A Note on Wandzura-Wilczek Relations

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Wandzura-Wilczek (WW) relations between matrix-elements of bilocal light-ray operators have recently regained interest in connection with off-forward scattering processes. Originally derived for matrix elements over leading-twist operators, their generalization to off-forward and exclusive processes gets complicated by the presence of higher-twist operators that are total derivatives of leading-twist ones and do not contribute to forward-scattering. We demonstrate that, for exclusive matrix-elements, the inclusion of these operators into WW-relations is essential for fulfilling constraints imposed by the conformal symmetry of massless QCD.

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The success of QCD as fundamental theory of strong interactions is intimately tied to its ability to describe hard exclusive and inclusive reactions, which has been tested in numerous experiments. In the corresponding kinematic regime, i.e. at large space-like virtualities, the relevant amplitudes are dominated by singularities on the light-cone, which, in the framework of a light-cone expansion [1], can be described in terms of contributions of definite twist. The notion of twist, for local operators, has been introduced originally by Gross and Treiman [2] as twist = dimension − spin; it uses the irreducible representations of the orthochronous Lorentz-group and, as such, is a Lorentz-invariant concept; its generalization to non-local operators has been derived, in a mathematical rigorous way, in [3, 4] for tensor operators up to rank 2. This "geometric" twist relies solely on geometry, i.e. the properties of space-time, and is independent of the dynamics of any underlying quantum field theory. An alternative approach to twist-counting in hard reactions is based on the light-cone quantization formalism [5]: quark fields ψ are decomposed into "good" and "bad" components, so that $\psi = \psi_+ + \psi_-$, with $\psi_+ = \frac{1}{2}\hat{p}\hat{z}\psi$ and $\psi_- = \frac{1}{2}\hat{z}\hat{p}\psi$ ($\hat{a} = a^\mu\gamma_\mu$ for arbitrary 4-vectors a; p and z are light-like vectors with $p \cdot z = 1$; z defines the light-cone). As discussed in [6], a "bad" component ψ_- introduces one unit of twist; the physical interpretation of that "dynamical" twist is that, in the infinite momentum frame, it counts the powers of $1/Q$, with which the corresponding matrix-elements appear in physical scattering amplitudes; albeit being convenient, it is not a Lorentz-invariant concept and does not agree with the geometric twist. The mismatch between dynamical and geometric twist becomes relevant once power-suppressed higher-twist contributions are included and leads to so-called Wandzura-Wilczek (WW) relations between matrix-elements of operators of different dynamical, but identical geometric twist, the prototype of which has been derived by Wandzura and Wilczek for the nucleon distribution functions (DFs) g_1 and g_2 [7]. A systematic study of WW-relations in forward-scattering has been done in Ref. [8]. The decomposition into operators of definite geometric twist has also been exploited by Nachtmann to calculate exactly target-mass corrections to Bjorken-scaling in deep-inelastic forward-scattering [9]. The successes of the purely geometric reasoning in applications to forward-scattering matrix-elements has prompted several authors to use it also for off-forward processes [10] and exclusive parton distribution amplitudes (DAs)[11, 12]. The purpose of this letter is to argue that, despite its apparent success in disentangling leading from higher-twist contributions to forward-scattering matrix-elements, the generalization of WW-relations to the non-forward and exclusive case requires the inclusion of operators of higher twist. These operators arise from the dynamics of the underlying quantum field theory or, more precisely, the equations of motion (EOM) and generate total translations. Their relevance for preserving gauge-invariance in deeply-virtual Compton-scattering has been discussed recently in [13]; in this letter we demonstrate that, for exclusive processes, the inclusion of these operators is essential for fulfilling the constraints posed by the dynamical symmetry of the theory, i.e. the invariance under collinear conformal transformations in the case of massless QCD on the light-cone.

