# Magnetic Oscillations in the Nambu – Jona-Lasinio model

**D. Ebert*** **and K.G. Klimenko**†

**Institut für Physik, Humboldt-Universität, D-10115 Berlin, Germany*

*†Institute for High Energy Physics, 142284 Protvino, Moscow Region, Russia*

**Abstract.** The phase structure of a simple Nambu–Jona-Lasinio model has been investigated at non-zero values of $\mu$ and $H$, where $H$ is an external magnetic field and $\mu$ is the chemical potential. On this basis magnetic oscillations effects were considered. It was shown that there are standard (periodic) van Alphen–de Haas magnetic oscillations of some thermodynamical quantities, including magnetization, pressure and particle density in the NJL system. Besides, we have found non-standard, i.e. non-periodic, magnetic oscillations, since the frequency of oscillations is a $H$-dependent quantity. Finally, there arises an oscillating behaviour not only for thermodynamical quantities, but also for a dynamical quantity like the quark mass.

Magnetic oscillations effects are well-known phenomena in condensed matter physics. In particular, the oscillation effect of the magnetization, which is called now the van Alphen–de Haas effect, was for the first time predicted by Landau and then experimentally observed in some non-relativistic systems (in metals) more than sixty years ago [1,2]. At present, a lot of the attention of researchers dealing with magnetic oscillations is focused on relativistic condensed matter systems (mainly on QED at nonzero values of the chemical potential $\mu$ and external magnetic field $H$), since the results of these studies may be applied to cosmology, astrophysics and high energy physics [3,4].

It was shown in the framework of QED that the thermodynamical potential $\Omega(\mu, H)$ of the system has in 1-loop approximation the following form $\Omega(\mu, H) = \Omega_{mon}(\mu, H) + \Omega_{osc}(\mu, H)$, where $\Omega_{mon}(\mu, H)$ is the monotonic part of $\Omega(\mu, H)$, and all magnetic oscillations are contained in the so-called oscillating part

$$\Omega_{osc}(\mu,H)=\sum_{k=1}^{\infty}[A_k(H)\cos(2\pi k\omega)+B_k(H)\sin(2\pi k\omega)], \qquad (1)$$

where $\omega=(\mu^2-m^2)/(2eH)$ ($e,m$ are electric charge and mass of fermions, respectively), and $A_k(H),B_k(H)$ are smoothly varying functions. Due to the presence of trigonometric functions, expression (1) obviously oscillates over the variable $(2eH)^{-1}$ with the frequency $(\mu^2-m^2)$, which is not an $H$-dependent quantity. In condensed matter physics such kind of oscillations are usually called periodic ones.

In the present talk magnetic oscillation effects are considered in the framework of quantum field theory with four-fermion interactions

$$L=\sum_{k=1}^{N}\bar q_k i\hat\partial q_k+\frac{G}{2N}[(\sum_{k=1}^{N}\bar q_k q_k)^2+(\sum_{k=1}^{N}\bar q_k i\gamma_5 q_k)^2], \qquad (2)$$

which is the $N$-fermionic extension of the simplest Nambu – Jona-Lasinio model (NJL) [5]. [1] Obviously, the model (2) is invariant under (global) $SU(N)$ and $U(1)_V$ transformations as well as continuous $U(1)_A$ chiral transformations: $q_k\to e^{i\theta\gamma_5}q_k\ ;\ (k=1,...,N)$.

We shall find the thermodynamic potential $\Omega(\mu,H)$, which is related to the corresponding effective potential $V_{H\mu}(\Sigma)$ of the NJL system (2) by

$$\Omega(\mu,H)=V_{\mu H}(\Sigma)\,\Big|_{\Sigma=\Sigma_{min}} \qquad (3)$$

and contains all the information about thermodynamical quantities such as magnetization, particle density, etc. In the relation (3), one should first of all calculate the effective potential $V_{H\mu}(\Sigma)$. So, before considering the magnetic oscillations, we can study the vacuum properties of the NJL model.

