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# Two-Loop QCD Helicity Amplitudes for $e^+e^- \to 3$ Jets

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**Abstract**

We compute the two-loop QCD helicity amplitudes for the process $e^+e^- \to q\bar{q}g$. The amplitudes are extracted in a scheme-independent manner from the coefficients appearing in the general tensorial structure for this process. The tensor coefficients are derived from the Feynman graph amplitudes by means of projectors, within the conventional dimensional regularization scheme. The actual calculation of the loop integrals is then performed by reducing all of them to a small set of known master integrals. The infrared pole structure of the renormalized helicity amplitudes agrees with the prediction made by Catani using an infrared factorization formula. We use this formula to structure our results for the finite part into terms arising from the expansion of the pole coefficients and a genuine finite remainder, which is independent of the scheme used to define the helicity amplitudes. The analytic result for the finite parts of the amplitudes is expressed in terms of one- and two-dimensional harmonic polylogarithms.

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# 1 Introduction

The three-jet production rate in electron–positron annihilation [1,2] and related event shape observables were measured to a very high precision at LEP, where they were used in particular for the determination of the strong coupling constant $\alpha_s$. At present, the error on the extraction of $\alpha_s$ from these data is dominated by the uncertainty inherent in the theoretical next-to-leading order (NLO) calculation [3–7] of the jet observables (see [8] for a review). The planned TESLA [9] linear $e^+e^-$ collider will allow precision QCD studies at energies even higher than at LEP. Given the projected luminosity of TESLA, one again expects the experimental errors to be well below the uncertainty of the NLO calculation.

The calculation of next-to-next-to-leading order (NNLO), i.e. $\mathcal{O}(\alpha_s^3)$, corrections to the three-jet rate in $e^+e^-$ annihilation has been considered as a highly important project for a long time [10]. In terms of matrix elements, it requires the computation of three contributions: the tree level $\gamma^* \to 5$ partons amplitudes [11–13], the one-loop corrections to the $\gamma^* \to 4$ partons amplitudes [14–17], and the two-loop (as well as the one-loop times one-loop) corrections to the $\gamma^* \to 3$ partons matrix elements. In a previous publication [18], we have derived both the interference of the tree and two-loop matrix elements and the self-interference of the one-loop amplitudes averaged over all external helicities. In the present work, we extend this calculation to compute the two-loop helicity amplitudes for the process $e^+e^- \to q\bar{q}g$.

The most precisely measured observables related to $e^+e^- \to 3$ jets are the jet production rate itself and a number of event-shape variables. The calculation of these phenomenologically most relevant applications, which also dominate the extraction of $\alpha_s$, at NNLO accuracy requires only the helicity averaged squared matrix element at the two-loop level derived in [18]. Nevertheless, the helicity amplitudes presented here are interesting for a number of reasons:

- Oriented event-shape observables, which measure the spatial orientation of the final-state jets relative to the direction of the incoming beams require, even for unpolarized beams, the calculation of the polarization tensor of the virtual photon mediating the interaction. This polarization tensor can be recovered from the helicity amplitudes.

- Likewise, to determine the direction of the decay leptons in the crossed process, $V + 1$ jet production at unpolarized hadron colliders, it is necessary to compute the polarization tensor of the vector boson.

- Polarization of the beams is an important option for the future linear $e^+e^-$ collider TESLA [9], thus providing a direct measurement of event-shape observables in polarized $e^+e^-$ annihilation.

- NNLO predictions for $(V + 1)$-jet production at the RHIC polarized proton–proton collider and for $(2 + 1)$ jet production at a currently discussed polarized upgrade of the HERA collider do require the calculation of the two-loop helicity amplitudes. These observables would then form part of a full NNLO determination of the polarized parton distribution functions in the proton.

- The study of formal aspects of two-loop matrix elements, such as their collinear limits or their high energy behaviour can be carried out more elegantly on the basis of the underlying helicity amplitudes.

Two-loop helicity amplitudes have up to now only been derived for $2 \to 2$ bosonic scattering processes with all external legs on-shell: for $gg \to \gamma\gamma$ [19], $\gamma\gamma \to \gamma\gamma$ [20,21] and $gg \to gg$ [22,23]. The latter calculation also confirmed earlier results for the squared two-loop $gg \to gg$ matrix element [24]. In the above calculations, which were all carried out within dimensional regularization [25–27] with $d = 4-2\epsilon$ space-time dimensions, two different methods were used to access the helicity structure of the matrix element: explicit contraction with the external polarization vectors [19,20,22,23] or projection onto the individual components of the Lorentz-invariant decomposition of the amplitude [21]. Once these are applied to expose the helicity structure, one is left with the task of computing a large number of two-loop integrals. Using integration-by-parts [27–29] and Lorentz-invariance [30] identities, these can be reduced [31] to a small number of so-called master integrals, which were derived for massless on-shell two-loop four-point functions in [32–37]. If an explicit contraction with the external polarization vectors is performed, one also has to compute two-loop integrals over the $(d-4)$ dimensional subspace of loop momenta, which reduce however to simple vacuum diagrams [23]. For $2 \to 2$ scattering processes with external fermions and all external legs on-shell ($e^+e^- \to e^+e^-$, $q\bar{q} \to q'\bar{q}'$, $q\bar{q} \to q\bar{q}$,

$q\bar{q} \to gg$, $q\bar{q} \to g\gamma$ and $q\bar{q} \to \gamma\gamma$), only the squared, helicity-averaged two-loop matrix elements were computed so far [38–41].

The method employed here to extract the two-loop helicity amplitudes for $e^+e^- \to q\bar{q}g$ is similar to the approach of [21] by applying projections on all components of the Lorentz-invariant decomposition of the amplitude. Using this approach, the corresponding one-loop helicity amplitudes were derived in [6]. The master integrals relevant in the present context are massless four-point functions with three legs on-shell and one leg off-shell. The complete set of these two-loop integrals was computed in [42], while earlier partial results had been presented in [43,44][1]. The master integrals in [42] are expressed in terms of two-dimensional harmonic polylogarithms (2dHPLs). The 2dHPLs are an extension of the harmonic polylogarithms (HPLs) of [47]. All HPLs and 2dHPLs that appear in the divergent parts of the planar master integrals have weight $\leq 3$ and can be related to the more commonly known Nielsen generalized polylogarithms [48, 49] of suitable arguments. The functions of weight 4 appearing in the finite parts of the master integrals can all be represented, by their very definition, as one-dimensional integrals over 2dHPLs of weight 3, hence of Nielsen's generalized polylogarithms of suitable arguments according to the above remark. A table with all relations is included in the appendix of [42]. Numerical routines providing an evaluation of the HPLs [50] and 2dHPLs [51] are available.

After carrying out ultraviolet renormalization of the amplitudes in the $\overline{\rm MS}$ scheme, one is left with poles which are purely of infrared origin. The infrared pole structure of the amplitudes can be predicted using Catani's infrared factorization formula [52]. We use this formalism to present the infrared poles and the finite parts of the helicity amplitudes in a compact form.

This paper is structured as follows: in Section 2, we outline the calculational method used to derive the helicity amplitudes and discuss the techniques used to extract the ultraviolet and infrared pole structure. We also elaborate on the relation to previous work. The two-loop helicity amplitudes are computed (in the Weyl–van der Waerden formalism, which is briefly described in the Appendix) in Section 3. Finally, Section 4 contains a discussion of the results and conclusions.

# 2 Method

## 2.1 Notation

We consider the production of a quark–antiquark–gluon system in electron–positron annihilation,

$$e^+(p_5) + e^-(p_6) \to \gamma^*(p_4) \longrightarrow q(p_1) + \bar{q}(p_2) + g(p_3) \,. \tag{2.1}$$

It is convenient to define the invariants

$$s_{12} = (p_1 + p_2)^2 \,, \qquad s_{13} = (p_1 + p_3)^2 \,, \qquad s_{23} = (p_2 + p_3)^2 \,, \tag{2.2}$$

which fulfil

$$p_4^2 = (p_1 + p_2 + p_3)^2 = s_{12} + s_{13} + s_{23} \equiv s_{123} \,, \tag{2.3}$$

as well as the dimensionless invariants

$$x = s_{12}/s_{123} \,, \qquad y = s_{13}/s_{123} \,, \qquad z = s_{23}/s_{123} \,, \tag{2.4}$$

which satisfy $x + y + z = 1$.

The renormalized amplitude $|\mathcal{M}\rangle$ can be written as

$$|\mathcal{M}\rangle = V^\mu \mathcal{S}_\mu(q; g; \bar{q}) \,, \tag{2.5}$$

[1]Note that an alternative approach avoiding the need to use the integration-by-parts and Lorentz-invariance identities to reduce the integrals appearing in the Feynman diagrams to a basis set has recently been proposed [45,46]. This method relies on obtaining analytic expressions for the basic topologies with arbitrary powers of the propagators and arbitrary dimensions, which can often be found in terms of nested sums involving Γ-functions. The Γ-functions can be directly expanded in $\epsilon$ and the nested sums related to multiple polylogarithms.

where $V^\mu$ represents the lepton current and $\mathcal{S}_\mu$ denotes the hadron current. In a previous paper [18], we have considered the unpolarized decay process

$$\gamma^*(p_4) \longrightarrow q(p_1) + \bar{q}(p_2) + g(p_3) \; . \tag{2.6}$$

for which the amplitude is obtained from Eq. (2.5) by replacing the lepton current by the polarization vector of the virtual photon $\epsilon_4^\mu$. The hadron current may be perturbatively decomposed as

$$\mathcal{S}_\mu(q;g;\bar{q}) = \sqrt{4\pi\alpha} e_q \sqrt{4\pi\alpha_s}\, \boldsymbol{T}^a_{ij} \left( \mathcal{S}_\mu^{(0)}(q;g;\bar{q}) + \left(\frac{\alpha_s}{2\pi}\right) \mathcal{S}_\mu^{(1)}(q;g;\bar{q}) + \left(\frac{\alpha_s}{2\pi}\right)^2 \mathcal{S}_\mu^{(2)}(q;g;\bar{q}) + \mathcal{O}(\alpha_s^3) \right), \tag{2.7}$$

where $e_q$ denotes the quark charge, $a$ is the adjoint colour index for the gluon and $i$ and $j$ are the colour indices for quark and antiquark. $\alpha_s$ is the QCD coupling constant at the renormalization scale $\mu$, and the $\mathcal{S}_\mu^{(i)}$ are the $i$-loop contributions to the renormalized amplitude. Renormalization of ultraviolet divergences is performed in the $\overline{\rm MS}$ scheme.

## 2.2 The general tensor

The most general tensor structure for the hadron current $\mathcal{S}_\mu(q;g;\bar{q})$ is

$$\begin{aligned}
\mathcal{S}_\mu(q;g;\bar{q}) &= \bar{u}(p_1)\not{p}_3 u(p_2)\,(A_{11}\epsilon_3.p_1\; p_{1\mu} + A_{12}\epsilon_3.p_1\; p_{2\mu} + A_{13}\epsilon_3.p_1\; p_{3\mu}) \\
&+ \bar{u}(p_1)\not{p}_3 u(p_2)\,(A_{21}\epsilon_3.p_2\; p_{1\mu} + A_{22}\epsilon_3.p_2\; p_{2\mu} + A_{23}\epsilon_3.p_2\; p_{3\mu}) \\
&+ \bar{u}(p_1)\gamma_\mu u(p_2)\,(B_1\epsilon_3.p_1 + B_2\epsilon_3.p_2) \\
&+ \bar{u}(p_1)\not{\epsilon}_3 u(p_2)\,(C_1 p_{1\mu} + C_2 p_{2\mu} + C_3 p_{3\mu}) \\
&+ D_1 \bar{u}(p_1)\not{\epsilon}_3\not{p}_3\gamma_\mu u(p_2) \\
&+ D_2 \bar{u}(p_1)\gamma_\mu\not{p}_3\not{\epsilon}_3 u(p_2) \; ,
\end{aligned} \tag{2.8}$$

where the constraint $\epsilon_3 \cdot p_3 = 0$ has been applied. All coefficients are functions of $s_{13}$, $s_{23}$ and $s_{123}$. The above tensor structure is a priori $d$-dimensional, since the dimensionality of the external states has not yet been specified. The hadron current is conserved and satisfies

$$\mathcal{S}_\mu(q;g;\bar{q})\; p_4^\mu = 0 \; ; \tag{2.9}$$

it must also obey the QCD Ward identity when the gluon polarization vector $\epsilon_3$ is replaced with the gluon momentum,

$$\mathcal{S}_\mu(q;g;\bar{q})(\epsilon_3 \to p_3) = 0. \tag{2.10}$$

These constraints yield relations amongst the 13 distinct tensor structures and applying these identities gives the gauge-invariant form of the tensor,

$$\begin{aligned}
\mathcal{S}_\mu(q;g;\bar{q}) &= A_{11}(s_{13},s_{23},s_{123})T_{11\mu} + A_{12}(s_{13},s_{23},s_{123})T_{12\mu} + A_{13}(s_{13},s_{23},s_{123})T_{13\mu} \\
&+ A_{21}(s_{13},s_{23},s_{123})T_{21\mu} + A_{22}(s_{13},s_{23},s_{123})T_{22\mu} + A_{23}(s_{13},s_{23},s_{123})T_{23\mu} \\
&+ B(s_{13},s_{23},s_{123})T_\mu,
\end{aligned} \tag{2.11}$$

where $A_{ij}$ and $B$ are gauge-independent functions and the tensor structures $T_{IJ\mu}$ and $T_\mu$ are given by

$$T_{1J\mu} = \bar{u}(p_1)\not{p}_3 u(p_2)\epsilon_3.p_1 p_{J\mu} - \frac{s_{13}}{2}\bar{u}(p_1)\not{\epsilon}_3 u(p_2) p_{J\mu} + \frac{s_{J4}}{4}\bar{u}(p_1)\not{\epsilon}_3\not{p}_3\gamma_\mu u(p_2), \tag{2.12}$$

$$T_{2J\mu} = \bar{u}(p_1)\not{p}_3 u(p_2)\epsilon_3.p_2 p_{J\mu} - \frac{s_{23}}{2}\bar{u}(p_1)\not{\epsilon}_3 u(p_2) p_{J\mu} + \frac{s_{J4}}{4}\bar{u}(p_1)\gamma_\mu\not{p}_3\not{\epsilon}_3 u(p_2), \tag{2.13}$$

$$\begin{aligned}
T_\mu &= s_{23}\left(\bar{u}(p_1)\gamma_\mu u(p_2)\epsilon_3.p_1 + \frac{1}{2}\bar{u}(p_1)\not{\epsilon}_3\not{p}_3\gamma_\mu u(p_2)\right) \\
&- s_{13}\left(\bar{u}(p_1)\gamma_\mu u(p_2)\epsilon_3.p_2 + \frac{1}{2}\bar{u}(p_1)\gamma_\mu\not{p}_3\not{\epsilon}_3 u(p_2)\right).
\end{aligned} \tag{2.14}$$

Each of the tensor structures satisfies both current conservation and the QCD Ward identity. The coefficients are further related by symmetry under the interchange of the quark and antiquark,

$$\begin{aligned} A_{21}(s_{13},s_{23},s_{123}) &= -A_{12}(s_{23},s_{13},s_{123}), \\ A_{22}(s_{13},s_{23},s_{123}) &= -A_{11}(s_{23},s_{13},s_{123}), \\ A_{23}(s_{13},s_{23},s_{123}) &= -A_{13}(s_{23},s_{13},s_{123}), \\ B(s_{13},s_{23},s_{123}) &= B(s_{23},s_{13},s_{123}). \end{aligned} \tag{2.15}$$

## 2.3 Projectors for the tensor coefficients

The coefficients $A_{IJ}$ and $B$ may be easily extracted from a Feynman diagram calculation, using projectors such that

$$\sum_{\text{spins}} \mathcal{P}(X)\ \epsilon_4^\mu \mathcal{S}_\mu(q;g;\bar{q}) = X(s_{13},s_{23},s_{123}). \tag{2.16}$$

