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Zero modes, beta functions and IR/UV interplay in higher-loop QED

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We analyze the relation between the short-distance behavior of quantum field theory and the strong-field limit of the background field formalism, for QED effective Lagrangians in self-dual backgrounds, at both one and two loop. The self-duality of the background leads to zero modes in the case of spinor QED, and these zero modes must be taken into account before comparing the perturbative β function coefficients and the coefficients of the strong-field limit of the effective Lagrangian. At one-loop this is familiar from instanton physics, but we find that at two-loop the role of the zero modes, and the interplay between IR and UV effects in the renormalization, is quite different. Our analysis is motivated in part by the remarkable simplicity of the two-loop QED effective Lagrangians for a self-dual constant background, and we also present here a new independent derivation of these two-loop results.

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I. INTRODUCTION

In quantum field theory there is a close connection between the short-distance behavior of renormalized Green's functions and the strong-field limit of associated quantities calculated using the background field method. This phenomenon can be interpreted as an IR/UV connection in the sense that the ultraviolet (UV) and infrared (IR) divergences are correlated. This correspondence leads, for example, to a direct relation between the perturbative β function and the strong-field asymptotics of the effective Lagrangian. Ritus derived this relation in the context of QED using the renormalization group, with the assumption that the strong-field limit of the renormalized effective Lagrangian is mass-independent [1, 2, 3]. Another, equivalent, derivation which invokes the scale anomaly [4, 5, 6] in a massless limit, has been given in various forms by many authors [7, 8, 9, 10, 11, 12, 13, 14]. Also, other approaches have been developed for extracting the β function from the effective Lagrangian, using either the operator product expansion [15] or the worldline formalism [16]. These issues have most often been investigated for magnetic or chromo-magnetic background fields, or for self-dual backgrounds (such as instantons) at one loop. Here, in this paper, we re-examine these issues at the two loop level for self-dual background fields. A self-dual background is special because it gives rise to zero modes in a spinor theory [17]. Also, for a self-dual background it is not possible to distinguish between the bare Lagrangian, $F_{\mu\nu}F^{\mu\nu}$, and the other Lorentz invariant combination, $F_{\mu\nu}\tilde{F}^{\mu\nu}$, which characterizes the zero-mode contributions, so it is necessary to identify and separate the zero mode contributions before taking the strong-field limit. Our analysis concentrates on QED, and is motivated in part by the recent results that the two-loop effective Lagrangian, in both spinor and scalar QED, for a constant self-dual background field has a remarkably simple closed form [18, 19, 20]. However, since many of the simplifications we find are due to the self-duality of the background, rather than due to the precise form of the background, another motivation is to learn which features might be applied to higher-loop calculations in other self-dual systems, such as for example QCD with instanton backgrounds.

In Section II we review the IR/UV correspondence between the perturbative β function and the strong-field limit of the effective Lagrangian. We show that a naive application of this correspondence fails for spinor QED with a self-dual background, at both one loop and two loop. In Section III we show how this apparent discrepancy is resolved at one loop by the separation of the zero mode contribution to the effective Lagrangian. In Section IV we show that at two loop the mechanism whereby the discrepancy is resolved is rather different, coming instead from a zero mode contribution to the mass renormalization. In this Section we also provide an independent derivation of the two loop effective Lagrangians for a constant self-dual background which were found previously [18, 19, 20] using the worldline formalism. The final section contains our conclusions and an appendix describes the calculation of the finite part of the mass renormalization.

II. STRONG-FIELD LIMITS AND BETA FUNCTIONS

A. General Argument

We begin by recalling the general argument [1, 2, 3, 7, 8, 9, 10, 11, 12, 13, 14] relating the strong-field asymptotic behavior of the effective Lagrangian to the perturbative β function. As we are interested here in QED, we present the argument for an abelian gauge theory, but it is more general. Consider an abelian gauge field coupled to spinor or scalar matter fields, which are either explicitly massless or which have a well-defined massless limit. Then the trace anomaly for the energy-momentum tensor states that [4, 5, 6]

$$\langle \Theta^{\mu}_{\mu} \rangle = \frac{\beta(\bar{e})}{2\bar{e}} \, \frac{e^2}{\bar{e}^2} \, (F_{\mu\nu})^2, \tag{2.1}$$

where $\bar{e}$ is the running coupling, and $\beta(\bar{e})$ is the β function, defined below in (2.5). The expectation value of the energy-momentum tensor can also be related to the effective Lagrangian for a constant background field strength $F_{\mu\nu}$:

$$\langle \Theta^{\mu\nu} \rangle = -\eta^{\mu\nu} \, \mathcal{L}_{\rm eff} + 2 \frac{\partial \mathcal{L}_{\rm eff}}{\partial \eta_{\mu\nu}}. \tag{2.2}$$

These two relations, (2.1) and (2.2), determine the effective Lagrangian to be of the form

$$\mathcal{L}_{\rm eff} = -\frac{1}{4} \, \frac{e^2}{\bar{e}^2(t)} \, F_{\mu\nu} F^{\mu\nu}, \tag{2.3}$$

where the "renormalization group time", t, is expressed in terms of the scale set by the field strength, serving as the renormalization scale parameter $\mu^2 \sim e|F|$,

$$t = \frac{1}{4} \, \ln \left(\frac{e^2 |F^2|}{\mu_0^4} \right), \tag{2.4}$$

and μ_0 denotes a fixed reference scale at which, for example, the value of the coupling may be measured. Note that in this argument the field strength plays the role which is usually associated with a momentum transfer Q^2. This already suggests at a very basic level why the strong-field and short-distance limits are related.

The β function is defined in terms of the running of the coupling as

$$\beta(\bar{e}(t)) \; = \; \frac{d\bar{e}(t)}{dt}. \tag{2.5}$$

To see how this solution (2.3) leads to an explicit connection between the strong-field asymptotics of $\mathcal{L}_{\rm eff}$ and the perturbative β function, note that (2.5) can also be expressed as

$$t = \int_e^{\bar{e}(t)} \frac{de'}{\beta(e')}, \tag{2.6}$$

where $e \equiv \bar{e}(0)$. Making a perturbative expansion of the β function

$$\beta(e) = \beta_1 e^3 + \beta_2 e^5 + \ldots \tag{2.7}$$

the relation (2.6) determines the running coupling, $\bar{e}(t)$, in terms of e as

$$\frac{1}{\bar{e}^2(t)} = \frac{1}{e^2} - 2\beta_1 \, t - 2\beta_2 e^2 \, t + O(e^4 t^2). \tag{2.8}$$

Inserting this into (2.3), the strong-field asymptotics of the effective Lagrangian is, to two-loop order,

$$\mathcal{L}_{\rm eff} \sim \frac{1}{16} \left(2\beta_1 e^2 + 2\beta_2 e^4 + \ldots \right) \, F_{\mu\nu} F^{\mu\nu} \, \ln \left(\frac{e^2 |F^2|}{\mu_0^4} \right), \tag{2.9}$$

where, as is conventional, we have subtracted the classical Lagrangian, $-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$, from $\mathcal{L}_{\rm eff}$.