Notice that special attention has been paid to the analysis of the vacuum structure of NJL-type models at non-zero temperature and chemical potential [6,7], in the presence of external (chromo-)magnetic fields [8–10], with allowance for curvature and non-trivial space-time topology [11,12]. The combined influence of external electromagnetic and gravitational

[1] For simplicity, we consider in the following fermions ("quarks") of equal electric charge.

fields on the dynamical chiral symmetry breaking (DCSB) effect in four-fermion field theories was investigated in [13,14]. However, the influence of both an external magnetic field $H$ and chemical potential $\mu$ on the phase structure of the NJL model was not considered up to now.

**Phase structure of the model.** The necessary information about the phase structure of a given field theoretical model is contained in the global minimum point of the corresponding effective potential. In the presence of $\mu, H$ the effective potential $V_{H\mu}(\Sigma)$ of the NJL model has in leading order of large $N$ the following form

$$V_{H\mu}(\Sigma) = V_H(\Sigma) - \frac{eH}{4\pi^2}\sum_{k=0}^{\infty}\alpha_k\theta(\mu - s_k)\left\{\mu\sqrt{\mu^2 - s_k^2} - s_k^2\ln\left[\frac{\mu + \sqrt{\mu^2 - s_k^2}}{s_k}\right]\right\}, \tag{4}$$

where $\alpha_k = 2 - \delta_{k0}$, $s_k = \sqrt{\Sigma^2 + 2eHk}$. $V_H(\Sigma)$ is the effective potential at $\mu = 0$, $H \neq 0$

$$V_H(\Sigma) = \frac{H^2}{2} + V_0(\Sigma) - \frac{(eH)^2}{2\pi^2}\left\{\zeta'(-1, x) - \frac{1}{2}[x^2 - x]\ln x + \frac{x^2}{4}\right\}, \tag{5}$$

where $x = \Sigma^2/(2eH)$, $\zeta(\nu, x)$ is the generalized Riemann zeta-function, $\zeta'(-1, x) = d\zeta(\nu, x)/d\nu|_{\nu=-1}$, and

$$V_0(\Sigma) = \frac{\Sigma^2}{2G} - \frac{1}{16\pi^2}\left\{\Lambda^4\ln\left(1 + \frac{\Sigma^2}{\Lambda^2}\right) + \Lambda^2\Sigma^2 - \Sigma^4\ln\left(1 + \frac{\Lambda^2}{\Sigma^2}\right)\right\} \tag{6}$$

is the effective potential at $H, \mu = 0$. In (6) $\Lambda$ is the ultraviolet cut off parameter. Finally, let us remark that $\Sigma$ is an auxiliary scalar field, which, at the tree level, is proportional to $\bar{q}q$ by the equations of motion. The global minimum point of the potential (4) defines the vacuum expectation value of $\Sigma$ and is equal to the dynamical quark mass.

At $\mu, H = 0$ and $G < G_c = 4\pi^2/\Lambda^2$ the global minimum point of $V_0(\Sigma)$ equals to the value $\Sigma = 0$. Hence, in this case quarks are massless and chiral symmetry remains intact. If $G > G_c$, the effective potential (6) has a nontrivial global minimum point, which we shall denote as $M$. (Evidently, $M$ depends on the values of $G$ and $\Lambda$ [7].)

At $\mu = 0$, $H \neq 0$ the chiral symmetry of the model is spontaneously broken for arbitrary values of the bare coupling constant $G$. This is due to the fact, that the global minimum point $\Sigma_0(H)$ of the potential $V_H(\Sigma)$ is unequal to zero [8,10].

In order to study the properties of the NJL model vacuum for the general case, when both $\mu$ and $H$ are nonzero, one should find all solutions of the stationarity equation

$$\frac{\partial}{\partial\Sigma}V_{H\mu}(\Sigma) = \frac{\partial}{\partial\Sigma}V_H(\Sigma) + \frac{2eH\Sigma}{4\pi^2}\sum_{k=0}^{\infty}\alpha_k\theta(\mu - s_k)\ln\left[\frac{\mu+\sqrt{\mu^2-s_k^2}}{s_k}\right] = 0 \qquad (7)$$

and select that one, at which the potential $V_{H\mu}(\Sigma)$ takes its smallest value. This is the global minimum point for the function (4). The properties of this point as a function of $\mu$ and $H$ give us a lot of information about the ground state. We omit here the detailed consideration of this procedure and present directly the phase structure description of the model (Figure 1).

