

Anatomy of Two-Loop Effective Action in Noncommutative Field Theories ¹

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abstract

We compute, at two-loop order, one-particle-irreducible Green functions and effective action in noncommutative $\lambda[\Phi^3]_\star$ -theory for both planar ($g=0, h=3$) and nonplanar ($g=1, h=1$) contributions. We adopt worldline formulation of the Feynman diagrammatics so that relation to string theory diagrammatics is made transparent in the Seiberg-Witten limit. We argue that the resulting two-loop effective action is expressible via open Wilson lines: one-particle-irreducible effective action is generating functional of connected diagrams for interacting open Wilson lines.

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1 Introduction

A notable feature of noncommutative field theories [1] is a sort of duality between the ultraviolet (UV) and the infrared (IR) scale physics — so-called UV-IR mixing [2, 3] — a phenomenon in sharp contrast to the commutative, local quantum field theories. The duality is particularly interesting, as it is reminiscent of the well-known ‘s-t channel duality’ present in string theories. There, infinite tower of point-particle states of string spectrum organizes in a modular invariant manner, and the ‘s-t channel duality’ interchanges the UV dynamics of open/closed strings with the IR dynamics of closed/open strings. In fact, it is now known that (a class of) noncommutative field theories arise quite naturally in the so-called Seiberg-Witten limit [4] of open string dynamics in the background of nonzero closed string two-form potential, B_{mn} . A viable interpretation would be that [5], in the Seiberg-Witten limit, the ‘s-t channel duality’ of underlying open string theories is partially retained and transmuted into the UV-IR mixing of resulting noncommutative field theories.

Intuitively, the UV-IR mixing is understood as the manifestation that, in a generic noncommutative field theory, low-energy spectrum includes, in addition to point-like ones, dipole-like excitations [6, 7]. The simplest illustration is provided by the ‘Mott exciton’ — electron-hole bound-state in metal under strong magnetic field [5]. Argument for existence of the dipoles is based solely on noncommutative geometry

$$[x^m, x^n] = i\theta^{mn} \tag{1.1}$$

and nothing else. These dipoles are induced by turning on the spacetime noncommutativity, Eq.(1.1). As such, its physical characteristics ought to depend on the noncommutativity parameter θ^{mn} , as is illustrated elegantly by the so-called dipole relation between dipole’s electric dipole moment Δx and center-of-mass momentum P :

$$\Delta x^m = \theta^{mn} P_n. \tag{1.2}$$

The UV-IR mixing then follows immediately from Eq.(1.2) – a given momentum scale P is mapped to, in addition to point-like excitations of characteristic scale $(\Delta x)_{\text{particle}} \sim \hbar/P$, dipole-like excitations of characteristic scale $(\Delta x)_{\text{dipole}} \sim \theta P$. Evidently, $(\Delta x)_{\text{particle}} \sim 1/(\Delta x)_{\text{dipole}}$, viz. what one might refer as UV and IR scales are excitation-dependent statement.

The above assertion implies that, in noncommutative field theories, the dipoles ought to be viewed as collective excitations, not as elementary excitations, caused by turning on the noncommutativity, Eq.(1.1). Then an interesting question would be whether the collective excitations – dipoles – are identifiable within noncommutative field theories as a sort of (a complete set of) interpolating composite operators. In particular, in view of the aforementioned universality, these operators ought to be present in all noncommutative field theories, be they

with gauge symmetry or not, or with Poincaré invariance or not. A conjecture has been put forward [5] that these operators are nothing but open Wilson lines, originally identified in the context of noncommutative gauge theories and S-duality therein [8]. In [9, 10], for noncommutative $\lambda[\Phi^3]_\star$ scalar field theory, the conjecture was confirmed at one-loop level — *scalar* open Wilson lines are the interpolating operators for noncommutative dipoles obeying the dipole relation Eq.(1.2), and sum up nonplanar part of the effective action.

In this paper, to substantiate the one-loop confirmation [9, 10] of the conjecture [5], we extend computation of the N-point, one-particle-irreducible Green functions and effective action thereof to two-loop order. We will be adopting the worldline formulation of the $\lambda[\Phi^3]_\star$ theory, extending the formulation constructed at one-loop order [10]. We work with the formulation, as it is particularly suited for detailed comparison with the Seiberg-Witten limit of string theory computations. Compared to the one-loop case, two-loop (and all higher-loop) computations exhibit nontrivial feature of the effective action: Green functions are classifiable into planar and nonplanar parts. At two-loop, the two parts correspond, in the string theory counterpart, to Riemann surfaces of genus-zero with three holes and genus-one with one hole, respectively. This is the feature that will lead eventually to interacting open Wilson lines, both at on-shell and off-shell.

This paper is organized as follows. In Section 2, we recapitulate aspects of two-loop worldline formulation for commutative $\lambda[\Phi^3]$ -theory relevant for noncommutative counterpart. In Section 3, we develop the worldline formulation of noncommutative $\lambda[\Phi^3]_\star$ -theory. In Section 4, we compute the N-point, one-particle-irreducible Green functions as integrals over the $(N+3)$ -dimensional moduli space of two-loop vacuum Feynman diagram with N marked points. We show explicitly that, in contrast to the commutative counterpart, the Green functions obtained from Φ -field insertions on planar vacuum diagrams are markedly different from those on nonplanar vacuum diagrams. Some tedious computational details are relegated to Appendix A. In Appendix B, via the standard Feynman diagrammatics, we confirm $\mathcal{O}(\theta^3)$ effects for nonplanar Green functions, an aspect markedly different from the planar ones. In section 5, we discuss briefly, upon summing over N-point Green functions, how the two-loop effective action is expressible entirely in terms of the scalar open Wilson lines.

Our notations are as follows. The d -dimensional spacetime is taken Euclidean, with metric g_{mn} . We use the shorthand notations as

$$p \cdot k = p_m g^{mn} k_n, \quad p \wedge k = p_m \theta^{mn} k_n \quad p \circ k = p_m (-\theta^2)^{mn} k_n.$$

Via Weyl-Moyal correspondence, products between fields are represented by Moyal's product:

$$A(x) \star B(y) = \exp\left(-\frac{i}{2}\theta^{mn}\partial_m^x \partial_n^y\right) A(x)B(y).$$

2 Two-loop effective action of $\lambda[\Phi^3]$ -theory

We begin with recapitulating worldline formulation of commutative $\lambda\Phi^3$ theory [11, 12, 13]. Specifically, we will compute two-loop part of the effective action, bearing in mind of extensions to noncommutative $\lambda[\Phi^3]_\star$ -theory in Section 3. We utilize the background field method and decompose $\Phi \rightarrow \Phi_0 + \varphi$, where Φ_0 is the background field and φ represents the quantum fluctuations. The generating functional, from which the Feynman rules are read off, is given by

$$Z[\Phi_0] = Z_0[\Phi_0] \int \mathcal{D}\varphi \exp \left(- \int d^d x \left\{ \frac{1}{2} \varphi(x) \left(-\partial_x^2 + m^2 + \lambda\Phi_0 \right) \varphi(x) + \frac{\lambda}{3!} \varphi^3 \right\} \right), \quad (2.1)$$

where Z_0 denotes the classical (tree) part. The resulting two-loop effective action is given, in terms of one-particle-irreducible vacuum diagram, by

$$\begin{aligned} \Gamma_2[\Phi] &= \frac{\lambda^2}{2 \cdot (3!)^2} \int \mathcal{D}\varphi d^d x_1 d^d x_2 \varphi^3(x_1) \varphi^3(x_2) \exp \left[-\frac{1}{2} \int d^d x \varphi(x) \left(-\partial_x^2 + m^2 + \lambda\Phi_0(x) \right) \varphi(x) \right] \\ &= \frac{\lambda^2}{2 \cdot 3!} \int d^d x_1 d^d x_2 \left\langle x_2 \left| \left(\partial_x^2 + m^2 + \lambda\Phi_0 \right)^{-1} \right| x_1 \right\rangle^3, \end{aligned} \quad (2.2)$$

where the second equality is obtained by applying the Wick contractions. The propagator in the last expression of Eq.(2.2) is expressible, via the worldline formulation, as:

$$\begin{aligned} \left\langle x_2 \left| \frac{1}{-\partial_x^2 + m^2 + \lambda\Phi_0(x)} \right| x_1 \right\rangle &= \int_0^\infty dT \int_{\substack{y(0)=x_1 \\ y(T)=x_2}} \mathcal{D}y(\tau) \mathcal{D}k(\tau) e^{-\int_0^T d\tau (k^2 + m^2 - ik \cdot \dot{y} + \lambda\Phi_0(y))} \\ &= \int_0^\infty dT \mathcal{N}(T) \int_{\substack{y(0)=x_1 \\ y(T)=x_2}} \mathcal{D}y(t) e^{-\int_0^T d\tau \left(\frac{1}{4} \dot{y}^2 + m^2 + \lambda\Phi_0(y) \right)}. \end{aligned} \quad (2.3)$$

Here, $\mathcal{N}(T)$ is the normalization factor coming from the k -integral and $\dot{y} \equiv dy(\tau)/d\tau$. The effective action Eq.(2.2) is then rewritable, after Taylor-expanding the background field interaction, in the following form:

$$\begin{aligned} \Gamma_2[\Phi_0] &= \frac{\lambda^2}{12} \sum_{N_1, N_2, N_3=0}^\infty \frac{(-\lambda)^{N_1+N_2+N_3}}{N_1! N_2! N_3!} \int d^d x_1 d^d x_2 \prod_{a=1}^3 \int_0^\infty dT_a e^{-m^2 T_a} \mathcal{N}(T_a) \\ &\times \int_{\substack{y_a(0)=x_1 \\ y_a(T_a)=x_2}} \mathcal{D}y_a(\tau^{(a)}) \exp \left(- \int_0^{T_a} d\tau^{(a)} \frac{1}{4} \dot{y}_a^2(\tau^{(a)}) \right) \left[\int_0^{T_a} d\tau^{(a)} \Phi_0 \left(y_a(\tau^{(a)}) \right) \right]^{N_a}. \end{aligned} \quad (2.4)$$

The three Feynman-Schwinger parameters, T_a ($a = 1, 2, 3$), specify moduli of the two-loop vacuum diagram. The number of background Φ_0 lines attached to the a -th internal propagator is denoted as N_a .

The N -point Green functions are extracted from Eq.(2.4) by replacing the background Φ_0 's with monochromatic plane-waves $\Phi_0(y(\tau)) \rightarrow \sum_{j=1}^N \exp[ip_j \cdot y(\tau)]$. The δ -functions imposing

energy-momentum conservation at the two interaction vertices follow evidently from integration over x_1, x_2 . Because of them, terms containing the same energy-momentum vector more than once can be discarded. Introduce ‘scalar vertex operator’:

$$V^{(a)}(p) := \int_0^{T_a} d\tau^{(a)} \exp \left[ip \cdot y_a \left(\tau^{(a)} \right) \right] ,$$

and replace the background Φ_0 ’s into monochromatic plane-waves:

$$\frac{1}{N_a!} \left[\int_0^{T_a} \Phi_0 \left(y_a(\tau^{(a)}) \right) d\tau^{(a)} \right]^{N_a} \longrightarrow \sum_{i_1 < i_2 < \dots < i_{N_a}} V_{i_1}^{(a)} \dots V_{i_{N_a}}^{(a)} ,$$

where each i_k ($k = 1, \dots, N_a$) runs from 1 to N in so far as the time-ordering is obeyed. We then find, from Eq.(2.4), the two-loop part of the one-particle-irreducible, N -point Green function:

$$\begin{aligned} \Gamma_N[\Phi_0] &= \frac{1}{12} (-\lambda)^{N+2} \sum_{N_1, N_2, N_3=0}^N \sum_{\sigma(N_1, N_2, N_3)} \int d^d x_1 d^d x_2 \prod_{a=1}^3 \int_0^\infty dT_a e^{-m^2 T_a} \mathcal{N}(T_a) \\ &\times \int_{\substack{y_a(0)=x_1 \\ y_a(T_a)=x_2}} \mathcal{D}y_a(\tau^{(a)}) \exp \left[- \int_0^{T_a} d\tau^{(a)} \frac{1}{4} \dot{y}_a^2 \right] \prod_{n=1}^{N_a} \int_0^{T_a} d\tau_n^{(a)} e^{ip_n^{(a)} \cdot y(\tau_n^{(a)})}. \end{aligned} \quad (2.5)$$

