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Gauge and BRST Generators for Space-Time Non-commutative U(1) Theory

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Abstract

The Hamiltonian (gauge) symmetry generators of non-local (gauge) theories are presented. The construction is based on the $d+1$ dimensional space-time formulation of d dimensional non-local theories. The procedure is applied to $U(1)$ space-time non-commutative gauge theory. In the Hamiltonian formalism the Hamiltonian and the gauge generator are constructed. The nilpotent BRST charge is also obtained. The Seiberg-Witten map between non-commutative and commutative theories is described by a canonical transformation in the superphase space and in the field-antifield space. The solutions of classical master equations for non-commutative and commutative theories are related by a canonical transformation in the antibracket sense.

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1 Introduction

Non-local theories are described by actions that contain an infinite number of temporal derivatives. There exists an equivalent formulation of those theories in a space-time of one dimension higher [1]. The space has two times and the dynamics in this space is described in such a way that the evolution is local with respect to one of the times. The Hamiltonian formalism is constructed in the d+1 dimensions as a local theory with respect to the evolution time [1][2][3][4]. The Euler-Lagrange equations appear as Hamiltonian constraints [2]. A characteristic feature is that there is no dynamics in the usual sense; *i.e.* the physical trajectories are not obtained as evolution of some given initial conditions.

In this paper we construct symmetry generators of non-local theories. Corresponding to symmetries of a non-local Lagrangian the symmetry generators are constructed in a natural way in d+1 dimensions and are conserved quantities. When original symmetries of the non-local theory are gauge symmetries the corresponding transformations are realized as rigid symmetries in the d+1 dimensions.

We analyze in detail the case of space-time non-commutative (NC) $U(1)$ gauge theory[1]. We study the relation between the gauge generators of the NC and commutative theories. The nilpotent Hamiltonian BRST charges are constructed. We also analyze the BRST symmetry at Lagrangian level using the field-antifield formalism. The Seiberg-Witten (SW) map [5] is extended to a canonical transformation in superphase space and in the field-antifield space. The solutions of the classical master equation for non-commutative and commutative theories are related by a canonical transformation in the antibracket sense.

The organization of the paper is as follows. In section 2 we study the general properties of symmetry generators of non-local theories. In section 3 we construct the gauge symmetry generator for $U(1)$ NC gauge theory. Section 4 is devoted to study the relation between the gauge generators of commutative and $U(1)$ NC gauge theories. In section 5 we construct the BRST generator. There is an appendix where the ordinary $U(1)$ local Maxwell theory is analyzed in terms of the d+1 dimensional formalism.

2 Hamiltonian formalism of non-local theories and symmetry generators

A non-local Lagrangian at time t depends not only on variables at time t but also on ones at different times. In other words it depends on an infinite number of time derivatives of the positions $q_i(t)$ [2]. The analogue of the tangent bundle for Lagrangians depending on positions and velocities is now infinite dimensional. It is the space of all possible trajectories. The action is

$$S[q] = \int dt \; L^{non}(t). \tag{2.1}$$

[1]Here we use the term "$U(1)$" for "rank one" gauge field. It is not abelian for the NC case.

[2]In this section we will explicitly consider the case of mechanics.

The Euler-Lagrange (EL) equation is obtained by taking the functional variation of (2.1),

$$\frac{\delta S[q]}{\delta q_i(t)} \;=\; \int dt' E^i(t',t;[q]) \;=\; 0, \qquad E^i(t',t;[q]) \;\equiv\; \frac{\delta L^{non}(t')}{\delta q_i(t)}. \tag{2.2}$$

In previous papers [1][2] we have developed a general Hamiltonian formalism of non-local theories [3] as one of a local theory in a space one dimension higher than the original one. This space has two times for the system of (2.1) and it is a two dimensional field theory. The Hamiltonian system is described in terms of the fields $\mathcal{Q}_i(t,\sigma)$ and their canonical momenta $\mathcal{P}^i(t,\sigma)$. The extra local coordinate σ has the signature of time, but here it is considered as 'spatial' from the point of view of the chosen evolution time t in $1+1$ dimensions.