The explicit forms for the seven projectors in $d$ space-time dimensions are,

$$\begin{aligned}
\mathcal{P}(A_{11}) &= \frac{(s_{23}s_{123}d+s_{13}s_{12}(d-2))}{2s_{13}^3s_{12}^2(d-3)s_{123}}T_{11}^\dagger\cdot\epsilon_4^* - \frac{(s_{13}+s_{23})(d-2)}{2s_{13}^2s_{12}^2(d-3)s_{123}}T_{12}^\dagger\cdot\epsilon_4^* \\
&-\frac{((s_{23}+s_{12})d+2s_{13})}{2s_{12}s_{13}^3(d-3)s_{123}}T_{13}^\dagger\cdot\epsilon_4^* - \frac{(s_{23}s_{123}(d-2)+s_{13}s_{12}(d-4))}{2s_{23}s_{12}^2s_{13}^2s_{123}(d-3)}T_{21}^\dagger\cdot\epsilon_4^* \\
&+\frac{(s_{13}+s_{23})(d-4)}{2(d-3)s_{12}^2s_{123}s_{13}s_{23}}T_{22}^\dagger\cdot\epsilon_4^* + \frac{(s_{23}+s_{12})(d-4)}{2s_{23}s_{12}s_{13}^2s_{123}(d-3)}T_{23}^\dagger\cdot\epsilon_4^* \\
&-\frac{1}{2s_{13}^2s_{12}^2(d-3)}T^\dagger\cdot\epsilon_4^*, \\
\mathcal{P}(A_{12}) &= -\frac{(s_{13}+s_{23})(d-2)}{2s_{13}^2s_{12}^2(d-3)s_{123}}T_{11}^\dagger\cdot\epsilon_4^* + \frac{(d-2)(s_{23}s_{12}(d-4)+s_{13}s_{123}(d-2))}{2s_{13}^2s_{12}^2s_{23}(d-3)s_{123}(d-4)}T_{12}^\dagger\cdot\epsilon_4^* \\
&-\frac{(d-2)(s_{13}+s_{12})}{2s_{13}^2s_{12}s_{23}(d-3)s_{123}}T_{13}^\dagger\cdot\epsilon_4^* + \frac{((D-6)(d-2)(s_{13}+s_{23})-4s_{12})}{2(d-4)s_{12}^2s_{13}s_{23}s_{123}(d-3)}T_{21}^\dagger\cdot\epsilon_4^* \\
&-\frac{(s_{23}s_{12}(d-4)+s_{13}s_{123}(d-2))}{2s_{12}^2s_{13}s_{23}^2(d-3)s_{123}}T_{22}^\dagger\cdot\epsilon_4^* + \frac{(2s_{23}+(s_{13}+s_{12})(d-2))}{2s_{12}s_{13}s_{23}^2s_{123}(d-3)}T_{23}^\dagger\cdot\epsilon_4^* \\
&-\frac{(d-2)}{2(d-4)s_{12}^2s_{13}s_{23}(d-3)}T^\dagger\cdot\epsilon_4^*, \\
\mathcal{P}(A_{13}) &= -\frac{((s_{23}+s_{12})d+2s_{13})}{2s_{12}s_{13}^3(d-3)s_{123}}T_{11}^\dagger\cdot\epsilon_4^* - \frac{(d-2)(s_{13}+s_{12})}{2s_{13}^2s_{12}s_{23}(d-3)s_{123}}T_{12}^\dagger\cdot\epsilon_4^* \\
&+\frac{(s_{13}s_{23}(d-2)+s_{12}s_{123}d)}{2s_{13}^3s_{12}s_{23}s_{123}(d-3)}T_{13}^\dagger\cdot\epsilon_4^* + \frac{((s_{12}+s_{23})(d-2)+2s_{13})}{2s_{13}^2s_{12}s_{23}(d-3)s_{123}}T_{21}^\dagger\cdot\epsilon_4^* \\
&+\frac{(s_{13}+s_{12})(d-4)}{2s_{12}s_{13}s_{23}^2s_{123}(d-3)}T_{22}^\dagger\cdot\epsilon_4^* - \frac{(s_{13}+s_{12})(s_{23}+s_{12})(d-4)}{2s_{13}^2s_{12}s_{23}^2s_{123}(d-3)}T_{23}^\dagger\cdot\epsilon_4^* \\
&+\frac{1}{2s_{23}s_{13}^2s_{12}(d-3)}T^\dagger\cdot\epsilon_4^*, \\
\mathcal{P}(A_{21}) &= -\frac{(s_{23}s_{12}(d-2)+s_{13}s_{12}(d-4))}{2s_{13}^2s_{12}^2s_{23}(d-3)s_{123}}T_{11}^\dagger\cdot\epsilon_4^* + \frac{(-4s_{12}+(s_{13}+s_{23})(D-6)(d-2))}{2(d-4)s_{12}^2s_{13}s_{23}s_{123}(d-3)}T_{12}^\dagger\cdot\epsilon_4^* \\
&+\frac{(s_{23}+s_{12})(d-2)+2s_{13})}{2s_{13}^2s_{12}s_{23}(d-3)s_{123}}T_{13}^\dagger\cdot\epsilon_4^* + \frac{(d-2)(s_{23}s_{123}(d-2)+s_{13}s_{12}(d-4))}{2s_{12}^2s_{13}s_{23}^2s_{123}(d-3)(d-4)}T_{21}^\dagger\cdot\epsilon_4^* \\
&-\frac{(s_{13}+s_{23})(d-2)}{2s_{12}^2s_{23}^2s_{123}(d-3)}T_{22}^\dagger\cdot\epsilon_4^* - \frac{(s_{23}+s_{12})(d-2)}{2s_{13}s_{23}^2s_{12}(d-3)s_{123}}T_{23}^\dagger\cdot\epsilon_4^* \\
&+\frac{(d-2)}{2(d-4)s_{12}^2s_{13}s_{23}(d-3)}T^\dagger\cdot\epsilon_4^*,
\end{aligned}$$

$$
\begin{aligned}
\mathcal{P}(A_{22}) &= \frac{(s_{13}+s_{23})(d-4)}{2s_{13}s_{12}^2s_{23}(d-3)s_{123}}T_{11}^\dagger\cdot\epsilon_4^* - \frac{(s_{23}s_{12}(d-4)+s_{13}s_{123}(d-2))}{2s_{12}^2s_{13}s_{23}^2(d-3)s_{123}}T_{12}^\dagger\cdot\epsilon_4^* \\
&+\frac{(s_{13}+s_{12})(d-4)}{2s_{13}s_{23}^2s_{12}(d-3)s_{123}}T_{13}^\dagger\cdot\epsilon_4^* - \frac{(s_{13}+s_{23})(d-2)}{2s_{12}^2s_{23}^2s_{123}(d-3)}T_{21}^\dagger\cdot\epsilon_4^* \\
&+\frac{(s_{23}s_{12}(d-2)+s_{13}s_{123}d)}{2s_{23}^3s_{12}^2(d-3)s_{123}}T_{22}^\dagger\cdot\epsilon_4^* - \frac{(s_{13}d+s_{12}d+2s_{23})}{2s_{12}s_{23}^3s_{123}(d-3)}T_{23}^\dagger\cdot\epsilon_4^* \\
&+\frac{1}{2s_{23}^2s_{12}^2(d-3)}T^\dagger\cdot\epsilon_4^*, \\
\mathcal{P}(A_{23}) &= \frac{(s_{23}+s_{12})(d-4)}{2s_{23}s_{12}s_{13}^2s_{123}(d-3)}T_{11}^\dagger\cdot\epsilon_4^* + \frac{(2s_{23}+(s_{13}+s_{12})(d-2))}{2s_{13}s_{23}^2s_{12}(d-3)s_{123}}T_{12}^\dagger\cdot\epsilon_4^* \\
&-\frac{(s_{13}+s_{12})(s_{23}+s_{12})(d-4)}{2s_{13}^2s_{12}s_{23}^2s_{123}(d-3)}T_{13}^\dagger\cdot\epsilon_4^* - \frac{(s_{23}+s_{12})(d-2)}{2s_{13}s_{23}^2s_{12}(d-3)s_{123}}T_{21}^\dagger\cdot\epsilon_4^* \\
&-\frac{((s_{13}+s_{12})d+2s_{23})}{2s_{12}s_{23}^3s_{123}(d-3)}T_{22}^\dagger\cdot\epsilon_4^* + \frac{(s_{13}s_{23}(d-2)+s_{12}s_{123}d)}{2s_{13}s_{12}s_{23}^3s_{123}(d-3)}T_{23}^\dagger\cdot\epsilon_4^* \\
&-\frac{1}{2s_{23}^2s_{13}s_{12}(d-3)}T^\dagger\cdot\epsilon_4^*, \\
\mathcal{P}(B) &= -\frac{1}{2(d-3)s_{12}^2s_{13}^2}T_{11}^\dagger\cdot\epsilon_4^* - \frac{(d-2)}{2(d-4)s_{12}^2s_{13}s_{23}(d-3)}T_{12}^\dagger\cdot\epsilon_4^* + \frac{1}{2s_{23}s_{12}(d-3)s_{13}^2}T_{13}^\dagger\cdot\epsilon_4^* \\
&+\frac{(d-2)}{2(d-4)s_{12}^2s_{13}s_{23}(d-3)}T_{21}^\dagger\cdot\epsilon_4^* + \frac{1}{2s_{23}^2s_{12}^2(d-3)}T_{22}^\dagger\cdot\epsilon_4^* - \frac{1}{2s_{13}s_{12}(d-3)s_{23}^2}T_{23}^\dagger\cdot\epsilon_4^* \\
&+\frac{1}{2(d-4)s_{12}^2s_{13}s_{23}}T^\dagger\cdot\epsilon_4^*. \qquad (2.17)
\end{aligned}
$$

## 2.4 The perturbative expansion of the tensor coefficients

Each of the unrenormalized coefficients $A_{IJ}$ and $B$ has a perturbative expansion of the form

$$
\begin{aligned}
A_{IJ}^{\rm un} &= \sqrt{4\pi\alpha}e_q\sqrt{4\pi\alpha_s}\,\boldsymbol{T}_{ij}^a\left[A_{IJ}^{(0),{\rm un}} + \left(\frac{\alpha_s}{2\pi}\right)A_{IJ}^{(1),{\rm un}} + \left(\frac{\alpha_s}{2\pi}\right)^2 A_{IJ}^{(2),{\rm un}} + \mathcal{O}(\alpha_s^3)\right], \\
B^{\rm un} &= \sqrt{4\pi\alpha}e_q\sqrt{4\pi\alpha_s}\,\boldsymbol{T}_{ij}^a\left[B^{(0),{\rm un}} + \left(\frac{\alpha_s}{2\pi}\right)B^{(1),{\rm un}} + \left(\frac{\alpha_s}{2\pi}\right)^2 B^{(2),{\rm un}} + \mathcal{O}(\alpha_s^3)\right], \qquad (2.18)
\end{aligned}
$$

where the dependence on $(s_{13}, s_{23}, s_{123})$ is implicit. At tree level,

$$
\begin{aligned}
A_{IJ}^{(0),{\rm un}}(s_{13},s_{23},s_{123}) &= 0, \qquad (2.19) \\
B^{(0),{\rm un}}(s_{13},s_{23},s_{123}) &= \frac{2}{s_{13}s_{23}}\,. \qquad (2.20)
\end{aligned}
$$

The one-loop contributions can be written in terms of the one-loop box integral in $d = 6-2\epsilon$ dimensions, $\mathrm{Box}^6(s_{ij}, s_{ik}, s_{ijk})$, and the one-loop bubble, $\mathrm{Bub}(s_{ij})$, as follows:

$$
\begin{aligned}
&A_{11}^{(1),{\rm un}}(s_{13},s_{23},s_{123}) = \\
&\quad N\Bigg(-\frac{(d-4)}{2(s_{13}+s_{12})s_{13}}\,\mathrm{Bub}(s_{123}) - \frac{(d-4)}{2s_{12}s_{13}}\,[\mathrm{Bub}(s_{13})-\mathrm{Bub}(s_{123})] \\
&\qquad -\frac{((d-2)s_{23}s_{12}+(d-4)s_{23}s_{13}+4s_{12}(s_{12}+s_{13}))}{2s_{12}s_{13}(s_{13}+s_{12})^2}\,[\mathrm{Bub}(s_{23})-\mathrm{Bub}(s_{123})] \\
&\qquad -\frac{(d-4)(4s_{12}+(d-2)s_{23})}{4s_{12}s_{13}}\,\mathrm{Box}^6(s_{13},s_{23},s_{123})\Bigg)
\end{aligned}
$$

$$+\frac{1}{N}\Bigg(\frac{(d-4)}{2(s_{13}+s_{12})s_{13}}\ \mathrm{Bub}(s_{123})+\frac{d}{2s_{13}^2}\ [\mathrm{Bub}(s_{12})-\mathrm{Bub}(s_{123})]$$
$$+\frac{(s_{12}+s_{13})(ds_{23}+4s_{13})+2s_{23}s_{13}}{2(s_{13}+s_{12})^2s_{13}^2}\ [\mathrm{Bub}(s_{23})-\mathrm{Bub}(s_{123})]$$
$$+\frac{(d-4)(d-6)}{4s_{13}}\ \mathrm{Box}^6(s_{12},s_{13},s_{123})+\frac{(d-2)(ds_{23}+4s_{13})}{4s_{13}^2}\ \mathrm{Box}^6(s_{12},s_{23},s_{123})\Bigg),\tag{2.21}$$

$$A_{12}^{(1),\mathrm{un}}(s_{13},s_{23},s_{123})=$$
$$N\Bigg(-\frac{(d-10)}{2s_{12}(s_{23}+s_{12})}\ [\mathrm{Bub}(s_{13})-\mathrm{Bub}(s_{123})]-\frac{((d-10)s_{13}-4s_{12})}{2s_{12}s_{13}(s_{13}+s_{12})}\ [\mathrm{Bub}(s_{23})-\mathrm{Bub}(s_{123})]$$
$$+\frac{(4(d-4)s_{12}-(d-2)(d-10)s_{13})}{4s_{12}s_{13}}\ \mathrm{Box}^6(s_{13},s_{23},s_{123})\Bigg)$$
$$+\frac{1}{N}\Bigg(\frac{(d-2)}{2s_{23}s_{13}}\ [\mathrm{Bub}(s_{12})-\mathrm{Bub}(s_{123})]+\frac{((d-2)s_{12}+2(d-6)s_{23})}{2s_{23}s_{12}(s_{23}+s_{12})}\ [\mathrm{Bub}(s_{13})-\mathrm{Bub}(s_{123})]$$
$$+\frac{(d-6)(s_{12}+2s_{13})}{2s_{12}s_{13}(s_{13}+s_{12})}\ [\mathrm{Bub}(s_{23})-\mathrm{Bub}(s_{123})]$$
$$+\frac{((d-2)^2s_{12}s_{13}+4(d-4)s_{12}s_{23}+2(d-4)(d-6)s_{13}s_{23})}{4s_{12}s_{13}s_{23}}\ \mathrm{Box}^6(s_{12},s_{13},s_{123})$$
$$+\frac{(d-6)((d-2)s_{12}+2(d-4)s_{13})}{4s_{12}s_{13}}\ \mathrm{Box}^6(s_{12},s_{23},s_{123})\Bigg),\tag{2.22}$$

$$A_{13}^{(1),\mathrm{un}}(s_{13},s_{23},s_{123})=$$
$$N\Bigg(\frac{(d-6)}{2(s_{13}+s_{12})s_{13}}\ [\mathrm{Bub}(s_{23})-\mathrm{Bub}(s_{123})]+\frac{(d-4)(d-6)}{4s_{13}}\ \mathrm{Box}^6(s_{13},s_{23},s_{123})\Bigg)$$
$$+\frac{1}{N}\Bigg(-\frac{(d-4)}{s_{13}(s_{23}+s_{13})}\ \mathrm{Bub}(s_{123})-\frac{(d-4)}{2s_{23}s_{13}}\ [\mathrm{Bub}(s_{13})-\mathrm{Bub}(s_{123})]$$
$$+\frac{(4s_{12}s_{13}^2-ds_{12}(s_{13}+s_{23})^2-2(d-2)s_{13}s_{23}(s_{13}+s_{23}))}{2(s_{23}+s_{13})^2s_{13}^2s_{23}}\ [\mathrm{Bub}(s_{12})-\mathrm{Bub}(s_{123})]$$
$$-\frac{(2(d-3)s_{13}+ds_{12})}{2s_{13}^2(s_{13}+s_{12})}\ [\mathrm{Bub}(s_{23})-\mathrm{Bub}(s_{123})]-\frac{(d-4)((d-2)s_{12}+4s_{23})}{4s_{23}s_{13}}\ \mathrm{Box}^6(s_{12},s_{13},s_{123})$$
$$-\frac{(d-2)(ds_{12}+2(d-4)s_{13})}{4s_{13}^2}\ \mathrm{Box}^6(s_{12},s_{23},s_{123})\Bigg),\tag{2.23}$$

$$B^{(1),\mathrm{un}}(s_{13},s_{23},s_{123})=$$
$$N\Bigg(\frac{d^2-3d+4}{4(d-4)s_{13}s_{23}}\ \mathrm{Bub}(s_{123})$$
$$+\frac{(4(d-3)s_{12}(s_{12}+s_{23})+(d-4)(d-7)s_{23}s_{13})}{2s_{12}s_{23}(s_{23}+s_{12})s_{13}(d-4)}\ [\mathrm{Bub}(s_{13})-\mathrm{Bub}(s_{123})]$$
$$+\frac{(4(d-3)s_{12}^2+(d-2)(d-7)s_{13}s_{23})}{8s_{12}s_{13}s_{23}}\ \mathrm{Box}^6(s_{13},s_{23},s_{123})\Bigg)$$
$$+\frac{1}{N}\Bigg(\frac{(7d-16-d^2)}{4(d-4)s_{13}s_{23}}\ \mathrm{Bub}(s_{123})+\frac{(16-5d)}{4(d-4)s_{13}s_{23}}\ [\mathrm{Bub}(s_{12})-\mathrm{Bub}(s_{123})]$$

$$-\frac{(s_{12}+(d-6)s_{23})}{2(s_{23}+s_{12})s_{23}s_{12}}\ [\mathrm{Bub}(s_{13})-\mathrm{Bub}(s_{123})]$$
$$-\frac{(4(d-3)s_{23}s_{12}+(d-4)(d-6)s_{13}s_{23}+(d-2)s_{12}s_{13})}{4s_{12}s_{13}s_{23}}\ \mathrm{Box}^6(s_{12},s_{13},s_{123})\Bigg)$$
$$+\Bigg\{s_{13}\leftrightarrow s_{23}\Bigg\}. \tag{2.24}$$

Explicit expansions of the one-loop integrals around $\epsilon \sim 0$ in terms of HPLs and 2dHPLs are listed in Appendix A of [18].

