KINETIC DESCRIPTION OF ELECTRON-PROTON INSTABILITY IN HIGH-INTENSITY LINACS AND STORAGE RINGS

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Abstract

The Vlasov-Maxwell equations are used to investigate properties of the electron-ion two-stream instability for a continuous, high-intensity ion beam propagating in the z-direction with directed axial velocity $V_b = \beta_b c$ through a background population of (stationary) electrons. The analysis is carried out for arbitrary beam intensity, consistent with transverse confinement of the beam particles, and arbitrary fractional charge neutralization. Stability properties are calculated for dipole perturbations with azimuthal mode number $\ell = 1$ about monoenergetic ion and electron distribution functions.

1 INTRODUCTION

Periodic focusing accelerators and transport systems[1, 2] have a wide range of applications ranging from basic scientific research, to applications such as spallation neutron sources, tritium production, and heavy ion fusion. For a *one-component* high-intensity beam, considerable progress has been made in describing the self-consistent evolution of the beam distribution function $f_b(\mathbf{x}, \mathbf{p}, t)$ and the self-generated electric and magnetic fields in kinetic analyses[1, 3, 4, 5, 6] based on the Vlasov-Maxwell equations. In many practical accelerator applications, however, an (unwanted) second charge component is present. When a second charge component is present, it has been recognized for many years[7, 8, 9, 10, 11, 12] that the relative streaming motion of the high-intensity beam particles through the background charge species provides the free energy to drive the classical *two-stream* instability[13].

In the present analysis, we apply the Vlasov-Maxwell equations[1, 14] to describe the self-consistent interaction of the ion and electron distribution functions with the applied field and the self-generated electric and magnetic fields. The analysis can be applied to ion beams ranging from the emittance-dominated, moderate-intensity proton beams in proton linacs and storage rings, to the low-emittance, space-charge-dominated ion beams in heavy ion fusion.

2 THEORETICAL MODEL AND ASSUMPTIONS

The present analysis [14] considers a continuous ion beam with distribution function $f_b(\mathbf{x}, \mathbf{p}, t)$, and characteristic radius r_b and axial momentum $\gamma_b m_b \beta_b c$ propagating in the z-direction through a background population of electrons with distribution function $f_e(\mathbf{x}, \mathbf{p}, t)$. The ions have directed axial velocity $V_b = \beta_b c$, and the background electrons are assumed to be nonrelativistic and stationary with $\int d^3p p_z f_e \simeq 0$ in the laboratory frame. The *applied* focusing force on a beam ion is modeled by

$$\mathbf{F}_{foc} = -\gamma_b m_b \omega_{\beta b}^2 \mathbf{x}_\perp \ ,\psi \tag{1}$$

where $\mathbf{x}_\perp = x\hat{\mathbf{e}}_x + y\hat{\mathbf{e}}_y$ is the transverse displacement, $(\gamma_b - 1)m_b c^2$ is the ion kinetic energy, m_b is the ion rest mass, c is the speed of light *in vacuo*, and $\omega_{\beta b} = const.$ is the effective betatron frequency for the applied focusing field. Assuming that the ion density exceeds the background electron density, the space-charge force on an electron, provides transverse confinement of the background electrons by the electrostatic potential $\phi(\mathbf{x}, t)$. It is further assumed that the ion motion in the beam frame is nonrelativistic. The electrostatic potential $\phi(\mathbf{x}, t)$ is determined self-consistently from Poisson's equation $\nabla^2\phi = -4\pi e(Z_b n_b - n_e)$, and the z-component of vector potential $A_z(\mathbf{x}, t)$ is determined self-consistently from $\nabla^2 A_z = -4\pi Z_b e \beta_b n_b$, where the electrons are assumed to carry zero axial current in the laboratory frame. Here, $n_b(\mathbf{x}, t) = \int d^3p f_b(\mathbf{x}, \mathbf{p}, t)$ and $n_e(\mathbf{x}, t) = \int d^3p f_e(\mathbf{x}, \mathbf{p}, t)$ are the ion and electron densities, respectively.