The Hamiltonian is introduced by

$$H(t) \;=\; \int d\sigma \; [\; \mathcal{P}^i(t,\sigma)\mathcal{Q}_i{}'(t,\sigma) \;-\; \delta(\sigma)\mathcal{L}(t,\sigma)\;], \tag{2.3}$$

where $\mathcal{Q}_i'(t,\sigma) \equiv \partial_\sigma \mathcal{Q}_i(t,\sigma)$. The "Lagrangian density" $\mathcal{L}(t,\sigma)$ is constructed from the non-local Lagrangian $L^{non}(t)$ by the following replacements,

$$q_i(t) \;\to\; \mathcal{Q}_i(t,\sigma), \quad \frac{d^n}{dt^n}q_i(t) \;\to\; \frac{\partial^n}{\partial\sigma^n}\mathcal{Q}_i(t,\sigma), \quad q_i(t+\rho) \;\to\; \mathcal{Q}_i(t,\sigma+\rho). \tag{2.4}$$

If the original Lagrangian depends explicitly on t we should replace the t by $(t+\sigma)$. Therefore the Lagrangian density $\mathcal{L}(t,\sigma)$ is *local* with respect the evolution time t. It depends on the fields $\mathcal{Q}_i(t,\sigma+\rho)$ and an infinite number of sigma derivatives of it, but not on any derivative with respect to the evolution time t. Thus the Hamiltonian (2.3) is indeed a phase space quantity.

The Hamilton equations are, denoting time (t) derivatives by "dots",

$$\dot{\mathcal{Q}}_i(t,\sigma) \;=\; \mathcal{Q}_i'(t,\sigma), \tag{2.5}$$

$$\dot{\mathcal{P}}^i(t,\sigma) \;=\; \mathcal{P}^{i\prime}(t,\sigma) + \frac{\delta\mathcal{L}(t,0)}{\delta\mathcal{Q}_i(t,\sigma)} = \mathcal{P}^{i\prime}(t,\sigma) + \mathcal{E}^i(t;0,\sigma), \tag{2.6}$$

where $\mathcal{E}(t;\sigma',\sigma)$ is defined by

$$\mathcal{E}^i(t;\sigma',\sigma) \;=\; \frac{\delta\mathcal{L}(t,\sigma')}{\delta\mathcal{Q}_i(t,\sigma)}. \tag{2.7}$$

From (2.5) the two dimensional fields $\mathcal{Q}_i(t,\sigma)$ depend only on a chiral combination of the two times $t+\sigma$ on shell. They are identified with the position variables $q_i(t)$ of the original system by

$$\mathcal{Q}_i(t,\sigma) \;=\; q_i(t+\sigma), \qquad i.e. \qquad q_i(t) \;=\; \mathcal{Q}_i(t,0). \tag{2.8}$$

The solutions of this 1+1 dimensional field equations are related to those of the EL equations (2.2) of the original non-local Lagrangian $L^{non}(t)$ if we impose a constraint on the momentum, [1]

$$\varphi^i(t,\sigma) \;=\; \mathcal{P}^i(t,\sigma) \;-\; \int d\sigma' \; \chi(\sigma,-\sigma') \; \mathcal{E}^i(t;\sigma',\sigma) \;\approx\; 0, \tag{2.9}$$

[3] See also [3][4]

where $\chi(\sigma, -\sigma')$ is defined by using the sign distribution $\epsilon(\sigma)$ as $\chi(\sigma, -\sigma') = \frac{\epsilon(\sigma)-\epsilon(\sigma')}{2}$. We use *weak equality* symbol "$\approx$" for equations that hold on the constraint surface [6].