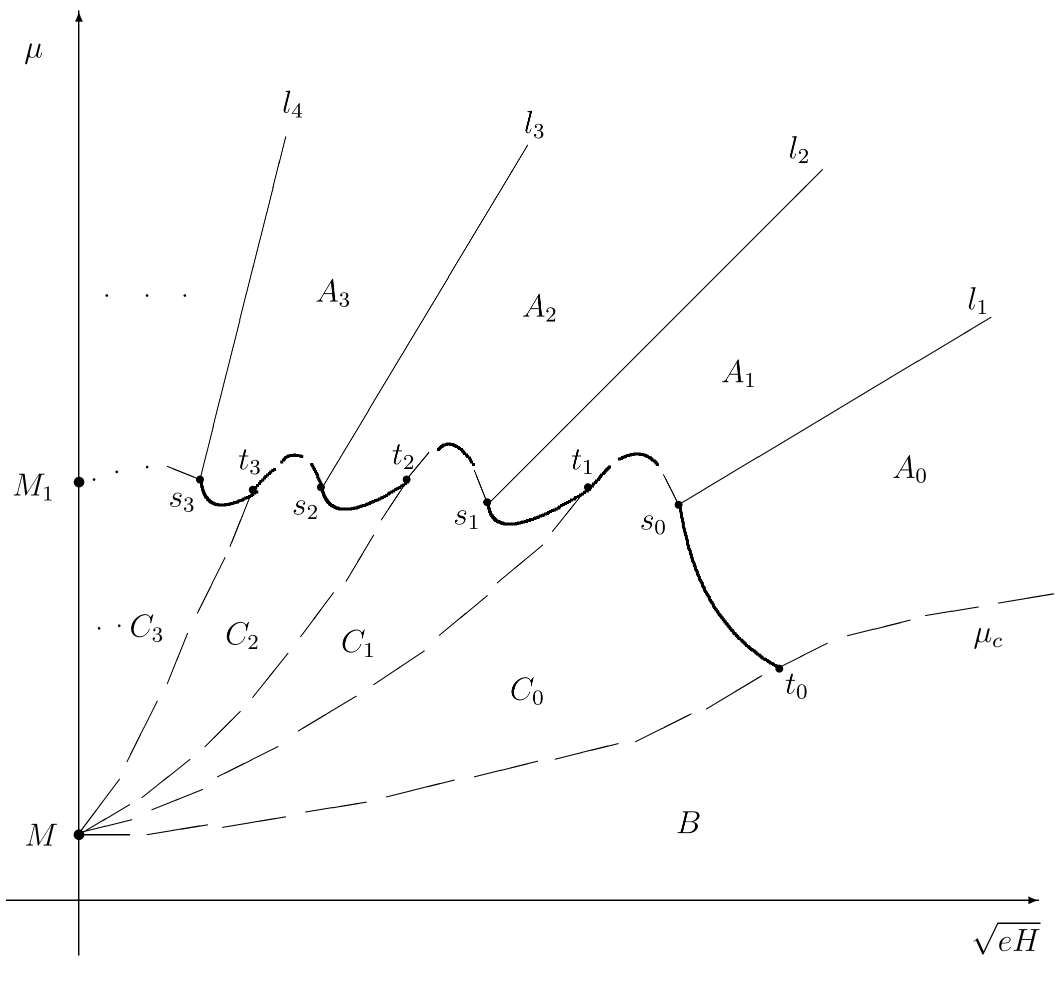


**FIGURE 1.** Phase structure of the NJL model. (Detailed description of the figure is given in the text.)

In this figure, in the plane $(\mu, \sqrt{eH})$ the phase portrait of the model is qualitatively represented for the case $G_c < G < (1.225...)G_c$, where $M$ is the quark mass at $\mu = H = 0$,