Here, $N_1 + N_2 + N_3 = N$, and $\sigma(N_1, N_2, N_3)$ refers to all possible graph-theoretic combinatorics for attaching N_a momenta to each of the three internal propagators, irrespective of ordering of the external Φ_0 lines. Expand $y_a(\tau)$ into normal modes

$$y_a(\tau) = x_1 + \frac{\tau}{T_a} (x_2 - x_1) + \sum_{m=1}^{\infty} y_m \sin \left(\frac{m\pi\tau}{T_a} \right)$$

and perform the functional integral over y_a . Integration over y_m ($m = 1, 2, \dots$) yields

$$\mathcal{N}(T) \int_{\substack{y(0)=x_1 \\ y(T)=x_2}} \mathcal{D}y(\tau) \exp \left[- \int_0^T d\tau \frac{1}{4} \dot{y}^2(\tau) \right] = \left(\frac{1}{4\pi T} \right)^{\frac{d}{2}} \exp \left[- \frac{(x_1 - x_2)^2}{4T} \right],$$

while subsequent integration over x_i ($i = 1, 2$) yields the N -point Green function:

$$\Gamma_N = \frac{1}{12} \sum_{\{N_a\}=0}^N \sum_{\{\sigma\}} (2\pi)^d \delta \left(\sum_{a=1}^3 \sum_{n=1}^{N_a} p_n^{(a)} \right) \Gamma^{(N_1, N_2, N_3)}$$

with

$$\begin{aligned} \Gamma^{(N_1, N_2, N_3)} &= \frac{(-\lambda)^{N+2}}{(4\pi)^d} \prod_{a=1}^3 \int_0^\infty dT_a e^{-m^2 T_a} \Delta^{\frac{d}{2}}(T) \int_0^{T_a} \prod_{n=1}^{N_a} d\tau_n^{(a)} \\ &\times \exp \left[\frac{1}{2} \sum_{a=1}^3 \sum_{j,k}^{N_a} p_j^{(a)} G_{aa}^{\text{sym}} \left(\tau_j^{(a)}, \tau_k^{(a)} \right) p_k^{(a)} + \sum_{a=1}^3 \sum_{j=1}^{N_a} \sum_{k=1}^{N_{a+1}} p_j^{(a)} G_{aa+1}^{\text{sym}} \left(\tau_j^{(a)}, \tau_k^{(a+1)} \right) p_k^{(a+1)} \right]. \end{aligned} \quad (2.6)$$

Here, $\Delta(\mathbb{T})$ denotes the following combination of the two-loop vacuum diagram moduli parameters:

$$\Delta(\mathbb{T}) = (\mathbb{T}_1\mathbb{T}_2 + \mathbb{T}_2\mathbb{T}_3 + \mathbb{T}_3\mathbb{T}_1)^{-1} .$$

Note that, in Eq.(4.14), the Green function is summarized in terms of the worldline correlators [11]:

$$G_{aa}^{\text{sym}}(x, y) = |x - y| - \Delta(\mathbb{T})(\mathbb{T}_{a+1} + \mathbb{T}_{a+2})(x - y)^2 , \quad (2.7)$$

$$G_{aa+1}^{\text{sym}}(x, y) = (x + y) - \Delta(\mathbb{T}) \left(x^2\mathbb{T}_{a+1} + y^2\mathbb{T}_a + (x + y)^2\mathbb{T}_{a+2} \right) . \quad (2.8)$$

The worldline correlators Eqs.(2.7, 2.8) coincide precisely with the string theory worldsheet correlators [13, 14] in the zero-slope limit, $\alpha' \rightarrow 0$. Likewise, the moduli-dependent measure factors $\left(\prod_{a=1}^3 e^{-m^2\mathbb{T}_a} \right) \Delta^{\frac{d}{2}}(\mathbb{T})$ originate from the integrand of the string worldsheet partition function in the same limit.

3 Two-Loop Effective Action of $\lambda[\Phi^3]_{\star}$ -theory

We now extend worldline formalism of the $\lambda[\Phi^3]_{\star}$ -theory investigated in [10]. Again, expand the action around a background field: $\Phi = \Phi_0 + \varphi$. After Wick rotation to the Euclidean space, it reads

$$\begin{aligned} S &= S^{1\text{-loop}} + S_{\text{int}} \\ &= \int d^d x \left(\frac{1}{2}(\partial\varphi)^2 + \frac{1}{2}m^2\varphi^2 + \frac{\lambda}{2}\varphi \star \Phi_0 \star \varphi \right) (x) + \int d^d x \left(\frac{\lambda}{3!}\varphi \star \varphi \star \varphi \right) (x) . \end{aligned}$$

Taking the momentum-space representation, with which the one-loop effective action computation has become simplified enormously [9], we obtain:

$$\begin{aligned} S^{1\text{-loop}} &= \frac{1}{2} \int \frac{d^d k_1}{(2\pi)^d} \frac{d^d k_2}{(2\pi)^d} \tilde{\varphi}(k_1) \tilde{\varphi}(k_2) \left[(k^2 + m^2)(2\pi)^d \delta^{(d)}(k_1 + k_2) \right. \\ &\quad \left. + \frac{\lambda}{2} \int \frac{d^d p}{(2\pi)^d} (2\pi)^d \delta^{(d)}(p + k_1 + k_2) \left(e^{\frac{i}{2}k_1 \wedge p} + e^{-\frac{i}{2}k_1 \wedge p} \right) \tilde{\Phi}_0(p) \right] \end{aligned} \quad (3.1)$$

$$S_{\text{int}} = \frac{\lambda}{3!} \int \prod_{a=1}^3 \frac{d^d k_a}{(2\pi)^d} (2\pi)^d \delta^{(d)} \left(\sum_{a=1}^3 k_a \right) e^{\frac{i}{2}k_1 \wedge k_2} \tilde{\varphi}(k_1) \tilde{\varphi}(k_2) \tilde{\varphi}(k_3) . \quad (3.2)$$

Schematically, the two-loop effective action is \star -product deformation of the commutative effective action, Eq.(2.2):

$$\begin{aligned} \Gamma^{2\text{-loop}}[\Phi_0] &= \int \mathcal{D}\varphi e^{-S_{1\text{-loop}}} \frac{i^2}{2!} (S_{\text{int}})^2 \\ &= \frac{\lambda^2}{2 \cdot (3!)^2} \int d^d x_1 d^d x_2 \left\langle \varphi_{\star}^3(x_1) \varphi_{\star}^3(x_2) \right\rangle_{\Phi_0} , \end{aligned} \quad (3.3)$$

where $\langle \rangle_{\Phi_0}$ denotes Wick contraction by the background-dependent propagator. Thus, the two-loop vacuum diagram is specified entirely by S_{int} , while insertion of external Φ_0 -background interactions is governed by $S_{1\text{-loop}}$. Using the momentum-space representation Eq.(3.2), we obtain

$$\begin{aligned} \Gamma^{2\text{-loop}}[\Phi_0] &= \frac{\lambda^2}{2 \cdot (3!)^2} \int \prod_{a=1}^3 \frac{d^d k_a}{(2\pi)^d} (2\pi)^d \delta^{(d)} \left(\sum k_a \right) e^{\frac{i}{2} k_1 \wedge k_2} \\ &\times \int \prod_{b=1}^3 \frac{d^d l_b}{(2\pi)^d} (2\pi)^d \delta^{(d)} \left(\sum l_b \right) e^{\frac{i}{2} l_1 \wedge l_2} \langle \tilde{\varphi}(k_1) \tilde{\varphi}(k_2) \tilde{\varphi}(k_3) \tilde{\varphi}(l_1) \tilde{\varphi}(l_2) \tilde{\varphi}(l_3) \rangle_{\Phi_0}. \end{aligned}$$

Begin with two-loop vacuum diagrams, for which we set $\Phi_0 = 0$. Thus, the propagator is simply given by

$$\langle \tilde{\varphi}(k_1) \tilde{\varphi}(l_1) \rangle_0 = (2\pi)^d \delta^{(d)}(k_1 + l_1) \frac{1}{(k_1^2 + m^2)}. \quad (3.4)$$

Working out the phase-space integrals for each of the six possible combinatorics (neglecting tadpoles), we obtain, after ℓ_a -integrals,

$$\begin{aligned} \Gamma^{2\text{-loop}}[0] &= \frac{3\lambda^2}{2 \cdot (3!)^2} \int d^d x_1 d^d x_2 \int \prod_{a=1}^3 \frac{d^d k_a}{(2\pi)^d} \frac{e^{i k_a \cdot (x_1 - x_2)}}{(k_a^2 + m^2)} \left(1 + e^{i k_1 \wedge k_2} \right) \\ &:= (\mathbf{P} + \mathbf{NP}), \end{aligned}$$

viz. a sum of two types of vacuum diagrams, **P** and **NP**. Diagrammatically, the phase-factor $e^{i k_1 \wedge k_2}$ in **NP** originates from crossing [2] two of the three internal propagators, as is evident from the noncommutative Feynman diagrammatics. We refer the two diagrams **P** and **NP** as planar and nonplanar vacuum diagrams, respectively. They are depicted in Fig. 1. As is evident from ‘thickening’ the internal propagators, they are mapped, at zero slope limit, to the following string worldsheet diagrams:

- planar vacuum diagram \longrightarrow sphere with three holes ($g=0, h=3$),
- nonplanar vacuum diagram \longrightarrow torus with a hole ($g=1, h=1$).

Actually, the two types of the vacuum Feynman diagrams are related each other. In the configuration-space Feynman diagrammatics, the nonplanar vacuum diagram **NP** is obtainable from the planar one **P** by inserting Moyal’s pseudo-differential phase-operator:

$$\begin{aligned} \Gamma^{2\text{-loop}}[0] &= \frac{\lambda^2}{24} \int d^d x_1 d^d x_2 (1 + e^{-i \partial_z \wedge \partial_w}) \\ &\times \int \prod_{a=1}^3 \frac{d^d k_a}{(2\pi)^d} \frac{e^{i k_1 \cdot (x_1 - z)} e^{i k_2 \cdot (x_1 - w)} e^{i k_3 \cdot (x_1 - x_2)}}{(k_1^2 + m^2)(k_2^2 + m^2)(k_3^2 + m^2)} \Big|_{z=w=x_2}. \end{aligned} \quad (3.5)$$

Having obtained the two-loop vacuum diagrams for $\Phi_0 = 0$, we readily obtain the vacuum diagrams for nonzero Φ_0 -background by replacing the free propagators in Eq.(3.5) into the full, background-dependent propagator.

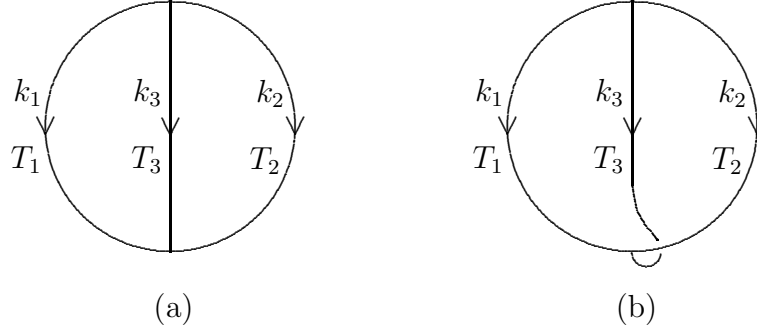


Figure 1: Two types of two-loop vacuum diagram in $\lambda[\Phi^3]_\star$ field theory are shown. The diagram (a) is a planar vacuum bubble and the diagram (b) is a nonplanar vacuum bubble.

Hence, we will first derive noncommutative counterpart of the scalar propagator ². Inferring from

$$\int d^d x \varphi(x) \star \Phi_0(x) \star \varphi(x) = \int \frac{d^d k_1}{(2\pi)^d} \frac{d^d k_2}{(2\pi)^d} \frac{d^d p}{(2\pi)^d} (2\pi)^d \delta^d(k_1 + k_2 + p) e^{-\frac{i}{2} k_1 \wedge p} \tilde{\varphi}(k_1) \tilde{\Phi}_0(p) \tilde{\varphi}(k_2),$$

when obtaining the background-dependent scalar propagator by taking functional derivative with respect to φ , a natural prescription of Moyal's phase-factor would be to strip off the $e^{-\frac{i}{2} k_1 \wedge p}$ factor from the interaction term in Eq.(3.1). Implicit to this prescription is that, upon expanded in powers of the background field, products of a string of $\tilde{\Phi}_0$'s is interpreted in terms of the \star -product. This will be elaborated further in the next section. We have verified consistency of this prescription for one-loop computation by comparing the result with that in [9, 10], and hence expect the consistency for two-loop computation as well. Being so, we express the background-dependent scalar propagator (noncommutative counterpart of Eq.(2.3)) in the worldline formalism as

$$\begin{aligned} \Delta_a(x_1, x_2) &= \left\langle x_2 \left| \frac{1}{-\partial_x^2 + m^2 + \frac{\lambda}{2} \Phi_0(x) \star} \right| x_1 \right\rangle \\ &= \int_0^\infty dT_{x_1} \int_{\substack{y_a(0)=x_2 \\ y_a(T_a)=w}} \mathcal{D}y_a(\tau^{(a)}) \mathcal{D}k_a(\tau^{(a)}) \\ &\quad \times \mathcal{P}_t \exp \left[- \int_0^{T_a} d\tau^{(a)} \left(k_a^2 + m^2 - ik_a \dot{y}_a + \frac{\lambda}{2} \int \frac{d^d p}{(2\pi)^d} (1 + e^{ik_a \wedge p}) \tilde{\Phi}_0(p) \right) \right]. \end{aligned} \quad (3.6)$$

²Our phase-factor convention is different from the analysis given in [9, 10]. The different conventions, however, affect intermediate steps only, and yield identical final result.