The stability condition of (2.9) requires

$$\dot{\varphi}^i(t,\sigma) \approx \delta(\sigma)\, [\int d\sigma'\ \mathcal{E}^i(t;\sigma',0)] \approx 0. \tag{2.10}$$

We should require further consistency conditions of this constraint. Repeating this we get an infinite set of Hamiltonian constraints which are expressed collectively as

$$\tilde{\varphi}^i(t,\sigma) = \int d\sigma' \mathcal{E}^i(t;\sigma',\sigma) \approx 0, \quad (-\infty < \sigma < \infty). \tag{2.11}$$

If we use (2.5) and (2.8) it reduces to the EL equation (2.2) of $q_i(t)$ obtained from $L^{non}(t)$.

In this way we can describe the non-local Lagrangian system as a $1+1$ dimensional local Hamiltonian system with the Hamiltonian (2.3) and the constraints (2.9) and (2.11). The formalism introduced here can be thought as a generalization of the Ostrogradski formalism [7] to the case of infinite derivative theories.

2.1 Hamiltonian symmetry generators

For local theories symmetry properties of the system are examined using the Nöether theorem [8]. In Hamiltonian formalism the relation between symmetry and conservation law has been discussed extensively for singular Lagrangian systems, for example [9][10][11]. Here we want to develop a formalism for the case of non-local theories.

Suppose we have a non-local Lagrangian , (2.1), which is invariant under some transformation $\delta q(t)$ up to a total derivative,

$$\delta L^{non}(t) = \int dt'\ \frac{\delta L^{non}(t)}{\delta q_i(t')}\ \delta q_i(t') = \frac{d}{dt} k(t). \tag{2.12}$$

Since our $1+1$ dimensional theory is local in the evolution time t, we can construct the corresponding symmetry generator in the Hamiltonian formalism as

$$G(t) = \int d\sigma\ [\ \mathcal{P}^i(t,\sigma)\delta\mathcal{Q}_i(t,\sigma) - \delta(\sigma)\mathcal{K}(t,\sigma)\ ], \tag{2.13}$$

where $\delta\mathcal{Q}_i(t,\sigma)$ and $\mathcal{K}(t,\sigma)$ are constructed from $\delta q(t)$ and $k(t)$ respectively by the same replacement (2.4), as $\mathcal{L}(t,\sigma)$ was obtained from $L^{non}(t)$. The quasi-invariance of the non-local Lagrangian (2.12) means

$$\int d\sigma'\ \frac{\delta\mathcal{L}(t,\sigma)}{\delta\mathcal{Q}_i(t,\sigma')}\delta\mathcal{Q}_i(t,\sigma') = \partial_\sigma\mathcal{K}(t,\sigma). \tag{2.14}$$

When the original non-local Lagrangian has a gauge symmetry the $\delta q_i(t)$ and $k(t)$ contain an arbitrary function of time $\lambda(t)$ and its t derivatives. In $\delta\mathcal{Q}_i(t,\sigma)$ and $\mathcal{K}(t,\sigma)$ the $\lambda(t)$ is replaced by $\Lambda(t,\sigma)$ in the same manner as $q_i(t)$ is replaced by $\mathcal{Q}_i(t,\sigma)$ in (2.4). However in order for the transformation generated by (2.13) to be a symmetry of the Hamilton equations, $\Lambda(t,\sigma)$ can not be an arbitrary function of t but should satisfy

$$\dot{\Lambda}(t,\sigma) = \Lambda'(t,\sigma) \tag{2.15}$$

as will be shown shortly. This restriction on the parameter function Λ means that the transformations generated by $G(t)$ in the d+1 dimensional Hamiltonian formalism are rigid transformations in contrast with the original ones for the non-local theory which are gauge transformations. In the appendix we will see how this rigid transformations in the d+1 dimensional Hamiltonian formalism are reduced to the usual gauge transformations in d dimension for the $U(1)$ Maxwell theory.