We center our discussion around the specific case of light vector-meson DAs of dynamical twist-3, which are relevant for describing light-cone dominated processes involving vector mesons like e.g. the DIS-exclusive process $\gamma^* + N \to V + N$ and can be expressed in terms of matrix-elements of gauge-invariant non-local operators sandwiched between the vacuum and the meson state, $\langle 0|\bar{u}(x)\Gamma[x,-x]d(-x)|\rho^-(P)\rangle$, where Γ is a generic Dirac-matrix structure and $[x,y]$ denotes the path-ordered gauge-factor along the straight line connecting the points

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x and y. Specifying to chiral-even DAs, one can decompose the relevant vector and axial-vector matrix-elements on the light-cone $z^2 = 0$ as [11, 12]

$$
\begin{aligned}
\langle 0|\bar u(z)\gamma_\mu[z,-z]d(-z)|\rho^-(P,\lambda)\rangle \; &= \; f_\rho m_\rho \int_{-1}^{1} d\xi \Big[p_\mu \frac{e^{(\lambda)}\cdot z}{p\cdot z}\hat\Phi^{(2)}(\xi) e_0(i\zeta\xi) \\
&+ e^{(\lambda)}_{\perp\mu}\Big\{\hat\Phi^{(2)}(\xi)e_1(i\zeta\xi) + \hat\Phi^{(3)}(\xi)[e_0(i\zeta\xi) - e_1(i\zeta\xi)]\Big\} - \frac{1}{2} z_\mu \frac{e^{(\lambda)}\cdot z}{(p\cdot z)^2} m_\rho^2 \Big\{ \hat\Phi^{(4)}(\xi)\Big[e_0(i\zeta\xi) - 3e_1(i\zeta\xi) + 2\int_0^1 dt\, e_1(i\zeta\xi t)\Big] \\
&- \hat\Phi^{(2)}(\xi)\Big[e_1(i\zeta\xi) - 2\int_0^1 dt\, e_1(i\zeta\xi t)\Big] + 4\hat\Phi^{(3)}\Big[e_1(i\zeta\xi) - \int_0^1 dt\, e_1(i\zeta\xi t)\Big]\Big\}\Big], \\
\langle 0|\bar u(z)\gamma_\mu\gamma_5[z,-z]d(-z)|\rho^-(P,\lambda)\rangle \; &= \; \frac{1}{2} f_\rho m_\rho \epsilon_\mu^{\ \nu\alpha\beta} e^{(\lambda)}_{\perp\nu} p_\alpha z_\beta \int_{-1}^{1} d\xi\, \hat\Xi^{(3)}(\xi) e_0(i\zeta\xi),
\end{aligned}
$$

where $\hat\Phi^{(d)}$ and $\hat\Xi^{(d)}$ contain only contributions from geometric twist-d and we use the following abbreviations:

$$
e^{(\lambda)}_{\perp\mu} = e^{(\lambda)}_\mu - p_\mu \frac{e^{(\lambda)}\cdot z}{p\cdot z}, \qquad \zeta = p\cdot z, \qquad e_0(i\zeta\xi) = e^{i\zeta\xi}, \qquad e_1(i\zeta\xi) = \int_0^1 dt\, e^{i\zeta\xi t}.
$$

The same matrix-elements can also be expressed in terms of contributions of definite dynamical twist as [14, 15]

$$
\langle 0|\bar u(z)\gamma_\mu[z,-z]d(-z)|\rho^-(P,\lambda)\rangle \; = \; f_\rho m_\rho \int_{-1}^{1} d\xi\, e^{i\xi p\cdot z}\left[p_\mu \frac{e^{(\lambda)}\cdot z}{p\cdot z}\hat\phi_{||}(\xi) + e^{(\lambda)}_{\perp\mu}\hat g^{(v)}_\perp(\xi) - \frac{1}{2} z_\mu \frac{e^{(\lambda)}\cdot z}{(p\cdot z)^2} m_\rho^2\, \hat g_3(\xi)\right], \tag{1}
$$

$$
\langle 0|\bar u(z)\gamma_\mu\gamma_5[z,-z]d(-z)|\rho^-(P,\lambda)\rangle \; = \; \frac{1}{2} f_\rho m_\rho \epsilon_\mu^{\ \nu\alpha\beta} e^{(\lambda)}_{\perp\nu} p_\alpha z_\beta \int_{-1}^{1} d\xi\, e^{i\xi p\cdot z}\hat g^{(a)}_\perp(\xi); \tag{2}
$$