In order to illustrate this correspondence explicitly, we recall that the QED β functions, for spinor and scalar QED, to two-loop order, are

$$\beta_{\rm spinor} = \frac{e^3}{12\pi^2} + \frac{e^5}{64\pi^4} + \ldots \tag{2.10}$$

$$\beta_{\rm scalar} = \frac{e^3}{48\pi^2} + \frac{e^5}{64\pi^4} + \ldots \tag{2.11}$$

The β function can also be expressed in terms of $\alpha = e^2/(4\pi)$ instead of e by a change of variables, $\beta^{(\alpha)} = \frac{d\alpha}{dt} = \frac{2e}{4\pi}\beta(e)\Big|_{e=\sqrt{4\pi\alpha}}$, leading to the form

$$\beta^{(\alpha)}_{\rm spinor} = \frac{2\alpha^2}{3\pi} + \frac{\alpha^3}{2\pi^2} + \ldots \tag{2.12}$$

$$\beta^{(\alpha)}_{\rm scalar} = \frac{\alpha^2}{6\pi} + \frac{\alpha^3}{2\pi^2} + \ldots \tag{2.13}$$

B. Explicit example: constant magnetic field background

Equation (2.9) gives a direct correspondence between the perturbative β function coefficients and the strong-field behavior of the effective Lagrangian. We now compare this with some explicit results where the effective Lagrangian is known. First, consider the Euler-Heisenberg effective Lagrangian for a constant background *magnetic* field, of strength B. At one loop, the on-shell renormalized effective Lagrangians, for spinor and scalar QED, are [22, 23]

$$\mathcal{L}^{(1)\,\rm magnetic}_{\rm spinor} = -\frac{e^2B^2}{8\pi^2}\int_0^\infty \frac{ds}{s}\, e^{-m^2 s/(eB)}\left[\frac{1}{s\,\tanh(s)} - \frac{1}{s^2} - \frac{1}{3}\right], \tag{2.14}$$

$$\mathcal{L}^{(1)\,\rm magnetic}_{\rm scalar} = \frac{e^2B^2}{16\pi^2}\int_0^\infty \frac{ds}{s}\, e^{-m^2 s/(eB)}\left[\frac{1}{s\,\sinh(s)} - \frac{1}{s^2} + \frac{1}{6}\right]. \tag{2.15}$$

The leading strong-field asymptotics is determined by the IR behavior of the propertime integrand for $s \to \infty$; at the one-loop level this yields

$$\mathcal{L}^{(1)\,\rm magnetic}_{\rm spinor} \sim \frac{e^2B^2}{24\pi^2}\ln\left(\frac{eB}{m^2}\right) + \ldots \tag{2.16}$$

$$\mathcal{L}^{(1)\,\rm magnetic}_{\rm scalar} \sim \frac{e^2B^2}{96\pi^2}\ln\left(\frac{eB}{m^2}\right) + \ldots \tag{2.17}$$

Noting that $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = -\frac{1}{2}B^2$, and comparing with the correspondence (2.9), we deduce that $\beta_1^{\rm spinor} = \frac{1}{12\pi^2}$ and $\beta_1^{\rm scalar} = \frac{1}{48\pi^2}$, in agreement with the one-loop β function coefficients quoted in (2.10) and (2.11).

The two-loop renormalized effective Lagrangians for a constant background field were derived by Ritus for both spinor [1] and scalar [2] QED. While the actual expressions for the effective Lagrangians are complicated double parameter integrals, it is nevertheless possible to extract the two-loop leading strong-field asymptotics for the constant magnetic field case [3]:

$$\mathcal{L}^{(2)\,\rm magnetic}_{\rm spinor} \sim \frac{e^4B^2}{128\pi^4}\ln\left(\frac{eB}{m^2}\right) + \ldots \tag{2.18}$$

$$\mathcal{L}^{(2)\,\rm magnetic}_{\rm scalar} \sim \frac{e^4B^2}{128\pi^4}\ln\left(\frac{eB}{m^2}\right) + \ldots \tag{2.19}$$

Once again, comparing with the correspondence (2.9), we deduce that $\beta_2^{\rm spinor} = \beta_2^{\rm scalar} = \frac{1}{64\pi^4}$, in agreement with the two-loop β function coefficients quoted in (2.10) and (2.11).

C. Explicit example: constant self-dual background

Another interesting solvable case is when the constant background field is self-dual:

$$F_{\mu\nu} = \tilde{F}_{\mu\nu} \equiv \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}. \tag{2.20}$$

It is well-known that self-dual backgrounds have special properties which often lead to dramatic simplifications. This can be traced to the fact that a self-dual background has definite helicity and the Dirac operator in such a background has a quantum mechanical supersymmetry [25, 26, 27]. Since a self-dual background has definite helicity, the effective Lagrangian for such a background can be used as a generating functional for amplitudes with all external lines having the same helicity. It is also well-known that many remarkable simplifications occur for such helicity amplitudes [28, 29]. Recently it has been found that analogous simplifications occur in the two-loop effective Lagrangian itself [18, 19, 20].

At one-loop, the on-shell renormalized effective Lagrangians for a constant self-dual background can be deduced from the results of Euler and Heisenberg [22] and Schwinger [23]:

$$\mathcal{L}^{(1)\,\mathrm{self-dual}}_{\mathrm{spinor}} = -\frac{e^2 f^2}{8\pi^2}\int_0^\infty \frac{ds}{s}\, e^{-m^2 s/(ef)}\left[\coth^2 s - \frac{1}{s^2} - \frac{2}{3}\right], \tag{2.21}$$

$$\mathcal{L}^{(1)\,\mathrm{self-dual}}_{\mathrm{scalar}} = \frac{e^2 f^2}{16\pi^2}\int_0^\infty \frac{ds}{s}\, e^{-m^2 s/(ef)}\,\left[\frac{1}{\sinh^2 s} - \frac{1}{s^2} + \frac{1}{3}\right]. \tag{2.22}$$

Here, f denotes the magnitude of the self-dual field strength,

$$\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \equiv f^2. \tag{2.23}$$

Note that the one-loop renormalized effective Lagrangians in (2.21) and (2.22) satisfy $\mathcal{L}^{(1)\,\mathrm{self-dual}}_{\mathrm{spinor}} = -2\mathcal{L}^{(1)\,\mathrm{self-dual}}_{\mathrm{scalar}}$, which is a consequence of the supersymmetry of the self-dual background [26, 27].

At two-loop, the on-shell renormalized effective Lagrangians for a constant self-dual background can be expressed in closed-form [18, 19, 20] in terms of the digamma function,

$$\mathcal{L}^{(2)\,\mathrm{self-dual}}_{\mathrm{spinor}} = -2\alpha\,\frac{m^4}{(4\pi)^3}\frac{1}{\kappa^2}\left[3\xi^2(\kappa) - \xi'(\kappa)\right], \tag{2.24}$$

$$\mathcal{L}^{(2)\,\mathrm{self-dual}}_{\mathrm{scalar}} = \alpha\,\frac{m^4}{(4\pi)^3}\frac{1}{\kappa^2}\left[\frac{3}{2}\xi^2(\kappa) - \xi'(\kappa)\right]. \tag{2.25}$$

Here the dimensionless parameter κ is defined as

$$\kappa = \frac{m^2}{2ef}, \tag{2.26}$$

and the function $\xi(\kappa)$ is

$$\xi(\kappa) = -\kappa\left(\psi(\kappa) - \ln\kappa + \frac{1}{2\kappa}\right). \tag{2.27}$$

Note that this function $\xi(\kappa)$ is essentially the digamma function, $\psi(\kappa) = \frac{d}{d\kappa}\ln\Gamma(\kappa)$, with the first two terms of its large κ asymptotic expansion subtracted (see Eq (6.3.18) in [30]). It is interesting to note also that the one-loop expressions (2.21) and (2.22) can be expressed [18] simply in terms of the function $\int^\kappa \xi$.

From (2.26) it is clear that the strong-field limit corresponds to the small κ limit. Thus, the strong-field behaviors can be deduced from the known series expansion of $\psi(\kappa)$ (see Eq (6.3.14) in [30]). For scalar QED in a constant self-dual background, the one- and two-loop leading strong-field behaviors of the effective Lagrangians (2.22) and (2.25) are [18]

$$\mathcal{L}^{(1)\,\mathrm{self-dual}}_{\mathrm{scalar}} \sim \frac{e^2}{48\pi^2}\,f^2\,\ln\left(\frac{ef}{m^2}\right) + \ldots \tag{2.28}$$

$$\mathcal{L}^{(2)\,\mathrm{self-dual}}_{\mathrm{scalar}} \sim \frac{e^4}{64\pi^4}\, f^2\, \ln\left(\frac{ef}{m^2}\right) + \dots \tag{2.29}$$

Once again, comparing with (2.9), we see that the coefficients of this leading behavior agree with the scalar QED β function coefficients at one- and two-loop in (2.11), as was noted already in [18].