Similarly, the unrenormalized two-loop $A_{IJ}^{(2),\mathrm{un}}$ and $B^{(2),\mathrm{un}}$ coefficients were obtained analytically (making extensive use of the computer algebra programs MAPLE [53], FORM2 [54] and FORM3 [55], where the latter two are particularly well suited for handling the large-size expressions arising at intermediate stages of the calculation) in terms of a basis set of two-loop master integrals. This basis set comprises 14 planar topologies and 5 non-planar topologies. Five of the topologies require more than one master integral, so that in total 24 master integrals are needed. A more detailed discussion can be found in Ref. [18]. However, we note that Laurent expansions for each of these master integrals have been derived in [42] by solving differential equations for the master integrals (equations that are differential with respect to the momentum scales involved in the diagram). The $\epsilon$-expansions of $A_{IJ}^{(2),\mathrm{un}}$ and $B^{(2),\mathrm{un}}$ can therefore be obtained by directly substituting the $\epsilon$-expansions of the individual master integrals.

## 2.5 Relation to previous work

We have considered the case where the correlations with the lepton current are ignored in a previous paper [18]. In this instance, the squared amplitude for the process $\gamma^* \to q\bar{q}g$, summed over spins, colours and quark flavours, was denoted by

$$\langle \mathcal{M}|\mathcal{M}\rangle = \sum |\epsilon_4 \cdot \mathcal{S}(q;g;\bar{q})|^2 = \mathcal{T}(x,y,z)\ . \tag{2.25}$$

The perturbative expansion of $\mathcal{T}(x,y,z)$ at renormalization scale $\mu^2 = q^2 = s_{123}$ reads:

$$\begin{aligned}\mathcal{T}(x,y,z) &= 16\pi^2\alpha\sum_q e_q^2\alpha_s(q^2)\Bigg[\mathcal{T}^{(2)}(x,y,z)+\left(\frac{\alpha_s(q^2)}{2\pi}\right)\mathcal{T}^{(4)}(x,y,z)\\ &\quad+\left(\frac{\alpha_s(q^2)}{2\pi}\right)^2\mathcal{T}^{(6)}(x,y,z)+\mathcal{O}(\alpha_s^3(q^2))\Bigg]\ ,\end{aligned} \tag{2.26}$$

where

$$\mathcal{T}^{(2)}(x,y,z) = \langle\mathcal{M}^{(0)}|\mathcal{M}^{(0)}\rangle = 4V(1-\epsilon)\left[(1-\epsilon)\left(\frac{y}{z}+\frac{z}{y}\right)+\frac{2(1-y-z)-2\epsilon yz}{yz}\right]\ , \tag{2.27}$$
$$\mathcal{T}^{(4)}(x,y,z) = \langle\mathcal{M}^{(0)}|\mathcal{M}^{(1)}\rangle + \langle\mathcal{M}^{(1)}|\mathcal{M}^{(0)}\rangle\ , \tag{2.28}$$
$$\mathcal{T}^{(6)}(x,y,z) = \langle\mathcal{M}^{(1)}|\mathcal{M}^{(1)}\rangle + \langle\mathcal{M}^{(0)}|\mathcal{M}^{(2)}\rangle + \langle\mathcal{M}^{(2)}|\mathcal{M}^{(0)}\rangle\ , \tag{2.29}$$

where $V = N^2-1$, with $N$ the number of colours. $\mathcal{T}^{(4)}(x,y,z)$ was first derived in [3,4] through to $\mathcal{O}(\epsilon^0)$ while an explicit expression for it to all orders in $\epsilon$ was given in [6]. The contribution to $\mathcal{T}^{(6)}(x,y,z)$ from the interference of two-loop and tree diagrams

$$\mathcal{T}^{(6,[2\times0])}(x,y,z) = \langle\mathcal{M}^{(0)}|\mathcal{M}^{(2)}\rangle + \langle\mathcal{M}^{(2)}|\mathcal{M}^{(0)}\rangle\ , \tag{2.30}$$

as well as the one-loop self-interference

$$\mathcal{T}^{(6,[1\times1])}(x,y,z) = \langle\mathcal{M}^{(1)}|\mathcal{M}^{(1)}\rangle \tag{2.31}$$

were first derived in [18].

It is straightforward to obtain the interference of the tree and $i$-loop amplitudes in terms of the tensor coefficients, $A_{IJ}$ and $B$. We find

$$\begin{aligned}
&\langle \mathcal{M}^{(0)} | \mathcal{M}^{(i)} \rangle = \\
&\quad \frac{V}{2} \Bigg\{ 2(1-\epsilon) \left( (s_{12}s_{123} + s_{12}s_{13} + s_{13}s_{23}) - \epsilon(s_{13}+s_{23})(s_{12}+s_{13}) \right) A_{11}^{(i)}(s_{13}, s_{23}, s_{123}) \\
&\qquad + \left( 2(s_{12}+s_{23})^2 - 2\epsilon \left( s_{123}s_{23} + (s_{12}+s_{23})^2 \right) + 2\epsilon^2 (s_{13}+s_{23})(s_{12}+s_{23}) \right) A_{12}^{(i)}(s_{13}, s_{23}, s_{123}) \\
&\qquad + 2 \left( s_{23} - \epsilon(s_{13}+s_{23}) \right) \left( s_{123} - \epsilon(s_{13}+s_{23}) \right) A_{13}^{(i)}(s_{13}, s_{23}, s_{123}) \\
&\qquad + 2 \left( s_{13}^2 + s_{23}^2 + 2s_{12}s_{123} - 2\epsilon(s_{123}^2 - s_{12}s_{13} - s_{12}s_{23} - s_{13}s_{23}) + \epsilon^2 (s_{13}+s_{23})^2 \right) B^{(i)}(s_{13}, s_{23}, s_{123}) \\
&\qquad\qquad + \{ p_1 \leftrightarrow p_2 \} \Bigg\}. \qquad (2.32)
\end{aligned}$$

The above relation holds for the unrenormalized as well as for the renormalized matrix element, involving the appropriate unrenormalized or renormalized tensor coefficients respectively. Similar, but more lengthy, expressions can easily be obtained for the interference of $i$- and $j$-loop amplitudes. We have checked that inserting the expressions for $A_{IJ}^{(i)}$ and $B^{(i)}$ into Eq. (2.32) reproduces our earlier results [18] at the one- and two-loop level both at the master integral level and after making an expansion in $\epsilon$.

## 2.6 Ultraviolet renormalization

The renormalization of the matrix element is carried out by replacing the bare coupling $\alpha_0$ with the renormalized coupling $\alpha_s \equiv \alpha_s(\mu^2)$, evaluated at the renormalization scale $\mu^2$

$$\alpha_0 \mu_0^{2\epsilon} S_\epsilon = \alpha_s \mu^{2\epsilon} \left[ 1 - \frac{\beta_0}{\epsilon} \left( \frac{\alpha_s}{2\pi} \right) + \left( \frac{\beta_0^2}{\epsilon^2} - \frac{\beta_1}{2\epsilon} \right) \left( \frac{\alpha_s}{2\pi} \right)^2 + \mathcal{O}(\alpha_s^3) \right] , \qquad (2.33)$$

where

$$S_\epsilon = (4\pi)^\epsilon e^{-\epsilon\gamma} \qquad \text{with Euler constant } \gamma = 0.5772\ldots$$

and $\mu_0^2$ is the mass parameter introduced in dimensional regularization [25–27] to maintain a dimensionless coupling in the bare QCD Lagrangian density; $\beta_0$ and $\beta_1$ are the first two coefficients of the QCD $\beta$-function:

$$\beta_0 = \frac{11C_A - 4T_R N_F}{6}, \qquad \beta_1 = \frac{17C_A^2 - 10C_A T_R N_F - 6C_F T_R N_F}{6} , \qquad (2.34)$$

with the QCD colour factors

$$C_A = N, \qquad C_F = \frac{N^2-1}{2N}, \qquad T_R = \frac{1}{2} . \qquad (2.35)$$

We denote the $i$-loop contribution to the unrenormalized coefficients by $A_{IJ}^{(i),\mathrm{un}}$ and $B^{(i),\mathrm{un}}$, using the same normalization as for the decomposition of the renormalized amplitude (2.7); the dependence on $(s_{13}, s_{23}, s_{123})$ is always understood implicitly. The renormalized coefficients are then obtained as

$$\begin{aligned}
A_{IJ}^{(0)} &= 0, \\
A_{IJ}^{(1)} &= S_\epsilon^{-1} A_{IJ}^{(1),\mathrm{un}}, \\
A_{IJ}^{(2)} &= S_\epsilon^{-2} A_{IJ}^{(2),\mathrm{un}} - \frac{3\beta_0}{2\epsilon} S_\epsilon^{-1} A_{IJ}^{(1),\mathrm{un}}, \qquad (2.36)
\end{aligned}$$

and

$$B^{(0)} = B^{(0),\mathrm{un}},$$

$$
\begin{aligned}
B^{(1)} &= S_\epsilon^{-1} B^{(1),\mathrm{un}} - \frac{\beta_0}{2\epsilon} B^{(0),\mathrm{un}}, \\
B^{(2)} &= S_\epsilon^{-2} B^{(2),\mathrm{un}} - \frac{3\beta_0}{2\epsilon} S_\epsilon^{-1} B^{(1),\mathrm{un}} - \left(\frac{\beta_1}{4\epsilon} - \frac{3\beta_0^2}{8\epsilon^2}\right) B^{(0),\mathrm{un}}.
\end{aligned} \tag{2.37}
$$

For the remainder of this paper we will set the renormalization scale $\mu^2 = q^2$. The full scale dependence of the tensor coefficients is given by

$$
\begin{aligned}
A_{IJ} &= \sqrt{4\pi\alpha} e_q \sqrt{4\pi\alpha_s}\, \boldsymbol{T}^a_{ij} \left\{ \left(\frac{\alpha_s(\mu^2)}{2\pi}\right) A^{(1)}_{IJ} + \left(\frac{\alpha_s(\mu^2)}{2\pi}\right)^2 \left[ A^{(2)}_{IJ} + 2\beta_0 A^{(1)}_{IJ} \ln\left(\frac{\mu^2}{q^2}\right) \right] + \mathcal{O}(\alpha_s^3) \right\}, \\
B &= \sqrt{4\pi\alpha} e_q \sqrt{4\pi\alpha_s}\, \boldsymbol{T}^a_{ij} \left\{ B^{(0)} + \left(\frac{\alpha_s(\mu^2)}{2\pi}\right) \left[ B^{(1)} + \beta_0 B^{(0)} \ln\left(\frac{\mu^2}{q^2}\right) \right] \right. \\
&\quad \left. + \left(\frac{\alpha_s(\mu^2)}{2\pi}\right)^2 \left[ B^{(2)} + \left(2\beta_0 B^{(1)} + \beta_1 B^{(0)}\right) \ln\left(\frac{\mu^2}{q^2}\right) + \beta_0^2 B^{(0)} \ln^2\left(\frac{\mu^2}{q^2}\right) \right] + \mathcal{O}(\alpha_s^3) \right\}.
\end{aligned} \tag{2.38}
$$

## 2.7 Infrared behaviour of the tensor coefficients

After performing ultraviolet renormalization, the amplitudes still contain singularities, which are of infrared origin and will be analytically cancelled by those occurring in radiative processes of the same order. Catani [52] has shown how to organize the infrared pole structure of the one- and two-loop contributions renormalized in the $\overline{\mathrm{MS}}$ scheme in terms of the tree and renormalized one-loop amplitudes. The same procedure applies to the tensor coefficients. In particular, the infrared behaviour of the one-loop coefficients is given by

$$
\begin{aligned}
A^{(1)}_{IJ} &= A^{(1),\mathrm{finite}}_{IJ}, \\
B^{(1)} &= \boldsymbol{I}^{(1)}(\epsilon) B^{(0)} + B^{(1),\mathrm{finite}},
\end{aligned} \tag{2.39}
$$

while the two-loop singularity structure is

$$
\begin{aligned}
A^{(2)}_{IJ} &= \boldsymbol{I}^{(1)}(\epsilon) A^{(1)}_{IJ} + A^{(2),\mathrm{finite}}_{IJ}, \\
B^{(2)} &= \left( -\frac{1}{2}\boldsymbol{I}^{(1)}(\epsilon)\boldsymbol{I}^{(1)}(\epsilon) - \frac{\beta_0}{\epsilon}\boldsymbol{I}^{(1)}(\epsilon) + e^{-\epsilon\gamma}\frac{\Gamma(1-2\epsilon)}{\Gamma(1-\epsilon)} \left(\frac{\beta_0}{\epsilon} + K\right) \boldsymbol{I}^{(1)}(2\epsilon) + \boldsymbol{H}^{(2)}(\epsilon) \right) B^{(0)} \\
&\quad + \boldsymbol{I}^{(1)}(\epsilon) B^{(1)} + B^{(2),\mathrm{finite}},
\end{aligned} \tag{2.40}
$$

where the constant $K$ is

$$
K = \left(\frac{67}{18} - \frac{\pi^2}{6}\right) C_A - \frac{10}{9} T_R N_F. \tag{2.41}
$$

The finite remainders $A^{(i),\mathrm{finite}}_{IJ}$ and $B^{(i),\mathrm{finite}}$ remain to be calculated.

For this particular process, there is only one colour structure present at tree level which, in terms of the gluon colour $a$ and the quark and antiquark colours $i$ and $j$, is simply $\boldsymbol{T}^a_{ij}$. Adding higher loops does not introduce additional colour structures, and the amplitudes are therefore vectors in a one-dimensional space. Similarly, the infrared singularity operator $\boldsymbol{I}^{(1)}(\epsilon)$ is a $1 \times 1$ matrix in the colour space and is given by

$$
\boldsymbol{I}^{(1)}(\epsilon) = -\frac{e^{\epsilon\gamma}}{2\Gamma(1-\epsilon)} \left[ N \left(\frac{1}{\epsilon^2} + \frac{3}{4\epsilon} + \frac{\beta_0}{2N\epsilon}\right) (\mathtt{S}_{13} + \mathtt{S}_{23}) - \frac{1}{N}\left(\frac{1}{\epsilon^2} + \frac{3}{2\epsilon}\right) \mathtt{S}_{12} \right], \tag{2.42}
$$

where (since we have set $\mu^2 = s_{123}$)

$$
\mathtt{S}_{ij} = \left(-\frac{s_{123}}{s_{ij}}\right)^\epsilon. \tag{2.43}
$$

Note that on expanding $\mathtt{S}_{ij}$, imaginary parts are generated, the sign of which is fixed by the small imaginary part $+i0$ of $s_{ij}$. The origin of the various terms in Eq. (2.42) is straightforward. Each parton pair $ij$ in the event forms a radiating antenna of scale $s_{ij}$. Terms proportional to $\mathtt{S}_{ij}$ are cancelled by real radiation

emitted from leg $i$ and absorbed by leg $j$. The soft singularities $\mathcal{O}(1/\epsilon^2)$ are independent of the identity of the participating partons and are universal. However, the collinear singularities depend on the identities of the participating partons. For each quark we find a contribution of $3N/(4\epsilon)$ and for each gluon we find a contribution of $\beta_0/(2\epsilon)$ coming from the integral over the collinear splitting function.

Finally, the term of Eq. (2.40) that involves $\boldsymbol{H}^{(2)}(\epsilon)$ produces only a single pole in $\epsilon$ and is given by

$$\boldsymbol{H}^{(2)}(\epsilon) = \frac{e^{\epsilon\gamma}}{4\,\epsilon\,\Gamma(1-\epsilon)} H^{(2)} \,, \tag{2.44}$$

where the constant $H^{(2)}$ is renormalization-scheme-dependent. As with the single-pole parts of $\boldsymbol{I}^{(1)}(\epsilon)$, the process-dependent $H^{(2)}$ can be constructed by counting the number of radiating partons present in the event. In our case, there is a quark–antiquark pair and a gluon present in the final state, so that

$$H^{(2)} = 2H_q^{(2)} + H_g^{(2)} \,, \tag{2.45}$$

where, in the $\overline{\rm MS}$ scheme:

$$H_g^{(2)} = \left(\frac{1}{2}\zeta_3 + \frac{5}{12} + \frac{11\pi^2}{144}\right) N^2 + \frac{5}{27}\,N_F^2 + \left(-\frac{\pi^2}{72} - \frac{89}{108}\right) N N_F - \frac{N_F}{4N}, \tag{2.46}$$

$$\begin{aligned} H_q^{(2)} &= \left(\frac{7}{4}\zeta_3 + \frac{409}{864} - \frac{11\pi^2}{96}\right) N^2 + \left(-\frac{1}{4}\zeta_3 - \frac{41}{108} - \frac{\pi^2}{96}\right) + \left(-\frac{3}{2}\zeta_3 - \frac{3}{32} + \frac{\pi^2}{8}\right) \frac{1}{N^2} \\ &\quad + \left(\frac{\pi^2}{48} - \frac{25}{216}\right) \frac{(N^2-1)N_F}{N} \,, \end{aligned} \tag{2.47}$$

so that

$$\begin{aligned} H^{(2)} &= \left(4\zeta_3 + \frac{589}{432} - \frac{11\pi^2}{72}\right) N^2 + \left(-\frac{1}{2}\zeta_3 - \frac{41}{54} - \frac{\pi^2}{48}\right) + \left(-3\zeta_3 - \frac{3}{16} + \frac{\pi^2}{4}\right) \frac{1}{N^2} \\ &\quad + \left(-\frac{19}{18} + \frac{\pi^2}{36}\right) N N_F + \left(-\frac{1}{54} - \frac{\pi^2}{24}\right) \frac{N_F}{N} + \frac{5}{27} N_F^2. \end{aligned} \tag{2.48}$$

The factors $H_q^{(2)}$ and $H_g^{(2)}$ are directly related to those found in gluon–gluon scattering [24], quark–quark scattering [39] and quark–gluon scattering [40] (which each involve four partons) as well as in the quark form factor [56–59] and gluon form factor [60]. We also note that (on purely dimensional grounds) one might expect terms of the type $\mathtt{S}_{ij}^2$ to be present in $H^{(2)}$. Of course such terms are $1 + \mathcal{O}(\epsilon)$ and therefore leave the pole part unchanged, only modifying the finite remainder. At present it is not known how to systematically include these effects.