$\hat\phi_{||}$ has twist-2, $\hat g_\perp^{(v,a)}$ (dynamical) twist-3 and $\hat g_3$ has dynamical twist-4. The relation between the two sets of DAs is given by [12]

$$
\begin{aligned}
\hat\phi_{||}(\xi) \; &\equiv \; \hat\Phi^{(2)}(\xi), \\
\hat g^{(v)}_\perp(\xi) \; &= \; \hat\Phi^{(3)}(\xi) + \int_\xi^{\mathrm{sign}(\xi)} \frac{d\omega}{\omega}\left(\hat\Phi^{(2)} - \hat\Phi^{(3)}\right)(\omega), \\
\hat g_3(\xi) \; &= \; \hat\Phi^{(4)}(\xi) - \int_\xi^{\mathrm{sign}(\xi)} \frac{d\omega}{\omega}\Big\{\left(\hat\Phi^{(2)} - 4\hat\Phi^{(3)} + 3\hat\Phi^{(4)}\right)(\omega) \\
&\quad + 2\ln\left(\frac{\xi}{\omega}\right)\left(\hat\Phi^{(2)} - 2\hat\Phi^{(3)} + \hat\Phi^{(4)}\right)(\omega)\Big\}, \\
\hat g^{(a)}_\perp(\xi) \; &\equiv \; \hat\Xi^{(3)}(\xi).
\end{aligned} \tag{3}
$$

Similar relations can be derived for chiral-odd DAs over the tensor and pseudoscalar currents.

At this point we do observe a one-to-one correspondence between the decomposition in terms of geometric twist DAs and DAs in dynamical twist: there are four functions each. A difference does occur, however, as soon as we include information on the dynamics of the theory. This information is twofold, and is encoded in the dynamical symmetries of the theory on the one hand and the equations of motion (EOM) on the other hand. As for massless QCD at light-like distances, the relevant symmetry is the invariance under collinear conformal transformations, i.e. the group SL(2,R)$\cong$SO(2,1), which is exact for the free theory and valid to leading order in the perturbative expansion [16]; for not too small renormalization scales, the corresponding quantum number "conformal spin", defined as 1/2 (dimension + spin projection onto the line z_μ), is thus a good quantum number and allows a partial-wave expansion of the corresponding amplitudes [15]; in the limit $\alpha_s \to 0$ one obtains the so-called asymptotic DAs, which are defined as the contribution with lowest conformal spin. Theoretical calculations of the non-asymptotic corrections to the ρ-meson DAs show that they are small already at scales ~ 1 GeV [14]. The EOM, on the other hand, allow one to establish relations between e.g. bilinear operators of higher twist and trilinear operators of leading twist and serve to identify the dynamically independent degrees of freedom of a given DA [15]. In particular it turns out that the basis of higher-twist DAs is overcomplete: the number of independent degrees of freedom is less than the number of independent Lorentz-structures. This observation is of course not new; to the best of our knowledge, the EOM have first been employed by Shuryak and Vainshtein [17] in relation with the WW-decomposition [7] of the (dynamical twist-3) spin-dependent nucleon DF g_2,

$$g_2(x_B) = -g_1(x_B) + \int_{x_B}^1 \frac{dy}{y}\, g_1(y) + \bar{g}_2(x_B), \qquad (4)$$

where g_1 is the leading-twist longitudinal spin DF and $\bar{g}_2$ is the geometric twist-3 part. The original derivation of Wandzura & Wilczek involved only quark-quark operators, but Shuryak and Vainshtein noted that, by virtue of the EOM, the operators relevant for $\bar{g}_2$ can be written as quark-quark-gluon operators whose matrix-elements are expected to be small and thus can be neglected.