The two types of interaction vertices, $(1 + e^{ik_a \wedge p})$, represent, in the double-line notation of the noncommutative Feynman diagrams, insertions on the ‘left’ and ‘right’ side (with crossing), respectively. One can verify that the one-loop effective action computed in [9, 10] is readily derived by imposing periodic boundary condition ($x_1 = x_2$) and taking into account of Jacobian associated with Killing symmetry, $1/T_a: dT_a \rightarrow dT_a/T_a$. The two-loop effective action is now obtained by Wick-contracting Eq.(3.3) and is expressible schematically as a sum of the planar and the nonplanar parts:

$$\Gamma^{2\text{-loop}}[\Phi_0] = \Gamma^{\mathbf{P}}[\Phi_0] + \Gamma^{\mathbf{NP}}[\Phi_0]$$

where

$$\Gamma^{\mathbf{P}}[\Phi_0] = \frac{\lambda^2}{24} \int d^d x_1 d^d x_2 \Delta_1(x_2, x_1) \star \Delta_2(x_2, x_1) \star \Delta_3(x_2, x_1) , \quad (3.7)$$

$$\Gamma^{\mathbf{NP}}[\Phi_0] = \frac{\lambda^2}{24} \int d^d x_1 d^d x_2 e^{-i\partial_z \wedge \partial_w} \Delta_1(z, x_1) \star \Delta_2(w, x_1) \star \Delta_3(x_2, x_1) \Big|_{z=w=x_2} . \quad (3.8)$$

The \star -product between the propagators summarizes the Filk phase-factors [16] associated with the *vacuum* Feynman diagram. The additional insertion of the pseudo-differential operator, $e^{-i\partial_z \wedge \partial_w}$, for the nonplanar part accounts for the extra phase-factor associated with the nonplanar crossing. See Fig. 1(b).

We end this section with a cautionary remark concerning our nomenclature. The planar part Eq.(3.7) of the effective action summarizes one-particle-irreducible Green functions arising from *planar* and *nonplanar* insertions of the background Φ_0 -fields on the *planar* vacuum diagram. Likewise, the nonplanar part Eq.(3.8) of the effective action summarizes one-particle-irreducible Green functions arising from arbitrary insertions of the background Φ_0 -fields on the *nonplanar* vacuum diagram ³.

4 The N-point One-Particle-Irreducible Green functions

In this section, we will compute the N-point, one-particle-irreducible Green functions, and express them as (N+3)-dimensional integral over the moduli space of two-loop Feynman diagram with N marked points. The expression will eventually enable us to resum them over N, and obtain the two-loop effective action ⁴. In computing the Green functions, we will pave essentially the same steps as Eq.(2.4) through Eq.(2.5) in section 2, taking, at the same time, proper care of Filk’s phase-factors resulting from insertions of the background Φ_0 field on each internal propagator.

³While notationally consistent, we trust this will be a source of profound confusion to the readers.

⁴Because of variety of technical complications, the resummation and computation of the two-loop effective action will be relegated to a separate paper [15].

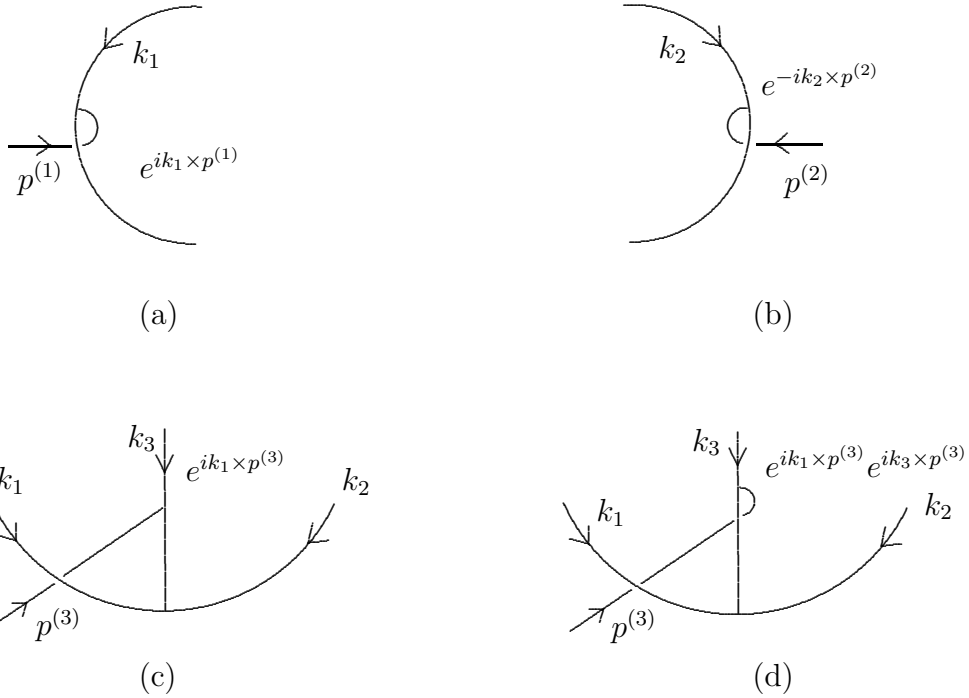


Figure 2: Assignments of phase factors resulting from the crossing of external insertions and internal lines.

4.1 Moyal's Phase-Factors

As Filk's phase-factors are the new ingredients in the noncommutative Feynman diagrammatics, we will begin with enumerating and fixing convention of them. Apart from insertion of the pseudo-differential operator, $e^{-i\partial_z \wedge \partial_w}$, the expressions Eqs.(3.7, 3.8) of $\Gamma^{\mathbf{P}}$ and $\Gamma^{\mathbf{NP}}$ comprise the same integrand – triple product of the Φ_0 -field dependent propagators. To extract Moyal's phase-factors, we first Taylor-expand each propagator in powers of Φ_0 's, precisely as in Eq.(2.4). By doing so, we readily observe that two types of interaction vertices are generated: $1 \cdot \widetilde{\Phi}_0$ for the planar Φ_0 -insertion, and $e^{ik_a \wedge p} \widetilde{\Phi}_0$ for the nonplanar Φ_0 -insertion. One might thus suppose that all the phase-factors are prescribable by introducing an insertion-specific phase-factor:

$$\exp[ik_a \wedge p_j^{(a)} \varepsilon_j^{(a)}], \quad j = 1, 2, \dots, N_a \quad (4.1)$$

where $\varepsilon_j^{(a)} = 0$ and 1 for planar and nonplanar Φ_0 -insertions, respectively. It turned out that this prescription is valid if and only if there is only one independent internal momentum, viz. one-loop diagram. For two-loop diagrams, a closer inspection of the diagrammatics depicted in Fig. 2 is imperative.

We will assign the Filk phase-factors beginning with ($a = 1$) internal propagator and sequentially with ($a = 2$), ($a = 3$) ones. For the ($a = 1$) internal propagator, the phase-factor assignment Eq.(4.1) is complete (Fig. 2(a)). Next, for the ($a = 2$) internal propagator, as (Fig. 2(b)) indicates, a Φ_0 -field insertion along the ‘left’ edge crosses the ($a = 2$) internal propagator carrying momentum k_2 . Because of the crossing, the insertion-specific phase-factor, Eq.(4.1), ought to be assigned with $\varepsilon_j^{(2)} = 0$ and -1 for planar and nonplanar insertions, respectively. Lastly, for the ($a = 3$) internal propagator, we will prescribe the phase-factor convention as follows. Let the insertions to the ‘right’ edge to cross the ($a = 3$) internal propagator (just as the insertions to the ‘right’ edge for the ($a = 1$) internal propagator), and assign the insertion-specific phase-factor, Eq.(4.1), with $\varepsilon^{(3)} = 0$ and 1 for the ‘left’ and the ‘right’ insertions, respectively. Also, let these insertions eventually cross the ($a = 1$) internal propagator — a convention giving rise to an extra phase-factor $e^{ik_1 \wedge p_j^{(3)}}$; $j = 1, 2, \dots, N_3$ for both ‘left’ and ‘right’ insertions along the ($a = 3$) internal propagator. See Figs. 2(c) and 2(d). The resulting phase-factor assignment ⁵ of $\varepsilon^{(a)}$ ’s is depicted in Fig. 3(a).

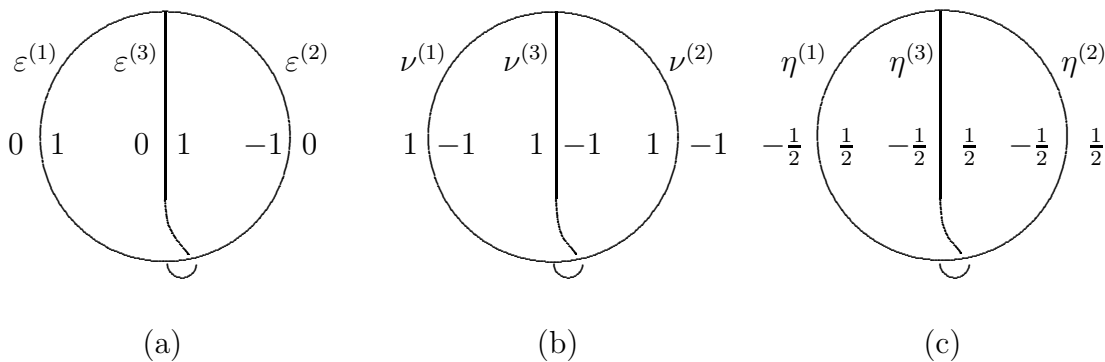


Figure 3: The assignments of $\varepsilon^{(a)}$, $\nu^{(a)}$ and $\eta^{(a)}$ for six possible types of external insertions. The assignments are identical for the planar vacuum diagrams.

Collecting all the phase-factors arising from the Taylor expansion, the term with N_1, N_2, N_3 powers of the Φ_0 -field attached to $a = 1, 2, 3$ internal propagators, respectively, is accompanied by

$$\left(\prod_{j=1}^{N_1} e^{ik_1 \wedge p_j^{(1)} \varepsilon_j^{(1)}} \right) \left(\prod_{j=1}^{N_2} e^{ik_2 \wedge p_j^{(2)} \varepsilon_j^{(2)}} \right) \left(\prod_{j=1}^{N_3} e^{ik_3 \wedge p_j^{(3)} \varepsilon_j^{(3)}} e^{ik_1 \wedge p_j^{(3)}} \right), \quad (4.2)$$

where

$$\varepsilon^{(1)} = 0, 1, \quad \varepsilon^{(2)} = 0, -1, \quad \varepsilon^{(3)} = 0, 1.$$

⁵Different crossing prescriptions may be adopted for the Φ_0 -field insertions, but they all turn out equivalent once the energy-momentum conservation conditions at each vertex are imposed.

Inferring from Figs. 4 and 5, we extract the overall Filk phase-factor (which ought to show up in the \star -products of Eq.(3.6)) as a product of three copies ($a = 1, 2, 3$) of

$$\mathcal{P}_t \frac{1}{N_a!} \left[\int_0^{T_a} d\tau^{(a)} \hat{\Phi}(p) \right]^{N_a} \rightarrow \sum_{\{\nu\}} \sum_{i_1 < i_2 < \dots < i_{N_a}} V_{i_1}^{(a)} \dots V_{i_{N_a}}^{(a)} \times \exp \left[\frac{i}{4} \sum_{k < l}^{N_a} p_{i_k}^{(a)} \wedge p_{i_l}^{(a)} (\nu_{i_k}^{(a)} + \nu_{i_l}^{(a)}) \epsilon (\tau_{i_k}^{(a)} - \tau_{i_l}^{(a)}) \right], \quad (4.3)$$

multiplied by the moduli-independent phase-factor

$$\Xi(p_i^{(a)}) = \exp \left[-\frac{i}{2} \sum_{a=1}^3 \left(\sum_{i,j=1}^{N_a} \left(\frac{1}{2} + \eta_i^{(a)} \right) \left(\frac{1}{2} - \eta_j^{(a)} \right) p_i^{(a)} \wedge p_j^{(a)} + \frac{1}{3} \sum_i^{N_a} \sum_j^{N_{a+1}} p_i^{(a)} \wedge p_j^{(a+1)} \right) \right]. \quad (4.4)$$

Here, as shown in Figs. 3(b) and 3(c), we have introduced another set of insertion-specific sign factors:

$$\nu_j^{(a)} = (-)^{a+1} - 2\varepsilon_j^{(a)} = -2\eta_j^{(a)}. \quad (4.5)$$

Note that (See Fig. 3(c))

$$\eta^{(1)} = \varepsilon^{(1)} - \frac{1}{2}, \quad \eta^{(2)} = \varepsilon^{(2)} + \frac{1}{2}, \quad \eta^{(3)} = \varepsilon^{(3)} - \frac{1}{2} \quad (4.6)$$

defines the most symmetric assignment of the sign factor. In Eq.(4.3), $\sum_{\{\nu\}}$ denotes summation over all possible 2^{N_a} choices of each $\nu_j^{(a)}$, corresponding two inequivalent ways (distinguished by different Moyal's phase-factor) of attaching Φ_0 -field on a -th internal propagator.