The generator $G(t)$ generates the transformation of $\mathcal{Q}_i(t,\sigma)$,

$$\delta\mathcal{Q}_i(t,\sigma) \;=\; \{\mathcal{Q}_i(t,\sigma),\; G(t)\}, \tag{2.16}$$

corresponding to the transformation $\delta q_i(t)$ in the non-local Lagrangian . The transformation of the momentum $\mathcal{P}^i(t,\sigma)$ is such that the Hamiltonian and the constraints are invariant on the phase space satisfying the constraints.

We first see that the generator $G(t)$ is a conserved quantity,

$$\begin{aligned}
\frac{d}{dt}G(t) \;&=\; \{G(t),H(t)\} \;+\; \frac{\partial}{\partial t}G(t) \qquad (2.17)\\
&=\; \int d\sigma d\sigma'\Big[\;\mathcal{P}^j(t,\sigma)\left(\frac{\delta(\delta\mathcal{Q}_j(t,\sigma))}{\delta\mathcal{Q}_i(t,\sigma')}\mathcal{Q}_j{}'(t,\sigma') - \partial_\sigma\delta(\sigma-\sigma')\delta\mathcal{Q}_j(t,\sigma')\right.\\
&\qquad \left. +\;\frac{\delta(\delta\mathcal{Q}_j(t,\sigma))}{\delta\Lambda(t,\sigma')}\dot\Lambda(t,\sigma')\right) \;-\; \delta(t,\sigma)\;\left(\frac{\delta\mathcal{K}(t,\sigma)}{\delta\mathcal{Q}_i(t,\sigma')}\mathcal{Q}_i{}'(t,\sigma')\right.\\
&\qquad \left.\left. -\;\frac{\delta(\mathcal{L}(t,\sigma))}{\delta\mathcal{Q}_i(t,\sigma')}\delta\mathcal{Q}_i(t,\sigma') + \frac{\delta\mathcal{K}(t,\sigma)}{\delta\Lambda(t,\sigma')}\dot\Lambda(t,\sigma')\right)\right] \;=\; 0. \qquad (2.18)
\end{aligned}$$

The last term of (2.17) is an explicit t derivative through $\Lambda(t,\sigma)$. In order to show (2.18) we need to use the symmetry condition (2.14) and the condition on $\Lambda(t,\sigma)$ in (2.15).

We can check the invariance of the constraint (2.11) under the symmetry transformations,

$$\begin{aligned}
\{\tilde\varphi^i(t,\sigma),G(t)\} \;&=\; \{\int d\sigma''\mathcal{E}^i(t,\sigma'',\sigma),\int d\sigma'\;[\;\mathcal{P}^j(t,\sigma')\delta\mathcal{Q}_j(t,\sigma') \;-\; \delta(\sigma')\mathcal{K}(t,\sigma')\;]\}\\
&=\; \int d\sigma' d\sigma''\frac{\delta^2\mathcal{L}(t,\sigma'')}{\delta\mathcal{Q}_j(t,\sigma')\delta\mathcal{Q}_i(t,\sigma)}\delta\mathcal{Q}_j(t,\sigma') \;=\; \int d\sigma'\frac{\delta\tilde\varphi^j(t,\sigma')}{\delta\mathcal{Q}_i(t,\sigma)}\delta\mathcal{Q}_j(t,\sigma')\\
&=\; -\int d\sigma'\tilde\varphi^j(t,\sigma')\frac{\delta(\delta\mathcal{Q}_j(t,\sigma'))}{\delta\mathcal{Q}_i(t,\sigma)} \;\approx\; 0,
\end{aligned}\tag{2.19}$$

where we have used an identity obtained from (2.14),

$$\int d\sigma d\sigma'\mathcal{E}^j(t,\sigma,\sigma')\delta\mathcal{Q}_j(t,\sigma') \;=\; \int d\sigma'\tilde\varphi^j(t,\sigma')\delta\mathcal{Q}_j(t,\sigma') \;=\; 0. \tag{2.20}$$