On the other hand, for spinor QED in a constant self-dual background, the leading strong-field behaviors of the effective Lagrangians (2.21) and (2.24) are

$$\mathcal{L}^{(1)\,\mathrm{self-dual}}_{\mathrm{spinor}} \sim -\frac{e^2}{24\pi^2}\, f^2\, \ln\left(\frac{ef}{m^2}\right) + \dots \tag{2.30}$$

$$\mathcal{L}^{(2)\,\mathrm{self-dual}}_{\mathrm{spinor}} \sim -\frac{e^4}{32\pi^4}\, f^2\, \ln\left(\frac{ef}{m^2}\right) + \dots \tag{2.31}$$

Comparing with (2.9), we see that the coefficients of these leading behaviors **do not** agree with the spinor QED β function coefficients in (2.10), at either one-loop or two-loop. This apparent mis-match is the issue which will be resolved in the following sections of this paper. As already hinted in the Introduction, the key is that for the spinor case (but not the scalar case) there are zero modes in a self-dual background, and these zero modes must be separated first, before making the strong-field comparison.

III. ONE-LOOP EULER-HEISENBERG LAGRANGIAN FOR A SELF-DUAL FIELD

The results of (2.30) and (2.31) make it clear that the general argument given in Section II A needs to be modified in some way for the case of spinor QED with a self-dual background. This modification must take account of the presence of zero modes for spinor QED in a self-dual background. Furthermore, this must be done at any loop order. At one-loop it is well-known that the small mass limit is complicated by the existence of normalizable zero modes for the massless Dirac equation, and that the resolution is known to involve the separation of a logarithmic term proportional to the number of zero modes [17, 25, 31, 32]. Here we are interested in the role of the zero modes at two loops. However, in order to proceed to the two-loop level in subsequent sections, we first briefly recap the evaluation and renormalization of the one-loop Euler-Heisenberg effective Lagrangian for a self-dual background. This will serve to explain the difference between scalar and spinor QED, in a self-dual background, with respect to the connection between the β function coefficients and the strong-field behavior of the on-shell renormalized effective Lagrangian.

A. One-loop scalar QED in a self-dual background

For scalar QED, the one-loop effective Lagrangian, $\mathcal{L}^{(1)}$, is defined as

$$\int d^4x\, \mathcal{L}^{(1)}_{\mathrm{scalar}} = -\frac{1}{2}\, \ln\det\left(-D^2+m^2\right), \tag{3.1}$$

where $D_\mu = \partial_\mu + ieA_\mu$ is the covariant derivative. This can also be expressed in terms of the scalar propagator, $G = \frac{1}{-D^2+m^2}$, for scalar particles in the given background. For a self-dual background, the scalar propagator has a simple position space representation (up to an unimportant gauge dependent phase),

$$G(x,x') = \left(\frac{ef}{4\pi}\right)^2 \int_0^\infty \frac{dt}{\sinh^2(eft)}\, \exp\left[-m^2 t - \frac{ef}{4}(x-x')^2 \coth(eft)\right]. \tag{3.2}$$

The unrenormalized effective Lagrangian is therefore

$$\mathcal{L}^{(1)\,\mathrm{self-dual}}_{\mathrm{scalar,unren.}} = \left(\frac{ef}{4\pi}\right)^2 \int_0^\infty \frac{dt}{t}\, e^{-m^2 t}\, \left[\frac{1}{\sinh^2(eft)}\right]. \tag{3.3}$$

To obtain the renormalized one-loop effective Lagrangian, $\mathcal{L}^{(1)}_R$, we first subtract the (divergent) zero-field contribution, and then introduce an ultraviolet cutoff Λ through a lower bound $\frac{1}{\Lambda^2}$ on the proper-time t integral. Furthermore, we introduce a (redundant) renormalization scale μ by writing

$$\begin{aligned}\mathcal{L}^{(1)\,\mathrm{self-dual}}_{\mathrm{scalar}} &= \left(\frac{ef}{4\pi}\right)^2 \int_0^\infty \frac{dt}{t}\, e^{-m^2 t}\left[\frac{1}{\sinh^2(eft)} - \frac{1}{(eft)^2} + \frac{1}{3}\right] + \frac{e^2}{48\pi^2} f^2 \left[\ln\left(\frac{m^2}{\mu^2}\right) + \gamma + \ln\left(\frac{\mu^2}{\Lambda^2}\right)\right] \\ &= \mathcal{L}^{(1)\,\mathrm{self-dual}}_{\mathrm{scalar,R}} + \frac{\alpha}{12\pi} f^2 \left[\ln\left(\frac{\mu^2}{\Lambda^2}\right) + \gamma\right]. \end{aligned} \tag{3.4}$$

Here γ is Euler's constant, and we dropped terms of $\mathcal{O}(m^2/\Lambda^2)$. In the last line, we separated the renormalized one-loop Lagrangian, $\mathcal{L}^{(1)}_R$, from the counterterm. The latter can be combined with the unrenormalized classical action, corresponding to charge and field strength renormalization, so that $\alpha = \alpha(\mu) = \frac{e^2(\mu)}{4\pi}$ becomes the running coupling [23]. For instance, implementing electron mass-shell renormalization conditions, $\mu = m$, so that $\alpha(m) \simeq 1/137$, the $\ln m^2/\mu^2$ term would drop out. However, since we are interested in the strong-field limit and its mass-(in-)dependence, let us keep the μ dependence.

For our purposes, it is important to observe that the strong-field limit of the effective Lagrangian [see Eq. (2.28)] comes from the $\frac{1}{3}$ term inside the square brackets in the first line of (3.4), while the one-loop β function coefficient is determined by the μ dependence in the logarithmic term on the second line of (3.4), which is the charge renormalization counterterm. These terms have the same coefficient, which illustrates the connection (2.9) between the strong-field limit of the one-loop effective Lagrangian and the one-loop β function. It also confirms the assumption [3] of mass-independence of the strong-field limit, since in the $m \to 0$ limit we can write

$$\mathcal{L}^{(1)\,\mathrm{self-dual}}_{\mathrm{scalar,R}} \sim \frac{\alpha}{12\pi} f^2 \ln\left(\frac{ef}{m^2}\right) + \frac{\alpha}{12\pi} f^2 \ln\left(\frac{m^2}{\mu^2}\right) = \frac{\alpha}{12\pi} f^2 \ln\left(\frac{ef}{\mu^2}\right), \tag{3.5}$$

which guarantees that the limit $m \to 0$ can be taken. This is important because the existence of a well-defined massless limit is a necessary prerequisite for the trace-anomaly argument described in Section IIA.