# 3 Helicity amplitudes

We can extend the results of the previous section to include $Z$ boson exchange,

$$e^+(p_5) + e^-(p_6) \to (Z^*, \gamma^*)\,(p_4) \longrightarrow q(p_1) + \bar{q}(p_2) + g(p_3) \,, \tag{3.1}$$

where the off-shell vector boson now distinguishes between left- and right-handed fermions by keeping track of the helicity of the final state quarks.[2] A convenient method to evaluate the helicity amplitudes is in terms of Weyl–van der Waerden spinors, which is described briefly in Appendix A and in detail in [61, 62].

It is also straightforward to include the spin-correlations with the initial state by contracting the hadronic current with the lepton current $V_\mu$ for fixed helicities of the initial state electron (and positron). Using the spinor calculus of Appendix A we can express the lepton current $V_\mu$ in terms of the helicities of the incident

[2] Note that the full matrix element for any process should be summed over both photon and $Z$-boson exchange.

$e^+$ and $e^-$ (with momenta $p_5$ and $p_6$ respectively). Explicitly,

$$V_\mu^\gamma(e^+\!+,e^-\!-) = e\sigma_\mu^{\dot{A}B} p_{6\dot{A}} p_{5B} \frac{L_{ee}^\gamma}{s}, \qquad V_\mu^Z(e^+\!+,e^-\!-) = e\sigma_\mu^{\dot{A}B} p_{6\dot{A}} p_{5B} \frac{L_{ee}^Z}{s-M_Z^2+i\Gamma_Z M_Z},$$
$$V_\mu^\gamma(e^+\!-,e^-\!+) = e\sigma_\mu^{\dot{A}B} p_{5\dot{A}} p_{6B} \frac{R_{ee}^\gamma}{s}, \qquad V_\mu^Z(e^+\!-,e^-\!+) = e\sigma_\mu^{\dot{A}B} p_{5\dot{A}} p_{6B} \frac{R_{ee}^Z}{s-M_Z^2+i\Gamma_Z M_Z}. \tag{3.2}$$

The hadronic current $\mathcal{S}_\mu$ is related to the fixed helicity currents, $\mathcal{S}_{\dot{A}B}$, by

$$\mathcal{S}_\mu(q+;g\lambda;\overline{q}-) = R_{f_1f_2}^V \sqrt{2}\,\sigma_\mu^{\dot{A}B} \mathcal{S}_{\dot{A}B}(q+;g\lambda;\overline{q}-), \tag{3.3}$$

$$\mathcal{S}_\mu(q-;g\lambda_1;\overline{q}+) = L_{f_1f_2}^V \sqrt{2}\,\sigma_\mu^{\dot{A}B} \mathcal{S}_{\dot{A}B}(q-;g\lambda;\overline{q}+). \tag{3.4}$$

As in Eq. (2.7), the gauge boson coupling is extracted from $\mathcal{S}_{\dot{A}B}$. As mentioned earlier, the left- and right-handed currents couple with a different strength when the vector boson is a $Z$.

The currents with the quark helicities flipped follow from parity conservation:

$$\mathcal{S}_{\dot{A}B}(q-;g\lambda;\overline{q}+) = (\mathcal{S}_{\dot{B}A}(q+;g(-\lambda);\overline{q}-))^* \ . \tag{3.5}$$

Charge conjugation implies the following relations between currents with different helicities:

$$\mathcal{S}_{\dot{A}B}(q\lambda_q;g\lambda;\overline{q}\lambda_{\overline{q}}) = (-1)\mathcal{S}_{\dot{A}B}(\overline{q}\lambda_{\overline{q}};g\lambda;q\lambda_q). \tag{3.6}$$

All helicity amplitudes are therefore related to the amplitudes with $\lambda_q = +$ and $\lambda_{\bar{q}} = -$.

Explicitly, we find

$$\begin{aligned}\mathcal{S}_{\dot{A}B}(q+;g+;\overline{q}-) &= \alpha(y,z)\ \frac{p_{1\dot{A}D}p_2^D p_{2B}}{\langle p_1p_3\rangle\langle p_3p_2\rangle} + \beta(y,z)\ \frac{p_{3\dot{A}D}p_2^D p_{2B}}{\langle p_1p_3\rangle\langle p_3p_2\rangle} + \gamma(y,z)\ \frac{p_{1\dot{C}B}p_3^{\dot{C}} p_{3\dot{A}}}{\langle p_1p_3\rangle\langle p_3p_2\rangle^*} \\ &\quad + \delta(y,z)\ \frac{\langle p_1p_3\rangle^*}{\langle p_1p_3\rangle\langle p_1p_2\rangle^*}\left(p_{1\dot{A}B} + p_{2\dot{A}B} + p_{3\dot{A}B}\right)\ .\end{aligned} \tag{3.7}$$

The other helicity amplitudes are obtained from $\mathcal{S}_{\dot{A}B}(q+;g+;\bar{q}-)$ by the above parity and charge conjugation relations, while the coefficients $\alpha$, $\beta$ and $\gamma$ are written in terms of the tensor coefficients:

$$\begin{aligned}\alpha(y,z) &= \frac{s_{23}s_{13}}{4}\Big(2B(s_{13},s_{23},s_{123}) + A_{12}(s_{13},s_{23},s_{123}) - A_{11}(s_{13},s_{23},s_{123})\Big), \\ \beta(y,z) &= \frac{s_{13}}{4}\Big(2s_{23}B(s_{13},s_{23},s_{123}) + 2(s_{12}+s_{13})A_{11}(s_{13},s_{23},s_{123}) \\ &\qquad + s_{23}\big(A_{12}(s_{13},s_{23},s_{123}) + A_{13}(s_{13},s_{23},s_{123})\big)\Big), \\ \gamma(y,z) &= \frac{s_{13}s_{23}}{4}\Big(A_{11}(s_{13},s_{23},s_{123}) - A_{13}(s_{13},s_{23},s_{123})\Big), \\ \delta(y,z) &= -\frac{s_{12}s_{13}}{4}A_{11}(s_{13},s_{23},s_{123}).\end{aligned} \tag{3.8}$$

When the hadron tensor is contracted with $\epsilon_4^\mu$ or the lepton current $V^\mu$, the final term of Eq. (3.7) vanishes[3]. Furthermore, current conservation implies the following relation between the four helicity coefficients,

$$\alpha(y,z) - \beta(y,z) - \gamma(y,z) - \frac{2s_{123}}{s_{12}}\ \delta(y,z) = 0. \tag{3.9}$$

This relation is fulfilled automatically once the tensor coefficients are inserted and does therefore not yield a further reduction of the tensor basis.

[3]And for this reason was omitted in Ref. [6].

As with the tensor coefficients, the helicity amplitude coefficients $\alpha$, $\beta$ and $\gamma$ are vectors in colour space and have perturbative expansions:

$$\Omega = \sqrt{4\pi\alpha}\sqrt{4\pi\alpha_s}\,\boldsymbol{T}^a_{ij}\,\left[\Omega^{(0)} + \left(\frac{\alpha_s}{2\pi}\right)\Omega^{(1)} + \left(\frac{\alpha_s}{2\pi}\right)^2\Omega^{(2)} + \mathcal{O}(\alpha_s^3)\right]\,, \tag{3.10}$$

for $\Omega = \alpha, \beta, \gamma$. The dependence on $(y, z)$ is again implicit.

The ultraviolet and infrared properties of the helicity coefficients match those of the tensor coefficients,

$$\begin{aligned}
\Omega^{(0)} &= \Omega^{(0),\mathrm{un}},\\
\Omega^{(1)} &= S_\epsilon^{-1}\Omega^{(1),\mathrm{un}} - \frac{\beta_0}{2\epsilon}\Omega^{(0),\mathrm{un}},\\
\Omega^{(2)} &= S_\epsilon^{-2}\Omega^{(2),\mathrm{un}} - \frac{3\beta_0}{2\epsilon}S_\epsilon^{-1}\Omega^{(1),\mathrm{un}} - \left(\frac{\beta_1}{4\epsilon} - \frac{3\beta_0^2}{8\epsilon^2}\right)\Omega^{(0),\mathrm{un}},
\end{aligned} \tag{3.11}$$

and

$$\begin{aligned}
\Omega^{(1)} &= \boldsymbol{I}^{(1)}(\epsilon)\Omega^{(0)} + \Omega^{(1),\mathrm{finite}},\\
\Omega^{(2)} &= \left(-\frac{1}{2}\boldsymbol{I}^{(1)}(\epsilon)\boldsymbol{I}^{(1)}(\epsilon) - \frac{\beta_0}{\epsilon}\boldsymbol{I}^{(1)}(\epsilon) + e^{-\epsilon\gamma}\frac{\Gamma(1-2\epsilon)}{\Gamma(1-\epsilon)}\left(\frac{\beta_0}{\epsilon} + K\right)\boldsymbol{I}^{(1)}(2\epsilon) + \boldsymbol{H}^{(2)}(\epsilon)\right)\Omega^{(0)}\\
&\quad + \boldsymbol{I}^{(1)}(\epsilon)\Omega^{(1)} + \Omega^{(2),\mathrm{finite}},
\end{aligned} \tag{3.12}$$

where $\boldsymbol{I}^{(1)}(\epsilon)$ and $\boldsymbol{H}^{(2)}(\epsilon)$ are defined in Eqs. (2.42) and (2.44) respectively.

At leading order

$$\alpha^{(0)}(y,z) = \beta^{(0)}(y,z) = 1 \qquad \text{and} \qquad \gamma^{(0)}(y,z) = 0. \tag{3.13}$$

The renormalized next-to-leading order helicity amplitude coefficients can be straightforwardly obtained to all orders in $\epsilon$ from the tensor coefficients using Eqs. (2.21)–(2.24). For practical purposes, they are needed through to $\mathcal{O}(\epsilon^2)$ in evaluating the infrared-divergent one-loop contribution to the two-loop amplitude, while only the finite piece is needed for the one-loop self-interference. They can be decomposed according to their colour structure as follows:

$$\Omega^{(1),\mathrm{finite}}(y,z) = N\,a_\Omega(y,z) + \frac{1}{N}\,b_\Omega(y,z) + \beta_0\,c_\Omega(y,z)\,. \tag{3.14}$$

The expansion of the coefficients through to $\epsilon^2$ yields HPLs and 2dHPLs up to weight 4 for $a_\Omega$, $b_\Omega$ and up to weight 3 for $c_\Omega$. The explicit expressions are of considerable size, such that we only quote the $\epsilon^0$-terms here (although these have been known already for a long time [6]). The expressions through to $\mathcal{O}(\epsilon^2)$ can be obtained in FORM format from the authors. An example of the size and structure of those coefficients can be found in [18], where we explicitly list the helicity-averaged one-loop times one-loop and tree times two-loop matrix elements. The one-loop coefficients read:

$$\begin{aligned}
a_\alpha(y,z) &= -\frac{7}{4} - \frac{\pi^2}{12} + \frac{3}{8}\mathrm{H}(0;z) - \frac{1}{2}\mathrm{H}(0,z)\mathrm{G}(0;y) - \frac{1}{2}\mathrm{H}(1,0;z) - \frac{3}{8}\mathrm{G}(0;y) + \frac{1}{2}\mathrm{G}(1,0;y)\\
&\quad - \frac{1}{4(1-z)^2}\mathrm{H}(0;z) - \frac{1}{4(1-z)}\Big(1 + 2\mathrm{H}(0;z)\Big) + \mathcal{O}(\epsilon)\,,\\
b_\alpha(y,z) &= \frac{z^2}{2y^2}\Big(\mathrm{H}(0;z)\mathrm{G}(1-z;y) + \mathrm{H}(1;z)\mathrm{G}(-z;y) - \mathrm{G}(-z,1-z;y)\Big) + \frac{z}{2y}\Big(-\mathrm{H}(0;z)\\
&\quad + 2\mathrm{H}(0;z)\mathrm{G}(1-z;y) - \mathrm{H}(1;z) + 2\mathrm{H}(1;z)\mathrm{G}(-z;y) + \mathrm{G}(1-z;y) - 2\mathrm{G}(-z,1-z;y)\Big)\\
&\quad + \frac{1}{2y(1-z)}\mathrm{H}(0;z) - \frac{1}{2y}\mathrm{H}(0;z) + \frac{1}{4(1-z)^2}\mathrm{H}(0;z) + \frac{1}{4(1-z)}\Big(1 + 2\mathrm{H}(0;z)\Big) + \frac{7}{4} - \frac{3}{4}\mathrm{H}(0;z)\\
&\quad + \frac{1}{2}\mathrm{H}(0;z)\mathrm{G}(1-z;y) + \frac{1}{2}\mathrm{H}(0,1;z) - \frac{3}{4}\mathrm{H}(1;z) + \mathrm{H}(1;z)\mathrm{G}(-z;y) - \frac{1}{2}\mathrm{H}(1;z)\mathrm{G}(0;y)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{3}{4}\mathrm{G}(1-z;y)+\frac{1}{2}\mathrm{G}(1-z,0;y)-\mathrm{G}(-z,1-z;y)+\frac{1}{2}\mathrm{G}(0,1-z;y)-\frac{1}{2}\mathrm{G}(1,0;y)+\mathcal{O}(\epsilon)\,,\\
c_\alpha(y,z) &= -\frac{1}{4}\mathrm{H}(0;z)-\frac{1}{4}\mathrm{G}(0;y)+\frac{i\pi}{2}+\mathcal{O}(\epsilon)\,,\\
a_\beta(y,z) &= -\frac{3}{2}-\frac{\pi^2}{12}+\frac{3}{8}\mathrm{H}(0;z)-\frac{1}{2}\mathrm{H}(0;z)\mathrm{G}(0;y)-\frac{1}{2}\mathrm{H}(1,0;z)-\frac{3}{8}\mathrm{G}(0;y)+\frac{1}{2}\mathrm{G}(1,0;y)\\
&\quad+\frac{1}{4(1-z)}\mathrm{H}(0;z)+\mathcal{O}(\epsilon)\,,\\
b_\beta(y,z) &= \frac{z(1-z)}{2y^2}\Big(-\mathrm{H}(0;z)\mathrm{G}(1-z;y)-\mathrm{H}(1;z)\mathrm{G}(-z;y)+\mathrm{G}(-z,1-z;y)\Big)+\frac{z}{2y}\Big(-\mathrm{H}(0;z)\\
&\quad+2\mathrm{H}(0;z)\mathrm{G}(1-z;y)-\mathrm{H}(1;z)+2\mathrm{H}(1;z)\mathrm{G}(-z;y)+\mathrm{G}(1-z;y)-2\mathrm{G}(-z,1-z;y)\Big)\\
&\quad+\frac{1}{2y}\Big(-2\mathrm{H}(0;z)\mathrm{G}(1-z;y)+\mathrm{H}(1;z)-2\mathrm{H}(1;z)\mathrm{G}(-z;y)-\mathrm{G}(1-z;y)+2\mathrm{G}(-z,1-z;y)\Big)\\
&\quad+\frac{z}{2(y+z)^2}\Big(-\mathrm{H}(1;z)+\mathrm{G}(1-z;y)\Big)+\frac{z}{2(y+z)}-\frac{1}{4(1-z)}\mathrm{H}(0;z)+\frac{1}{2(y+z)}\Big(\mathrm{H}(1;z)\\
&\quad-\mathrm{G}(1-z;y)\Big)+\frac{3}{2}-\frac{3}{4}\mathrm{H}(0;z)+\frac{1}{2}\mathrm{H}(0;z)\mathrm{G}(1-z;y)+\frac{1}{2}\mathrm{H}(0,1;z)-\frac{3}{4}\mathrm{H}(1;z)\\
&\quad+\mathrm{H}(1;z)\mathrm{G}(-z;y)-\frac{1}{2}\mathrm{H}(1;z)\mathrm{G}(0;y)+\frac{3}{4}\mathrm{G}(1-z;y)+\frac{1}{2}\mathrm{G}(1-z,0;y)-\mathrm{G}(-z,1-z;y)\\
&\quad+\frac{1}{2}\mathrm{G}(0,1-z;y)-\frac{1}{2}\mathrm{G}(1,0;y)\Big)+\mathcal{O}(\epsilon)\,,\\
c_\beta(y,z) &= -\frac{1}{4}\mathrm{H}(0;z)-\frac{1}{4}\mathrm{G}(0;y)+\frac{i\pi}{2}+\mathcal{O}(\epsilon)\,,\\
a_\gamma(y,z) &= -\frac{1}{4}+\frac{1}{4(1-z)^2}\mathrm{H}(0;z)+\frac{1}{4(1-z)}\Big(1-\mathrm{H}(0;z)\Big)+\mathcal{O}(\epsilon)\,,\\
b_\gamma(y,z) &= \frac{1}{4}+\frac{z}{2y^2}\Big(-\mathrm{H}(0;z)\mathrm{G}(1-z;y)-\mathrm{H}(1;z)\mathrm{G}(-z;y)+\mathrm{G}(-z,1-z;y)\Big)-\frac{1}{2y(1-z)}\mathrm{H}(0;z)\\
&\quad+\frac{1}{2y}\Big(\mathrm{H}(0;z)+\mathrm{H}(1;z)-\mathrm{G}(1-z;y)\Big)+\frac{z}{2(y+z)^2}\Big(\mathrm{H}(1;z)-\mathrm{G}(1-z;y)\Big)-\frac{z}{2(y+z)}\\
&\quad+\frac{1}{2(y+z)}\Big(-\mathrm{H}(1;z)+\mathrm{G}(1-z;y)\Big)-\frac{1}{4(1-z)^2}\mathrm{H}(0;z)+\frac{1}{4(1-z)}\Big(-1+\mathrm{H}(0;z)\Big)+\mathcal{O}(\epsilon)\,,\\
c_\gamma(y,z) &= 0\,.
\end{aligned}
\tag{3.15}
$$

It should be noted that these finite pieces of the one-loop coefficients can equally well be written in terms of ordinary logarithms and dilogarithms, see [3, 6]. The reason to express them in terms of HPLs and 2dHPLs here is their usage in the infrared counter-term of the two-loop coefficients, which cannot be fully expressed in terms of logarithmic and polylogarithmic functions.

