and its derivatives.

It is not difficult to show the following relation

$$T^a(x)\, U^b(y) - T^b(y)\, U^a(x) = t^{abc}\, U^c(y)\, \delta(x-y) \tag{4.28}$$

This is the so called Well-Zumino consistency condition.

Let us introduce operators

$$X^a(x) = T^a(x) + i\mu\, U^a(x) \tag{4.29}$$

where μ are real. We have

$$\left[X^a(x), X^b(y)\right] = t_{abc}\, X^c(y)\, \delta(x-y) \tag{4.30}$$

We can say that the operators $\chi^a(x)$ give us some new representation of gauge group. Formula (4.27) implies that $Z[A]$ is invariant under this new representation, if we put $\mu = -1$.

These results imply the interpretation of the anomaly as the infinitezimal 1-cocycle. It is not suprising that it can be expressed via geometrical constructions such as the second Chern-Simons class.

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