$M_1 = (\Lambda^2/2 - 2\pi^2/G)^{1/2}$. Here one can see infinite sets of symmetric massless $A_0, A_1, ...$ phases, as well as massive phases $C_0, C_1, ...$ with DCSB. In addition, there is another massive phase $B$. Dashed and solid lines in Figure 1 are critical curves of first- and second-order phase transitions, respectively. One can also see on this phase portrait infinitely many tricritical points $t_k, s_k$ ($k = 0, 1, 2, ...$) which lie on the boundary between massless and massive phases (chiral boundary). (A point of the phase diagram is called a tricritical one if, in an arbitrarily small vicinity of it, there are first- as well as second-order phase transitions.) Numerical investigation gives the following values of the external magnetic field corresponding to tricritical points $t_0$ and $s_0$ at different values of the bare coupling constant $G$: $eH_{t_0}/\Lambda^2$=0.01...; 0.08...; 0.13... as well as $eH_{s_0}/\Lambda^2$=0.006...; 0.056...; 0.103... for $G/G_c$=1.01; 1.1; 1.2, respectively. We should also remark that the part $\overbrace{t_0\mu_c(H)}$ of the chiral boundary is described by the equation $V_{H\mu}(0) = V_{H\mu}(\Sigma_0(H))$.

Points $(\mu, H)$ of the phase diagram, lying above the chiral boundary, correspond to the chirally symmetric ground state of the NJL model. One-fermion excitations of this vacuum have zero masses. At first sight, it might seem that the properties of this symmetric vacuum are slightly varied, when parameters $\mu$ and $H$ are changed. However, this is not the case, and in this region, as was mentioned above, we have infinitely many massless symmetric phases of the theory corresponding to infinitely many Landau levels, as well as a variety of critical curves of second-order phase transitions. Let us next show this.

It is well-known that the state of thermodynamic equilibrium (the ground state) of an arbitrary quantum system is described by the thermodynamic potential (TDP) $\Omega$, which is just the value of the effective potential at its global minimum point (see (3)). In the case under consideration, the TDP $\Omega(\mu, H)$ at $\mu > M_1$ (see Figure 1) has the form

$$\Omega(\mu,H) \equiv V_{H\mu}(0) = V_H(0) - \\ -\frac{eH}{4\pi^2}\sum_{k=0}^{\infty}\alpha_k\theta(\mu-\epsilon_k)\Big\{\mu\sqrt{\mu^2-\epsilon_k^2} - \epsilon_k^2\ln\left[\left(\sqrt{\mu^2-\epsilon_k^2}+\mu\right)/\epsilon_k\right]\Big\}, \tag{8}$$

where $\epsilon_k = \sqrt{2eHk}$. We shall use the following criterion of phase transitions: if at least one first (second) partial derivative of $\Omega(\mu, H)$ is a discontinuous function at some point, then this is a point of a first- (second-) order phase transition.

Using this criterion, let us show that lines $l_k = \{(\mu, H) : \mu = \sqrt{2eHk}\}$ $(k = 1, 2, ...)$, are critical lines of second-order phase transitions.

Indeed, from (8) one easily finds

$$\left.\frac{\partial\Omega}{\partial\mu}\right|_{(\mu,H)\to l_{k+}} - \left.\frac{\partial\Omega}{\partial\mu}\right|_{(\mu,H)\to l_{k-}} = 0, \tag{9}$$

as well as:

$$\left.\frac{\partial^2\Omega}{(\partial\mu)^2}\right|_{(\mu,H)\to l_{k+}} - \left.\frac{\partial^2\Omega}{(\partial\mu)^2}\right|_{(\mu,H)\to l_{k-}} = -\left.\frac{eH\mu}{2\pi^2\sqrt{\mu^2 - \epsilon_k^2}}\right|_{\mu\to\epsilon_{k+}} \to -\infty. \tag{10}$$

Equation (9) means that the first derivative $\partial\Omega/\partial\mu$ is a continuous function on all lines $l_k$. However, the second derivative $\partial^2\Omega/(\partial\mu)^2$ has an infinite jump on each line $l_k$ (see (10)), so these lines are critical curves of second-order phase transitions. (Similarly, we can prove the discontinuity of $\partial^2\Omega/(\partial H)^2$ and $\partial^2\Omega/\partial\mu\partial H$ on all lines $l_n$.)