In general, however, the EOM do not only involve quark-quark-gluon operators, but also quark-quark operators with total derivatives, schematically $O^{(3)} = \tilde{O}^{(3),qqG} + \hat{\partial}^\mu O^{(2)}_\mu$, where $\tilde{O}^{(3),qqG}$ is an interaction-dependent operator and $O^{(2,3)}$ are quark-quark-correlation operators of geometric twist-2 and 3, respectively. The important point is now that, although the above relation of course respects the geometric twist, matrix-elements can blur this decomposition, which happens whenever the matrix-elements over total derivatives do not vanish, i.e. for exclusive processes and off-forward scattering, where the total derivative turns into a momentum (transfer). To be specific, let us quote the formulas for the geometric twist-3 part of the vector and axial-vector currents [14, 18]

$$\begin{aligned}\Big[\bar{u}(-z)\gamma_\mu(\gamma_5)d(z)\Big]_{\text{twist 3}} &= -g_s\int_0^1 du\int_{-u}^{u} dv\, \bar{u}(-uz)\Big[u\tilde{G}_{\mu\nu}(vz)z^\nu \not{z}\gamma_5(\gamma_5) - ivG_{\mu\nu}(vz)z^\nu\not{z}(\gamma_5)\Big]d(uz)\\ &\quad + i\epsilon_\mu{}^{\nu\alpha\beta}\int_0^1 udu\, z_\nu\hat{\partial}_\alpha\Big[\bar{u}(-uz)\gamma_\beta\gamma_5(\gamma_5)d(uz)\Big], \qquad (5)\end{aligned}$$

where $G_{\mu\nu}$ is the gluonic field strength, $\tilde{G}_{\mu\nu} = (1/2)\epsilon_{\mu\nu\alpha\beta}G^{\alpha\beta}$ its dual, and $\hat{\partial}_\alpha$ is the derivative over the total translation:

$$\hat{\partial}_\alpha\Big[\bar{u}(-z)\gamma_\beta\, d(z)\Big] \equiv \frac{\partial}{\partial y_\alpha}\Big[\bar{u}(-z+y)\gamma_\beta\, d(z+y)\Big]\Big|_{y\to 0}.$$

We would like to stress that, from the group-theoretical point of view, Eq. (5) is a genuine geometric twist-3 relation. Taking matrix-elements and neglecting the manifestly interaction-dependent quark-quark-gluon operators, whose numerical contribution is small [15], one finds the "WW-type" relations [14]

$$\begin{aligned}\hat{g}_\perp^{(v),\text{dWW}}(\xi) &= \int_{-1}^{\xi} d\omega\,\frac{\hat{\phi}_{||}(\omega)}{1-\omega} + \int_{\xi}^{1} d\omega\,\frac{\hat{\phi}_{||}(\omega)}{1+\omega},\\ \hat{g}_\perp^{(a),\text{dWW}}(\xi) &= \int_{-1}^{\xi} d\omega\,\frac{1-\xi}{1-\omega}\,\hat{\phi}_{||}(\omega) + \int_{\xi}^{1} d\omega\,\frac{1+\xi}{1+\omega}\,\hat{\phi}_{||}(\omega).\end{aligned} \qquad (6)$$

A comparison with (3) reveals that $\hat{g}_\perp^{(a)}$, although manifestly of geometric twist-3, has "inherited" twist-2 contributions from the total derivative; also $\hat{g}_\perp^{(v)}$ contains such terms. The important point to note is that the above WW-relations are consistent with the conformal expansion in the sense that (a) inserting the asymptotic DA $\hat{\phi}_{||}^{\text{as}} = 3(1-\xi^2)/4$ yields the asymptotic DAs

$$g_\perp^{(v),\text{as}} = \frac{3}{4}\,(1+\xi^2)\,, \quad g_\perp^{(a),\text{as}} = \frac{3}{4}\,(1-\xi^2), \qquad (7)$$

as required by conformal expansion, cf. Ref. [15], and that (b) contributions of higher conformal spin to $\hat{\phi}_{||}$ translate into contributions of the same conformal spin to $\hat{g}_\perp^{(v,a)}$.The interaction-dependent operators only contribute at non-leading conformal spin.