The invariance of the constraint (2.9) is, using (2.14) and (2.20) ,

$$\begin{aligned}
&\{\varphi^i(t,\sigma),G(t)\} =\\
&= -\int d\sigma'\varphi^j(t,\sigma')\frac{\delta(\delta\mathcal{Q}_j(t,\sigma'))}{\delta\mathcal{Q}_i(t,\sigma)} - \int d\sigma'[\int d\sigma''\chi(\sigma',-\sigma'')\mathcal{E}^j(t;\sigma'',\sigma')\frac{\delta(\delta\mathcal{Q}_j(t,\sigma'))}{\delta\mathcal{Q}_i(t,\sigma)}\\
&\qquad - \delta(\sigma')\frac{\delta(\mathcal{K}(t,\sigma'))}{\delta\mathcal{Q}_i(t,\sigma)} + \int d\sigma''\chi(\sigma,-\sigma'')\frac{\delta\mathcal{E}^i(t;\sigma'',\sigma)}{\delta\mathcal{Q}_j(t,\sigma')}\delta\mathcal{Q}_j(t,\sigma')]\\
&= -\int d\sigma'\varphi^j(t,\sigma')\frac{\delta(\delta\mathcal{Q}_j(t,\sigma'))}{\delta\mathcal{Q}_i(t,\sigma)} + \int d\sigma'\chi(\sigma,-\sigma')\tilde\varphi^j(t,\sigma')\frac{\delta(\delta\mathcal{Q}_j(t,\sigma'))}{\delta\mathcal{Q}_i(t,\sigma)} \;\approx\; 0.
\end{aligned}\tag{2.21}$$

Thus we have shown that the constraint surface defined by $\varphi \approx \tilde{\varphi} \approx 0$ is invariant under the transformations generated by $G(t)$.

When the non-local Lagrangian in (2.1) does not depend on t explicitly the time translation is a symmetry of the Lagrangian. The generator is the Hamiltonian H in (2.3) and it is conserved. We should recover its expression (2.3) from the general form of the generator (2.13). The Lagrangian changes as $\delta L^{non} = \varepsilon \dot{L}^{non}$ under time translation $\delta q_i(t) = \varepsilon \dot{q}_i(t)$. The corresponding generator in the present formalism is

$$G_H(t) \;=\; \int d\sigma \; [\; \mathcal{P}^i(t,\sigma)(\varepsilon \mathcal{Q}_i'(t,\sigma)) \;-\; \delta(\sigma)(\varepsilon \mathcal{L}(t,\sigma)) \;], \qquad (2.22)$$

which is ε times the Hamiltonian (2.3). In this case the conservation of the constraints (2.9) and (2.10) is understood also from (2.21) and (2.19).

Summarizing, we have constructed the Hamiltonian symmetry generators of a general non-local theory working in a d+1 dimensional space. In this formulation original gauge symmetries in d dimensions are rigid symmetries in the d+1 dimensional space. This way of understanding of gauge symmetries is also useful for ordinary higher derivative theories, see appendix and [12].

3 $U(1)$ non-commutative gauge theory

The magnetic $U(1)$ non-commutative (NC) gauge theory appears in the decoupling limit of D-p branes in the presence of a constant NS-NS two form [5]. The theory could formally be extended to the electric case. However in this case the field theory is acausal [13][14] and non-unitary [15][16]. In terms of strings this is because there is an obstruction to the decoupling limit in the case of an electromagnetic background [17][18][19][20][21]. Here we are interested in the general case of *space-time* non-commutativity with $\theta^{0i} \neq 0$.