B. One-loop spinor QED in a self-dual background

For spinor QED, the one-loop effective Lagrangian, $\mathcal{L}^{(1)}$, is defined as

$$\int d^4x\, \mathcal{L}^{(1)}_{\mathrm{spinor}} = \ln\det\left(D\!\!\!/ + m\right) = \frac{1}{2}\ln\det\left(-D\!\!\!/^{\,2} + m^2\right). \tag{3.6}$$

A self-dual background has definite helicity [17, 25, 26, 27], which has the consequence that

$$D\!\!\!/^{\,2}\, P_L = D^2\, P_L, \tag{3.7}$$

where $P_L = \frac{1}{2}(1+\gamma_5)$ is the projector onto positive helicity states. It follows that the spinor propagator S can be expressed in terms of the scalar propagator G [31, 32, 33],

$$\begin{aligned} S &= \frac{1}{D\!\!\!/ + m} \\ &= -\left(D\!\!\!/ - m\right) G\, P_L - G\, D\!\!\!/\, P_R + \frac{1}{m}\left(1 + D\!\!\!/\, G D\!\!\!/\right) P_R, \end{aligned} \tag{3.8}$$

where $P_R = \frac{1}{2}(1-\gamma_5)$. This can also be expressed in proper-time form as [33]

$$S = -\int_0^\infty dt\, e^{-m^2 t}\left[\left(D\!\!\!/ - m\right) e^{D^2 t}\, P_L + e^{D^2 t}\, D\!\!\!/\, P_R - m D\!\!\!/\, \frac{1}{D^2} e^{D^2 t} D\!\!\!/\, P_R - m\, P\right]. \tag{3.9}$$

The last term in (3.9) involves the projector, P, onto the zero modes,

$$P = \left(1 + D\!\!\!/\, G_0 D\!\!\!/\right) P_R, \tag{3.10}$$

where $G_0 = \lim_{m\to 0} G$ denotes the massless scalar propagator. After some straightforward Dirac traces, one finds that the one-loop spinor effective Lagrangian (3.6) can be expressed in terms of the scalar effective Lagrangian, up to a zero-mode projection contribution,

$$\mathcal{L}^{(1)\,\mathrm{self-dual}}_{\mathrm{spinor}} = -\frac{1}{2}\frac{1}{V}\int_0^\infty \frac{dt}{t}\, e^{-m^2 t}\left\{\mathrm{Tr}_x\left[4\, e^{D^2 t}\right] + \mathrm{Tr}_x\, \mathrm{tr}_{\mathrm{Dirac}}\, P\right\}, \tag{3.11}$$

where V denotes the spacetime volume. The first term on the RHS of Eq. (3.11) is just $-2\mathcal{L}_{\rm scalar}^{(1)\,\rm self-dual}$, which is, as mentioned above, a direct reflection of the supersymmetry of the self-dual background at one-loop. The second term on the RHS of Eq. (3.11) counts the number (density) of zero modes

$$\begin{aligned} n_{\rm F} &\equiv \frac{1}{V}\,{\rm Tr}_x\,{\rm tr}_{\rm Dirac}P \\ &= \left(\frac{ef}{2\pi}\right)^2 . \end{aligned} \tag{3.12}$$

Note that $n_{\rm F}$ is just the square of the usual 2d Landau degeneracy factor, since the 4d self-dual system factorizes into two orthogonal 2d Landau systems [9].

Thus, the one-loop spinor effective Lagrangian can be expressed in proper-time form as

$$\begin{aligned} \mathcal{L}_{\rm spinor}^{(1)\,\rm self-dual} &= -\frac{(ef)^2}{8\pi^2}\int_{1/\Lambda^2}^{\infty}\frac{dt}{t}e^{-m^2 t}\left[\frac{1}{\sinh^2(eft)}-\frac{1}{t^2}+1-\frac{2}{3}+\frac{2}{3}\right] \\ &= -\frac{(ef)^2}{8\pi^2}\int_0^\infty\frac{dt}{t}\,e^{-m^2 t}\left[\frac{1}{\sinh^2(eft)}-\frac{1}{t^2}+\frac{1}{3}\right]+\frac{e^2}{12\pi^2}f^2\left[\ln\left(\frac{m^2}{\mu^2}\right)+\gamma+\ln\left(\frac{\mu^2}{\Lambda^2}\right)\right] \\ &= \mathcal{L}_{\rm spinor,R}^{(1)\,\rm self-dual}+\frac{\alpha}{3\pi}\,f^2\left[\ln\left(\frac{\mu^2}{\Lambda^2}\right)+\gamma\right]. \end{aligned} \tag{3.13}$$

On the first line of (3.13), the 1 refers to the zero mode contribution, and $\frac{2}{3}$ is added and subtracted to achieve the charge renormalization. As in the scalar case (3.4), the strong-field limit is read off from the last term, $\frac{1}{3}$, inside the square brackets on the second line of (3.13), while the one-loop β function coefficient is read off from the coefficient of the logarithmic terms, which are responsible for charge renormalization. In this spinor case, in contrast to the scalar case, the coefficients of these two terms are different. It is clear that the source of the difference is precisely the zero mode contribution. This explains the mis-match, at one-loop, between the strong-field asymptotics of the one-loop effective Lagrangian and the one-loop β function coefficient in the spinor case with the self-dual background field. This mis-match, due to the zero mode contribution, also prevents the strong-field limit from becoming mass-independent [as it was in the scalar case: see Eq. (3.5)], and therefore violates the assumptions of the trace-anomaly argument given in Section IIA.

There is another useful perspective on this mis-match in the one-loop spinor self-dual case: note that the *unrenormalized* one-loop Lagrangian in the scalar case (3.3) is infrared (IR) finite even in the massless limit. However, the *renormalized* one-loop Lagrangian (3.4) has an IR divergence at the upper bound of the proper-time integral in the massless limit (which is actually cancelled by the $\ln m^2/\mu^2$ term). It is precisely the charge renormalization subtraction removing the logarithmic UV divergence which introduces the IR divergence. And it is the IR divergence which dominates the strong-field limit. Hence, we observe an UV/IR connection in QED: the strong-field limit results from the IR behavior of the proper-time integral which receives contributions from the counter-terms controlling the UV behavior. This means that if the *unrenormalized* Lagrangian is IR finite (as it is in magnetic backgrounds for both scalar and spinor QED, and in the scalar self-dual case), the β function and the strong-field limit coincide. The presence of zero modes, in the spinor self-dual case, obviously spoils this UV/IR connection, because it leads to an additional IR divergence of the unrenormalized Lagrangian.

IV. TWO-LOOP EULER-HEISENBERG LAGRANGIAN FOR A SELF-DUAL FIELD

We now turn to a two-loop analysis of the Euler-Heisenberg Lagrangian for a self-dual field in both spinor and scalar QED. We concentrate again on the role of the zero modes. Interestingly, this role will turn out to be of a different nature than at one loop. The unrenormalized two-loop spinor Lagrangian can be written in coordinate space as

$$\mathcal{L}_{\rm spinor}^{(2)}=\frac{e^2}{2}\int d^4x'\,\mathcal{D}(x-x')\,{\rm tr}_{\rm Dirac}\left[\gamma_\mu\langle x|S|x'\rangle\gamma_\mu\langle x'|S|x\rangle\right], \tag{4.1}$$

where we have introduced the photon propagator $\mathcal{D}(x-x')=[4\pi^2(x-x')^2]^{-1}$ and work in the Feynman gauge for convenience. Furthermore, we have used bracket notation for the propagators, $S(x,x')\equiv\langle x|S|x'\rangle$.

The influence of the zero-mode contribution can conveniently be studied with the aid of representation (3.8,3.9) of the spinor propagator S, which allows for a separation of the zero-mode contribution,

$$\mathcal{L}_{\rm spinor,\,z.m.}^{(2)}=2e^2\int d^4x'\,\mathcal{D}(x-x')\int_0^\infty dt\,{\rm e}^{-m^2 t}\,\langle x|{\rm e}^{D^2 t}|x'\rangle\,{\rm tr}_{\rm Dirac}\langle x'|P|x\rangle. \tag{4.2}$$

Here the first matrix element corresponds to the proper-time integrand of the massless scalar propagator (3.2), and the second matrix element contains the projector P onto the zero modes, defined in (3.10). For a constant self-dual background

$$\langle x|e^{D^2 t}|x'\rangle = \frac{1}{(4\pi)^2}\frac{(ef)^2}{\sinh^2(eft)}\,\exp\left[-\frac{ef}{4}(x-x')^2\coth(eft)\right]. \tag{4.3}$$