Eqs.(4.3, 4.4) are structurally similar to Filk's phase-factors encountered in N-point Green functions at one-loop [10]. As we will elaborate in section 5, Eqs.(4.3, 4.4) are intimately tied with the generalized \star -product and, upon summing over N, re-expression of the two-loop effective action in terms of the scalar open Wilson lines.

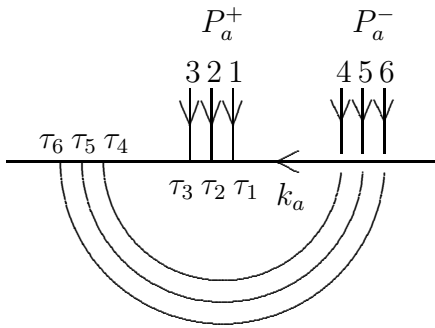


Figure 4: On an internal line, Filk's phase-factor for the planar external insertions has an opposite sign to the one for nonplanar external insertions.

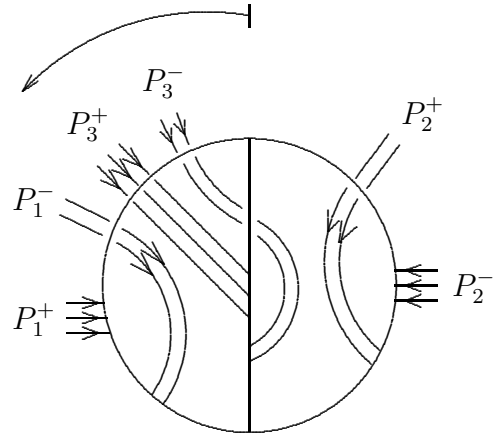


Figure 5: Assignment of the overall Filk's phase-factor.

The phase-factors Eqs.(4.3,4.4) are accounted for as follows. First, according to the phase-factor prescription adopted as above (see Fig. 4), one ought to arrange crossing of nonplanar insertions (nonzero $\varepsilon^{(a)}$) *prior to* any planar insertions. Moreover, lack of phase-factors for cyclically ordered Φ_0 -field insertions implies that, graphically, these insertions do not cross one another, explaining the origin of the phase-factor in Eq.(4.3) [17, 18, 9, 10], which involves the ordering factor $\epsilon\left(\tau_{i_k}^{(a)} - \tau_{i_l}^{(a)}\right)$. Next, as readily understood from Fig. 5, additional Filk's phase-factors among the partial sum of momenta P_a^\pm 's,

$$P_a^\pm = \sum_{j=1}^{N_a} \left(\frac{1 \pm \nu_j^{(a)}}{2} \right) p_j^{(a)} \quad \text{and} \quad P_a = P_a^+ + P_a^- = \sum_{j=1}^{N_a} p_j^{(a)},$$

needs to be included. Again imposing the condition that insertions on each boundary do not cross one another, ordering of P_a^\pm 's, and hence the Filk's phase-factors are well-defined. For instance, in Fig. 5, we have prescribed the Filk ordering as $P_3^-, P_3^+, P_1^-, P_1^+, P_2^-,$ and P_2^+ . The resulting phase-factors are collected into

$$\Xi\left(p_i^{(a)}\right) = \exp \left[-\frac{i}{2} \sum_{a=1}^3 \left((-1)^a P_a^+ \wedge P_a^- + \frac{1}{3} P_a \wedge P_{a+1} \right) \right], \quad (4.7)$$

yielding, when expanded in terms of individual $p_i^{(a)}$'s, precisely the Ξ -factor, Eq.(4.4). Note that the Ξ phase-factor vanishes identically at one-loop.

Putting together all the Filk phase-factors, the N-point Green function is expressible as:

$$\Gamma_N = \frac{1}{24} \sum_{N=0}^{\infty} (-\lambda)^{2+N} \sum_{N_1, N_2, N_3=0}^N \sum_{\sigma(N_1, N_2, N_3)} \sum_{\{\nu\}} C_{\{N\}} \Gamma_{(N_1, N_2, N_3)},$$

where $C_{\{N\}}$ denotes the combinatoric factor associated with marked N points on the two-loop graph, $N_1 + N_2 + N_3 = N$, and

$$\begin{aligned} \Gamma_{(N_1, N_2, N_3)} &= \prod_{a=1}^3 \int_0^\infty d\Gamma_a \int_0^{\Gamma_a} \prod_{i=1}^{N_a} d\tau_i^{(a)} \Xi\left(p_i^{(a)}\right) \exp \left[\frac{i}{4} \sum_{k < l}^{N_a} p_k^{(a)} \wedge p_l^{(a)} \left(\nu_k^{(a)} + \nu_l^{(a)} \right) \epsilon\left(\tau_k^{(a)} - \tau_l^{(a)}\right) \right] \\ &\times \int d^d x_1 d^d x_2 \int_{\substack{y_a(0)=x_1 \\ y_a(\Gamma)=X_a}} \mathcal{D}y_a\left(\tau^{(a)}\right) \exp \left[i \sum_{j=1}^{N_a} p_j^{(a)} y_a\left(\tau_j^{(a)}\right) \right] \\ &\times \int \mathcal{D}k_a\left(\tau^{(a)}\right) \exp \left[-\int_0^{\Gamma_a} \left(k_a^2 + m^2 - ik_a \dot{y}_a \right) d\tau^{(a)} \right] \prod_{j=1}^{N_a} \exp \left[ik_a\left(\tau_j^{(a)}\right) \wedge p_j^{(a)} \varepsilon_j^{(a)} \right]. \end{aligned} \quad (4.8)$$

Here, for the planar part, we set $X_1 = X_2 = X_3 = x_2$, and, for nonplanar part, we take $(X_1, X_2, X_3) = (z, w, x_2)$ first and, after acting the pseudo-differential operator $e^{-i\partial_z \wedge \partial_w}$, set $z = w = x_2$. Because of the phase-factor convention adopted, the k_a -integration in the last line of Eq.(4.8) take a slightly different form for $a = 1$ and for $a = 2, 3$. For $a = 1$,

$$\int \mathcal{D}k_1 \exp \left[-\int_0^{\Gamma_1} \left(k_1^2(\tau) - ik_1 \cdot \dot{y}_1(\tau) - ik_1(\tau) \wedge \left(\sum_{j=1}^{N_1} p_j^{(1)} \varepsilon_j^{(1)} + \sum_{j=1}^{N_3} p_j^{(3)} \right) \delta\left(\tau - \tau_j^{(1)}\right) \right) d\tau \right],$$

while, for $a = 2, 3$,

$$\int \mathcal{D}k_a \exp \left[- \int_0^{T_a} \left(k_a^2(\tau) - ik_a \cdot \dot{y}_a(\tau) - ik_a(\tau) \wedge \left(\sum_{j=1}^{N_a} p_j^{(a)} \varepsilon_j^{(a)} \right) \delta(\tau - \tau_j^{(a)}) \right) d\tau \right]. \quad (4.9)$$

The notation Π' in Eq.(4.8) is to emphasize the extra contribution $\prod_{j=1}^{N_3} e^{ik_1 \wedge p_j^{(3)}}$ originating from the last phase-factor in Eq.(4.2).

After performing k_a integrations, we obtain

$$\begin{aligned} \Gamma_{(N_1, N_2, N_3)} &= \prod_{a=1}^3 \int_0^\infty dT_a \int_0^{T_a} \prod_{i=1}^{N_a} d\tau_i^{(a)} \Xi(p_i^{(a)}) \exp \left[\frac{i}{4} \sum_{k<l}^{N_a} p_k^{(a)} \wedge p_l^{(a)} (\nu_k^{(a)} + \nu_l^{(a)}) \epsilon(\tau_k^{(a)} - \tau_l^{(a)}) \right] \\ &\times \int d^d x_1 d^d x_2 \int_{\substack{y_a(0)=x_1 \\ y_a(T)=x_2}} \mathcal{D}y_a \exp \left[i \sum_{j=1}^{N_a} p_j^{(a)} y_a(\tau_j^{(a)}) \right] \prod_{j=1}^{N_a} \exp \left[-\frac{1}{2} \dot{y}_a(\tau_j^{(a)}) \wedge p_j^{(a)} \varepsilon_j^{(a)} \right]. \end{aligned}$$

The functional integration over y^a demonstrates manifestly that the ‘noncommutative’ scalar vertex operator ought to be identified with

$$V_j^{(a)} = \int_0^{T_a} d\tau \exp \left[ip_j \cdot \left(x^{(a)}(\tau) - \frac{i}{2} \varepsilon_j \wedge \dot{x}^{(a)}(\tau) \right) \right].$$

Performing the y_a integral as in section 2, we finally obtain ⁶

$$\begin{aligned} \Gamma_{(N_1, N_2, N_3)} &= \prod_{a=1}^3 \int_0^\infty dT_a \int_0^{T_a} \prod_{i=1}^{N_a} d\tau_i^{(a)} \Xi(p_i^{(a)}) \exp \left[\frac{i}{4} \sum_{k<l}^{N_a} p_k^{(a)} \wedge p_l^{(a)} (\nu_k^{(a)} + \nu_l^{(a)}) \epsilon(\tau_k^{(a)} - \tau_l^{(a)}) \right] \\ &\times \int d^d x_1 d^d x_2 \left(\frac{1}{4\pi T_a} \right)^{\frac{d}{2}} \exp \left[\frac{-(X_a - x_1)^2}{4T_a} \right] \exp \left[i \sum_{j=1}^{N_a} p_j^{(a)} x_1 \right] \\ &\times \exp \left[i(X_a - x_1) \sum_{j=1}^{N_a} \frac{1}{T_a} \left(\tau_j^{(a)} \cdot p_j^{(a)} + \frac{i}{2} \wedge p_j^{(a)} \varepsilon_j^{(a)} \right) \right] \\ &\times \exp \left[\frac{1}{2} \sum_{i,j=1}^{N_a} p_i^{(a)} \cdot p_j^{(a)} \left(|\tau_i^{(a)} - \tau_j^{(a)}| - (\tau_i^{(a)} + \tau_j^{(a)}) + \frac{2\tau_i^{(a)} \tau_j^{(a)}}{T_a} \right) \right. \\ &\quad \left. - \frac{i}{2} \sum_{i,j=1}^{N_a} p_i^{(a)} \wedge p_j^{(a)} \left(\varepsilon_j^{(a)} \left(1 - \frac{\tau_i^{(a)} - \tau_j^{(a)}}{T_a} \right) + \varepsilon_j^{(a)} \left(1 - \frac{\tau_i^{(a)} + \tau_j^{(a)}}{T_a} \right) \right) \right. \\ &\quad \left. - \frac{1}{4T_a} \sum_{i,j=1}^{N_a} p_i^{(a)} \circ p_j^{(a)} \varepsilon_i^{(a)} \varepsilon_j^{(a)} \right]. \quad (4.10) \end{aligned}$$

It now remains to perform the x_1, x_2 integrals in Eqs.(3.7, 3.8). In the next subsections, we will do so for the planar and the nonplanar cases separately.

⁶The expression is arrangeable cyclically symmetrically if the η 's defined in Eq.(4.6) is used for the sign factors.