We consider the $U(1)$ (rank one) *NC* Maxwell theory in d dimensions with the action

$$S \;=\; \int d^d x \; (-\frac{1}{4} \; \widehat{F}_{\mu\nu} \widehat{F}^{\mu\nu}), \qquad (3.1)$$

where $\widehat{F}_{\mu\nu}$ is the field strength of the $U(1)$ *NC* gauge potential $\widehat{A}_\mu$ defined by[4]

$$\widehat{F}_{\mu\nu} \;=\; \partial_\mu \widehat{A}_\nu - \partial_\nu \widehat{A}_\mu - i[\widehat{A}_\mu, \widehat{A}_\nu]. \qquad (3.2)$$

The commutators in this paper are defined by the Moyal * product as

$$[f, g] \;\equiv\; f * g - g * f, \qquad f(x) * g(x) = [e^{i\frac{\theta^{\mu\nu}}{2}\partial_{\alpha^\mu}\partial_{\beta^\nu}} f(x+\alpha) g(x+\beta)]_{\alpha=\beta=0}. \quad (3.3)$$

The EL equation of motion is

$$\widehat{D}_\mu \widehat{F}^{\mu\nu} \;=\; 0, \qquad (3.4)$$

where the covariant derivative is defined by $\widehat{D} = \partial - i[\widehat{A}, \;\;]$.

[4] We put "hats" on the quantities of the *NC* theory.

The gauge transformation is

$$\delta \widehat{A}_\mu \;=\; \widehat{D}_\mu \lambda \tag{3.5}$$

and satisfies a non-Abelian gauge algebra,

$$(\delta_\lambda \delta_{\lambda'} - \delta_{\lambda'} \delta_\lambda) \widehat{A}_\mu \;=\; -i \widehat{D}_\mu [\lambda, \lambda']. \tag{3.6}$$

Since the field strength transforms covariantly as

$$\delta \widehat{F}_{\mu\nu} \;=\; -i [\widehat{F}_{\mu\nu}, \lambda] \tag{3.7}$$

the Lagrangian density of (3.1) transforms as

$$\delta(-\frac{1}{4}\, \widehat{F}_{\mu\nu} \widehat{F}^{\mu\nu}) \;=\; \frac{i}{2}\, [\widehat{F}_{\mu\nu}, \lambda]\; \widehat{F}^{\mu\nu}. \tag{3.8}$$

Using $\int dx (f * g) = \int dx (fg)$ and the associativity of the *star* product (3.8) becomes a total divergence. The action (3.1) is invariant under the $U(1)$ *NC* transformations (3.5).

The Lagrangian (3.1) contains time derivatives of infinite order and is non-local. The *NC* gauge transformation (3.5) is also non-local since, for electric backgrounds ($\theta^{0i} \neq 0$), it contains time derivatives of infinite order of λ. Here we construct the Hamiltonian and the generator for the $U(1)$ *NC* theory using the formalism introduced in the last section. The canonical structure is realized in the d+1 dimensional formalism. Corresponding to the d dimensional gauge potential $\widehat{A}_\mu(t, \mathbf{x})$, we denote the gauge potential in d+1 dimensional one as $\widehat{\mathcal{A}}_\mu(t, \sigma, \mathbf{x})$. We regard t as the evolution "time" and $(\sigma, \mathbf{x}^i) \equiv x^\mu$ as "spatial" coordinates. Now $x^0 = \sigma$ is the coordinate denoted by σ of $q_i(t, \sigma)$ in the last section. The $(d-1)$ spatial coordinates $\mathbf{x}$ are corresponding to the indices i of $q_i(t, \sigma)$. The signature of d+1 space is $(-, -, +, +, ..., +)$.

The canonical system equivalent to the non-local action (3.1) is defined by the Hamiltonian (2.3) and two constraints, (2.9) and (2.10). The Hamiltonian is

$$H(t) \;=\; \int d^d x\; [\widehat{\mathit{\Pi}}^\nu(t, x) \partial_{x^0} \widehat{\mathcal{A}}_\nu(t, x) - \delta(x^0) \mathcal{L}(t, x)], \tag{3.9}$$

where $\widehat{\mathit{\Pi}}^\nu$ is a momentum for $\widehat{\mathcal{A}}_\nu$ and

$$\mathcal{L}(t, x) \;=\; -\frac{1}{4}\, \widehat{\mathcal{F}}_{\mu\nu}(t, x) \widehat{\mathcal{F}}^{\mu\nu}(t, x), \tag{3.10}$$

$$\widehat{\mathcal{F}}_{\mu\nu}(t, x) \;=\; \partial_\mu \widehat{\mathcal{A}}_\nu(t, x) - \partial_\nu \widehat{\mathcal{A}}_\mu(t, x) - i[\widehat{\mathcal{A}}_\mu(t, x), \widehat{\mathcal{A}}_\nu(t, x)]. \tag{3.11}$$