The finite two-loop remainder is obtained by subtracting the predicted infrared structure (expanded through to $\mathcal{O}(\epsilon^0)$) from the renormalized helicity coefficient. We further decompose the finite remainder according to the colour structure, as follows:

$$
\begin{aligned}
\Omega^{(2),\mathrm{finite}}(y,z) &= N^2 A_\Omega(y,z)+B_\Omega(y,z)+\frac{1}{N^2}C_\Omega(y,z)+NN_F D_\Omega(y,z)\\
&\quad+\frac{N_F}{N}E_\Omega(y,z)+N_F^2 F_\Omega(y,z)+N_{F,V}\left(\frac{4}{N}-N\right)G_\Omega(y,z)\,,
\end{aligned}
\tag{3.16}
$$

where the last term is generated by graphs where the virtual gauge boson does not couple directly to the final-state quarks. This contribution is denoted by $N_{F,V}$ and is proportional to the charge weighted sum of the quark flavours. In the case of purely electromagnetic interactions we find,

$$
N_{F,\gamma}=\frac{\sum_q e_q}{e_q}\,. \tag{3.17}
$$

Including $Z$-interactions, the same class of diagrams yields not only a contribution from the vector component of the $Z$, which for the right-handed quark amplitude is given by

$$N_{F,Z} = \frac{\sum_q \left(L^Z_{qq} + R^Z_{qq}\right)}{2R^Z_{qq}} \,, \tag{3.18}$$

but also a contribution involving the axial couplings of the $Z$ [63]. This contribution vanishes if summed over isospin doublets. The large mass splitting of the third quark family induces a non-vanishing contribution from this class of diagrams, which can however not be computed within the framework of massless QCD employed here, but can only be obtained within an effective theory with large top-quark mass. In contrast to the vector contribution from these diagrams, which is finite, one could expect divergences in the axial vector contribution, which would be cancelled by the single unresolved limits of the corresponding axial contributions to four-parton final states [14, 15]. Results from the four-parton final states show that this axial contribution is numerically very small [64].

The helicity coefficients contain HPLs and 2dHPLs up to weight 4 in the $A, B, C, G$-terms, up to weight 3 in the $D, E$-terms (which do moreover contain only a limited subset of purely planar master integrals) and up to weight 2 in the $F$-term. The size of each helicity coefficient is comparable to the size of the helicity-averaged tree times two-loop matrix element quoted in [18]. We do therefore only quote the $A$- and $D$-terms of each coefficient, which form the leading colour contributions, and which turn out to be numerically dominant, approximating the full expressions to an accuracy of about 20%. The complete set of coefficients in FORM format can be obtained from the authors.

These leading colour terms are:

$$\begin{aligned}
A_\alpha(y,z) =\\
&\frac{1}{48y(1-z)}\Big[\pi^2 - 13\mathrm{H}(0;z) + 6\mathrm{H}(1,0;z) + 6\mathrm{G}(1,0;y)\Big] - \frac{1}{48y(1-y-z)}\Big[\pi^2 - 13\mathrm{H}(0;z) + 6\mathrm{H}(1,0;z)\\
&+6\mathrm{G}(1,0;y)\Big] - \frac{z}{16(1-y)^2}\mathrm{G}(0;y) - \frac{z}{16(1-y)} + \frac{z}{12(1-y-z)^2}\Big[-\frac{5\pi^2}{6} - 5\mathrm{H}(0;z)\mathrm{G}(0;y) - 5\mathrm{H}(1,0;z)\\
&+5\mathrm{G}(1,0;y)\Big] + \frac{z}{16(1-y-z)}\Big[\frac{14\pi^2}{3} - 11\mathrm{H}(0;z) + 28\mathrm{H}(0;z)\mathrm{G}(0;y) + 28\mathrm{H}(1,0;z) + 11\mathrm{G}(0;y)\\
&-28\mathrm{G}(1,0;y)\Big] + \frac{z^2}{16(1-y-z)^2}\Big[\frac{11\pi^2}{6} + 11\mathrm{H}(0;z)\mathrm{G}(0;y) + 11\mathrm{H}(1,0;z) - 11\mathrm{G}(1,0;y)\Big]\\
&+\frac{1}{3(1-y)}\mathrm{G}(0;y) + \frac{1}{48(1-z)^2}\Big[-\frac{\pi^2}{6} + \pi^2\big(3\mathrm{H}(0;z) + 3\mathrm{H}(1;z) - \mathrm{G}(1-z;y) + \mathrm{G}(0;y)\big) + 6\zeta_3\\
&-\frac{355}{6}\mathrm{H}(0;z) - 6\mathrm{H}(0;z)\mathrm{G}(1-z,0;y) + 10\mathrm{H}(0;z)\mathrm{G}(0;y) + 45\mathrm{H}(0,0;z) + 12\mathrm{H}(0,0;z)\mathrm{G}(0;y)\\
&+18\mathrm{H}(0,1,0;z) - \mathrm{H}(1,0;z) - 6\mathrm{H}(1,0;z)\mathrm{G}(1-z;y) + 6\mathrm{H}(1,0;z)\mathrm{G}(0;y) + 12\mathrm{H}(1,0,0;z) + 18\mathrm{H}(1,1,0;z)\\
&+6\mathrm{G}(1-z,1,0;y) - 6\mathrm{G}(0,1,0;y)\Big] + \frac{1}{72(1-z)}\Big[\pi^2\big(-8 + 9\mathrm{H}(0;z) + 9\mathrm{H}(1;z) - 3\mathrm{G}(1-z;y)\\
&+3\mathrm{G}(0;y)\big) + 18\zeta_3 - \frac{277}{4} - 65\mathrm{H}(0;z) - 18\mathrm{H}(0;z)\mathrm{G}(1-z,0;y) + 39\mathrm{H}(0;z)\mathrm{G}(0;y) + 81\mathrm{H}(0,0;z)\\
&+36\mathrm{H}(0,0;z)\mathrm{G}(0;y) + 54\mathrm{H}(0,1,0;z) - 48\mathrm{H}(1,0;z) - 18\mathrm{H}(1,0;z)\mathrm{G}(1-z;y) + 18\mathrm{H}(1,0;z)\mathrm{G}(0;y)\\
&+36\mathrm{H}(1,0,0;z) + 54\mathrm{H}(1,1,0;z) + 18\mathrm{G}(1-z,1,0;y) + 15\mathrm{G}(0;y) - 18\mathrm{G}(0,1,0;y) - 9\mathrm{G}(1,0;y)\Big]\\
&+\frac{1}{48(1-y-z)^2}\Big[-2\pi^2\mathrm{H}(1;z) - \pi^2\mathrm{G}(0;y) + 12\zeta_3 - 6\mathrm{H}(1,0;z)\mathrm{G}(0;y) - 12\mathrm{H}(1,1,0;z) - 6\mathrm{G}(0,1,0;y)\Big]\\
&+\frac{1}{48(1-y-z)}\Big[-4\pi^2\mathrm{H}(1;z) - 2\pi^2\mathrm{G}(0;y) + 24\zeta_3 - 13\mathrm{H}(0;z) - 12\mathrm{H}(1,0;z)\mathrm{G}(0;y) - 24\mathrm{H}(1,1,0;z)\\
&-20\mathrm{G}(0;y) - 12\mathrm{G}(0,1,0;y)\Big] + \frac{\pi^2}{288}\Big[-\frac{928}{3} - 5\mathrm{H}(0;z) + 12\mathrm{H}(0;z)\mathrm{G}(1-z;y) + 36\mathrm{H}(0;z)\mathrm{G}(0;y)\\
&-12\mathrm{H}(0;z)\mathrm{G}(1;y) + 24\mathrm{H}(0,1;z) + 24\mathrm{H}(1;z)\mathrm{G}(1-z;y) - 24\mathrm{H}(1;z)\mathrm{G}(-z;y) - 12\mathrm{H}(1;z)\mathrm{G}(1;y)
\end{aligned}$$

$$
\begin{aligned}
&+24\mathrm{H}(1,0;z)+12\mathrm{H}(1,1;z)-44\mathrm{G}(1-z;y)+12\mathrm{G}(1-z,0;y)-24\mathrm{G}(1-z,1;y)+24\mathrm{G}(-z,1-z;y)\\
&-24\mathrm{G}(0,1-z;y)+49\mathrm{G}(0;y)-24\mathrm{G}(0,1;y)+12\mathrm{G}(1,1-z;y)+44\mathrm{G}(1;y)-36\mathrm{G}(1,0;y)+24\mathrm{G}(1,1;y)\Big]\\
&+\frac{\zeta_3}{72}\Big[317-18\mathrm{H}(0;z)+90\mathrm{H}(1;z)-72\mathrm{G}(1-z;y)-18\mathrm{G}(0;y)-18\mathrm{G}(1;y)\Big]+\frac{11\pi^4}{360}+\frac{1}{72}\Big[-\frac{89959}{144}\\
&+\frac{2149}{12}\mathrm{H}(0;z)-66\mathrm{H}(0;z)\mathrm{G}(1-z,0;y)-18\mathrm{H}(0;z)\mathrm{G}(1-z,1,0;y)+36\mathrm{H}(0;z)\mathrm{G}(-z,1-z,0;y)\\
&-36\mathrm{H}(0;z)\mathrm{G}(0,1-z,0;y)-66\mathrm{H}(0;z)\mathrm{G}(0;y)+126\mathrm{H}(0;z)\mathrm{G}(0,0;y)-18\mathrm{H}(0;z)\mathrm{G}(0,1,0;y)\\
&+18\mathrm{H}(0;z)\mathrm{G}(1,1-z,0;y)-3\mathrm{H}(0;z)\mathrm{G}(1,0;y)-36\mathrm{H}(0;z)\mathrm{G}(1,0,0;y)+\frac{23}{2}\mathrm{H}(0,0;z)\\
&+72\mathrm{H}(0,0;z)\mathrm{G}(0;y)+36\mathrm{H}(0,0;z)\mathrm{G}(0,0;y)+72\mathrm{H}(0,0,1,0;z)+3\mathrm{H}(0,1,0;z)-18\mathrm{H}(0,1,0;z)\mathrm{G}(1-z;y)\\
&+36\mathrm{H}(0,1,0;z)\mathrm{G}(-z;y)+18\mathrm{H}(0,1,0;z)\mathrm{G}(0;y)-18\mathrm{H}(0,1,0;z)\mathrm{G}(1;y)+36\mathrm{H}(0,1,1,0;z)\\
&-71\mathrm{H}(1,0;z)-66\mathrm{H}(1,0;z)\mathrm{G}(1-z;y)+18\mathrm{H}(1,0;z)\mathrm{G}(1-z,0;y)+36\mathrm{H}(1,0;z)\mathrm{G}(-z,1-z;y)\\
&-36\mathrm{H}(1,0;z)\mathrm{G}(-z,0;y)-36\mathrm{H}(1,0;z)\mathrm{G}(0,1-z;y)+96\mathrm{H}(1,0;z)\mathrm{G}(0;y)+18\mathrm{H}(1,0;z)\mathrm{G}(1,1-z;y)\\
&-18\mathrm{H}(1,0;z)\mathrm{G}(1,0;y)+72\mathrm{H}(1,0,0;z)+36\mathrm{H}(1,0,0;z)\mathrm{G}(0;y)+72\mathrm{H}(1,0,1,0;z)\\
&+36\mathrm{H}(1,1,0;z)\mathrm{G}(1-z;y)-36\mathrm{H}(1,1,0;z)\mathrm{G}(-z;y)-18\mathrm{H}(1,1,0;z)\mathrm{G}(1;y)+36\mathrm{H}(1,1,0,0;z)\\
&+18\mathrm{H}(1,1,1,0;z)+18\mathrm{G}(1-z,0,1,0;y)+66\mathrm{G}(1-z,1,0;y)+36\mathrm{G}(1-z,1,1,0;y)\\
&-36\mathrm{G}(-z,1-z,1,0;y)-36\mathrm{G}(-z,0,1,0;y)+36\mathrm{G}(0,1-z,1,0;y)+\frac{49}{3}\mathrm{G}(0;y)+160\mathrm{G}(0,0;y)\\
&-36\mathrm{G}(0,0,1,0;y)-30\mathrm{G}(0,1,0;y)+36\mathrm{G}(0,1,1,0;y)-18\mathrm{G}(1,1-z,1,0;y)+71\mathrm{G}(1,0;y)\\
&-126\mathrm{G}(1,0,0;y)+54\mathrm{G}(1,0,1,0;y)-66\mathrm{G}(1,1,0;y)+36\mathrm{G}(1,1,0,0;y)-36\mathrm{G}(1,1,1,0;y)\Big]\\
&+i\pi\Bigg\{-\frac{11}{16(1-z)^2}H(0;z)+\frac{1}{16(1-z)}\Big[-11-22\mathrm{H}(0;z)\Big]+2\zeta_3+\frac{1}{48}\Big[-\frac{44\pi^2}{3}-\frac{2345}{18}-11\mathrm{H}(0;z)\\
&\qquad-66\mathrm{H}(0;z)\mathrm{G}(0;y)-66\mathrm{H}(1,0;z)-110\mathrm{G}(0;y)+66\mathrm{G}(1,0;y)\Big]\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
D_\alpha(y,z)=\;&\\
&\frac{1}{12y(1-z)}\mathrm{H}(0;z)-\frac{1}{12y(1-y-z)}\mathrm{H}(0;z)+\frac{z}{6(1-y-z)^2}\Big[\frac{\pi^2}{6}+\mathrm{H}(0;z)\mathrm{G}(0;y)+\mathrm{H}(1,0;z)\\
&-\mathrm{G}(1,0;y)\Big]+\frac{z}{4(1-y-z)}\Big[-\frac{\pi^2}{3}+\mathrm{H}(0;z)-2\mathrm{H}(0;z)\mathrm{G}(0;y)-2\mathrm{H}(1,0;z)-\mathrm{G}(0;y)+2\mathrm{G}(1,0;y)\Big]\\
&+\frac{z^2}{4(1-y-z)^2}\Big[-\frac{\pi^2}{6}-\mathrm{H}(0;z)\mathrm{G}(0;y)-\mathrm{H}(1,0;z)+\mathrm{G}(1,0;y)\Big]-\frac{1}{12(1-y)}\mathrm{G}(0;y)\\
&+\frac{1}{72(1-z)^2}\Big[\pi^2+25\mathrm{H}(0;z)-\frac{3}{2}\mathrm{H}(0;z)\mathrm{G}(0;y)-9\mathrm{H}(0,0;z)+6\mathrm{H}(1,0;z)\Big]\\
&+\frac{1}{144(1-z)}\Big[4\pi^2+38+37\mathrm{H}(0;z)-6\mathrm{H}(0;z)\mathrm{G}(0;y)-36\mathrm{H}(0,0;z)+24\mathrm{H}(1,0;z)-3\mathrm{G}(0;y)\Big]\\
&+\frac{1}{12(1-y-z)}\Big[\mathrm{H}(0;z)+2\mathrm{G}(0;y)\Big]+\frac{\pi^2}{72}\Big[\frac{395}{12}-\mathrm{H}(0;z)+2\mathrm{G}(1-z;y)-\mathrm{G}(0;y)-2\mathrm{G}(1;y)\Big]\\
&-\frac{19}{36}\zeta_3+\frac{1}{144}\Big[\frac{3661}{18}-25\mathrm{H}(0;z)+24\mathrm{H}(0;z)\mathrm{G}(1-z,0;y)+29\mathrm{H}(0;z)\mathrm{G}(0;y)-36\mathrm{H}(0;z)\mathrm{G}(0,0;y)\\
&+6\mathrm{H}(0;z)\mathrm{G}(1,0;y)-28\mathrm{H}(0,0;z)-36\mathrm{H}(0,0;z)\mathrm{G}(0;y)-6\mathrm{H}(0,1,0;z)+40\mathrm{H}(1,0;z)\\
&+24\mathrm{H}(1,0;z)\mathrm{G}(1-z;y)-30\mathrm{H}(1,0;z)\mathrm{G}(0;y)-36\mathrm{H}(1,0,0;z)-24\mathrm{G}(1-z,1,0;y)+53\mathrm{G}(0;y)\\
&-82\mathrm{G}(0,0;y)+6\mathrm{G}(0,1,0;y)-40\mathrm{G}(1,0;y)+36\mathrm{G}(1,0,0;y)+24\mathrm{G}(1,1,0;y)\Big]\\
&+i\pi\Bigg\{\frac{1}{8(1-z)^2}\mathrm{H}(0;z)+\frac{1}{8(1-z)}\Big[1+2\mathrm{H}(0;z)\Big]+\frac{1}{48}\Big[\frac{8\pi^2}{3}-\frac{28}{3}+13\mathrm{H}(0;z)+12\mathrm{H}(0;z)\mathrm{G}(0;y)
\end{aligned}
$$