4.2 Planar Part

For the planar contribution, we set $z = w = x_2$, and perform the x_1, x_2 integrations ⁷. We obtain the result as

$$\Gamma_N^{\mathbf{P}} = \frac{1}{24}(-\lambda)^{2+N} \sum_{N_1, N_2, N_3=0}^N \sum_{\sigma} \sum_{\{\nu\}} (2\pi)^d \delta \left(\sum_{a=1}^3 \sum_{i=1}^{N_a} p_i^{(a)} \right) C_{\{N\}}^{\mathbf{P}} \Gamma_{(N_1, N_2, N_3)}^{\mathbf{P}} \quad (4.11)$$

with

$$\begin{aligned} \Gamma_{(N_1, N_2, N_3)}^{\mathbf{P}} &= \frac{1}{(4\pi)^d} \int_0^\infty \prod_{a=1}^3 dT_a e^{-m^2 T_a} \Delta^{\frac{d}{2}}(\mathbb{T}) \prod_{a=1}^3 \int_0^{T_a} \prod_{i=1}^{N_a} d\tau_i^{(a)} \Xi \left(p_i^{(a)} \right) \\ &\times \exp \left[\frac{i}{4} \sum_{k<l}^{N_a} p_k^{(a)} \wedge p_l^{(a)} \left(\nu_k^{(a)} + \nu_l^{(a)} \right) \epsilon \left(\tau_k^{(a)} - \tau_l^{(a)} \right) \right] \\ &\times \exp \left[\frac{1}{2} \sum_{a=1}^3 \sum_{i,j}^{N_a} p_i^{(a)} \cdot G_{aa}^{\mathbf{P}} \left(\tau_i^{(a)}, \tau_j^{(a)}; \eta_i^{(a)}, \eta_j^{(a)} \right) p_j^{(a)} \right. \\ &\left. + \sum_{a=1}^3 \sum_i^{N_a} \sum_j^{N_{a+1}} p_i^{(a)} G_{aa+1}^{\mathbf{P}} \left(\tau_i^{(a)}, \tau_j^{(a+1)}; \eta_i^{(a)}, \eta_j^{(a+1)} \right) p_j^{(a+1)} \right]. \quad (4.12) \end{aligned}$$

Here, the worldline correlators for the planar vacuum diagram are given by

$$\begin{aligned} G_{aa}^{\mathbf{P}mn} \left(\tau_i^{(a)}, \tau_j^{(a)}; \eta_i^{(a)}, \eta_j^{(a)} \right) &= g^{mn} G_{aa}^{\text{sym}} \left(\tau_i^{(a)}, \tau_j^{(a)} \right) - 2i\Delta(\mathbb{T})\theta^{mn} (\mathbb{T}_{a+1} + \mathbb{T}_{a+2}) \tau_i^{(a)} \eta_j^{(a)} \\ &\quad + \frac{1}{4}\Delta(\mathbb{T})(-\theta^2)^{mn} (\mathbb{T}_{a+1} + \mathbb{T}_{a+2}) \left(\eta_i^{(a)} - \eta_j^{(a)} \right)^2, \end{aligned}$$

and

$$\begin{aligned} G_{aa+1}^{\mathbf{P}mn} \left(\tau_i^{(a)}, \tau_j^{(a+1)}; \eta_i^{(a)}, \eta_j^{(a+1)} \right) &= g^{mn} G_{aa+1}^{\text{sym}} \left(\tau_i^{(a)}, \tau_j^{(a+1)} \right) \\ &\quad + i\Delta(\mathbb{T})\theta^{mn} \left[\mathbb{T}_{a+2} \left(\tau_i^{(a)} \eta_j^{(a+1)} - \eta_i^{(a)} \tau_j^{(a+1)} \right) + \frac{1}{2}\mathbb{T}_{a+1} \tau_i^{(a)} + \frac{1}{2}\mathbb{T}_a \tau_j^{(a+1)} \right] \\ &\quad + \frac{1}{4}\Delta(\mathbb{T})(-\theta^2)^{mn} \left[\left(\eta_i^{(a)} + \frac{1}{2} \right)^2 \mathbb{T}_{a+1} + \left(\eta_j^{(a+1)} - \frac{1}{2} \right)^2 \mathbb{T}_a + \left(\eta_i^{(a)} + \eta_j^{(a+1)} \right)^2 \mathbb{T}_{a+2} \right]. \end{aligned}$$

In this expression, as explained in Appendix A, we have reversed ordering of the τ -variable via $\tau^{(a)} \rightarrow \mathbb{T}_a - \tau^{(a)}$. Consequently, the sign of the $\epsilon \left(\tau_k^{(a)} - \tau_\ell^{(a)} \right)$ -dependent phase in Eq.(4.12) is reversed compared to, for example, that in Eq.(4.11). It is to be noted that the last term, proportional to $(-\theta^2)^{mn}$, is structurally the same as the G_{aa+1}^{sym} worldline correlator, Eq.(2.8), except that the Φ_0 -insertion dependence is through the sign-factors (instead of the insertion moduli parameters).

⁷Details are sketched in Appendix A.

4.3 Nonplanar Part

In this case, we need to insert the pseudo-differential operator $e^{-i\partial_z \wedge \partial_w}$ first, and then perform the x_1, x_2 integrations. The operator insertion amounts effectively to the \star -product among the background-dependent propagators. See Appendix A for details. The insertion also causes the $\Delta(\mathbb{T}_a)$ factor being replaced by $\Delta_\theta(\mathbb{T}_a)$:

$$\Delta_\theta(\mathbb{T}_a) = \left(\mathbb{T}_1 \mathbb{T}_2 + \mathbb{T}_2 \mathbb{T}_3 + \mathbb{T}_3 \mathbb{T}_1 - \frac{\theta^2}{4} \right)^{-1}.$$

The final result is expressible as follows:

$$\Gamma_{\mathbb{N}}^{\text{NP}} = \frac{1}{24} (-\lambda)^{2+\mathbb{N}} \sum_{\mathbb{N}_1, \mathbb{N}_2, \mathbb{N}_3=0}^{\mathbb{N}} \sum_{\sigma} \sum_{\{\nu\}} (2\pi)^d \delta \left(\sum_{a=1}^3 \sum_{n=1}^{\mathbb{N}_a} p_n^{(a)} \right) C_{\{\mathbb{N}\}}^{\text{NP}} \Gamma_{(\mathbb{N}_1, \mathbb{N}_2, \mathbb{N}_3)}^{\text{NP}} \quad (4.13)$$

with

$$\begin{aligned} \Gamma_{(\mathbb{N}_1, \mathbb{N}_2, \mathbb{N}_3)}^{\text{NP}} &= \frac{1}{(4\pi)^d} \int_0^\infty \prod_{a=1}^3 d\mathbb{T}_a e^{-m^2 \mathbb{T}_a} (\Delta_\theta)^{\frac{d}{2}}(\mathbb{T}_a) \prod_{a=1}^3 \int_0^{\mathbb{T}_a} \prod_{i=1}^{\mathbb{N}_a} d\tau_i^{(a)} \Xi(p_i^{(a)}) \\ &\times \exp \left[\frac{i}{4} \sum_{k<l}^{\mathbb{N}_a} p_k^{(a)} \wedge p_l^{(a)} (\nu_k^{(a)} + \nu_l^{(a)}) \epsilon(\tau_k^{(a)} - \tau_l^{(a)}) \right] \\ &\times \exp \left[\frac{1}{2} \sum_{a=1}^3 \sum_{i,j}^{\mathbb{N}_a} p_i^{(a)} G_{aa}^{\text{NP}}(\tau_i^{(a)}, \tau_j^{(a)}; \eta_i^{(a)}, \eta_j^{(a)}) p_j^{(a)} \right. \\ &\quad \left. + \sum_{a=1}^3 \sum_i^{\mathbb{N}_a} \sum_j^{\mathbb{N}_{a+1}} p_i^{(a)} G_{aa+1}^{\text{NP}}(\tau_i^{(a)}, \tau_j^{(a+1)}; \eta_i^{(a)}, \eta_j^{(a+1)}) p_j^{(a+1)\nu} \right]. \quad (4.14) \end{aligned}$$

Here, the worldline correlators for the nonplanar two-loop vacuum diagram are defined as:

$$\begin{aligned} G_{aa}^{\text{NP}mn}(\tau_i^{(a)}, \tau_j^{(a)}; \eta_i^{(a)}, \eta_j^{(a)}) &= g^{mn} G_{\theta aa}^{\text{sym}}(\tau_i^{(a)}, \tau_j^{(a)}) - 2i \Delta_\theta(\mathbb{T}) \theta^{mn} (\mathbb{T}_{a+1} + \mathbb{T}_{a+2}) \tau_i^{(a)} \eta_j^{(a)} \\ &+ \frac{1}{4} \Delta_\theta(\mathbb{T}) (-\theta^2)^{mn} (\mathbb{T}_{a+1} + \mathbb{T}_{a+2}) (\eta_i^{(a)} - \eta_j^{(a)})^2 - \frac{i}{4} \Delta_\theta(\mathbb{T}) (\theta^3)^{mn} (\eta_i^{(a)} - \eta_j^{(a)}), \quad (4.15) \end{aligned}$$

and

$$\begin{aligned} G_{aa+1}^{\text{NP}mn}(\tau_i^{(a)}, \tau_j^{(a+1)}; \eta_i^{(a)}, \eta_j^{(a+1)}) &= g^{mn} G_{\theta aa+1}^{\text{sym}}(\tau_i^{(a)}, \tau_j^{(a+1)}) \\ &+ i \Delta_\theta(\mathbb{T}) \theta^{mn} \left[\mathbb{T}_{a+2} (\tau_i^{(a)} \eta_j^{(a+1)} - \eta_i^{(a)} \tau_j^{(a+1)}) - \frac{1}{2} \mathbb{T}_{a+1} \tau_i^{(a)} - \frac{1}{2} \mathbb{T}_a \tau_j^{(a+1)} + \tau_i^{(a)} \tau_j^{(a+1)} \right] \\ &+ \frac{1}{4} \Delta_\theta(\mathbb{T}) (-\theta^2)^{mn} \left[\left(\eta_i^{(a)} - \frac{1}{2} \right)^2 \mathbb{T}_{a+1} + \left(\eta_j^{(a+1)} + \frac{1}{2} \right)^2 \mathbb{T}_a + (\eta_i^{(a)} + \eta_j^{(a+1)})^2 \mathbb{T}_{a+2} \right] \\ &+ \frac{1}{2} \Delta_\theta(\mathbb{T}) (-\theta^2)^{mn} \left[\left(\eta_i^{(a)} - \frac{1}{2} \right) \tau_j^{(a+1)} - \left(\eta_j^{(a+1)} + \frac{1}{2} \tau_i^{(a)} \right) \right] \\ &+ \frac{i}{4} \Delta_\theta(\mathbb{T}) (+\theta^3)^{mn} \left[\left(\eta_i^{(a)} + \frac{1}{2} \right) \left(\eta_j^{(a+1)} - \frac{1}{2} \right) + \frac{1}{3} \right] + \frac{i}{3} \theta^{mn}. \quad (4.16) \end{aligned}$$

Moreover, $G_{\theta ab}^{\text{sym}}$ are defined by replacing $\Delta(\text{T})$ by $\Delta_{\theta}(\text{T})$ in G_{ab}^{sym} .

A notable difference of the nonplanar case, as compared to the planar case, is that the worldline propagators comprise of $\mathcal{O}(\theta^3)$ contribution, and the contribution is independent of the Φ_0 -field insertion moduli. In Appendix B, we illustrate the contribution explicitly by computing several two-loop nonplanar Feynman diagrams.

5 Generalized Star Products and Open Wilson Lines

In the last section, we have computed systematically the two-loop contribution to the N-point, one-particle-irreducible Green functions in noncommutative $\lambda[\Phi^3]_{\star}$ -theory. We have adopted the worldline formulation for the computation in order to make contact with the Seiberg-Witten limit from the underlying string theories.

According to the conjecture [5], once resummed over N with appropriate combinatoric factors, the resulting two-loop effective action ought to be expressible in terms of open Wilson lines, viz. the effective action represents interactions among the noncommutative dipoles. A prerequisite to emergence of such structures is that the effective action ought to be a functional defined in terms of generalized \star -products of the elementary field Φ , much as in the one-loop effective action. There, the generalized \star -product was expressible in momentum-space representation in terms of the following kernel:

$$J_N := \int_0^1 \cdots \int_0^1 d\tau_1 \cdots d\tau_N \exp \left[-i \sum_{i=1}^N \tau_i p_i \wedge k + \frac{i}{2} \sum_{i < j=1}^N \epsilon(\tau_i - \tau_j) p_i \wedge p_j \right]. \quad (5.1)$$

In this section, for both the planar and the nonplanar contributions, we will indicate briefly how the generalized \star -products come about and in what sense they are resumable into open Wilson lines.