Here the *star* product is defined with respect to $x^\mu = (\sigma, \mathbf{x}^i)$ in place of $x^\mu = (t, \mathbf{x}^i)$ in (3.3). Thus it contains infinite order of spatial derivatives but no time derivative. The Hamiltonian contains no derivative with respect to t and is a function of the canonical pairs $(\widehat{\mathcal{A}}_\mu(t, x), \widehat{\mathit{\Pi}}^\mu(t, x))$ with the Poisson bracket

$$\{\widehat{\mathcal{A}}_\mu(t, x), \widehat{\mathit{\Pi}}^\nu(t, x')\} \;=\; \delta_\mu{}^\nu\; \delta^{(d)}(x - x'). \tag{3.12}$$

The momentum constraint (2.9) is

$$\begin{aligned}\varphi^\nu(t,x) &= \widehat{\Pi}^\nu(t,x) + \int dy\ \chi(x^0,-y^0)\ \widehat{\mathcal{F}}^{\mu\nu}(t,y)\ \widehat{\mathcal{D}}^y_\mu \delta(x-y) \\ &= \widehat{\Pi}^\nu(t,x) + \delta(x^0)\widehat{\mathcal{F}}^{0\nu}(t,x) - \frac{i}{2}\Big(\epsilon(x^0)[\widehat{\mathcal{F}}^{\mu\nu},\widehat{\mathcal{A}}_\mu] - [\epsilon(x^0)\widehat{\mathcal{F}}^{\mu\nu},\widehat{\mathcal{A}}_\mu]\Big) \approx 0.\end{aligned} \tag{3.13}$$

The constraint (2.10) obtained from the consistency of the above constraint is

$$\tilde{\varphi}^\nu(t,x) = \widehat{\mathcal{D}}_\mu \widehat{\mathcal{F}}^{\mu\nu}(t,x) \approx 0. \tag{3.14}$$

Note that these constraints are reducible $\widehat{\mathcal{D}}_\mu\tilde{\varphi}^\mu \equiv 0$. They reproduce the EL equation of motion (3.4) using the Hamilton equation (2.5),

$$\partial_t \widehat{\mathcal{A}}_\mu(t,x) = \{\widehat{\mathcal{A}}_\mu(t,x), H(t)\} = \partial_{x^0}\widehat{\mathcal{A}}_\mu(t,x) \tag{3.15}$$

and the identification (2.8), $\widehat{\mathcal{A}}_\mu(t,x^\nu) = \widehat{A}_\mu(t+x^0,\mathbf{x})$. Since the Lagrangian of (3.1) has translational invariance the Hamiltonian (3.9) as well as the constraints (3.13) and (3.14) are conserved.

The generator of the $U(1)$ NC transformation, (2.13), is

$$G[\Lambda] = \int dx[\ \widehat{\Pi}^\mu \delta\widehat{\mathcal{A}}_\mu - \delta(x^0)\ \mathcal{K}^0\ ], \tag{3.16}$$

where the last term is evaluated from

$$\int dx[-\delta(x^0)\ \mathcal{K}^0\ ] = \int dx[\frac{\epsilon(x^0)}{2}\ \partial_\mu \mathcal{K}^\mu\ ] = \int dx[\frac{\epsilon(x^0)}{2}\ \delta\mathcal{L}\ ]. \tag{3.17}$$