Thus, we can write

$$\mathcal{L}^{(2)}_{\rm spinor,\,z.m.} = \frac{e^2}{32\pi^4}\,(ef)\int_0^\infty \frac{ds}{\sinh^2 s}e^{-m^2 s/(ef)}\int d^4x'\frac{1}{(x-x')^2}\,\exp\left[-\frac{ef}{4}(x-x')^2\coth s\right]\,{\rm tr}_{\rm Dirac}\langle x'|P|x\rangle. \tag{4.4}$$

In the strong-field limit, $\frac{ef}{m^2}\to\infty$, we observe that

$$\frac{1}{4\pi^2(x-x')^2}\,\exp\left[-\frac{ef}{4}(x-x')^2\coth s\right]\;\to\;\frac{1}{ef\coth s}\delta(x-x'). \tag{4.5}$$

Therefore, in the strong-field limit the zero mode contribution is

$$\mathcal{L}^{(2)}_{\rm spinor,z.m.}\to\frac{\alpha}{2\pi}\int_0^\infty ds\,\frac{{\rm e}^{-\frac{m^2}{ef}s}}{\sinh s\cosh s}\,n_{\rm F}. \tag{4.6}$$

Our main observation here is that the zero-mode contribution (4.6) to the unrenormalized two-loop Lagrangian is IR finite even in the massless limit. This should be contrasted with the one-loop spinor case [see Eqs. (3.11) and (3.12)], where the zero-mode contribution is the source of the IR divergence. At two-loops, even though Eq. (4.6) has a UV divergence to be absorbed in charge renormalization, the corresponding subtraction contributes equally to the β function and the strong-field limit by virtue of the UV/IR connection discussed at the end of Section III. Therefore, the zero-mode contribution identified in Eq. (4.6) is **not** the source of the difference between the β function coefficients and the strong-field limit coefficients.

In order to pinpoint the actual source of the mis-match at the two-loop level, let us perform the calculation in a straightforward way, starting from Eq. (4.1) and using the relations (3.8) and (3.9) to trade the spinor propagators S for a representation in terms of the scalar propagator G. (This derivation complements the two-loop derivations in [18, 19], which were done using the world-line formalism.) After taking the Dirac trace, we arrive at a simple form in terms of matrix elements of G [19]:

$$\mathcal{L}^{(2)}_{\rm spinor} = \frac{e^2}{2}\int d^4x'\,\mathcal{D}(x-x')\left[-8\,\langle x|D_\alpha G|x'\rangle\langle x'|D_\alpha G|x\rangle + 16\,\langle x|G|x'\rangle\langle x'|D_\alpha G D_\alpha|x\rangle + 16\,\langle x|x'\rangle\langle x'|G|x\rangle\right]. \tag{4.7}$$

The last term corresponds to a "tadpole" diagram in the scalar language, suggesting a quadratic divergence. However, this is only seemingly the case; in fact, there is a cancellation of the last term with a corresponding divergence in the second term. This can be seen from the identity

$$\langle x|D_\alpha G D_\alpha|x'\rangle = -\langle x|x'\rangle + m^2\langle x|G|x'\rangle + \frac{(ef)^2}{2}(x-x')^2\langle x|G|x'\rangle, \tag{4.8}$$

which makes it clear that the "tadpole" terms cancel in spinor QED (as they should).

The scalar two-loop Lagrangian is also written in a form analogous to Eq. (4.7), but with different coefficients:

$$\mathcal{L}^{(2)}_{\rm scalar} = -e^2\int d^4x'\,\mathcal{D}(x-x')\left[\langle x|D_\alpha G|x'\rangle\langle x'|D_\alpha G|x\rangle + \langle x|G|x'\rangle\langle x'|D_\alpha G D_\alpha|x\rangle + 4\,\langle x|x'\rangle\langle x'|G|x\rangle\right]. \tag{4.9}$$

In this case, the "tadpole" terms no longer cancel, and a quadratic divergence remains. This is exactly as expected in the scalar case, since this quadratic divergence reflects the presence of a relevant operator, the scalar mass term, in scalar QED. In fact, as we shall confirm below, the complete tadpole term, including its divergence, can be absorbed into the mass renormalization.

Before we proceed with the evaluation of these expressions, let us comment that we derived Eqs.(4.7,4.9) without recourse to the explicit form of the background field. Only the self-duality of the background has been used, so that we expect the similarities between Eqs.(4.7) and (4.9) to hold also in a more general context.

A. Calculation of the two-loop Lagrangians

The representations (4.7) and (4.9) of the spinor and scalar two-loop Lagrangians can be constructed from two basic terms (we drop "tadpole" terms in the scalar case from now on, since they only modify the mass renormalization as discussed above):

$$\mathcal{L}^{(2)} = \frac{\alpha^2}{(4\pi)^2}\, f^2 \Big(A\, I_1 + B\, I_2\Big), \tag{4.10}$$

where the numerical coefficients are $A = -4$ and $B = 8$ in the spinor case, and $A = B = -1$ in the scalar case. The integrals I_1 and I_2 are

$$\begin{aligned} I_1 &:= \frac{(4\pi)^3}{\alpha f^2} \int d^4x'\, \mathcal{D}(x-x')\, \langle x|D_\alpha G|x'\rangle\langle x'|D_\alpha G|x\rangle \\ &= \int_0^\infty y dy\, \mathrm{e}^{-\frac{m^2}{ef}y} \left[\frac{2}{\sinh^2 y} - \frac{\coth y}{\sinh^2 y}\int_0^1 du\big(\coth yu + \coth y(1-u)\big)\right], \qquad (4.11) \\ I_2 &:= \frac{(4\pi)^3}{\alpha f^2} \int d^4x'\, \mathcal{D}(x-x')\, \Big(\langle x|G|x'\rangle\langle x'|D_\alpha G D_\alpha|x\rangle + \langle x|G|x'\rangle\langle x|x'\rangle\Big) \\ &= \int_0^\infty y dy\, \mathrm{e}^{-\frac{m^2}{ef}y} \left[\frac{2}{\sinh^2 y} + \frac{m^2}{ef}\frac{1}{\sinh y}\int_0^1 \frac{du}{(\sinh yu)(\sinh y(1-u))}\right], \qquad (4.12) \end{aligned}$$

where we have inserted the proper-time form of the scalar propagator (3.2), leading to a proper-time double integral. Furthermore, we have rescaled the proper-time parameter as $s = eft$, and then performed the substitutions $y = s+s'$, $u = s'/(s+s')$. Both integrals I_1 and I_2 are IR finite but UV divergent and require regularization. As at one loop, we introduce an UV proper-time cutoff for each proper-time integral which implies

$$\int_0^\infty dy \to \int_{\frac{2ef}{\Lambda^2}}^\infty dy \quad ; \quad \int_0^1 du \to \int_{\frac{ef}{\Lambda^2 y}}^{1-\frac{ef}{\Lambda^2 y}} du. \tag{4.13}$$

Now the u integrations can be performed, and we arrive at representations for I_1 and I_2 which are similar:

$$\begin{aligned} I_1 &= \int_{\frac{2ef}{\Lambda^2}}^\infty dy\, \mathrm{e}^{-\frac{m^2}{ef}y} \left\{\frac{2y}{\sinh^2 y} + \frac{m^2}{ef}\frac{1}{\sinh^2 y}\left(\ln\left[\frac{\sinh(y-\frac{ef}{\Lambda^2})}{\sinh(\frac{ef}{\Lambda^2})}\right] - \frac{ef}{m^2}\coth(y-\frac{ef}{\Lambda^2})\right)\right\}, \qquad (4.14) \\ I_2 &= \int_{\frac{2ef}{\Lambda^2}}^\infty dy\, \mathrm{e}^{-\frac{m^2}{ef}y} \left\{\frac{2y}{\sinh^2 y} + \frac{2m^2}{ef}\frac{1}{\sinh^2 y}\, \ln\left[\frac{\sinh(y-\frac{ef}{\Lambda^2})}{\sinh(\frac{ef}{\Lambda^2})}\right]\right\}. \qquad (4.15) \end{aligned}$$