Begin with the the planar contribution, Eq.(4.11), corresponding in underlying string theory to the worldsheet of genus zero with three holes. Were if it reexpressible as representing cubic interaction among dipoles, each boundary of the hole ought to be identified with an open Wilson line whose size in spacetime is proportional to the total momentum inserted along the boundary. Thus, introduce the a -th boundary momentum as:

$$k_a = \sum_{i=1}^{N_a} \left(\frac{1}{2} - \eta_i^{(a)} \right) p_i^{(a)} + \sum_{i=1}^{N_{a+1}} \left(\frac{1}{2} + \eta_i^{(a+1)} \right) p_i^{(a+1)}, \quad (a = 1, 2, 3).$$

They constitute a direct two-loop counterpart of the momentum k in Eq.(5.1). Remarkably, we have found that the second exponential in Eq.(4.12) is simplified into a suggestive form:

$$\exp \left[\Delta(\text{T}) \sum_{a=1}^3 \left\{ T_{a+2} k_a \wedge \left(\sum_{i=1}^{N_{a+1}} \tau_i^{(a+1)} p_i^{(a+1)} - \sum_{i=1}^{N_a} \tau_i^{(a)} p_i^{(a)} \right) - \frac{1}{4} T_{a+2} k_a \circ k_a \right\} \right]$$

$$\rightarrow \prod_{a=1}^3 \exp \left[-i\Delta(\mathbb{T}) \sum_{i=1}^{N_a} \tau_i^{(a)} p_i^{(a)} \wedge (\mathbb{T}_{a+2} k_a - \mathbb{T}_a k_{a+1}) \right] \exp \left[-\frac{1}{4} \Delta(\mathbb{T}) \mathbb{T}_{a+2} k_a \circ k_a \right], \quad (5.2)$$

where we have first taken the low-energy and large noncommutativity limit, considered already in [9]. In the last expression, the phase-factor of the first exponential is readily seen as the counterpart of the first term in Eq.(5.1). Thus, combining the first exponential of Eq.(5.2) with the Ξ -factor and the first exponential in Eq.(4.12), we readily find that kernels reminiscent of Eq.(5.1) ought to follow. On the other hand, the second exponential of Eq.(5.2) is quadratic in $\theta^{mn} k_n$, and hence act as the Gaussian damping factor turning, for each a -th boundary, UV dynamics into IR dynamics. Results including these aspects will be reported in a separate paper elsewhere [15].

The nonplanar contribution, Eq.(4.13), also exposes new features. The contribution corresponds to, in underlying string theory, the worldsheet of genus one with a hole, yielding homology cycles with nontrivial intersections. As such, compared to the planar contribution Eq.(4.11) (for which there is no nontrivial homology cycle), all extra terms present in Eq.(4.13) would be attributable to the existence of the intersecting homology cycles [19]. An immediate question is whether the open Wilson lines play an important role — for example, whether the nonplanar contribution, Eq.(4.13), is also expressible entirely in terms of the open Wilson lines. If so, these homology cycles ought to describe ‘worldsheet’ of *virtual* open Wilson lines, and hence would disclose *quantum aspects* of the noncommutative dipoles in noncommutative field theories. We are currently investigating various aspects including these issues and will report them elsewhere.

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Appendix

A Derivation of Eq.(4.12) and Eq.(4.14)

As a starting point, we rewrite Eq.(4.10) and apply the change of variable $Y = X_a - x_1$ for the x_2 integration (the z and w dependences are retained for the nonplanar case). In what follows, we will omit the overall energy-momentum δ -function arising from

$$\prod_{a=1}^3 \prod_{j=1}^{N_a} \int d^d x_1 e^{ip_j^{(a)} x_1} = (2\pi)^d \delta \left(\sum_{a=1}^3 \sum_{j=1}^{N_a} p_j^{(a)} \right) .$$

Introduce functions f_a ; $a = 1, 2, 3$:

$$f_a(Y) = \exp \left[-\frac{1}{4T_a} Y^2 + iY \cdot q_a \right] \quad \text{where} \quad q_a^m = \sum_{j=1}^{N_a} \frac{1}{T_a} \left(\tau_j^{(a)} p_j^{(a)m} + \frac{i}{2} \theta^{mn} p_j^{(a)n} \varepsilon_j^{(a)} \right) .$$

Note that, for notational simplicity, we have suppressed dependence of f_a 's on the insertion moduli, $\tau_j^{(a)}$'s. Introduce also functions, H_0 and H_1 :

$$H_0 = \frac{i}{4} \sum_{a=1}^3 \sum_{k<l}^{N_a} p_k^{(a)} \wedge p_l^{(a)} \left(\nu_k^{(a)} + \nu_l^{(a)} \right) \epsilon \left(\tau_k^{(a)} - \tau_l^{(a)} \right) ,$$

and

$$\begin{aligned} H_1 &= \frac{1}{2} \sum_{a=1}^3 \sum_{i,j=1}^{N_a} p_i^{(a)} \cdot p_j^{(a)} \left(\left| \tau_i^{(a)} - \tau_j^{(a)} \right| - \left(\tau_i^{(a)} + \tau_j^{(a)} \right) + \frac{2\tau_i^{(a)} \tau_j^{(a)}}{T_a} \right) \\ &- \frac{i}{2} \sum_{a=1}^3 \sum_{i,j=1}^{N_a} p_i^{(a)} \wedge p_j^{(a)} \left(\varepsilon_j^{(a)} \left[1 - \frac{\tau_i^{(a)} - \tau_j^{(a)}}{T_a} \right] + \varepsilon_j^{(a)} \left[1 - \frac{\tau_i^{(a)} + \tau_j^{(a)}}{T_a} \right] \right) \\ &- \sum_{a=1}^3 \sum_{i,j=1}^{N_a} \frac{1}{4T_a} p_i^{(a)} \circ p_j^{(a)} \varepsilon_i^{(a)} \varepsilon_j^{(a)} . \end{aligned}$$

We were then able to express the N-point Green functions defined in Eqs.(4.11, 4.13) as

$$\Gamma_{(N_1, N_2, N_3)}^{\mathbf{P}} = \prod_{a=1}^3 \int_0^\infty dT_a \left(\frac{1}{4\pi T_a} \right)^{\frac{d}{2}} e^{-m^2 T_a} \int \prod_{i=1}^{T_a N_a} d\tau_i^{(a)} e^{H_0 + H_1} \int d^d Y f_1(Y) f_2(Y) f_3(Y) , \quad (\text{A.3})$$

and

$$\Gamma_{(N_1, N_2, N_3)}^{\mathbf{NP}} = \prod_{a=1}^3 \int_0^\infty dT_a \left(\frac{1}{4\pi T_a} \right)^{\frac{d}{2}} e^{-m^2 T_a} \int \prod_{i=1}^{T_a N_a} d\tau_i^{(a)} e^{H_0 + H_1} \int d^d Y (f_1(Y) \star f_2(Y)) f_3(Y) . \quad (\text{A.4})$$

A.1 Planar Case Eq.(4.12)

For the planar case, Eq.(A.3), integration over Y^m is expressible as:

$$\Gamma_{(N_1, N_2, N_3)}^n = \prod_{a=1}^3 \int_0^\infty dT_a \left(\frac{1}{4\pi T_a} \right)^{\frac{d}{2}} e^{-m^2 T_a} \int_0^{T_a} \prod_{i=1}^{N_a} d\tau_i^{(a)} e^{H_0} e^{H_1 + H_2} ,$$

where

$$H_2 = -\Delta(T) \sum_{a=1}^3 \sum_{b=1}^3 \sum_{j=1}^{N_a} \sum_{k=1}^{N_b} \frac{T_1 T_2 T_3}{T_a T_b} \left(p_j^{(a)} \cdot p_k^{(b)} \tau_j^{(a)} \tau_k^{(b)} + p_j^{(a)} \wedge p_k^{(b)} \tau_j^{(a)} \varepsilon_k^{(b)} - \frac{1}{4} p_j^{(a)} \circ p_k^{(b)} \varepsilon_j^{(a)} \varepsilon_k^{(b)} \right) .$$

The e^{H_0} part reproduces the first exponential in Eq.(4.12). The exponent of $e^{H_1 + H_2}$ is expandable explicitly as:

$$\begin{aligned} (H_1 + H_2) = & \frac{1}{2} \sum_{a=1}^3 \sum_{i,j=1}^{N_a} \left\{ p_i^{(a)} \cdot p_j^{(a)} G_{aa}^{\text{sym}}(\tau_i^{(a)}, \tau_j^{(a)}) - 2i p_i^{(a)} \wedge p_j^{(a)} \varepsilon_j^{(a)} \right. \\ & + 2i \Delta(T) p_i^{(a)} \wedge p_j^{(a)} (T_{a+1} + T_{a+2}) \tau_i^{(a)} \varepsilon_j^{(a)} \\ & \left. + \frac{1}{4} \Delta(T) p_i^{(a)} \circ p_j^{(a)} (T_{a+1} + T_{a+2}) (\varepsilon_i^{(a)} - \varepsilon_j^{(a)})^2 \right\} \\ & + \sum_{a=1}^3 \sum_{i=1}^{N_a} \sum_{j=1}^{N_{a+1}} \left\{ p_i^{(a)} p_j^{(a+1)} G_{aa+1}^{\text{sym}}(\tau_i^{(a)}, \tau_j^{(a+1)}) \right. \\ & - i \Delta(T) p_i^{(a)} \wedge p_j^{(a+1)} (\tau_i^{(a)} \varepsilon_j^{(a+1)} - \varepsilon_i^{(a)} \tau_j^{(a+1)}) T_{a+2} \\ & \left. - \frac{1}{4} \Delta(T) p_i^{(a)} \circ p_j^{(a+1)} [(\varepsilon_i^{(a)})^2 T_{a+1} + (\varepsilon_j^{(a+1)})^2 T_a + (\varepsilon_i^{(a)} + \varepsilon_j^{(a+1)})^2 T_{a+2}] \right\} \\ & - i \Delta(T) \sum_{i=1}^{N_3} \sum_{j=1}^{N_1} p_i^{(3)} \wedge p_j^{(1)} [(T_2 + T_3) \tau_j^{(1)} - T_2 \tau_i^{(3)}] \\ & - i \Delta(T) \sum_{i=1}^{N_2} \sum_{j=1}^{N_3} p_i^{(2)} \wedge p_j^{(3)} (T_3 \tau_i^{(2)} + T_2 \tau_j^{(3)}) - i \sum_{i=1}^{N_1} \sum_{j=1}^{N_3} p_i^{(1)} \wedge p_j^{(3)} \\ & + \frac{1}{2} \Delta(T) \sum_{i=1}^{N_1} \sum_{j=1}^{N_3} p_i^{(1)} \circ p_j^{(3)} \left[-(T_2 + T_3) \varepsilon_i^{(1)} - T_2 \varepsilon_j^{(3)} + \frac{1}{2} (T_2 + T_3) \right] \\ & + \frac{1}{2} \Delta(T) \sum_{i=1}^{N_2} \sum_{j=1}^{N_3} p_i^{(2)} \circ p_j^{(3)} \left[T_3 \varepsilon_i^{(2)} - T_2 \varepsilon_j^{(3)} + \frac{1}{2} (T_2 + T_3) \right] . \end{aligned} \quad (\text{A.5})$$

Here, the G_{ab}^{sym} propagators are precisely those defined in Eqs.(2.7, 2.8) and, at appropriate places, we have applied the following energy-momentum conservation identity:

$$\begin{aligned} & \sum_{a=1}^3 \sum_{i,j=1}^{N_a} p_i^{(a)} \theta^M p_j^{(a)} C_a (\tau_i^{(a)})^N = \\ & \sum_{a=1}^3 \sum_{i=1}^{N_a} \sum_{j=1}^{N_{a+1}} p_i^{(a)} \theta^M p_j^{(a+1)} \left(-C_a (\tau_i^{(a)})^N + (-1)^{M+1} C_{a+1} (\tau_j^{(a+1)})^N \right) , \end{aligned} \quad (\text{A.6})$$

where M and N are integer-valued, and the C_a ($a = 1, 2, 3$) are τ -independent constants. The second exponential in Eq.(4.12) is then obtained, after straightforward and tedious algebra, from reversing the insertion moduli as $\tau^{(a)} \rightarrow T_a - \tau^{(a)}$, applying the formula, Eq.(A.6), and rearranging them into cyclically symmetric form. Finally, the factor Ξ in Eq.(4.12) follows from moduli-independent phase-factors in Eq.(A.5).