Finally, the stability analysis assumes perturbations with sufficiently long axial wavelength and high frequency that $k_z^2 r_b^2 \ll 1$, $|\omega/k_z - \beta_b c| \gg v_{Tbz}$, and $|\omega/k_z| \gg v_{Tez}$. Here, $v_{Tbz} = (2T_{bz}/\gamma_b m_b)^{1/2}$ and $v_{Tez} = (2T_{ez}/m_e)^{1/2}$ are the axial thermal speeds of the beam ions and the background electrons, respectively. Furthermore, the perturbed axial forces are treated as negligibly small, and the analysis neglects the effects of Landau damping due to axial momentum spread.

We make use of these assumptions to simplify the theoretical model[14]. First, we introduce the reduced distribution functions defined by $F_j(\mathbf{x}, \mathbf{p}_\perp, t) = \int dp_z f_j(\mathbf{x}, \mathbf{p}, t)$

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for $j = b, e$. Because $\int dp_z p_z f_e \simeq 0$ for the electrons, the nonlinear Vlasov equation for $F_e(\mathbf{x}, \mathbf{p}_\perp, t)$ is given by

$$\Big\{\frac{\partial}{\partial t} + \frac{\mathbf{p}_\perp}{m_e}\cdot\frac{\partial}{\partial \mathbf{x}_\perp} + (e\nabla_\perp\phi)\cdot\frac{\partial}{\partial \mathbf{p}_\perp}\Big\}F_e(\mathbf{x}, \mathbf{p}_\perp, t) = 0\ , \tag{2}$$

where $-e$ is the electron charge, and $\nabla_\perp \equiv \hat{\mathbf{e}}_x\partial/\partial x + \hat{\mathbf{e}}_y\partial/\partial y$. The ions, however, have large directed axial velocity $V_b \simeq \beta_b c$, and the Vlasov equation for $F_b(\mathbf{x}, \mathbf{p}_\perp, t)$ becomes

$$\begin{aligned}&\Big\{\frac{\partial}{\partial t} + V_b\frac{\partial}{\partial z} + \frac{\mathbf{p}_\perp}{\gamma_b m_b}\cdot\frac{\partial}{\partial \mathbf{x}_\perp} - \big(\gamma_b m_b {\omega_{\beta b}}^2\mathbf{x}_\perp \\ &+ Z_b e\nabla_\perp\psi\big)\cdot\frac{\partial}{\partial \mathbf{p}_\perp}\Big\}F_b(\mathbf{x}, \mathbf{p}_\perp, t) = 0\ .\end{aligned} \tag{3}$$

Here, $+Z_b e$ is the ion charge, and $\psi(\mathbf{x}, t) \equiv \phi(\mathbf{x}, t) - \beta_b A_z(\mathbf{x}, t)$. The self-field potentials $\phi(\mathbf{x}, t)$, and $\psi(\mathbf{x}, t)$ in Eqs. (2) and (3) are determined self-consistently from

$$\begin{aligned}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi &= -4\pi e\left(Z_b\int d^2p F_b - \int d^2p F_e\right)\ ,\\ \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\psi &= -4\pi e\left(\frac{Z_b}{\gamma_b^2}\int d^2p F_b - \int d^2p F_e\right)\ .\end{aligned} \tag{4}$$

We assume that the beam propagates axially through a perfectly conducting cylindrical pipe with radius $r = r_w$. Enforcing $[E_\theta^s]_{r=r_w} = [E_z^s]_{r=r_w} = [B_r^s]_{r=r_w} = 0$ readily gives $\phi(r = r_w, \theta, z, t) = 0$, and $\psi(r = r_w, \theta, z, t) = 0$. Here, the constant values of the potentials at $r = r_w$ have been taken equal to zero.