$$+12\mathrm{H}(1,0;z)+31\mathrm{G}(0;y)-12\mathrm{G}(1,0;y)\Big]\Big\}$$

$$A_\beta(y,z)=$$
$$-\frac{z}{16(1-y)^2}\mathrm{G}(0;y)-\frac{z}{16(1-y)}+\frac{z}{16(y+z)^2}\Big[\frac{47\pi^2}{3}\mathrm{H}(1;z)-\frac{47\pi^2}{3}\mathrm{G}(1-z;y)-94\mathrm{H}(0;z)\mathrm{G}(1-z,0;y)$$
$$-94\mathrm{H}(0,1,0;z)-99\mathrm{H}(1,0;z)-94\mathrm{H}(1,0;z)\mathrm{G}(1-z;y)+94\mathrm{H}(1,0;z)\mathrm{G}(0;y)+94\mathrm{H}(1,1,0;z)$$
$$+94\mathrm{G}(1-z,1,0;y)+94\mathrm{G}(0,1,0;y)-99\mathrm{G}(1,0;y)\Big]+\frac{z}{16(y+z)}\Big[-\frac{47\pi^2}{3}+11+44\mathrm{H}(0;z)$$
$$-94\mathrm{H}(0;z)\mathrm{G}(0;y)-94\mathrm{H}(1,0;z)-55\mathrm{G}(0;y)+94\mathrm{G}(1,0;y)\Big]+\frac{z}{12(1-y-z)^2}\Big[\frac{5\pi^2}{6}-\frac{\pi^2}{2}\mathrm{H}(1;z)$$
$$-\frac{\pi^2}{4}\mathrm{G}(0;y)+3\zeta_3+5\mathrm{H}(0;z)\mathrm{G}(0;y)+5\mathrm{H}(1,0;z)-\frac{3}{2}\mathrm{H}(1,0;z)\mathrm{G}(0;y)-3\mathrm{H}(1,1,0;z)-\frac{3}{2}\mathrm{G}(0,1,0;y)$$
$$-5\mathrm{G}(1,0;y)\Big]+\frac{z}{12(1-y-z)}\Big[-\frac{19\pi^2}{3}+5\mathrm{H}(0;z)-38\mathrm{H}(0;z)\mathrm{G}(0;y)-38\mathrm{H}(1,0;z)-\frac{53}{4}\mathrm{G}(0;y)$$
$$+38\mathrm{G}(1,0;y)\Big]+\frac{z^2}{8(y+z)^3}\Big[-11\pi^2\mathrm{H}(1;z)+11\pi^2\mathrm{G}(1-z;y)+66\mathrm{H}(0;z)\mathrm{G}(1-z,0;y)+66\mathrm{H}(0,1,0;z)$$
$$+33\mathrm{H}(1,0;z)+66\mathrm{H}(1,0;z)\mathrm{G}(1-z;y)-66\mathrm{H}(1,0;z)\mathrm{G}(0;y)-66\mathrm{H}(1,1,0;z)-66\mathrm{G}(1-z,1,0;y)$$
$$-66\mathrm{G}(0,1,0;y)+33\mathrm{G}(1,0;y)\Big]+\frac{z^2}{16(y+z)^2}\Big[+22\pi^2-33\mathrm{H}(0;z)+132\mathrm{H}(0;z)\mathrm{G}(0;y)+132\mathrm{H}(1,0;z)$$
$$+33\mathrm{G}(0;y)-132\mathrm{G}(1,0;y)\Big]+\frac{z^2}{16(y+z)}\Big[11\pi^2-11\mathrm{H}(0;z)+66\mathrm{H}(0;z)\mathrm{G}(0;y)+66\mathrm{H}(1,0;z)$$
$$+11\mathrm{G}(0;y)-66\mathrm{G}(1,0;y)\Big]+\frac{z^2}{48(1-y-z)^2}\Big[-\frac{53\pi^2}{6}-53\mathrm{H}(0;z)\mathrm{G}(0;y)-53\mathrm{H}(1,0;z)+53\mathrm{G}(1,0;y)\Big]$$
$$+\frac{z^2}{16(1-y-z)}\Big[11\pi^2-11\mathrm{H}(0;z)+66\mathrm{H}(0;z)\mathrm{G}(0;y)+66\mathrm{H}(1,0;z)+11\mathrm{G}(0;y)-66\mathrm{G}(1,0;y)\Big]$$
$$+\frac{z^3}{8(y+z)^4}\Big[\frac{11\pi^2}{2}\mathrm{H}(1;z)-\frac{11\pi^2}{2}\mathrm{G}(1-z;y)-33\mathrm{H}(0;z)\mathrm{G}(1-z,0;y)-33\mathrm{H}(0,1,0;z)$$
$$-33\mathrm{H}(1,0;z)\mathrm{G}(1-z;y)+33\mathrm{H}(1,0;z)\mathrm{G}(0;y)+33\mathrm{H}(1,1,0;z)+33\mathrm{G}(1-z,1,0;y)+33\mathrm{G}(0,1,0;y)\Big]$$
$$+\frac{z^3}{8(y+z)^3}\Big[-\frac{11\pi^2}{2}-33\mathrm{H}(0;z)\mathrm{G}(0;y)-33\mathrm{H}(1,0;z)+33\mathrm{G}(1,0;y)\Big]+\frac{z^3}{16(y+z)^2}\Big[-\frac{11\pi^2}{2}$$
$$-33\mathrm{H}(0;z)\mathrm{G}(0;y)-33\mathrm{H}(1,0;z)+33\mathrm{G}(1,0;y)\Big]+\frac{z^3}{8(y+z)}\Big[-\frac{11\pi^2}{6}-11\mathrm{H}(0;z)\mathrm{G}(0;y)-11\mathrm{H}(1,0;z)$$
$$+11\mathrm{G}(1,0;y)\Big]+\frac{z^3}{16(1-y-z)^2}\Big[\frac{11\pi^2}{6}+11\mathrm{H}(0;z)\mathrm{G}(0;y)+11\mathrm{H}(1,0;z)-11\mathrm{G}(1,0;y)\Big]$$
$$+\frac{z^3}{8(1-y-z)}\Big[-\frac{11\pi^2}{6}-11\mathrm{H}(0;z)\mathrm{G}(0;y)-11\mathrm{H}(1,0;z)+11\mathrm{G}(1,0;y)\Big]+\frac{5}{24(1-y)}\mathrm{G}(0;y)$$
$$+\frac{1}{48(1-z)}\Big[\pi^2\big(\frac{1}{6}-3\mathrm{H}(0;z)-3\mathrm{H}(1;z)+\mathrm{G}(1-z;y)-\mathrm{G}(0;y)\big)-6\zeta_3+\frac{355}{6}\mathrm{H}(0;z)$$
$$+6\mathrm{H}(0;z)\mathrm{G}(1-z,0;y)-10\mathrm{H}(0;z)\mathrm{G}(0;y)-45\mathrm{H}(0,0;z)-12\mathrm{H}(0,0;z)\mathrm{G}(0;y)-18\mathrm{H}(0,1,0;z)$$
$$+\mathrm{H}(1,0;z)+6\mathrm{H}(1,0;z)\mathrm{G}(1-z;y)-6\mathrm{H}(1,0;z)\mathrm{G}(0;y)-12\mathrm{H}(1,0,0;z)-18\mathrm{H}(1,1,0;z)$$
$$-6\mathrm{G}(1-z,1,0;y)+6\mathrm{G}(0,1,0;y)\Big]+\frac{1}{8(y+z)}\Big[-\frac{7\pi^2}{3}\mathrm{H}(1;z)+\frac{7\pi^2}{3}\mathrm{G}(1-z;y)+14\mathrm{H}(0;z)\mathrm{G}(1-z,0;y)$$
$$+14\mathrm{H}(0,1,0;z)+25\mathrm{H}(1,0;z)+14\mathrm{H}(1,0;z)\mathrm{G}(1-z;y)-14\mathrm{H}(1,0;z)\mathrm{G}(0;y)-14\mathrm{H}(1,1,0;z)$$
$$-14\mathrm{G}(1-z,1,0;y)-14\mathrm{G}(0,1,0;y)+25\mathrm{G}(1,0;y)\Big]+\frac{1}{8(1-y-z)^2}\Big[\frac{\pi^2}{3}\mathrm{H}(1;z)+\frac{\pi^2}{6}\mathrm{G}(0;y)-2\zeta_3$$

$$
\begin{aligned}
&+\mathrm{H}(1,0;z)\mathrm{G}(0;y)+2\mathrm{H}(1,1,0;z)+\mathrm{G}(0,1,0;y)\Big]+\frac{1}{24(1-y-z)}\Big[\frac{13\pi^2}{6}+10\mathrm{H}(0;z)\mathrm{G}(0;y)+13\mathrm{H}(1,0;z)\\
&+10\mathrm{G}(0;y)-7\mathrm{G}(1,0;y)\Big]+\frac{\pi^2}{288}\Big[-\frac{946}{3}-5\mathrm{H}(0;z)+12\mathrm{H}(0;z)\mathrm{G}(1-z;y)+36\mathrm{H}(0;z)\mathrm{G}(0;y)\\
&-12\mathrm{H}(0;z)\mathrm{G}(1;y)+24\mathrm{H}(0,1;z)+24\mathrm{H}(1;z)\mathrm{G}(1-z;y)-24\mathrm{H}(1;z)\mathrm{G}(-z;y)-12\mathrm{H}(1;z)\mathrm{G}(1;y)\\
&+24\mathrm{H}(1,0;z)+12\mathrm{H}(1,1;z)-44\mathrm{G}(1-z;y)+12\mathrm{G}(1-z,0;y)-24\mathrm{G}(1-z,1;y)+24\mathrm{G}(-z,1-z;y)\\
&-24\mathrm{G}(0,1-z;y)+49\mathrm{G}(0;y)-24\mathrm{G}(0,1;y)+12\mathrm{G}(1,1-z;y)+44\mathrm{G}(1;y)-36\mathrm{G}(1,0;y)+24\mathrm{G}(1,1;y)\Big]\\
&+\frac{11\pi^4}{360}+\frac{\zeta_3}{72}\Big[317-18\mathrm{H}(0;z)+90\mathrm{H}(1;z)-72\mathrm{G}(1-z;y)-18\mathrm{G}(0;y)-18\mathrm{G}(1;y)\Big]+\frac{1}{144}\Big[-\frac{79987}{72}\\
&+\frac{1735}{6}\mathrm{H}(0;z)-132\mathrm{H}(0;z)\mathrm{G}(1-z,0;y)-36\mathrm{H}(0;z)\mathrm{G}(1-z,1,0;y)+72\mathrm{H}(0;z)\mathrm{G}(-z,1-z,0;y)\\
&-72\mathrm{H}(0;z)\mathrm{G}(0,1-z,0;y)-150\mathrm{H}(0;z)\mathrm{G}(0;y)+252\mathrm{H}(0;z)\mathrm{G}(0,0;y)-36\mathrm{H}(0;z)\mathrm{G}(0,1,0;y)\\
&+36\mathrm{H}(0;z)\mathrm{G}(1,1-z,0;y)-6\mathrm{H}(0;z)\mathrm{G}(1,0;y)-72\mathrm{H}(0;z)\mathrm{G}(1,0,0;y)+23\mathrm{H}(0,0;z)\\
&+144\mathrm{H}(0,0;z)\mathrm{G}(0;y)+72\mathrm{H}(0,0;z)\mathrm{G}(0,0;y)+144\mathrm{H}(0,0,1,0;z)+6\mathrm{H}(0,1,0;z)\\
&-36\mathrm{H}(0,1,0;z)\mathrm{G}(1-z;y)+72\mathrm{H}(0,1,0;z)\mathrm{G}(-z;y)+36\mathrm{H}(0,1,0;z)\mathrm{G}(0;y)-36\mathrm{H}(0,1,0;z)\mathrm{G}(1;y)\\
&+72\mathrm{H}(0,1,1,0;z)-160\mathrm{H}(1,0;z)-132\mathrm{H}(1,0;z)\mathrm{G}(1-z;y)+36\mathrm{H}(1,0;z)\mathrm{G}(1-z,0;y)\\
&+72\mathrm{H}(1,0;z)\mathrm{G}(-z,1-z;y)-72\mathrm{H}(1,0;z)\mathrm{G}(-z,0;y)-72\mathrm{H}(1,0;z)\mathrm{G}(0,1-z;y)+192\mathrm{H}(1,0;z)\mathrm{G}(0;y)\\
&+36\mathrm{H}(1,0;z)\mathrm{G}(1,1-z;y)-36\mathrm{H}(1,0;z)\mathrm{G}(1,0;y)+144\mathrm{H}(1,0,0;z)+72\mathrm{H}(1,0,0;z)\mathrm{G}(0;y)\\
&+144\mathrm{H}(1,0,1,0;z)+72\mathrm{H}(1,1,0;z)\mathrm{G}(1-z;y)-72\mathrm{H}(1,1,0;z)\mathrm{G}(-z;y)-36\mathrm{H}(1,1,0;z)\mathrm{G}(1;y)\\
&+72\mathrm{H}(1,1,0,0;z)+36\mathrm{H}(1,1,1,0;z)+36\mathrm{G}(1-z,0,1,0;y)+132\mathrm{G}(1-z,1,0;y)+72\mathrm{G}(1-z,1,1,0;y)\\
&-72\mathrm{G}(-z,1-z,1,0;y)-72\mathrm{G}(-z,0,1,0;y)+72\mathrm{G}(0,1-z,1,0;y)+\frac{8}{3}\mathrm{G}(0;y)+320\mathrm{G}(0,0;y)\\
&-72\mathrm{G}(0,0,1,0;y)-60\mathrm{G}(0,1,0;y)+72\mathrm{G}(0,1,1,0;y)-36\mathrm{G}(1,1-z,1,0;y)+160\mathrm{G}(1,0;y)\\
&-252\mathrm{G}(1,0,0;y)+108\mathrm{G}(1,0,1,0;y)-132\mathrm{G}(1,1,0;y)+72\mathrm{G}(1,1,0,0;y)-72\mathrm{G}(1,1,1,0;y)\Big]\\
&+i\pi\Bigg\{\frac{11}{16(1-z)}\mathrm{H}(0;z)+2\zeta_3+\frac{1}{48}\Big[-\frac{44\pi^2}{3}-\frac{1751}{18}-11\mathrm{H}(0;z)-66\mathrm{H}(0;z)\mathrm{G}(0;y)-66\mathrm{H}(1,0;z)\\
&\qquad-110\mathrm{G}(0;y)+66\mathrm{G}(1,0;y)\Big]\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
D_\beta(y,z)=\;&\\
&\frac{z}{4(y+z)^2}\Big[-\frac{4\pi^2}{3}\mathrm{H}(1;z)+\frac{4\pi^2}{3}\mathrm{G}(1-z;y)+8\mathrm{H}(0;z)\mathrm{G}(1-z,0;y)+8\mathrm{H}(0,1,0;z)+9\mathrm{H}(1,0;z)\\
&+8\mathrm{H}(1,0;z)\mathrm{G}(1-z;y)-8\mathrm{H}(1,0;z)\mathrm{G}(0;y)-8\mathrm{H}(1,1,0;z)-8\mathrm{G}(1-z,1,0;y)-8\mathrm{G}(0,1,0;y)\\
&+9\mathrm{G}(1,0;y)\Big]+\frac{z}{4(y+z)}\Big[\frac{4\pi^2}{3}-1-4\mathrm{H}(0;z)+8\mathrm{H}(0;z)\mathrm{G}(0;y)+8\mathrm{H}(1,0;z)+5\mathrm{G}(0;y)-8\mathrm{G}(1,0;y)\Big]\\
&+\frac{z}{6(1-y-z)^2}\Big[-\frac{\pi^2}{6}-\mathrm{H}(0;z)\mathrm{G}(0;y)-\mathrm{H}(1,0;z)+\mathrm{G}(1,0;y)\Big]+\frac{z}{12(1-y-z)}\Big[\frac{7\pi^2}{3}-2\mathrm{H}(0;z)\\
&+14\mathrm{H}(0;z)\mathrm{G}(0;y)+14\mathrm{H}(1,0;z)+5\mathrm{G}(0;y)-14\mathrm{G}(1,0;y)\Big]+\frac{z^2}{2(y+z)^3}\Big[\pi^2\mathrm{H}(1;z)-\pi^2\mathrm{G}(1-z;y)\\
&-6\mathrm{H}(0;z)\mathrm{G}(1-z,0;y)-6\mathrm{H}(0,1,0;z)-3\mathrm{H}(1,0;z)-6\mathrm{H}(1,0;z)\mathrm{G}(1-z;y)+6\mathrm{H}(1,0;z)\mathrm{G}(0;y)\\
&+6\mathrm{H}(1,1,0;z)+6\mathrm{G}(1-z,1,0;y)+6\mathrm{G}(0,1,0;y)-3\mathrm{G}(1,0;y)\Big]+\frac{z^2}{4(y+z)^2}\Big[-2\pi^2+3\mathrm{H}(0;z)\\
&-12\mathrm{H}(0;z)\mathrm{G}(0;y)-12\mathrm{H}(1,0;z)-3\mathrm{G}(0;y)+12\mathrm{G}(1,0;y)\Big]+\frac{z^2}{4(y+z)}\Big[-\pi^2+\mathrm{H}(0;z)
\end{aligned}
$$