A.2 Nonplanar Case Eq.(4.14)

For the nonplanar case Eq.(A.4), we'll need first to act the pseudo-differential operator $e^{-i\partial_z \wedge \partial_w}$ acting on f_a 's:

$$W \equiv f_1(Y) \star f_2(Y) = f_1(z^m - i\theta^{mn}\partial_w^n) f_2(w)|_{z=w=Y} .$$

We have computed this by taking the Fourier integral representation:

$$f_2(w) = \left(\frac{T_2}{\pi}\right)^{\frac{d}{2}} \int_{-\infty}^{\infty} d^d k e^{-T_2(k+q_2)^2} e^{-ik \cdot w} .$$

The result turns out expressible as

$$W = (T_2)^{\frac{d}{2}} (\det A)^{-1/2} e^{\frac{1}{4} J^T A^{-1} J} e^{-T_2 q_2^2} f_1(Y) ,$$

where A and J refer to $(d \times d)$ matrix and d -component vector, respectively:

$$A^{mn} = T_2 \delta^{mn} - \frac{1}{4T_1} (\theta^2)^{mn} ,$$

$$J^m = \left(-2T_2 q_2^m + i\theta^{mn} q_1^n\right) + \left(\frac{-\theta}{2T_1} - i\right)^{mn} Y^n \equiv j^m + C^{mn} Y^n . \quad (\text{A.7})$$

Inserting these expressions into Eq.(A.4), and then integrating over Y 's, we obtain

$$\Gamma_{(N_1, N_2, N_3)}^{NP} = \prod_{a=1}^3 \int_0^{\infty} dT_a \left(\frac{1}{4\pi T_a}\right)^{\frac{d}{2}} e^{-m^2 T_a} \int \prod_{i=1}^{T_a N_a} d\tau_i^{(a)} e^{H_0} e^{H_1 + H'_2} (\pi T_2)^{\frac{d}{2}} (\det A)^{-1/2} (\det \tilde{A})^{-1/2} , \quad (\text{A.8})$$

where the new exponent is

$$H'_2 = \frac{1}{4} j^T A^{-1} j + \tilde{J}^T \tilde{A}^{-1} \tilde{J} - T_2 q_2^2 ,$$

and \tilde{A} and \tilde{J} refer to newly combined $(d \times d)$ matrix and d -component vector, respectively:

$$\begin{aligned} \tilde{A} &= A_0 - \frac{1}{4} C^T A^{-1} C \quad \text{where} \quad A_0 = \frac{1}{4} (T_1^{-1} + T_3^{-1}) \mathbf{I} , \\ \tilde{J}^m &= \frac{1}{4} (C^T A^{-1} j)^m + \frac{i}{2} (q_1 + q_3)^m . \end{aligned} \quad (\text{A.9})$$

Using the relation

$$A\tilde{A} = A_0A - \frac{1}{4}CC^T = (4T_1T_3\Delta_\theta)^{-1}\mathbf{I}$$

and the fact that the matrices A , A_0 and C commute one another, we have derived the following algebraic identities:

$$A^{-1}\left(1 + T_1T_3C\Delta_\theta(\mathbb{T})C^T\right) = (T_1 + T_3)\Delta_\theta(\mathbb{T}) ,$$

$$\tilde{A}^{-1} = 4T_1T_2T_3\Delta_\theta(\mathbb{T}) \left(1 - \frac{\theta^2}{4T_1T_2}\right) , \quad (\text{A.10})$$

$$(\det A)(\det \tilde{A}) = (4T_1T_3\Delta_\theta)^{-d} . \quad (\text{A.11})$$

In fact, due to Eq.(A.11), the weight factor in Eq.(A.8) is proportional to $(\Delta_\theta(\mathbb{T}))^{\frac{d}{2}}$, viz.

$$\left(\prod_{a=1}^3 \frac{1}{4\pi T_a}\right)^{\frac{d}{2}} (\pi T_2)^{\frac{d}{2}} (\det A)^{-1/2} (\det \tilde{A})^{-1/2} = \frac{1}{(4\pi)^d} (\Delta_\theta(\mathbb{T}))^{\frac{d}{2}} .$$

Using Eqs.(A.10), (A.10) and (A.6), the exponent $(H_1 + H'_2)$ of the second exponential is expandable explicitly as follows:

$$\begin{aligned} (H_1 + H'_2) &= \frac{1}{2} \sum_{a=1}^3 \sum_{i,j=1}^{N_a} \left\{ p_i^{(a)} \cdot p_j^{(a)} \left[|\tau_i^{(a)} - \tau_j^{(a)}| - \Delta_\theta(\mathbb{T})(T_{a+1} + T_{a+2}) (\tau_i^{(a)} - \tau_j^{(a)})^2 \right] \right. \\ &\quad \left. - \frac{1}{4} p_i^{(a)} \theta^2 p_j^{(a)} \Delta_\theta(\mathbb{T})(T_{a+1} + T_{a+2}) (\varepsilon_i^{(a)} - \varepsilon_j^{(a)})^2 \right\} \\ &+ \sum_{a=1}^3 \sum_{i=1}^{N_a} \sum_{j=1}^{N_{a+1}} \left\{ p_i^{(a)} \cdot p_j^{(a+1)} \left[(\tau_i^{(a)} + \tau_j^{(a+1)}) \right. \right. \\ &\quad \left. \left. - \Delta_\theta(\mathbb{T}) \left(2T_{a+2}\tau_i^{(a)}\tau_j^{(a+1)} + (T_{a+1} + T_{a+2})(\tau_i^{(a)})^2 + (T_{a+2} + T_a) (\tau_j^{(a+1)})^2 \right) \right] \right. \\ &\quad \left. - \frac{1}{4} p_i^{(a)} \theta^2 p_j^{(a+1)} \Delta_\theta(\mathbb{T}) \left[(\varepsilon_i^{(a)})^2 T_{a+1} + (\varepsilon_j^{(a+1)})^2 T_a + (\varepsilon_i^{(a)} + \varepsilon_j^{(a+1)})^2 T_{a+2} \right] \right. \\ &\quad \left. - \frac{1}{2} p_i^{(a)} \theta^2 p_j^{(a+1)} \Delta_\theta(\mathbb{T}) \left[\tau_i^{(a)} \varepsilon_j^{(a+1)} - \varepsilon_i^{(a)} \tau_j^{(a+1)} \right] \right\} \\ &- i \sum_{a=1}^3 \sum_{ij}^{N_a} p_i^{(a)} \wedge p_j^{(a)} \varepsilon_j^{(a)} - i \sum_{i=1}^{N_1} \sum_{j=1}^{N_3} p_i^{(1)} \wedge p_j^{(3)} \\ &+ i \Delta_\theta(\mathbb{T}) \sum_{a=1}^3 \sum_{ij}^{N_a} p_i^{(a)} \wedge p_j^{(a)} (T_{a+1} + T_{a+2}) \tau_i^{(a)} \varepsilon_j^{(a)} \\ &- i \Delta_\theta(\mathbb{T}) \sum_{a=1}^3 \sum_{i=1}^{N_a} \sum_{j=1}^{N_{a+1}} p_i^{(a)} \wedge p_j^{(a+1)} \left[T_{a+2} (\tau_i^{(a)} \varepsilon_j^{(a+1)} - \varepsilon_i^{(a)} \tau_j^{(a+1)}) - \tau_i^{(a)} \tau_j^{(a+1)} \right] \\ &+ i \Delta_\theta(\mathbb{T}) \sum_{i=1}^{N_1} \sum_{j=1}^{N_3} p_i^{(1)} \wedge p_j^{(3)} (T_2 + T_3) \tau_i^{(1)} + i \Delta_\theta(\mathbb{T}) \sum_{i=1}^{N_3} \sum_{j=1}^{N_2} p_i^{(3)} \wedge p_j^{(2)} T_3 \tau_j^{(2)} \end{aligned}$$

$$\begin{aligned}
& -i\Delta_\theta(\mathbb{T}) \sum_{i=1}^{N_3} \sum_{j=1}^{N_3} p_i^{(3)} \wedge p_j^{(3)} \mathbb{T}_2 \tau_i^{(3)} + \frac{i}{4} \Delta_\theta(\mathbb{T}) \sum_{a=1}^3 \sum_{i=1}^{N_a} \sum_{j=1}^{N_{a+1}} p_i^{(a)} \theta^3 p_j^{(a+1)} \varepsilon_i^{(a)} \varepsilon_j^{(a+1)} \\
& + \frac{1}{2} \Delta_\theta(\mathbb{T}) \sum_{i=1}^{N_1} \sum_{j=1}^{N_3} p_i^{(1)} \circ p_j^{(3)} \left[-(\mathbb{T}_2 + \mathbb{T}_3) \varepsilon_i^{(1)} - \mathbb{T}_2 \varepsilon_j^{(3)} + \frac{1}{2} (\mathbb{T}_2 + \mathbb{T}_3) \right] \\
& + \frac{1}{2} \Delta_\theta(\mathbb{T}) \sum_{i=1}^{N_2} \sum_{j=1}^{N_3} p_i^{(2)} \circ p_j^{(3)} \left[\mathbb{T}_3 \varepsilon_i^{(2)} - \mathbb{T}_2 \varepsilon_j^{(3)} + \frac{1}{2} (\mathbb{T}_2 + \mathbb{T}_3) \right] \\
& - \frac{1}{2} \Delta_\theta(\mathbb{T}) \left(\sum_{i=1}^{N_3} \sum_{j=1}^{N_2} p_i^{(3)} \circ p_j^{(2)} \tau_j^{(2)} - \sum_{i=1}^{N_3} \sum_{j=1}^{N_3} p_i^{(3)} \circ p_j^{(3)} \tau_i^{(3)} \right) \\
& + \frac{i}{4} \left(\Delta_\theta(\mathbb{T}) \sum_{i=1}^{N_3} \sum_{j=1}^{N_2} p_i^{(3)} \theta^3 p_j^{(2)} \varepsilon_j^{(2)} + \sum_{i=1}^{N_3} \sum_{j=1}^{N_3} p_i^{(3)} \theta^3 p_j^{(3)} \varepsilon_i^{(3)} \right). \tag{A.12}
\end{aligned}$$

Again, after straightforward and tedious algebra, we can regroup terms into cyclically symmetric combinations and obtain the second exponential and Ξ -factor of Eq.(4.14), where the worldline correlators are given by

$$\begin{aligned}
G_{aa}^{\text{NP}^{mn}} \left(\tau_i^{(a)}, \tau_j^{(a)}; \eta_i^{(a)}, \eta_j^{(a)} \right) &= g^{mn} G_{\theta aa}^{\text{sym}} \left(\tau_i^{(a)}, \tau_j^{(a)} \right) - 2i \eta_j^{(a)} \theta^{mn} \\
&+ 2i \Delta_\theta(\mathbb{T}) \theta^{mn} (\mathbb{T}_{a+1} + \mathbb{T}_{a+2}) \tau_i^{(a)} \eta_j^{(a)} \\
&+ \frac{1}{4} \Delta_\theta(\mathbb{T}) (-\theta^2)^{mn} (\mathbb{T}_{a+1} + \mathbb{T}_{a+2}) \left(\eta_i^{(a)} - \eta_j^{(a)} \right)^2,
\end{aligned}$$

and

$$\begin{aligned}
G_{aa+1}^{\text{NP}^{mn}} \left(\tau_i^{(a)}, \tau_j^{(a+1)}; \eta_i^{(a)}, \eta_j^{(a+1)} \right) &= g^{mn} G_{\theta aa+1}^{\text{sym}} \left(\tau_i^{(a)}, \tau_j^{(a+1)} \right) + \frac{i}{3} \theta^{mn} \\
&+ i \Delta_\theta(\mathbb{T}) \theta^{mn} \left[\mathbb{T}_{a+2} \left(\tau_i^{(a)} \eta_j^{(a+1)} - \eta_i^{(a)} \tau_j^{(a+1)} \right) + \frac{1}{2} \mathbb{T}_{a+1} \tau_i^{(a)} + \frac{1}{2} \mathbb{T}_a \tau_j^{(a+1)} - \tau_i^{(a)} \tau_j^{(a+1)} \right] \\
&+ \frac{1}{4} \Delta_\theta(\mathbb{T}) (-\theta^2)^{mn} \left[\left(\eta_i^{(a)} + \frac{1}{2} \right)^2 \mathbb{T}_{a+1} + \left(\eta_j^{(a+1)} - \frac{1}{2} \right)^2 \mathbb{T}_a + \left(\eta_i^{(a)} + \eta_j^{(a+1)} \right)^2 \mathbb{T}_{a+2} \right] \\
&- \frac{1}{2} \Delta_\theta(\mathbb{T}) (-\theta^2)^{mn} \left[\left(\eta_i^{(a)} + \frac{1}{2} \right) \tau_j^{(a+1)} - \tau_i^{(a)} \left(\eta_j^{(a+1)} - \frac{1}{2} \right) \right] \\
&+ \frac{i}{4} \Delta_\theta(\mathbb{T}) (\theta^3)^{mn} \left[\left(\eta_i^{(a)} + \frac{1}{2} \right) \left(\eta_j^{(a+1)} - \frac{1}{2} \right) + \frac{1}{3} \right].
\end{aligned}$$

while $G_{\theta ab}^{\text{sym}}$ are defined by replacing $\Delta(\mathbb{T})$ by $\Delta_\theta(\mathbb{T})$ in G_{ab}^{sym} . Note that the above expression of the worldline correlators are related to Eqs.(4.15, 4.16) by the the worldline moduli inversion, $\tau^{(a)} \rightarrow \mathbb{T}_a - \tau^{(a)}$.

B Nonplanar Feynman diagrams

In this appendix, we independently check our worldline computations, especially the existence of $\mathcal{O}(\theta^3)$ effects in the nonplanar case. While these terms are invisible in the N=2 Green

functions and hence left unnoticed in earlier work [20], they begin to be visible for $N \geq 3$ Green function. For simplicity in illustrating this point, we will consider a massless $\lambda[\Phi^3]_\star$ -theory.