3 EQUILIBRIUM PROPERTIES

Under quasisteady equilibrium conditions with $\partial/\partial t = 0$, we assume axisymmetric beam propagation and negligible variation with axial coordinate ($\partial/\partial\theta = 0 = \partial/\partial z$). The equilibrium distribution functions for the beam ions and background electrons are of the general form $F_b^0 = F_b^0(H_{\perp b})$ and $F_e^0 = F_e^0(H_{\perp e})$, where

$$\begin{aligned}H_{\perp b} &= \frac{1}{2\gamma_b m_b}\mathbf{p}_\perp^2 + \frac{1}{2}\gamma_b m_b\omega_{\beta b}^2 r^2 + Z_b e[\psi^0(r) - \hat{\psi}^0]\ ,\\ H_{\perp e} &= \frac{1}{2m_e}\mathbf{p}_\perp^2 - e[\phi^0(r) - \hat{\phi}^0]\ .\end{aligned} \tag{5}$$

Here, $r = (x^2+y^2)^{1/2}$ is the radial distance from the beam axis, $H_{\perp b}$ and $H_{\perp e}$ are exact single-particle constants of the motion, and $\hat{\psi}^0 \equiv \psi^0(r = 0)$ and $\hat{\phi}^0 \equiv \phi^0(r = 0)$ are constants.

There is wide latitude in specifying the functional forms of the equilibrium distribution functions[14]. In the present analysis, we assume monoenergetic ions and electrons, with distribution functions

$$\begin{aligned}F_b^0(H_{\perp b}) &= \frac{\hat{n}_b}{2\pi\gamma_b m_b}\delta(H_{\perp b} - \hat{T}_{\perp b})\ ,\\ F_e^0(H_{\perp e}) &= \frac{\hat{n}_e}{2\pi m_e}\delta(H_{\perp e} - \hat{T}_{\perp e})\ .\end{aligned} \tag{6}$$

Here, $\hat{n}_b$ and $\hat{n}_e \equiv fZ_b\hat{n}_b$ are positive constants corresponding to the ion and electron densities, $f = const.$ is the fractional charge neutralization, and $\hat{T}_{\perp b}$ and $\hat{T}_{\perp e}$ are constants corresponding to the on-axis ($r = 0$) values of the transverse ion and electron temperatures, respectively. Without presenting details[14], some algebraic manipulation of Eqs. (4) – (6) gives the step-function density profiles $n_j^0(r) = \hat{n}_j = const.$, for $0 \le r < r_b$, and $n_j^0(r) = 0$ for $r_b < r \le r_w$, and $j = b, e$. Here, the beam radius r_b is related to other equilibrium parameters by $\hat{\nu}_b^2 r_b^2 = 2\hat{T}_{\perp b}/\gamma_b m_b$ and $\hat{\nu}_e^2 r_b^2 = 2\hat{T}_{\perp e}/m_e$, where the (depressed) betatron frequencies $\hat{\nu}_b$ and $\hat{\nu}_e$ are defined by

$$\begin{aligned}\hat{\nu}_b^2 &= \omega_{\beta b}^2 - \frac{1}{2}\left(\frac{1}{\gamma_b^2} - f\right)\hat{\omega}_{pb}^2 = const.,\\ \hat{\nu}_e^2 &= \frac{1}{2}\frac{\gamma_b m_b}{Z_b m_e}(1-f)\hat{\omega}_{pb}^2 = const.\ ,\end{aligned} \tag{7}$$

and $\hat{\omega}_{pb}^2 = 4\pi\hat{n}_b Z_b^2 e^2/\gamma_b m_b$ is the ion plasma frequency-squared.

4 STABILITY ANALYSIS AND DISPERSION RELATION

For small-amplitude perturbations, a stability analysis proceeds by linearizing Eqs. (2)–(4). Perturbed quantities are expressed as $\delta\psi(\mathbf{x}, t) = \delta\hat{\psi}^l(r)\exp(ik_z z + il\theta - i\omega t)$, $\delta F_b(\mathbf{x}, \mathbf{p}_\perp, t) = \delta\hat{F}_b^l(r, \mathbf{p}_\perp)\exp(ik_z z + il\theta - i\omega t)$, etc., where $Im\omega > 0$ is assumed, corresponding to instability (temporal growth), k_z is the axial wavenumber, and l is the azimuthal harmonic number. The linearized Vlasov equations are formally integrated by using the method of characteristics[14]. For perturbations about the monoenergetic ion and electron distribution functions in Eq. (6), we obtain[14] the kinetic dispersion relation