The presence of an infinite set of massive phases $C_k$ on the phase portrait is conditioned by a special structure of the stationarity equation (7). Analytical and numerical considerations of it show that below the chiral boundary the effective potential global minimum point $\Sigma(\mu, H)$, which is identical to the quark mass, has $\mu$ and $H$ dependences. The function $\Sigma(\mu, H)$ is a continuous one inside each of regions $C_k$. However, it is a discontinuous one on each of the curves $\overset{\frown}{Mt_k}$, where the quark mass changes its value by a jump. That is why boundaries between $C_k$-regions are the first order phase transition lines. In contrast, in the phase $B$, the global minimum point is equal to $\Sigma_0(H)$ ($\equiv$ quark mass in the case $\mu = 0, H \neq 0$), which is a $\mu$-independent quantity. This means that the particle density $n \equiv -\partial\Omega/\partial\mu$ in the ground state of the phase $B$ is identically equal to zero, whereas in each phase $C_k$ this quantity differs from zero.

**Magnetic oscillations.** Now we want to show that there arise, from the presence of infinite sets of massless $A_k$ phases as well as of massive $C_k$ ones, magnetic oscillations (the so-called van Alphen–de Haas-type effect) of some physical parameters in the NJL model gauged by an external magnetic field.

Let the chemical potential be fixed, i.e. $\mu = \text{const} > \mathrm{M}_1$ (see Figure 1). Then on the plane $(\mu, \sqrt{eH})$ (see Figure 1) we have a line that crosses the critical lines $l_1, l_2, ...$ at points corresponding to some values $H_1, H_2, ...$ of the external magnetic field. The particle density $n$ and the magnetization $m$ of any thermodynamic system are defined by the TDP in the following way: $n = -\partial\Omega/\partial\mu$, $m = -\partial\Omega/\partial H$. At $\mu = \text{const}$ these quantities are continuous functions of the external magnetic field only, i.e. $n \equiv n(H)$, $m \equiv m(H)$. We know that all the second derivatives of $\Omega(\mu, H)$ are discontinuous on every critical line $l_n$ (see (10)). The functions $n(H)$ and $m(H)$, being continuous in the interval $H \in (0, \infty)$, therefore have first derivatives that are discontinuous on an infinite set of points $H_1, ..., H_k, ...$ Such a behaviour manifests itself a phenomenon usually called oscillations.

Analogously to QED and condensed-matter physics [1,2], let us again separate the expression for a physical quantity with oscillations into two parts: the first monotonic one does not contain any oscillations, whereas the second part, which is of particular interest here, contains all the oscillations. Following this rule, we can write down, say, the TDP (8) of the NJL model in the form $\Omega(\mu, H) = \Omega_{mon}(\mu, H) + \Omega_{osc}(\mu, H)$. In order to present the oscillating part $\Omega_{osc}(\mu, H)$ in an analytical form, we shall use the technique elaborated in [4], where manifestly analytical expressions for this quantity were found in the case of a perfect relativistic electron–positron gas. This technique can be used without any difficulties in our case, too. So, applying in (8) the Poisson summation formula [1]

$$\sum_{n=0}^{\infty} \alpha_n \Phi(n) = 2\sum_{k=0}^{\infty} \alpha_k \int_0^{\infty} \Phi(x)\cos(2\pi kx)dx, \tag{11}$$

where $\alpha_n = 2 - \delta_{n0}$, one can get for $\Omega_{osc}(\mu, H)$ the following expression