$$
\begin{aligned}
&-6\mathrm{H}(0;z)\mathrm{G}(0;y) - 6\mathrm{H}(1,0;z) - \mathrm{G}(0;y) + 6\mathrm{G}(1,0;y)\Big] + \frac{z^2}{12(1-y-z)^2}\Big[\frac{5\pi^2}{6} + 5\mathrm{H}(0;z)\mathrm{G}(0;y)\\
&+5\mathrm{H}(1,0;z) - 5\mathrm{G}(1,0;y)\Big] + \frac{z^2}{4(1-y-z)}\Big[-\pi^2 + \mathrm{H}(0;z) - 6\mathrm{H}(0;z)\mathrm{G}(0;y) - 6\mathrm{H}(1,0;z) - \mathrm{G}(0;y)\\
&+6\mathrm{G}(1,0;y)\Big] + \frac{z^3}{2(y+z)^4}\Big[-\frac{\pi^2}{2}\mathrm{H}(1;z) + \frac{\pi^2}{2}\mathrm{G}(1-z;y) + 3\mathrm{H}(0;z)\mathrm{G}(1-z,0;y) + 3\mathrm{H}(0,1,0;z)\\
&+3\mathrm{H}(1,0;z)\mathrm{G}(1-z;y) - 3\mathrm{H}(1,0;z)\mathrm{G}(0;y) - 3\mathrm{H}(1,1,0;z) - 3\mathrm{G}(1-z,1,0;y) - 3\mathrm{G}(0,1,0;y)\Big]\\
&+\frac{z^3}{2(y+z)^3}\Big[\frac{\pi^2}{2} + 3\mathrm{H}(0;z)\mathrm{G}(0;y) + 3\mathrm{H}(1,0;z) - 3\mathrm{G}(1,0;y)\Big] + \frac{z^3}{4(y+z)^2}\Big[\frac{\pi^2}{2} + 3\mathrm{H}(0;z)\mathrm{G}(0;y)\\
&+3\mathrm{H}(1,0;z) - 3\mathrm{G}(1,0;y)\Big] + \frac{z^3}{2(y+z)}\Big[\frac{\pi^2}{6} + \mathrm{H}(0;z)\mathrm{G}(0;y) + \mathrm{H}(1,0;z) - \mathrm{G}(1,0;y)\Big]\\
&+\frac{z^3}{4(1-y-z)^2}\Big[-\frac{\pi^2}{6} - \mathrm{H}(0;z)\mathrm{G}(0;y) - \mathrm{H}(1,0;z) + \mathrm{G}(1,0;y)\Big] + \frac{z^3}{2(1-y-z)}\Big[\frac{\pi^2}{6} + \mathrm{H}(0;z)\mathrm{G}(0;y)\\
&+\mathrm{H}(1,0;z) - \mathrm{G}(1,0;y)\Big] - \frac{1}{12(1-y)}\mathrm{G}(0;y) + \frac{1}{72(1-z)}\Big[-\pi^2 - 25\mathrm{H}(0;z) + \frac{3}{2}\mathrm{H}(0;z)\mathrm{G}(0;y)\\
&+9\mathrm{H}(0,0;z) - 6\mathrm{H}(1,0;z)\Big] + \frac{1}{2(y+z)}\Big[\frac{\pi^2}{6}\mathrm{H}(1;z) - \frac{\pi^2}{6}\mathrm{G}(1-z;y) - \mathrm{H}(0;z)\mathrm{G}(1-z,0;y) - \mathrm{H}(0,1,0;z)\\
&-2\mathrm{H}(1,0;z) - \mathrm{H}(1,0;z)\mathrm{G}(1-z;y) + \mathrm{H}(1,0;z)\mathrm{G}(0;y) + \mathrm{H}(1,1,0;z) + \mathrm{G}(1-z,1,0;y) + \mathrm{G}(0,1,0;y)\\
&-2\mathrm{G}(1,0;y)\Big] + \frac{1}{6(1-y-z)}\Big[-\frac{\pi^2}{6} - \mathrm{H}(0;z)\mathrm{G}(0;y) - \mathrm{H}(1,0;z) - \mathrm{G}(0;y) + \mathrm{G}(1,0;y)\Big]\\
&+\frac{\pi^2}{72}\Big[\frac{395}{12} - \mathrm{H}(0;z) + 2\mathrm{G}(1-z;y) - \mathrm{G}(0;y) - 2\mathrm{G}(1;y)\Big] - \frac{19}{36}\zeta_3 + \frac{1}{144}\Big[\frac{2977}{18} - 10\mathrm{H}(0;z)\\
&+24\mathrm{H}(0;z)\mathrm{G}(1-z,0;y) + 29\mathrm{H}(0;z)\mathrm{G}(0;y) - 36\mathrm{H}(0;z)\mathrm{G}(0,0;y) + 6\mathrm{H}(0;z)\mathrm{G}(1,0;y) - 28\mathrm{H}(0,0;z)\\
&-36\mathrm{H}(0,0;z)\mathrm{G}(0;y) - 6\mathrm{H}(0,1,0;z) + 40\mathrm{H}(1,0;z) + 24\mathrm{H}(1,0;z)\mathrm{G}(1-z;y) - 30\mathrm{H}(1,0;z)\mathrm{G}(0;y)\\
&-36\mathrm{H}(1,0,0;z) - 24\mathrm{G}(1-z,1,0;y) + 56\mathrm{G}(0;y) - 82\mathrm{G}(0,0;y) + 6\mathrm{G}(0,1,0;y) - 40\mathrm{G}(1,0;y)\\
&+36\mathrm{G}(1,0,0;y) + 24\mathrm{G}(1,1,0;y)\Big]\\
&+i\pi\Bigg\{-\frac{1}{8(1-z)}\mathrm{H}(0;z) + \frac{1}{48}\Big[\frac{8\pi^2}{3} - \frac{46}{3} + 13\mathrm{H}(0;z) + 12\mathrm{H}(0;z)\mathrm{G}(0,y) + 12\mathrm{H}(1,0;z) + 31\mathrm{G}(0;y)\\
&\qquad -12\mathrm{G}(1,0;y)\Big]\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
A_\gamma(y,z) =\ &+\frac{1}{48y(1-z)}\Big[-\pi^2 + 13\mathrm{H}(0;z) - 6\mathrm{H}(1,0;z) - 6\mathrm{G}(1,0;y)\Big] + \frac{1}{48y(1-y-z)}\Big[\pi^2 - 13\mathrm{H}(0;z)\\
&+6\mathrm{H}(1,0;z) + 6\mathrm{G}(1,0;y)\Big] + \frac{z}{16(y+z)^2}\Big[-\frac{11\pi^2}{3}\mathrm{H}(1;z) + \frac{11\pi^2}{3}\mathrm{G}(1-z;y) + 22\mathrm{H}(0;z)\mathrm{G}(1-z,0;y)\\
&+22\mathrm{H}(0,1,0;z) + 55\mathrm{H}(1,0;z) + 22\mathrm{H}(1,0;z)\mathrm{G}(1-z;y) - 22\mathrm{H}(1,0;z)\mathrm{G}(0;y) - 22\mathrm{H}(1,1,0;z)\\
&-22\mathrm{G}(1-z,1,0;y) - 22\mathrm{G}(0,1,0;y) + 55\mathrm{G}(1,0;y)\Big] + \frac{z}{16(y+z)}\Big[\frac{11\pi^2}{3} - 11 - 22\mathrm{H}(0;z)\\
&+22\mathrm{H}(0;z)\mathrm{G}(0;y) + 22\mathrm{H}(1,0;z) + 33\mathrm{G}(0;y) - 22\mathrm{G}(1,0;y)\Big] + \frac{z}{8(1-y-z)^2}\Big[\frac{\pi^2}{3}\mathrm{H}(1;z) + \frac{\pi^2}{6}\mathrm{G}(0;y)\\
&-2\zeta_3 + \mathrm{H}(1,0;z)\mathrm{G}(0;y) + 2\mathrm{H}(1,1,0;z) + \mathrm{G}(0,1,0;y)\Big] + \frac{z}{48(1-y-z)}\Big[\frac{10\pi^2}{3} + 13\mathrm{H}(0;z)\\
&+20\mathrm{H}(0;z)\mathrm{G}(0;y) + 20\mathrm{H}(1,0;z) + 20\mathrm{G}(0;y) - 20\mathrm{G}(1,0;y)\Big] + \frac{z^2}{8(y+z)^3}\Big[\frac{22\pi^2}{3}\mathrm{H}(1;z)
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{22\pi^2}{3}\mathrm{G}(1-z;y)-44\mathrm{H}(0;z)\mathrm{G}(1-z,0;y)-44\mathrm{H}(0,1,0;z)-33\mathrm{H}(1,0;z)-44\mathrm{H}(1,0;z)\mathrm{G}(1-z;y)\\
&+44\mathrm{H}(1,0;z)\mathrm{G}(0;y)+44\mathrm{H}(1,1,0;z)+44\mathrm{G}(1-z,1,0;y)+44\mathrm{G}(0,1,0;y)-33\mathrm{G}(1,0;y)\Big]\\
&+\frac{z^2}{16(y+z)^2}\Big[-\frac{44\pi^2}{3}+33\mathrm{H}(0;z)-88\mathrm{H}(0;z)\mathrm{G}(0;y)-88\mathrm{H}(1,0;z)-33\mathrm{G}(0;y)+88\mathrm{G}(1,0;y)\Big]\\
&+\frac{z^2}{16(y+z)}\Big[-\frac{22\pi^2}{3}+11\mathrm{H}(0;z)-44\mathrm{H}(0;z)\mathrm{G}(0;y)-44\mathrm{H}(1,0;z)-11\mathrm{G}(0;y)+44\mathrm{G}(1,0;y)\Big]\\
&+\frac{z^2}{12(1-y-z)^2}\Big[\frac{5\pi^2}{6}+5\mathrm{H}(0;z)\mathrm{G}(0;y)+5\mathrm{H}(1,0;z)-5\mathrm{G}(1,0;y)\Big]+\frac{z^2}{16(1-y-z)}\Big[-\frac{22\pi^2}{3}\\
&+11\mathrm{H}(0;z)-44\mathrm{H}(0;z)\mathrm{G}(0;y)-44\mathrm{H}(1,0;z)-11\mathrm{G}(0;y)+44\mathrm{G}(1,0;y)\Big]+\frac{z^3}{8(y+z)^4}\Big[-\frac{11\pi^2}{2}\mathrm{H}(1;z)\\
&+\frac{11\pi^2}{2}\mathrm{G}(1-z;y)+33\mathrm{H}(0;z)\mathrm{G}(1-z,0;y)+33\mathrm{H}(0,1,0;z)+33\mathrm{H}(1,0;z)\mathrm{G}(1-z;y)\\
&-33\mathrm{H}(1,0;z)\mathrm{G}(0;y)-33\mathrm{H}(1,1,0;z)-33\mathrm{G}(1-z,1,0;y)-33\mathrm{G}(0,1,0;y)\Big]+\frac{z^3}{8(y+z)^3}\Big[\frac{11\pi^2}{2}\\
&+33\mathrm{H}(0;z)\mathrm{G}(0;y)+33\mathrm{H}(1,0;z)-33\mathrm{G}(1,0;y)\Big]+\frac{z^3}{16(y+z)^2}\Big[\frac{11\pi^2}{2}+33\mathrm{H}(0;z)\mathrm{G}(0;y)+33\mathrm{H}(1,0;z)\\
&-33\mathrm{G}(1,0;y)\Big]+\frac{z^3}{8(y+z)}\Big[\frac{11\pi^2}{6}+11\mathrm{H}(0;z)\mathrm{G}(0;y)+11\mathrm{H}(1,0;z)-11\mathrm{G}(1,0;y)\Big]\\
&+\frac{z^3}{16(1-y-z)^2}\Big[-\frac{11\pi^2}{6}-11\mathrm{H}(0;z)\mathrm{G}(0;y)-11\mathrm{H}(1,0;z)+11\mathrm{G}(1,0;y)\Big]\\
&+\frac{z^3}{8(1-y-z)}\Big[\frac{11\pi^2}{6}+11\mathrm{H}(0;z)\mathrm{G}(0;y)+11\mathrm{H}(1,0;z)-11\mathrm{G}(1,0;y)\Big]+\frac{1}{48(1-z)^2}\Big[\pi^2\big(\frac{1}{6}-3\mathrm{H}(0;z)\\
&-3\mathrm{H}(1;z)+\mathrm{G}(1-z;y)-\mathrm{G}(0;y)\big)-6\zeta_3+\frac{355}{6}\mathrm{H}(0;z)+6\mathrm{H}(0;z)\mathrm{G}(1-z,0;y)-10\mathrm{H}(0;z)\mathrm{G}(0;y)\\
&-45\mathrm{H}(0,0;z)-12\mathrm{H}(0,0;z)\mathrm{G}(0;y)-18\mathrm{H}(0,1,0;z)+\mathrm{H}(1,0;z)+6\mathrm{H}(1,0;z)\mathrm{G}(1-z;y)\\
&-6\mathrm{H}(1,0;z)\mathrm{G}(0;y)-12\mathrm{H}(1,0,0;z)-18\mathrm{H}(1,1,0;z)-6\mathrm{G}(1-z,1,0;y)+6\mathrm{G}(0,1,0;y)\Big]\\
&+\frac{1}{48(1-z)}\Big[\pi^2\big(-\frac{7}{6}+3\mathrm{H}(0;z)+3\mathrm{H}(1;z)-\mathrm{G}(1-z;y)+\mathrm{G}(0;y)\big)+6\zeta_3+\frac{277}{6}-\frac{493}{6}\mathrm{H}(0;z)\\
&-6\mathrm{H}(0;z)\mathrm{G}(1-z,0;y)+4\mathrm{H}(0;z)\mathrm{G}(0;y)+45\mathrm{H}(0,0;z)+12\mathrm{H}(0,0;z)\mathrm{G}(0;y)+18\mathrm{H}(0,1,0;z)\\
&-7\mathrm{H}(1,0;z)-6\mathrm{H}(1,0;z)\mathrm{G}(1-z;y)+6\mathrm{H}(1,0;z)\mathrm{G}(0;y)+12\mathrm{H}(1,0,0;z)+18\mathrm{H}(1,1,0;z)\\
&+6\mathrm{G}(1-z,1,0;y)-10\mathrm{G}(0;y)-6\mathrm{G}(0,1,0;y)+6\mathrm{G}(1,0;y)\Big]+\frac{1}{48(1-y-z)}\Big[-\pi^2+13\mathrm{H}(0;z)\\
&-6\mathrm{H}(1,0;z)-6\mathrm{G}(1,0;y)\Big]+\frac{1}{48}\Big[\pi^2-\frac{277}{6}+23\mathrm{H}(0;z)+6\mathrm{H}(0;z)\mathrm{G}(0;y)+6\mathrm{H}(1,0;z)+10\mathrm{G}(0;y)\\
&-6\mathrm{G}(1,0;y)\Big]+i\pi\bigg\{\frac{11}{16(1-z)^2}\mathrm{H}(0;z)+\frac{11}{16(1-z)}\Big[1-\mathrm{H}(0;z)\Big]-\frac{11}{16}\bigg\}
\end{aligned}
$$

$$
\begin{aligned}
D_\gamma(y;z)=\\
&-\frac{1}{12y(1-z)}\mathrm{H}(0;z)+\frac{1}{12y(1-y-z)}\mathrm{H}(0;z)+\frac{z}{4(y+z)^2}\Big[\frac{\pi^2}{3}\mathrm{H}(1;z)-\frac{\pi^2}{3}\mathrm{G}(1-z;y)\\
&-2\mathrm{H}(0;z)\mathrm{G}(1-z,0;y)-2\mathrm{H}(0,1,0;z)-5\mathrm{H}(1,0;z)-2\mathrm{H}(1,0;z)\mathrm{G}(1-z;y)+2\mathrm{H}(1,0;z)\mathrm{G}(0;y)\\
&+2\mathrm{H}(1,1,0;z)+2\mathrm{G}(1-z,1,0;y)+2\mathrm{G}(0,1,0;y)-5\mathrm{G}(1,0;y)\Big]+\frac{z}{4(y+z)}\Big[-\frac{\pi^2}{3}+1+2\mathrm{H}(0;z)\\
&-2\mathrm{H}(0;z)\mathrm{G}(0;y)-2\mathrm{H}(1,0;z)-3\mathrm{G}(0;y)+2\mathrm{G}(1,0;y)\Big]+\frac{z}{12(1-y-z)}\Big[-\frac{\pi^2}{3}-\mathrm{H}(0;z)
\end{aligned}
$$