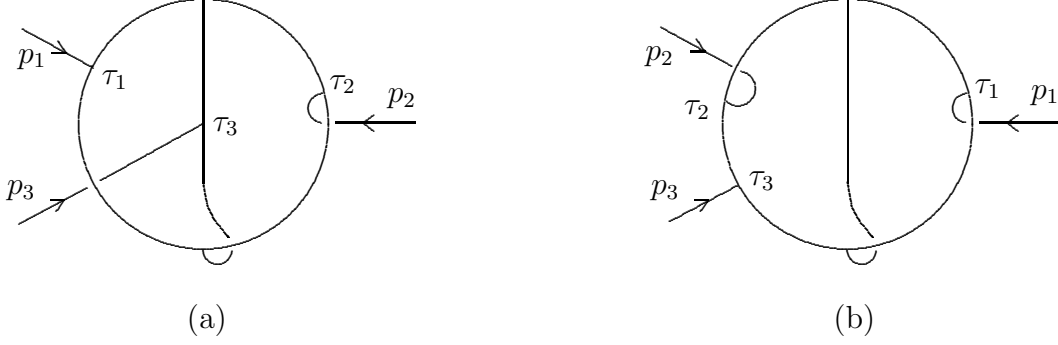


Figure 6: Nonplanar Feynman diagrams computed in Appendix B.

The first example is the case of Fig. 6(a), where all η^a values are equal to $-1/2$, and the overall Filk's phase-factor is equal to $e^{\frac{i}{2}p_1 \wedge p_2}$:

$$\Lambda_1 = e^{\frac{i}{2}p_1 \wedge p_2} \int \frac{d^d k_1}{(2\pi)^d} \frac{d^d k_2}{(2\pi)^d} \frac{d^d k_3}{(2\pi)^d} \frac{\delta(k_1 + k_2 + k_3) e^{i(k_1 + p_1) \wedge p_3} e^{-ik_2 \wedge p_2} e^{i(k_2 + p_2) \wedge (k_3 + p_3)}}{k_1^2 (k_1 + p_1)^2 k_2^2 (k_2 + p_2)^2 k_3^2 (k_3 + p_3)^2} .$$

Introducing d -dimensional Y^m integrations for the δ -function and the parametric integrals such as

$$\frac{1}{k_1^2 (k_1 + p_1)^2} = \int_0^\infty dT_1 \int_0^{T_1} d\tau_1 \exp \left[-\tau_1 k_1^2 - (T_1 - \tau_1) (k_1 + p_1)^2 \right] ,$$

we perform k_a ; $a = 1, 2, 3$, and Y integrals in due course. After applying the changes of variables $\tau_a \rightarrow T_a - \tau_a$, we then have

$$\Lambda_1 = \frac{1}{(4\pi)^d} \left(\prod_{a=1}^3 \int_0^\infty dT_a \int_0^{T_a} d\tau_a \right) (\Delta_\theta(T))^{d/2} \exp \left[\tilde{K}^T \tilde{A}^{-1} \tilde{K} + \frac{1}{4} K^T A^{-1} K + \tilde{H}_1 \right] , \quad (\text{B.1})$$

with

$$\tilde{K} = \frac{1}{4} (C^T A^{-1} K) + \frac{i}{2} Q , \quad (\text{B.2})$$

$$K = i\theta \cdot p_3 + 2\tau_3 p_3 - i \frac{\tau_2}{T_2} \theta \cdot p_2 , \quad (\text{B.3})$$

$$Q = -\frac{\tau_1}{T_1} p_1 - \frac{\tau_2}{T_2} p_2 + \frac{i}{2T_1} \theta \cdot p_2 ,$$

$$\begin{aligned} \tilde{H}_1 = & -p_1^2 \left(\tau_1 - \frac{\tau_1^2}{T_1} \right) - p_2^2 \left(\tau_2 - \frac{\tau_2^2}{T_2} \right) - p_3^2 \tau_3 \\ & - i \frac{\tau_1}{T_1} p_1 \wedge p_2 - \frac{1}{4T_1} p_2 \circ p_2 + \frac{i}{2} p_1 \wedge p_2 , \end{aligned}$$

where A and C are defined in Eq.(A.7), modified by the cyclic permutation $T_1 \rightarrow T_2 \rightarrow T_3$. Imposing the momentum conservation; $p_1 + p_2 + p_3 = 0$, we can rearrange the exponent in Eq.(B.1) in the following form

$$\exp \left[-\frac{i}{2} p_1 \wedge p_2 \right] \exp \left[\sum_{a=1}^3 p_a^m p_{a+1}^n \mathcal{G}_{aa+1}^{mn}(\tau_a, \tau_{a+1}) \right] ,$$

where

$$\begin{aligned} \mathcal{G}_{aa+1}^{mn}(\tau_a, \tau_{a+1}) &= g^{mn} G_{\theta aa+1}^{\text{sym}}(\tau_a, \tau_{a+1}) - \frac{1}{4} \Delta_\theta(\mathbb{T}) (-\theta^2)^{mn} (\mathbb{T}_{a+1} + \mathbb{T}_{a+2} - 2\tau_{a+1}) \\ &\quad - \frac{i}{2} \Delta_\theta(\mathbb{T}) \theta^{mn} \left[T_{a+2}(\tau_a - \tau_{a+1}) + \mathbb{T}_{a+1} \tau_a + \mathbb{T}_a \tau_{a+1} - 2\tau_a \tau_{a+1} \right] \\ &\quad + \frac{i}{3} \left(\frac{1}{4} \Delta_\theta(\mathbb{T}) \theta^3 + \theta \right)^{mn} . \end{aligned}$$

The functions \mathcal{G}_{aa+1} ; $a = 1, 2, 3$, coincide with the quantity Eq.(4.16) when setting $\eta^{(a)} = -\frac{1}{2}$, and the extra exponential factor is nothing but Ξ define in Eq.(4.4).

In a similar fashion, we calculate the second diagram (Fig. 6(b)), where the overall Filk's phase-factor amounts to $e^{-\frac{i}{2} p_1 \wedge p_2}$ and the assignments of external variables are $p_1^{(1)} = p_2$, $p_2^{(1)} = p_3$, $p_1^{(2)} = p_1$, $\eta_1^{(1)} = +\frac{1}{2}$, $\eta_2^{(1)} = -\frac{1}{2}$, $\eta_1^{(2)} = -\frac{1}{2}$:

$$\begin{aligned} \Lambda_2 &= e^{-\frac{i}{2} p_1 \wedge p_2} \int \frac{d^d k_1}{(2\pi)^d} \frac{d^d k_2}{(2\pi)^d} \frac{d^d k_3}{(2\pi)^d} \frac{\delta(k_1 + k_2 + k_3) e^{ik_1 \wedge p_2} e^{-ik_2 \wedge p_1} e^{i(k_2 + p_1) \wedge k_3}}{k_1^2 (k_1 + p_2)^2 (k_1 + p_2 + p_3)^2 k_3^2 k_2^2 (k_2 + p_1)^2} \\ &= \frac{1}{(4\pi)^d} \int_0^\infty d\mathbb{T}_3 \int_0^\infty d\mathbb{T}_2 \int_0^{\mathbb{T}_2} d\tau_2 \int_0^\infty d\mathbb{T}_1 \int_0^{\mathbb{T}_1} d\tau_1 \int_0^{\tau_2} d\tau_1 (\Delta_\theta(\mathbb{T}))^{\frac{d}{2}} \\ &\quad \times \exp \left[\tilde{L}^T \tilde{A}^{-1} \tilde{L} + \frac{1}{4} L^T A^{-1} L + \mathcal{H}_1 \right] , \end{aligned} \tag{B.4}$$

where A is again the permuted one (as well as C), and

$$\begin{aligned} \tilde{L} &= \frac{1}{4} (C^T A^{-1} L) + \frac{i}{2} \mathcal{Q} , \\ L &= i \frac{\tau_1}{\mathbb{T}_2} \theta \cdot p_1 + \frac{1}{2\mathbb{T}_2} \theta^2 \cdot p_1 , \\ \mathcal{Q} &= \left(\frac{\tau_1}{\mathbb{T}_2} - \frac{i\theta}{2\mathbb{T}_2} \right) p_1 + \left(\frac{\tau_2}{\mathbb{T}_1} + \frac{i\theta}{2\mathbb{T}_1} \right) p_2 + \frac{\tau_3}{\mathbb{T}_1} p_3 , \\ \mathcal{H}_1 &= -p_1^2 \left(\tau_1 - \frac{\tau_1^2}{\mathbb{T}_2} \right) - p_2^2 \left(\tau_2 - \frac{\tau_2^2}{\mathbb{T}_1} \right) - p_3^2 \left(\tau_3 - \frac{\tau_3^2}{\mathbb{T}_1} \right) - 2p_2 p_3 \left(\tau_2 - \frac{\tau_2 \tau_3}{\mathbb{T}_1} \right) \\ &\quad + i \left(1 - \frac{\tau_3}{\mathbb{T}_1} \right) p_1 \wedge p_2 - \frac{1}{4\mathbb{T}_1} p_2 \circ p_2 - \frac{1}{4\mathbb{T}_2} p_1 \circ p_1 - \frac{i}{2} p_1 \wedge p_2 . \end{aligned}$$

Imposing the overall energy-momentum conservation, and changing the moduli variables $\tau_2 \rightarrow \mathbb{T}_2 - \tau_2$; $\tau_i \rightarrow \mathbb{T}_1 - \tau_i$; $i = 2, 3$, we can rearrange the exponential in Eq.(B.4) as

$$\exp \left[-\frac{i}{2} p_1 \wedge p_2 \right] \exp \left[\frac{1}{2} \sum_{i,j=2}^3 p_i \cdot \mathcal{G}_{11}(\tau_i, \tau_j) \cdot p_j + \sum_{j=2}^3 p_j \cdot \mathcal{G}_{12}(\tau_j, \tau_1) \cdot p_1 \right] , \tag{B.5}$$

where

$$\begin{aligned}
\mathcal{G}_{11}^{mn}(\tau_2, \tau_3) &= g^{mn} G_{\theta 11}^{\text{sym}}(\tau_2, \tau_3) + i\theta^{mn}(\mathbb{T}_2 + \mathbb{T}_3)\tau_2 + \frac{1}{4}\Delta_\theta(\mathbb{T})(-\theta^2)^{mn}(\mathbb{T}_2 + \mathbb{T}_3) - \frac{i}{4}\Delta_\theta(\mathbb{T})(\theta^3)^{mn}, \\
\mathcal{G}_{11}^{mn}(\tau_3, \tau_2) &= g^{mn} G_{\theta 11}^{\text{sym}}(\tau_2, \tau_3) - i\theta^{mn}(\mathbb{T}_2 + \mathbb{T}_3)\tau_3 + \frac{1}{4}\Delta_\theta(\mathbb{T})(-\theta^2)^{mn}(\mathbb{T}_2 + \mathbb{T}_3) + \frac{i}{4}\Delta_\theta(\mathbb{T})(\theta^3)^{mn}, \\
\mathcal{G}_{12}^{mn}(\tau_2, \tau_1) &= g^{mn} G_{\theta 12}^{\text{sym}}(\tau_2, \tau_1) - \frac{i}{2}\Delta_\theta(\mathbb{T})\theta^{mn} \left[\mathbb{T}_3(\tau_2 + \tau_1) + \mathbb{T}_2\tau_2 + \mathbb{T}_1\tau_1 - 2\tau_2\tau_1 \right] \\
&\quad + \frac{i}{3} \left(\theta - \frac{1}{2}\Delta_\theta(\mathbb{T})\theta^3 \right)^{mn}, \\
\mathcal{G}_{12}^{mn}(\tau_3, \tau_1) &= g^{mn} G_{\theta 12}^{\text{sym}}(\tau_3, \tau_1) - \frac{i}{2}\Delta_\theta(\mathbb{T})\theta^{mn} \left[\mathbb{T}_3(\tau_3 - \tau_1) + \mathbb{T}_2\tau_3 + \mathbb{T}_1\tau_1 - 2\tau_3\tau_1 \right] \\
&\quad + \frac{1}{4}\Delta_\theta(\mathbb{T})(-\theta^2)^{mn}(\mathbb{T}_2 + \mathbb{T}_3 - 2\tau_1) + \frac{i}{3} \left(\theta - \frac{1}{4}\Delta_\theta(\mathbb{T})\theta^3 \right)^{mn}.
\end{aligned}$$

These quantities coincide with Eqs.(4.15, 4.16) for the present values of $\eta^{(a)}$ mentioned above. The extra factor appearing in Eq.(B.5) is nothing but Ξ defined by Eq.(4.4).

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