$$\Omega_{osc} = \frac{\mu}{4\pi^{3/2}} \sum_{k=1}^{\infty} \left(\frac{eH}{\pi k}\right)^{3/2} [Q(\pi k\nu)\cos(\pi k\nu + \pi/4) + P(\pi k\nu)\cos(\pi k\nu - \pi/4)], \tag{12}$$

where $\nu = \mu^2/(eH)$. The functions $P(x)$ and $Q(x)$ in (12) are connected with the Fresnel integrals $C(x)$ and $S(x)$ [16]: $C(x) = \frac{1}{2} + \sqrt{\frac{x}{2\pi}}[P(x)\sin x + Q(x)\cos x]$, $S(x) = \frac{1}{2} - \sqrt{\frac{x}{2\pi}}[P(x)\cos x - Q(x)\sin x]$. They have, at $x \to \infty$, the following asymptotics [16]: $P(x) = x^{-1} - 3x^{-3}/4 + ...$, $Q(x) = -x^{-2}/2 + 15x^{-4}/8 + ...$ Formula (12) presents, in a manifestly analytical form, the oscillating part of the TDP (8) for the NJL model at $\mu > M_1$. In the case under consideration, since the TDP is proportional to the pressure of the system, one can conclude that the pressure in the NJL model oscillates when $H \to 0$, too. It follows from (12) that the frequency of oscillations over the parameter $(eH)^{-1}$ equals $\mu^2/2$ and does not depend on $H$. So, in this case we have periodic magnetic oscillations. Then, starting from (12), one can easily find the corresponding expressions for the oscillating parts of $n(H)$ and $m(H)$. These quantities oscillate at $H \to 0$ with the same frequency $\mu^2/2$ and have a rather involved form, so we do not present them here.

Finally, we should note that the character of magnetic oscillations in the NJL model at $\mu > M_1$ resembles the magnetic oscillations in massless quantum electrodynamics [3,4]. Indeed, in this case in both models one can find periodic magnetic oscillations of some thermodynamic parameters.

Now let us show that at a fixed value of the chemical potential and $M < \mu < M_1$ the character of magnetic oscillations is changed. In this case on the plane $(\mu, \sqrt{eH})$ we have a line drawn through an infinite set of the $C_k$-phases. Hence, the thermodynamic potential of the NJL system has the following form: $\Omega(\mu, H) = V_{H\mu}(\Sigma(\mu, H))$, where $\Sigma(\mu, H)$ is the global minimum point of the potential $V_{H\mu}(\Sigma)$. Applying in (4) again the formula (11), one can find the following expression for the oscillating part of TDP

$$\Omega_{osc} \sim \sum_{k=1}^{\infty} \left(\frac{eH}{\pi k}\right)^{3/2} [Q(\pi k\nu)\cos(2\pi k\omega + \pi/4) + P(\pi k\nu)\cos(2\pi k\omega - \pi/4)], \tag{13}$$

where $\nu = \mu^2/(eH)$, $\omega = (\mu^2 - \Sigma^2(\mu, H))/(2eH)$. From (13) one can see that the TDP $\Omega(\mu, H)$ oscillates with frequency $(\mu^2 - \Sigma^2(\mu, H))/2$ if the variable $(eH)^{-1}$ tends to infinity. Since $\Omega(\mu, H)$ is, up to a sign, equal to the pressure in the ground state of the system, also in the present case the pressure in the NJL model is an oscillating quantity. Moreover, other thermodynamic quantities such as particle density $n = -\partial\Omega/\partial\mu$ and magnetization $m = -\partial\Omega/\partial H$ oscillate with the same frequency.

Here we should do an important remark. In the NJL model at $M < \mu < M_1$, in contrast to QED, the magnetic oscillation frequency is a $H$-dependent quantity. (Since the quark mass $\Sigma(\mu, H)$ has $H$-dependency.) So, strictly speaking, in the NJL model magnetic oscillations are not periodic ones. Recently, similar peculiarities of magnetic oscillations are observed in some ferromagnetic semiconductive materials such as $HgCr_2Se_4$ [17], where non-periodic magnetic oscillations over the variable $(eH)^{-1}$ were found to exist for electric conductivity as well as magnetization.