$$
-2\mathrm{H}(0;z)\mathrm{G}(0;y) - 2\mathrm{H}(1,0;z) - 2\mathrm{G}(0;y) + 2\mathrm{G}(1,0;y)\Big] + \frac{z^2}{2(y+z)^3}\Big[ - \frac{2\pi^2}{3}\mathrm{H}(1;z) + \frac{2\pi^2}{3}\mathrm{G}(1-z;y)
$$
$$
+4\mathrm{H}(0;z)\mathrm{G}(1-z,0;y) + 4\mathrm{H}(0,1,0;z) + 3\mathrm{H}(1,0;z) + 4\mathrm{H}(1,0;z)\mathrm{G}(1-z;y) - 4\mathrm{H}(1,0;z)\mathrm{G}(0;y)
$$
$$
-4\mathrm{H}(1,1,0;z) - 4\mathrm{G}(1-z,1,0;y) - 4\mathrm{G}(0,1,0;y) + 3\mathrm{G}(1,0;y)\Big] + \frac{z^2}{4(y+z)^2}\Big[\frac{4\pi^2}{3} - 3\mathrm{H}(0;z)
$$
$$
+8\mathrm{H}(0;z)\mathrm{G}(0;y) + 8\mathrm{H}(1,0;z) + 3\mathrm{G}(0;y) - 8\mathrm{G}(1,0;y)\Big] + \frac{z^2}{4(y+z)}\Big[\frac{2\pi^2}{3} - \mathrm{H}(0;z) + 4\mathrm{H}(0;z)\mathrm{G}(0;y)
$$
$$
+4\mathrm{H}(1,0;z) + \mathrm{G}(0;y) - 4\mathrm{G}(1,0;y)\Big] + \frac{z^2}{6(1-y-z)^2}\Big[ - \frac{\pi^2}{6} - \mathrm{H}(0;z)\mathrm{G}(0;y) - \mathrm{H}(1,0;z) + \mathrm{G}(1,0;y)\Big]
$$
$$
+\frac{z^2}{4(1-y-z)}\Big[\frac{2\pi^2}{3} - \mathrm{H}(0;z) + 4\mathrm{H}(0;z)\mathrm{G}(0;y) + 4\mathrm{H}(1,0;z) + \mathrm{G}(0;y) - 4\mathrm{G}(1,0;y)\Big]
$$
$$
+\frac{z^3}{2(y+z)^4}\Big[\frac{\pi^2}{2}\mathrm{H}(1;z) - \frac{\pi^2}{2}\mathrm{G}(1-z;y) - 3\mathrm{H}(0;z)\mathrm{G}(1-z,0;y) - 3\mathrm{H}(0,1,0;z) - 3\mathrm{H}(1,0;z)\mathrm{G}(1-z;y)
$$
$$
+3\mathrm{H}(1,0;z)\mathrm{G}(0;y) + 3\mathrm{H}(1,1,0;z) + 3\mathrm{G}(1-z,1,0;y) + 3\mathrm{G}(0,1,0;y)\Big] + \frac{z^3}{2(y+z)^3}\Big[ - \frac{\pi^2}{2}
$$
$$
-3\mathrm{H}(0;z)\mathrm{G}(0;y) - 3\mathrm{H}(1,0;z) + 3\mathrm{G}(1,0;y)\Big] + \frac{z^3}{4(y+z)^2}\Big[ - \frac{\pi^2}{2} - 3\mathrm{H}(0;z)\mathrm{G}(0;y) - 3\mathrm{H}(1,0;z)
$$
$$
+3\mathrm{G}(1,0;y)\Big] + \frac{z^3}{2(y+z)}\Big[ - \frac{\pi^2}{6} - \mathrm{H}(0;z)\mathrm{G}(0;y) - \mathrm{H}(1,0;z) + \mathrm{G}(1,0;y)\Big] + \frac{z^3}{4(1-y-z)^2}\Big[\frac{\pi^2}{6}
$$
$$
+\mathrm{H}(0;z)\mathrm{G}(0;y) + \mathrm{H}(1,0;z) - \mathrm{G}(1,0;y)\Big] + \frac{z^3}{2(1-y-z)}\Big[ - \frac{\pi^2}{6} - \mathrm{H}(0;z)\mathrm{G}(0;y) - \mathrm{H}(1,0;z) + \mathrm{G}(1,0;y)\Big]
$$
$$
+\frac{1}{72(1-z)^2}\Big[ - \pi^2 - 25\mathrm{H}(0;z) + \frac{3}{2}\mathrm{H}(0;z)\mathrm{G}(0;y) + 9\mathrm{H}(0,0;z) - 6\mathrm{H}(1,0;z)\Big] + \frac{1}{144(1-z)}\Big[2\pi^2 - 38
$$
$$
+65\mathrm{H}(0;z) - 3\mathrm{H}(0;z)\mathrm{G}(0;y) - 18\mathrm{H}(0,0;z) + 12\mathrm{H}(1,0;z) + 3\mathrm{G}(0;y)\Big] - \frac{1}{12(1-y-z)}\mathrm{H}(0;z) + \frac{19}{72}
$$
$$
-\frac{5}{48}\mathrm{H}(0;z) - \frac{1}{48}\mathrm{G}(0;y) + i\pi\Bigg\{ - \frac{1}{8(1-z)^2}\mathrm{H}(0;z) + \frac{1}{8(1-z)}\Big[ - 1 + \mathrm{H}(0;z)\Big] + \frac{1}{8}\Bigg\} \tag{3.19}
$$

¿From the $\Omega^{(1),\mathrm{finite}}$ and $\Omega^{(2),\mathrm{finite}}$, it is possible to recover the finite pieces of the helicity-averaged tree times two-loop and one-loop squared matrix elements by squaring (3.7):

$$
\begin{aligned}
\mathcal{F}inite^{(2\times 0)}(x,y,z) &= 8\,V\,\mathcal{R}\Bigg[\frac{(1-y)(1-y-z)}{yz}\alpha^{(2),\mathrm{finite}}(y,z) + \frac{1-y}{y}\beta^{(2),\mathrm{finite}}(y,z) \\
&\qquad -\gamma^{(2),\mathrm{finite}}(y,z) + (y \leftrightarrow z)\Bigg]\,, \\
\mathcal{F}inite^{(1\times 1)}(x,y,z) &= 4\,V\,\mathcal{R}\Bigg[(1-y-z)\left(\frac{(1-y-z)}{yz} + \frac{1}{2}\right)\,|\alpha^{(1),\mathrm{finite}}(y,z)|^2 \\
&\qquad + \left(\frac{1-y-z}{2} + \frac{z}{y}\right)\,|\beta^{(1),\mathrm{finite}}(y,z)|^2 + \left(\frac{1-y-z}{2} + \frac{y}{z}\right)\,|\gamma^{(1),\mathrm{finite}}(y,z)|^2 \\
&\qquad + \left(-3 + y + z + \frac{2-2z}{y}\right)\,\alpha^{(1),\mathrm{finite}}(y,z)\beta^{*(1),\mathrm{finite}}(y,z) \\
&\qquad -(1-y-z)\,\alpha^{(1),\mathrm{finite}}(y,z)\gamma^{*(1),\mathrm{finite}}(y,z) \\
&\qquad -(1+y+z)\,\beta^{(1),\mathrm{finite}}(y,z)\gamma^{*(1),\mathrm{finite}}(y,z) + (y \leftrightarrow z)\Bigg]\,.
\end{aligned} \tag{3.20}
$$

It is important to notice that (3.20) corresponds, by the very nature of the Weyl–van der Waerden helicity

formalism, to a scheme with external momenta and polarization vectors in four dimensions (internal states are always taken to be $d$-dimensional), which is sometimes called the 't Hooft–Veltman scheme [27]. This scheme is different from the conventional dimensional regularization used in [18], where all external momenta and polarization vectors are $d$-dimensional. Nevertheless, one obtains from (3.20) the same $\mathcal{F}inite^{(2\times0)}(x,y,z)$ as in Eq. (4.17) and $\mathcal{F}inite^{(1\times1)}(x,y,z)$ as in Eq. (4.25) of [18], since all scheme-dependent terms are correctly accounted for by the finite contributions arising from expanding the tree level and one-loop contributions in the renormalization and infrared factorization formulae.

It should also be noted that only the $\mathcal{O}(\epsilon^0)$ terms of $\Omega^{(1),\text{finite}}$ contribute to $\mathcal{F}inite^{(1\times1)}(x,y,z)$, terms subleading in $\epsilon$ are not required, since no term is multiplied with a divergent factor. Comparing the size of these $\mathcal{O}(\epsilon^0)$ terms (3.15) with the size of $\mathcal{F}inite^{(1\times1)}(x,y,z)$ in [18], it becomes clear that the squared one-loop amplitude can be evaluated much more elegantly by squaring the finite remainders of the helicity amplitudes than by computing the squared matrix element.

# 4 Conclusions and Outlook

In this paper, we have presented analytic formulae for the one- and two-loop virtual helicity amplitudes to the process $\gamma^* \to q\bar{q}g$. These amplitudes have been derived by defining projectors, which isolate the coefficients of the most general tensorial structure of the matrix element at any order in perturbation theory. Once the general tensor is known, the helicity amplitudes follow in a straightforward manner – they are linear combinations of the tensor coefficients. We applied the projectors directly to the Feynman diagrams and used the conventional approach of relating the ensuing tensor integrals to a basis set of master integrals. This latter step is identical to that employed to evaluate the interference of tree- and two-loop graphs in Ref. [18], apart from the fact that the projector is no longer the tree-level amplitude. As anticipated, the finite remainder from the interference of tree- and two-loop amplitudes can be reconstructed from the appropriate helicity amplitudes, with the difference between treating the external states in $d$ dimensions or four dimensions being isolated in the infrared-singular terms.

The results presented here therefore complement the earlier calculation of the interference of tree- and two-loop graphs in Ref. [18]. Knowledge of the helicity amplitudes allows additional information on the scattering process. In particular, observables that require knowledge of the polarization tensor of the virtual photon, such as oriented event shapes in unpolarized $e^+e^-$ scattering or event shapes in polarized $e^+e^-$ scattering, can be described at two-loop order.

Similar results can in principle be obtained for $(2+1)$-jet production in deep inelastic $ep$ scattering or $(V+1)$-jet production in hadron–hadron collisions. However, the rather different domains of convergence of the HPLs and 2dHPLs makes this a non-trivial task, and we defer this to a later paper. Nevertheless, the helicity approach will provide information on the direction of the decay leptons in $(V+1)$-jet production (with or without polarized protons). Determination of the polarized parton distribution functions in polarized electron–proton scattering will also benefit from the knowledge of the two-loop helicity amplitudes in the appropriate kinematic region.

Even though the evaluation of two-loop QCD matrix elements is becoming well established, the virtual corrections form only part of a full NNLO calculation. They must be combined with the one-loop corrections to $\gamma^* \to 4$ partons [14–17], where one of the partons becomes collinear or soft, as well as tree-level processes $\gamma^* \to 5$ partons [11–13] with two soft or collinear partons in a way that allows all of the infrared singularities to cancel one another. This task is far from trivial, even though the factorization properties of both the one-loop, one-unresolved-parton contribution [65–70] and the tree-level, two-unresolved-parton contributions [71–74] have been studied. Early studies for the case of photon-plus-one-jet final states in electron–positron annihilation in [75, 76], which involves both double radiation and single radiation from one-loop graphs, indicate the feasibility of developing a numerical NNLO program implementing the experimental definition of jet observables and event-shape variables, and significant progress is anticipated in the near future.

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# A Weyl–van der Waerden spinor calculus

The basic quantity is the two-spinor $\psi_A$ or $\psi^A$ and its complex conjugate $\psi_{\dot{A}}$ or $\psi^{\dot{A}}$. Raising and lowering of indices is done with the antisymmetric tensor $\varepsilon$,

$$\varepsilon_{AB} = \varepsilon^{AB} = \varepsilon_{\dot{A}\dot{B}} = \varepsilon^{\dot{A}\dot{B}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{A.1}$$

We define a antisymmetric spinorial "inner product":

$$\langle\psi_1\psi_2\rangle = \psi_{1A}\varepsilon^{BA}\psi_{2B} = \psi_{1A}\psi_2^A = -\psi_1^A\psi_{2A} = -\langle\psi_2\psi_1\rangle, \tag{A.2}$$

and

$$\langle\psi_1\psi_2\rangle^* = \psi_{1\dot{A}}\psi_2^{\dot{A}}. \tag{A.3}$$

Any momentum vector $k_\mu$ gets a bispinor representation by contraction with $\sigma^\mu$:

$$k_{\dot{A}B} = \sigma^\mu_{\dot{A}B}k_\mu = \begin{pmatrix} k_0 + k_3 & k_1 + ik_2 \\ k_1 - ik_2 & k_0 - k_3 \end{pmatrix}, \tag{A.4}$$

where $\sigma^0$ is the unit matrix and $\sigma_i$ are the Pauli matrices. Since

$$\sigma^\mu_{\dot{A}B}\sigma^{\nu\dot{A}B} = 2g^{\mu\nu}, \tag{A.5}$$

we have

$$k_{\dot{A}B}p^{\dot{A}B} = 2k\cdot p. \tag{A.6}$$

For light-like vectors one can show that

$$k_{\dot{A}B} = k_{\dot{A}}k_B, \tag{A.7}$$

where

$$k_A = \begin{pmatrix} (k_1 - ik_2)/\sqrt{k_0 - k_3} \\ \sqrt{k_0 - k_3} \end{pmatrix}, \tag{A.8}$$

so that for light-like vectors we have

$$2k\cdot p = \langle kp\rangle\langle kp\rangle^* = |\langle kp\rangle|^2. \tag{A.9}$$

The following relation is often useful:

$$\sigma^\mu_{\dot{A}B}\sigma_\mu^{\dot{C}D} = 2\delta_{\dot{A}}{}^{\dot{C}}\delta_B{}^D\ . \tag{A.10}$$

For massless spin-$\frac{1}{2}$ particles the four-spinors can be expressed in two-spinors as follows:

$$\begin{aligned} u_+(p) &= v_-(p) = \begin{pmatrix} p_B \\ 0 \end{pmatrix}, \\ u_-(p) &= v_+(p) = \begin{pmatrix} 0 \\ p^{\dot{B}} \end{pmatrix}, \\ \bar{u}_+(q) &= \bar{v}_-(q) = \begin{pmatrix} 0, & -iq_{\dot{A}} \end{pmatrix}, \\ \bar{u}_-(q) &= \bar{v}_+(q) = \begin{pmatrix} iq^A, & 0 \end{pmatrix}. \end{aligned} \tag{A.11}$$

The Dirac $\gamma$ matrices now become

$$\gamma^\mu = \begin{pmatrix} 0 & -i\sigma^\mu_{\dot{B}A} \\ i\sigma^{\mu\dot{A}B} & 0 \end{pmatrix}, \tag{A.12}$$

so that, for example:

$$\bar{u}_+(q)\gamma^\mu v_-(p) = q_{\dot{A}}\sigma^{\mu\dot{A}B}p_B \ . \tag{A.13}$$

The general electroweak vertex for vector boson $V$ coupling to two fermions is denoted by $ie\delta_{ij}\Gamma_\mu^{Vf_1f_2}$, where $i$ and $j$ are the colour labels associated with the fermions $f_1$ and $f_2$ respectively. The vertex contains left- and right-handed couplings,

$$\Gamma_\mu^{V,f_1f_2} = L^V_{f_1f_2}\gamma_\mu\left(\frac{1-\gamma_5}{2}\right) + R^V_{f_1f_2}\gamma_\mu\left(\frac{1+\gamma_5}{2}\right), \tag{A.14}$$

where for a photon,

$$L^\gamma_{f_1f_2} = R^\gamma_{f_1f_2} = -e_{f_1}\delta_{f_1f_2}, \tag{A.15}$$

and for a $Z$-boson,

$$L^Z_{f_1f_2} = \frac{I_3^{f_1} - \sin^2\theta_W e_{f_1}}{\sin\theta_W\cos\theta_W}\delta_{f_1f_2}, \qquad R^Z_{f_1f_2} = \frac{-\sin\theta_W e_{f_1}}{\cos\theta_W}\delta_{f_1f_2}. \tag{A.16}$$

Here, $e_f$ represents the fractional electric charge, $I_3^f$ the weak isospin and $\theta_W$ the weak mixing angle. In the Weyl–van der Waerden notation, the vertex $\Gamma_\mu^{V,f_1f_2}$ becomes,

$$\Gamma_\mu^{V,f_1f_2} = \begin{pmatrix} 0 & -iL^V_{f_1f_2}\sigma_{\mu\dot{B}A} \\ iR^V_{f_1f_2}\sigma_\mu^{\dot{A}B} & 0 \end{pmatrix}. \tag{A.17}$$

For the polarization vectors of outgoing gluons and photons we use the spinorial quantities

$$e^+_{\dot{A}B}(k) = \sqrt{2}\frac{k_{\dot{A}}b_B}{\langle bk\rangle} \ , \tag{A.18}$$

$$e^-_{\dot{A}B}(k) = \sqrt{2}\frac{b_{\dot{A}}k_B}{\langle bk\rangle^*} \ . \tag{A.19}$$

The gauge spinor $b$ is arbitrary and can be chosen differently in each gauge-invariant expression. A suitable choice can often simplify the calculation.

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