The above relations include the contribution of the twist-3 total-derivative operator in (5). What happens if, in the original spirit of WW, one only includes geometric twist-2 operators in the WW-relations? As shown in [12], one finds

$$\begin{aligned}\hat{g}_\perp^{(v),\text{gWW}}(\xi) &= \int_\xi^{\text{sign}(\xi)} \frac{d\omega}{\omega}\,\hat{\phi}_{||}(\omega),\\ \hat{g}_\perp^{(a),\text{gWW}}(\xi) &= 0,\end{aligned} \qquad (8)$$

which, inserting $\hat{\phi}_{||}^{\text{as}}$, yields

$$\hat{g}_\perp^{(v),\text{gWW}}(\xi) = \frac{3}{8}\left(\xi^2 - 1 - 2\ln\frac{\xi}{\text{sign}(\xi)}\right), \qquad (9)$$

which exhibits a logarithmic singularity at $\xi = 0$.

These results are quite different from those obtained from conformal expansion. The argument of WW for neglecting the twist-3 operators, corroborated by Shuryak and Vainshtein, was that they are equivalent to quark-quark-gluon operators whose matrix-elements can be neglected numerically. This argument, however, does no longer hold in exclusive kinematics, where twist-3 operators with total derivatives induce contributions that are as large as those from twist-2. Thus, the contributions $\hat{g}_\perp^{(v),\text{tw3}}(\xi) = \hat{g}_\perp^{(v)}(\xi) - \hat{g}_\perp^{(v),\text{gWW}}(\xi)$ and $\hat{g}_\perp^{(a),\text{tw3}}(\xi) = \hat{g}_\perp^{(a)}(\xi)$ are not small numerically.

In addition we have demonstrated above that the geometric WW-relations violate the restrictions imposed by conformal symmetry and yield (artificial) singularities that are cancelled *exactly* by total-derivative operators of geometric twist-3. We conclude that analyses based on geometric twist-2 are of rather limited use in deriving WW-relations between DAs of different dynamical twist and that it is essential to include twist-3 operators containing total derivatives in order to preserve the symmetries of the theory. This result is complementary to that obtained in Ref. [13] that twist-3 total-derivative operators are needed to restore gauge-invariance of physical amplitudes in off-forward kinematics.

Let us finally also comment on the possible extension of WW-relations to twist-4. At this order in the twist-expansion, trace-subtractions of leading twist-2 operators become relevant and give rise to two different types of relations: one gives the geometric twist-2 part of the dynamical twist-4 DA $\hat{g}_3$ and was obtained in [12]:

$$\hat{g}_3^{\rm tw2}(\xi) = -\int_\xi^{{\rm sign}(\xi)} \frac{d\omega}{\omega}\left\{1+2\ln\left(\frac{\xi}{\omega}\right)\right\}\hat{\phi}_{||}(\omega). \quad (10)$$

This expression shows again a logarithmic singularity that is not present in the full expression for $\hat{g}_3$ obtained in [19] using the EOM. Trace-subtractions of the twist-2 operator also give rise to so-called "kinematical" target-mass corrections. For the forward-scattering case, contributions of this type have been considered by Nachtmann. For the exclusive case, the corresponding operators have been considered in [20, 21], and for off-forward scattering, the resummation has been done in [22]. The relevance of such a procedure remains, however, unclear. For, in addition to the geometric mass-corrections, one also has also "dynamic" mass-corrections from operators $\sim x^{2n}\partial^{2n}O^{(2)}$ which are of geometric twist-$(2n+2)$ and such that are "hidden" in the twist-4 quark-quark-gluon operators entering by the EOM. This is the exact analogue of what happens at twist-3: decomposition of the relevant operators in terms of irreducible representations of the Lorentz-group gives only part of the information: the EOM have to be applied in order to obtain gauge- and conformal-invariant results. Indeed, the authors of [22] observe that their results for mass-corrections violate gauge-invariance. Dynamical mass-corrections have so far only been considered for the exclusive case, Ref. [19]. Again, it was not possible to formulate the twist-4 analogue of Eq. (5), with a clean separation of interaction-dependent and total-derivative terms; instead, one had to rely on a cumbersome local expansion that was used to obtain results for the next-to-leading order in the conformal expansion. Numerically, these corrections turned out to be relevant. We conclude that, at least for exclusive vector-meson DAs, a resummation of mass-corrections induced by trace-subtractions in the leading twist matrix element and the higher-twist operators containing total derivatives are relevant for a good approximation.

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