Finally, we should remark that in the NJL model not only thermodynamic quantities oscillate, but some dynamical parameters of the system do as well. This concerns, in particular, oscillations of the dynamical quark mass. In fact, by applying the Poisson summation formula (11) to the stationarity equation (7) and searching for the solution $\Sigma(\mu, H)$ of this equation in the form $\Sigma(\mu, H) = \Sigma_{mon} + \Sigma_{osc}$, one can easily find the following expressions for $H \to 0$:

$$\Sigma_{osc}(\mu, H) \sim \frac{(eH)^{3/2}}{\mu\tilde{M}} \sum_{k=1}^{\infty} \frac{\sin(2\pi k\tilde{\omega} - \pi/4)}{k^{3/2}}, \tag{14}$$

where $\tilde{\omega} = (\mu^2 - \tilde{M}^2)/(2eH)$, and $\tilde{M} \equiv M(\mu)$ is the quark mass at $H = 0$, $\mu \neq 0$.

**Conclusions:** Let us point out once more that for strongly correlated fermionic systems there is a possibility to observe nonperiodic magnetic oscillations. Moreover, in such systems in the presence of an external magnetic field some dynamical quantities (for example, fermion masses) should oscillate as well. Our results may be applicable in astrophysics, in the physics of neutron stars etc, where one should take into account the relativistic character of different phenomena.

Note that the strength of the surface magnetic field of a neutron star is about $10^{12}$ G and in the interior it is probably $10^{18}$ G [18]. Our numerical estimates of the $H_{s_0}$ values using $\Lambda = 700$ Mev show that the magnetic field corresponding to the tricritical point $s_0$ varies in the interval $10^{17}$ G$\div$ $10^{18}$ G, when $1.01 < G/G_c < 1.2$. Hence, the typical neutron star magnetic field strengths are much smaller, than the value of $H_{s_0}$, and are located in the oscillation region of the NJL model (see Figure 1). So, the $H$-dependency of different physical parameters (such as particle density, magnetization, quark mass, etc) inside neutron stars possibly has a nonperiodic oscillating character.

Despite the relativistic character of our investigations, we believe that qualitatively the presented results are valid for nonrelativistic electronic systems, and may be applicable in condensed matter physics, too.

More complete information about phase structure as well as magnetic oscillations in several NJL-type models one can find in our recent paper [19].


**ACKNOWLEDGMENTS**

We would like to thank David Blaschke for useful discussions. One of the authors (D.E.) gratefully acknowledges the kind support and warm hospitality of the colleagues of the Theory Division at CERN. This work was supported in part by the Russian Fund for Fundamental Research, project 98-02-16690, and by DFG-project 436 RUS 113/477.


---


[1] Landau, L.D., *Collections of Works*, Moscow: Nauka, 1969, vol.1 [in Russian]; L.D. Landau, L.D., and Lifshitz, E.M., *Statistical Physics*, Moscow: Nauka, 1976, vol.1 [in Russian]; Lifshitz, E.M., and Pitaevski, L.P., *Statistical Physics*, Oxford: Pergamon Press, 1980.

[2] De Haas, W.J., and Van Alphen, P.M., *Proc. Amsterdam Acad.* **33**, 1106 (1936); Shoenberg, D., *Magnetic Oscillations in Metals*, Cambridge: Cambridge University Press, 1984; Lifshitz, I.M., *Selected Works. Electronic Theory of Metals, Physics of Polymers and Bipolymers*, Moscow: Nauka, 1994 [in Russian].