An important observation is that each term in these expressions for I_1 and I_2 can be naturally expressed in terms of the function $\xi(\kappa)$ which was defined in (2.26) and (2.27). This function $\xi(\kappa)$ has the following integral representation

$$\xi \;=\; -\frac{1}{2}\int_0^\infty dy\, \mathrm{e}^{-\frac{m^2}{ef}y}\left(\frac{1}{\sinh^2 y} - \frac{1}{y^2}\right) \tag{4.16}$$

Recalling that $\kappa = m^2/(2ef)$, it follows that

$$\xi' \;=\; \int_0^\infty dy\, \mathrm{e}^{-\frac{m^2}{ef}y}\, y\left(\frac{1}{\sinh^2 y} - \frac{1}{y^2}\right) \tag{4.17}$$

Thus, the first term in each of I_1 and I_2 can be expressed in terms of ξ':

$$\begin{aligned} \int_{\frac{2ef}{\Lambda^2}}^\infty dy\, \mathrm{e}^{-\frac{m^2}{ef}y}\, \frac{y}{\sinh^2 y} &= \int_{\frac{2ef}{\Lambda^2}}^\infty dy\, \mathrm{e}^{-\frac{m^2}{ef}y}\, y\left(\frac{1}{\sinh^2 y} - \frac{1}{y^2}\right) + \int_{\frac{2ef}{\Lambda^2}}^\infty dy\, \mathrm{e}^{-\frac{m^2}{ef}y}\, \frac{1}{y} \\ &= \xi'(\kappa) + \left(-\gamma - \ln\left(\frac{2m^2}{\Lambda^2}\right)\right) \qquad (4.18) \end{aligned}$$

where we have dropped terms that vanish as the cutoff is removed (i.e., as $\Lambda \to \infty$).

Similarly, by considering the integral representation for $\xi^2(\kappa)$ we find that the log terms in the expressions (4.14) and (4.15) for I_1 and I_2 can also be expressed in terms of $\xi(\kappa)$ as

$$\frac{m^2}{ef}\int_{\frac{2ef}{\Lambda^2}}^{\infty} dy\, \mathrm{e}^{-\frac{m^2}{ef}y}\frac{1}{\sinh^2 y}\ln\left[\frac{\sinh(y-\frac{ef}{\Lambda^2})}{\sinh(\frac{ef}{\Lambda^2})}\right] = \frac{1}{2}-2\xi^2(\kappa)-2\frac{m^2}{ef}\left[\ln\left(\frac{\Lambda^2}{m^2}\right)+1-\gamma\right]\xi(\kappa) -2\kappa^2\left[\left(\ln\left(\frac{\Lambda^2}{m^2}\right)+1-\gamma\right)^2+1-\frac{2\Lambda^2}{m^2}\ln 2\right] \tag{4.19}$$

where once again we have dropped terms which vanish as the cutoff Λ is removed. Also notice that the last parenthesis term in (4.19) is proportional to κ^2, and so when inserted into the two-loop effective Lagrangian in (4.10) this term gives a field-independent contribution to the effective Lagrangian. Thus, we neglect this term, since it cancels when we subtract the zero field effective Lagrangian.

The remaining term in I_1 can also be written in terms of $\xi(\kappa)$, as

$$\int_{\frac{2ef}{\Lambda^2}}^{\infty} dy\, \mathrm{e}^{-\frac{m^2}{ef}y}\frac{\coth\left(y-\frac{ef}{\Lambda^2}\right)}{\sinh^2 y} = -\frac{1}{3}(1+\ln 2)+2\kappa\,\xi(\kappa) +\kappa^2\left[\left(3-2\gamma+2\ln\left(\frac{\Lambda^2}{m^2}\right)\right)-4\frac{\Lambda^2}{m^2}\ln 2+(4\ln 2-2)\,\frac{\Lambda^4}{m^4}\right] \tag{4.20}$$

up to terms vanishing as the cutoff is removed. Note that the final term in (4.20) is proportional to κ^2, and so can be dropped as it leads to a field-independent contribution to the effective Lagrangian.

So, putting everything together, we see that the entire two-loop effective Lagrangian (4.10) can be written in terms of the function $\xi(\kappa)$:

$$\mathcal{L}^{(2)} = \frac{\alpha^2}{(4\pi)^2}f^2\left\{2(A+B)\xi'-2(A+2B)\xi^2-4\kappa\,\xi\left[(A+2B)\left(\ln\left(\frac{\Lambda^2}{m^2}\right)-\gamma\right)+\left(\frac{3}{2}A+2B\right)\right] +\left[2(A+B)\left(\ln\left(\frac{\Lambda^2}{m^2}\right)-\gamma\right)+\left(\frac{5}{6}A+B\right)-\left(\frac{5}{3}A+2B\right)\ln 2\right]\right\} \tag{4.21}$$

The last term on the RHS of (4.21) is proportional to the bare Maxwell Lagrangian f^2, and so corresponds to the charge renormalization counterterm. However, even after doing this charge renormalization there remains on the RHS of (4.21) a logarithmic UV divergence with a nontrivial field dependence $\sim \xi(\kappa)$. This term can be seen to contribute to mass renormalization by noting that

$$-4\kappa\,\xi = 8\pi^2\left[\left\{\begin{matrix}1\\-2\end{matrix}\right\}\frac{m^2}{(ef)^2}\frac{\partial}{\partial m^2}\mathcal{L}^{(1)\,\mathrm{ren}}_{\left\{\substack{\mathrm{sp}\\ \mathrm{sc}}\right\}}-\frac{1}{8\pi^2}\left\{\begin{matrix}1\\0\end{matrix}\right\}\right], \tag{4.22}$$

where we have used a combined notation for spinor (upper) and scalar (lower) QED. The important difference arises from the last term in Eq. (4.22): in the scalar case this term is zero, and the ξ function is the only contribution required for a mass renormalization of the one-loop Lagrangian,

$$\mathcal{L}^{(1)}(ef,m_{\mathrm{R}}^2)=\mathcal{L}^{(1)}(ef,m^2)+\frac{\partial\mathcal{L}^{(1)}(ef,m^2)}{\partial m^2}\delta m^2, \tag{4.23}$$

where m_{R} denotes the renormalized mass, and δm^2 is the mass renormalization counterterm. However, in the spinor case, the ξ function is not sufficient, but has to be supplemented by the last term of Eq. (4.22) which accounts for the zero-mode contribution in $\mathcal{L}^{(1)}_{\mathrm{spinor}}$.

Inserting the mass renormalization representation (4.22) for $\kappa\,\xi$ into (4.21), we find that the unrenormalized two-loop Lagrangian can finally be written as

$$\mathcal{L}^{(2)}_{\left\{\substack{\mathrm{sp}\\ \mathrm{sc}}\right\}} = \frac{\alpha^2}{(4\pi)^2}f^2\left[2(A+B)\xi'-2(A+2B)\xi^2\right] +\frac{\alpha}{8\pi}\left\{\begin{matrix}1\\-2\end{matrix}\right\}\left[(A+2B)\left(\ln\left(\frac{\Lambda^2}{m^2}\right)-\gamma\right)+\left(\frac{3A}{2}+2B\right)\right]m^2\frac{\partial}{\partial m^2}\mathcal{L}^{(1)\,\mathrm{ren}}_{\left\{\substack{\mathrm{sp}\\ \mathrm{sc}}\right\}} +\frac{\alpha^2}{(4\pi)^2}f^2\left[\left\{\begin{matrix}A\\2(A+B)\end{matrix}\right\}\left(\ln\left(\frac{\Lambda^2}{m^2}\right)-\gamma\right)+\left\{\begin{matrix}-\frac{2A}{3}-B\\ \frac{5A}{6}+B\end{matrix}\right\}-\left(\frac{5A}{3}+2B\right)\ln 2\right] \tag{4.24}$$

This is our final result for the bare regularized two-loop Lagrangian, written in a transparent way such that renormalization is almost self-evident. The renormalized two-loop Lagrangian corresponds to the first term in (4.24); inserting $A = -4$ and $B = 8$ in the spinor case, and $A = B = -1$ in the scalar case, we rediscover the results of [18, 19, 20] quoted in Eqs. (2.24,2.25). Note that the results of [18, 19, 20] were derived using the world-line representation of the effective Lagrangian, so the result (4.24) provides an independent confirmation.