$$\begin{aligned}&\Big[\frac{2}{1-(r_b/r_w)^{2\ell}} + \frac{\hat{\omega}_{pb}^2}{\ell\gamma_b^2\hat{\nu}_b^2}\Gamma_b^\ell(\omega - k_z V_b)\Big]\Big[\frac{2}{1-(r_b/r_w)^{2\ell}}\\ &+ \frac{\hat{\omega}_{pe}^2}{\ell\hat{\nu}_e^2}\Gamma_e^\ell(\omega)\Big] = \frac{\hat{\omega}_{pe}^2}{\ell\hat{\nu}_e^2}\cdot\frac{\hat{\omega}_{pb}^2}{\ell\hat{\nu}_b^2}\Gamma_e^\ell(\omega)\Gamma_b^\ell(\omega - k_z V_b)\ .\end{aligned} \tag{8}$$

where $\hat{\omega}_{pe}^2 = 4\pi\hat{n}_e e^2/m_e$. Here, the ion and electron susceptibilities are defined by[14]

$$\begin{aligned}&\Gamma_b^\ell(\omega - k_z V_b) = -\frac{1}{2^\ell}\sum_{m=0}^{\ell}\frac{\ell!}{m!(\ell-m)!}\times\\ &\frac{(\ell-2m)\hat{\nu}_b}{[(\omega - k_z V_b) - (\ell-2m)\hat{\nu}_b]}\ ,\\ &\Gamma_e^\ell(\omega) = -\frac{1}{2^\ell}\sum_{m=0}^{\ell}\frac{\ell!}{m!(\ell-m)!}\frac{(\ell-2m)\hat{\nu}_e}{[\omega - (\ell-2m)\hat{\nu}_e]}\ ,\end{aligned} \tag{9}$$

for general azimuthal harmonic number l.

A careful examination[14] of Eq. (8) for $\hat{n}_e \neq 0$ shows that the strongest instability (largest growth rate) occurs for azimuthal mode number $\ell = 1$, corresponding to a simple (dipole) displacement of the beam ions and the background electrons. For $\ell = 1$, we find $\Gamma_e^1(\omega) = -\hat{\nu}_e^2/[\omega^2 - \hat{\nu}_e^2]$ and $\Gamma_b^1(\omega - k_z V_b) = -\hat{\nu}_b^2/[(\omega - k_z V_b)^2 - \hat{\nu}_b^2]$, and introduce the collective oscillation frequencies defined by

$$\omega_e^2 \equiv \hat{\nu}_e^2 + \frac{1}{2}\hat{\omega}_{pe}^2 \left(1 - \frac{r_b^2}{r_w^2}\right) = \frac{1}{2}\frac{\gamma_b m_b}{Z_b m_e}\hat{\omega}_{pb}^2 \left(1 - f\frac{r_b^2}{r_w^2}\right)$$
$$\omega_b^2 \equiv \hat{\nu}_b^2 + \frac{\hat{\omega}_{pb}^2}{2\gamma_b^2}\left(1 - \frac{r_b^2}{r_w^2}\right) = \omega_{\beta b}^2 + \frac{1}{2}\hat{\omega}_{pb}^2\left(f - \frac{1}{\gamma_b^2}\frac{r_b^2}{r_w^2}\right) \qquad (10)$$

where use is made of $\hat{\omega}_{pe}^2 = (\gamma_b m_b / Z_b m_e) f \hat{\omega}_{pb}^2$. Substituting into Eq. (8) and rearranging terms, the $\ell = 1$ dispersion relation can be expressed in the compact form

$$[(\omega - k_z V_b)^2 - \omega_b^2][\omega^2 - \omega_e^2] = \omega_f^4 \ , \qquad (11)$$

where ω_f is defined by

$$\omega_f^4 \equiv \frac{1}{4} f \left(1 - \frac{r_b^2}{r_w^2}\right)^2 \frac{\gamma_b m_b}{Z_b m_e}\hat{\omega}_{pb}^4 \ . \qquad (12)$$