[3] Elmfors, P., Persson, D., and Skagerstam, B.-S., *Phys. Rev. Lett.* **71**, 480 (1993); *Astropart. Phys.* **2**, 299 (1994); Persson, D., and Zeitlin, V., *Phys. Rev.* **D 51**, 2026 (1995); Andersen, J.O., and Haugset, T., *Phys. Rev.* **D 51**, 3073 (1995); Zhukovsky, V.Ch., Shoniya, T.L., and Eminov, P.A., *J. Exp. Theor. Phys.* **80**, 158 (1995); Zhukovsky, V.Ch., Vshivtsev, A.S., and Eminov, P.A., *Phys. Atom. Nucl.* **58**, 1195 (1995); Khalilov, V.R., *Phys. Atom. Nucl.* **61**, 1520 (1998).

[4] Vshivtsev, A.S., and Klimenko, K.G., *J. Exp. Theor. Phys.* **82**, 514 (1996).

[5] Nambu, Y., and Jona-Lasinio, G., *Phys. Rev.* **122**, 345 (1961); **124**, 246 (1961).

[6] Kawati, S., and Miyata, H., *Phys. Rev.* **D 23**, 3010 (1981); Bernard, V., Meissner, U.-G., and Zahed, I., *Phys. Rev.* **D 36**, 819 (1987); Christov, Chr.V., and Goeke, K., *Acta Phys. Pol.* **B 22**, 187 (1991); Ebert, D., Kalinovsky, Yu.L., Münchow, L., and Volkov, M.K., *Int. J. Mod. Phys.* **A 8**, 1295 (1993).

[7] Vshivtsev, A.S., Zhukovsky, V.Ch., and Klimenko, K.G., *J. Exp. Theor. Phys.* **84**, 1047 (1997).

[8] Klevansky, S.P., and Lemmer, R.H., *Phys. Rev.* **D 39**, 3478 (1989).

[9] Ebert, D., and Volkov, M.K., *Phys. Lett.* **B 272**, 86 (1991); Shovkovy, I.A., and Turkowski, V.M., *Phys. Lett.* **B 367**, 213 (1995); Ebert, D., and Zhukovsky, V.Ch., *Mod. Phys. Lett.* **A 12**, 2567 (1997); Babansky, A.Yu., Gorbar, E.V., and Shchepanyuk, G.V., *Phys. Lett.* **B 419**, 272 (1998).

[10] Gusynin, V.P., Miransky, V.A., and Shovkovy, I.A., *Phys. Lett.* **B 349**, 477 (1995).

[11] Inagaki, T., Muta, T., and Odintsov, S.D., *Mod. Phys. Lett.* **A 8**, 2117 (1993); Kim, D.K., and Koh, I.G., *Phys. Rev.* **D 51**, 4573 (1995); Ferrer, E.J., Gusynin, V.P., and de la Incera, V., *Phys. Lett.* **B 455**, 217 (1999).

[12] Vshivtsev, A.S., Vdovichenko, M.A., and Klimenko, K.G., *J. Exp. Theor. Phys.* **87**, 229 (1998).

[13] Inagaki, T., Muta, T., and Odintsov, S.D., *Progr. Theor. Phys. Suppl.* **127**, 93 (1997).

[14] Geyer, B., Granda, L.N., and Odintsov, S.D., *Mod. Phys. Lett.* **A 11**, 2053 (1996); Elizalde, E., Shil'nov, Yu.I., and Chitov, V.V., *Class. Quant. Grav.* **15**, 735 (1998).

[15] Prudnikov, A.P., Brychkov, Yu.A., and Marichev, O.I., *Integrals and Series*, New York: Gordon and Breach, 1986.

[16] Bateman, H., and Erdeyi, A., *Higher Transcendental Functions,* New York: McGrawHill, 1953.

[17] Balaev, A.D., et al., *Zh. Eksp. Teor. Fiz.* **113**, 1877 (1998) [in Russian].

[18] Shapiro, S.L., and Teukolsky, S.A., *Black Holes, White Dwarfs and Neutron Stars, The Physics of Compact Objects*, New York: Wiley, 1983.

[19] Ebert, D., Klimenko, K.G., Vdovichenko, M.A., and Vshivtsev, A.S., preprint CERN-TH/99-113; hep-ph/9905253 (to be published in Phys. Rev. D).