The second term in (4.24) represents the mass renormalization counter-term that has to be added to the one-loop Lagrangian in the spirit of Eq. (4.23). Here we can also read off the one-loop mass shift (apart from "tadpole" contribution for the scalar case, as discussed above),

$$\delta m^2_{\left\{ {\rm sp} \atop {\rm sc} \right\}} = \left\{ {1 \atop 1/2} \right\} \frac{3\alpha}{2\pi} \left[\left(\ln\left(\frac{\Lambda^2}{m^2}\right) - \gamma \right) + \left\{ {5/6 \atop 7/6} \right\} \right] m^2, \tag{4.25}$$

which agrees with independent one-loop computations using a proper-time cutoff [23, 34].

The last term in (4.24) correponds to the charge renormalization counterterm which has to be added to the Maxwell Lagrangian in order to renormalize the coupling and field strength (as in the one-loop case, we can trade the UV cutoff scale Λ for an arbitrary renormalization scale μ). Inserting the appropriate values for the coefficients A and B we find

$$\delta\mathcal{L}^{(2)\text{charge ren.}}_{\left\{ {\rm sp} \atop {\rm sc} \right\}} = -\frac{e^4}{64\pi^4} f^2 \left[\left(\ln\left(\frac{\Lambda^2}{m^2}\right) - \gamma \right) + \left\{ {4/3 \atop 11/24} \right\} + \left\{ {7/3 \atop -11/12} \right\} \ln 2 \right]. \tag{4.26}$$

As expected, from these charge renormalization terms we can read off the correct two-loop β function coefficients quoted in Eqs.(2.10,2.11).

The two-loop origin of the mis-match (in the spinor case with a self-dual background) between the β function coefficient and the strong-field behavior becomes clear now: although the zero-mode contribution exerts no direct influence on the IR behavior of the unrenormalized Lagrangian (cf. Eq. (4.6)), the zero-mode contribution to the mass renormalization term in Eq. (4.22) introduces another UV divergence which, together with the overall UV divergence of the unrenormalized Lagrangian, leads to the correct β function. Whereas the overall UV divergence contributes equally to the strong-field limit by the UV/IR connection, the zero-mode UV divergence from the mass renormalization does not affect the strong field limit. This is the subtle source of the mis-match in the spinor case at two-loop. We stress again that this is very different from the more familiar role of the zero-modes at one-loop, as described in Section IIIB.

V. CONCLUSIONS

We have analyzed the relation between the short-distance behavior and the strong-field limit of QED with electromagnetic backgrounds. On the one hand, the strong-field asymptotics of a renormalized QED effective Lagrangian is generally determined by its infrared behavior. Since, on the other hand, the terms which are relevant for the ultraviolet behavior affect also the infrared simply for dimensional reasons, quantum fluctuations induce an IR/UV interplay. In many instances, this mechanism leads to an exact IR/UV correspondence between the strong-field limit and the β function. For instance in the case of magnetic backgrounds or scalar QED, the strong-field limit can be computed from the β function and vice versa, as is also suggested by an argument involving the trace anomaly. The necessary condition for this exact IR/UV correspondence as well as the trace-anomaly argument is the mass independence of the strong-field limit or, phrased differently, the validity of the theory in the massless limit.

In the case of spinor QED in a self-dual background, an apparent discrepancy arises from a comparison of the strong-field behavior of the effective Lagrangian with the behavior predicted by the perturbative β function and a naive application of the trace-anomaly argument. The key to the resolution is the appearance of zero modes for spinor QED in a self-dual background which invalidate a direct massless limit. This is well understood at one-loop, but we found that the role of the zero modes is rather different at two loop. Indeed, one way to understand our two loop results is that there is, in fact, only really a one-loop effect: at the two-loop level, the zero modes do not introduce a new IR divergence, but enter instead through the inevitable reappearance of the one-loop effective Lagrangian via mass renormalization.

One motivation for our work is to prepare for future studies of higher-loop calculations in QCD for quarks in a self-dual instanton background. A great deal is known about this at one-loop [25, 32, 33, 35], but not at the two loop level. Many features of our QED analysis generalize to the instanton case because it was primarily the self-duality of the background, rather than its spacetime independence, which was most important. However, one major difference is that in QED the internal photon propagator does not feel the background field, while for the corresponding QCD diagram the internal gluon propagator *does* couple to the background (instanton) field. Here it would be interesting to make connection with the QED and QCD analysis of the one-loop polarization operator $\Pi_{\mu\nu}(Q^2)$ in a self-dual background, where the role of the zero modes has also been studied [36, 37].

Finally, we conclude with a discussion of a renormalization group (RG) interpretation of our results, which gives another perspective to the IR/UV connection in the renormalization of spinor and scalar QED in these self-dual backgrounds. The discrepancy between the strong-field limit and the β function coefficients can be viewed from a different perspective with the aid of a renormalization group (RG) equation for the effective Lagrangian. The RG equation can be derived from the statement that the renormalized Lagrangian is independent of the renormalization scale μ,

$$\mu \frac{d}{d\mu}\,\mathcal{L}(eF,\alpha,m;\mu) = 0, \tag{5.1}$$

where all quantities are assumed to be renormalized. Equation (5.1) states that any shift in μ is compensated for by corresponding shifts of the renormalized parameters. Since the product eF is RG invariant, it acts only as a spectator in the following considerations and can be omitted from now on. Introducing the anomalous mass dimension

$$\gamma_m = -\frac{\mu}{m}\frac{\partial m}{\partial \mu}, \tag{5.2}$$

the RG equation can be written as

$$\left(\mu\frac{\partial}{\partial\mu} + \beta^{(\alpha)}\frac{\partial}{\partial\alpha} - \gamma_m\, m\frac{\partial}{\partial m}\right)\mathcal{L}(\alpha,m;\mu) = 0. \tag{5.3}$$

If the strong-field limit was mass-independent, we could drop the term $\sim \gamma_m$ in Eq. (5.3) and read off the β function from this limit. In the self-dual spinor case, however, the mass-dependence induced by the zero modes forces us to keep this term even in the strong-field limit where $\mathcal{L}$ at one-loop is given by

$$\mathcal{L} = -\frac{(ef)^2}{4\pi\alpha} - \frac{(ef)^2}{24\pi^2}\ln\frac{ef}{m^2} + \frac{(ef)^2}{12\pi^2}\ln\frac{m^2}{\mu^2}, \quad \text{for } \frac{ef}{m^2}\to\infty. \tag{5.4}$$

The first term is simply the renormalized Maxwell term. Inserting Eq. (5.4) into Eq. (5.3) leads us to

$$\beta_m^{(\alpha)} = \frac{2}{3}\frac{\alpha^2}{\pi} + \gamma_m\frac{\alpha^2}{\pi} + \ldots, \tag{5.5}$$

where the dots represent higher-loop contributions. Here we appended the subscript m to the β function in order to indicate the mass dependence. Since γ_m is of order α, namely

$$\gamma_m = -\frac{\mu}{m}\frac{\partial m}{\partial\mu} = -\frac{1}{2}\frac{\mu}{m^2}\frac{\partial\delta m^2}{\partial\mu} = -\frac{3}{2}\frac{\alpha}{\pi} + \ldots, \tag{5.6}$$

as can be read off from Eq. (4.25) by trading Λ for μ, the mass dependence induces contributions to the β_m function at the two-loop level and higher. Adding the standard two-loop coefficient as obtained within a mass-independent scheme, we find

$$\beta_m^{(\alpha)} = \frac{2}{3}\frac{\alpha^2}{\pi} + \frac{1}{2}\frac{\alpha^3}{\pi^2} - \frac{3}{2}\frac{\alpha^3}{\pi^2} + \mathcal{O}(\alpha^4) = \frac{2}{3}\frac{\alpha^2}{\pi} - \frac{\alpha^3}{\pi^2} + \mathcal{O}(\alpha^4), \tag{5.7}$$

so that the two-loop coefficient in the nomenclature used in Sect. II reads $\beta_{m,2} = -1/(32\pi^4)$. This coefficient matches perfectly with the two-loop strong-field limit of the self-dual spinor case given in Eq. (2.31).