In the absence of background electrons ($f = 0$ and $\omega_f = 0$), Eq. (11) gives stable collective oscillations of the ion beam with frequency $\omega - k_z V_b = \pm\omega_b$, where ω_b is defined in Eq. (11). For $f \neq 0$, however, the ion and electron terms on the left-hand side of Eq. (11) are coupled by the ω_f^4 term on the right-hand side, leading to one unstable solution with $Im\omega > 0$ for a certain range of axial wavenumber k_z. It is important to recognize that the dispersion relation (11) is applicable over a wide range of normalized beam intensity and fractional charge neutralization. That is, Eq. (11) can be applied to the emittance-dominated, moderate-intensity ion beams ($\hat{\omega}_{pb}^2/\omega_{\beta b}^2 \lesssim 0.2$, say) in proton linacs and storage rings. On the other hand, Eq. (11) can also be applied to the low-emittance, very high-intensity ion beams ($\hat{\omega}_{pb}^2/\omega_{\beta b}^2$ approaching $2\gamma_b^2$, for $f = 0$) envisioned for heavy ion fusion.

A careful examination of Eq. (11) shows that the unstable, positive-frequency branch has frequency and wavenumber (ω, k_z) closely tuned to the values (ω_0, k_{z0}) defined by $\omega_0 = +\omega_e$ and $\omega_0 - k_{z0}V_b = -\omega_b$. In this regime, expressing $\omega = \omega_0 + \delta\omega$ and $k_z = k_{z0} + \delta k_z$, and assuming $|\delta\omega| \ll 2\omega_e$, the dispersion relation (11) is given to good approximation by

$$\delta\omega(\delta\omega - \delta k_z V_b)[1 - (\delta\omega - \delta k_z V_b)/2\omega_b] = -\Gamma_0^2 \equiv -\frac{\omega_f^4}{4\omega_e\omega_b} \ . \qquad (13)$$

At moderate beam intensities with $\Gamma_0^2 \ll 1$, the unstable solution to Eq. (13) satisfies $|\delta\omega - \delta k_z V_b| \ll 2\omega_b$. In this regime, Eq. (13) can be approximated by the quadratic form $\delta\omega(\delta\omega - \delta k_z V_b) = -\Gamma_0^2 \equiv -\omega_f^4/4\omega_e\omega_b$. This quadratic dispersion relation supports an unstable solution with growth rate $Im\delta\omega = \Gamma_0[1 - (\delta k_z V_b/2\Gamma_0)^2]^{1/2}$ for δk_z in the (symmetric) interval, $-2\Gamma_0 < \delta k_z V_b < 2\Gamma_0$. The maximum growth rate is $(Im\delta\omega)_{max} = \Gamma_0 \equiv \omega_f^2/2(\omega_e\omega_b)^{1/2}$, which occurs for $\delta k_z = 0$. For example, for a proton beam ($Z_b = 1$, $m_b/m_e = 1836$) with relativistic mass factor $\gamma_b = 1.85$, a moderate value of normalized beam intensity $\hat{\omega}_{pb}^2/\omega_{\beta b}^2 = 0.1$, large wall radius $r_w/r_b \to \infty$ and fractional charge neutralization $f = 0.1$, we obtain $(Im\delta\omega)_{max} = 0.127\omega_{\beta b}$, corresponding to a particularly virulent growth rate for the electron-proton (e-p) instability. For this choice of system parameters, the central oscillation frequency and wavenumber are $\omega_0 = 13.03\omega_{\beta b}$ and $k_{z0}V_b = 14.03\omega_{\beta b}$.

5 CONCLUSIONS

The general kinetic formalism[14] outlined here can also be applied to perturbations about a wide range of non-monoenergetic equilibrium distribution functions. A detailed, self-consistent stability analysis based on Eqs. (2)–(4) for continuously varying profiles is beyond the scope of the present article. It is sufficient to note that the spread in (depressed) betatron frequencies[7, 14] associated with continuously varying profiles is expected to lead to a *threshold* in beam intensity and/or fractional charge neutralization for the onset of instability.

6 ACKNOWLEDGEMENT

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