We can interpret the coincidence in the following way: there is, in fact, a correspondence between the strong-field limit and the β function in the self-dual spinor case at two-loop; but this correspondence applies only to the β function β_m of an implicitly electron-mass-dependent regularization scheme. (Note that the standard argument [38] of scheme-independence of the two-loop coefficient holds for mass-independent schemes only.) This mass-dependent scheme is natural in the self-dual spinor case because of the presence of the zero modes which inhibit a direct massless limit.

This analysis can be performed at any loop order. In those cases where the strong-field limit is mass-independent such as a magnetic background, this analysis connects the β function with the strong-field limit coefficients and is well understood [1, 2, 3, 24]. In the present case, however, such an analysis connects the mass-dependent β function, the anomalous mass dimension and the strong-field limit with each other. For instance, if the strong-field limit at n-loop order and the anomalous dimension at (n-1)-loop order are known, we can extract the n-loop mass-dependent

β_m function and also the mass-independent β function by virtue of the n-loop analogue of Eq. (5.5). Aiming at an n-loop computation of the β function, this is the same amount of information required as for a magnetic background, but the computation for a self-dual background will be much simpler.

As a further remark, let us point out that we have discussed possible massless limits of QED always as continuous limits of massive theories in this work. In this sense, a massless limit of the self-dual spinor case does not exist because of the zero modes. This does not imply that a massless formulation of the self-dual spinor case does not exist at all. On the contrary, it is well possible that a massless formulation exists but requires a different treatment similar to the case of massless gluonic fluctuations in a self-dual Yang-Mills background [9, 25]. For this, the integration over the fermions has to be decomposed into zero-mode and non-zero-mode fluctuations. The non-zero modes have to be integrated out first in a background consisting of the constant self-dual field plus zero-mode fluctuations. We expect that the non-zero-mode integration "dresses" the zero modes in such a way that they acquire a mass. Contrary to the gluonic case, this mechanism requires a two-loop calculation, so that an effective four-fermion coupling between the zero and nonzero modes can be generated by photon exchange. In a self-dual background, the non-zero modes will form a condensate which then gives a mass to the zero modes because of this four-fermion interaction. A strong evidence for this scenario is given by our observation that the zero modes do not induce an IR divergence at the two-loop level. Once the zero modes are lifted by this effective mass, they can finally be integrated out. Since there is no further scale in this formulation, the standard relation between the strong-field limit and the beta function can be expected to hold for this theory once the zero modes are integrated out. This explains also why the massless limit of the massive case cannot be continuous because the strong-field limit coefficient changes discontinuously in this limit.

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VI. APPENDIX

In this appendix we comment on the *finite* part of the mass renormalization in (4.25). This finite part is not relevant for the main discussion of this paper, as it does not affect either the β function or the strong-field limit. However, it is crucial for the derivation of the finite renormalized two-loop effective Lagrangians (2.24) and (2.25) for spinor and scalar QED, respectively, in a constant self-dual background. These two-loop results were first derived in [18, 19] using the worldline formalism, and here we have given an independent derivation in Section IV of this current paper using a conventional field theory diagrammatic approach.

In order to fix the finite part of the mass renormalization mass shift, one approach is to compare (4.25) with an independent calculation of the UV properties of the mass operator [23, 34], done in the same regularization scheme. For a constant magnetic background, this scheme dependence of the finite mass shift has been studied at two-loop in [39]. Another approach, implicit in [1, 2, 3, 40], is to demand that the leading growth rate of the coefficients of the weak-field expansion of the two-loop renormalized effective Lagrangian coincides (up to a factor of $\alpha\pi$) with the leading growth rate of the coefficients of the weak-field expansion of the one-loop renormalized effective Lagrangian. This ensures that the leading imaginary part of the $\mathcal{L}_{\rm eff}$, when the field is analytically continued to an unstable regime, involves the same physical electron mass at two-loop as at one-loop. This is because the leading imaginary parts go like $\exp[-m^2\pi/(e|f|)]$, and these nonperturbative factors are related to the leading divergence rate of the perturbative coefficients of the (divergent) weak-field expansion in the standard way . Therefore, any mis-match between the leading growth rates at one-loop and two-loop corresponds to a shifted value of m^2, and vice versa. This gives an interesting "nonperturbative" definition of the renormalized mass, which is completely compatible with the standard definition of the mass through a renormalized perturbative Green's function [1, 2, 3]. The correspondence of the leading growth rates of the one-loop and two-loop weak-field expansions has been confirmed numerically in [41] for the case of a constant electric background, and has been confirmed analytically in [20] for the case of a constant self-dual background (with f analytically continued $f \to if$).

From the calculation presented in Section IV, it is easy to see from (4.23) and (4.24) that any finite shift in the finite parts (5/6 and 7/6 for spinor and scalar, respectively) of the mass shift in (4.25) would introduce into the renormalized two-loop effective Lagrangians on the first line of (4.24) an additional term of the form

$$\delta\mathcal{L}^{(2)} \;\sim\; \alpha\, m^2 \frac{\partial}{\partial m^2}\mathcal{L}^{(1)} \sim \alpha\,\kappa \frac{\partial}{\partial\kappa}\mathcal{L}^{(1)}. \tag{A.1}$$

Now the one-loop effective Lagrangians in (2.21) and (2.22) have divergent weak-field expansions of the form

$$\mathcal{L}^{(1)} = m^4 \sum_{n=2}^{\infty} \frac{c_n^{(1)}}{\kappa^{2n}}, \tag{A.2}$$

where the magnitude of the expansion coefficients grows factorially as [20]

$$|c_n^{(1)}| \sim \frac{\Gamma(2n-1)}{(2\pi)^{2n}}. \tag{A.3}$$

Similarly, the two-loop effective Lagrangians in (2.24) and (2.25) have divergent weak-field expansions of the form

$$\mathcal{L}^{(2)} = \alpha\pi\, m^4 \sum_{n=2}^{\infty} \frac{c_n^{(2)}}{\kappa^{2n}}, \tag{A.4}$$

where the magnitude of the expansion coefficients grows factorially as [20]

$$|c_n^{(2)}| \sim \frac{\Gamma(2n-1)}{(2\pi)^{2n}}. \tag{A.5}$$

This leading growth rate is precisely the same as the one-loop rate in (A.3), confirming Ritus's criterion at this order.

The two-loop results (2.24) and (2.25) for the on-shell renormalized effective Lagrangians appear in the first line of (4.24), when the finite parts of the mass shifts are as specified in (4.25). If these finite parts were shifted, then the renormalized effective Lagrangians would acquire a further shift as in (A.1). However, this extra piece clearly has the wrong growth rate, with the magintude of the expansion coefficients now growing like $\Gamma(2n)/(2\pi)^{2n}$, which is faster than the one-loop growth rate in (A.3). Thus we see that we can indeed uniquely implement Ritus's nonperturbative criterion as a means to fix the physical renormalized mass, including the finite part of the mass shift. And the result is completely consistent with the finite parts found by standard perturbative means [23, 34]. We believe that the method described here is not only of theoretical interest, but at higher loop orders might actually be technically preferable to a direct calculation of the mass shift.

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