Using (3.8) the $U(1)$ generator becomes

$$G[\Lambda] = \int dx\left[\widehat{\Pi}^\mu\widehat{\mathcal{D}}_\mu\Lambda + \frac{i}{4}\epsilon(x^0)\widehat{\mathcal{F}}_{\mu\nu}\ [\widehat{\mathcal{F}}^{\mu\nu},\Lambda]\right], \tag{3.18}$$

where $\Lambda(t,x^\mu)$ is an arbitrary function satisfying (2.15),

$$\dot{\Lambda}(t,x^\mu) = \partial_{x^0}\Lambda(t,x^\mu) \tag{3.19}$$

It can be expressed as a linear combination of the constraints,

$$G[\Lambda] = \int dx\ \Lambda\left[-(\widehat{\mathcal{D}}_\mu\varphi^\mu) - \delta(x^0)\tilde{\varphi}^0 + \frac{i}{2}\Big(\epsilon(x^0)[\tilde{\varphi}^\nu,\widehat{\mathcal{A}}_\nu] - [\epsilon(x^0)\tilde{\varphi}^\nu,\widehat{\mathcal{A}}_\nu]\Big)\right]. \tag{3.20}$$

The fact that the generator (3.20) is a sum of constraints shows explicitly the conservation of the generator on the constraint surface. It also means the $U(1)$ invariance of the Hamiltonian on the constraint surface. Furthermore $G[\Lambda]$ is conserved, without using constraints, for $\Lambda(t,x)$ satisfying (3.19),

$$\frac{d}{dt}G[\Lambda] = \{G[\Lambda],H\} + \frac{\partial}{\partial t}G[\Lambda] = 0 \tag{3.21}$$

in agreement with (2.18).

The Hamiltonian can also be written as

$$H \;=\; G[\widehat{\mathcal{A}}_0] \;+\; \int dx\, \varphi^i\, \widehat{\mathcal{F}}_{0i} \;+\; E_L, \tag{3.22}$$

where the first term is the $U(1)$ generator (3.20) of the parameter $\Lambda = \widehat{\mathcal{A}}_0$. The last term E_L is the non weakly zero part of the Hamiltonian,

$$\begin{aligned} E_L \;=\; & \int dx\, \delta(x^0)\, \{\frac{1}{2}\widehat{\mathcal{F}}_{0i}^2 + \frac{1}{4}\widehat{\mathcal{F}}_{ij}^2\} \\ & +\frac{i}{2}\int dx\, \widehat{\mathcal{A}}_0 \left(\frac{1}{2}[\widehat{\mathcal{F}}^{ij}, \epsilon(x^0)\widehat{\mathcal{F}}_{ij}] - [\widehat{\mathcal{F}}^{0i}, \epsilon(x^0)\widehat{\mathcal{F}}_{0i}]\right) \\ & +\frac{i}{2}\int dx\, \widehat{\mathcal{A}}_j \left([\widehat{\mathcal{F}}_{0i}, \epsilon(x^0)\widehat{\mathcal{F}}^{ij}] - [\epsilon(x^0)\widehat{\mathcal{F}}_{0i}, \widehat{\mathcal{F}}^{ij}]\right). \end{aligned} \tag{3.23}$$

This expression is useful, for example, to evaluate the energy of classical configurations of the theory. The two terms in the first line have the same form as the "energy" of the commutative $U(1)$ theory. The last two lines are non-local contributions. However they vanish in two cases, (1) in $\theta^{0i} = 0$ (magnetic) background and (2) for t independent solutions of $\mathcal{A}_\mu$.

4 Seiberg-Witten map, gauge generators and Hamiltonians

Seiberg and Witten [5] have introduced a map between the gauge potential A_μ in an $U(1)$ commutative and $\widehat{A}_\mu$ in an $U(1)$ *NC* theories. Here we discuss the Seiberg-Witten (SW) map for the space-time $U(1)$ *NC* theories as a *canonical transformation* in the Hamiltonian formalism in d+1 dimensions. We also show how $U(1)$ generator in the *NC* theory is mapped to the one of the commutative theory.

The SW map from the $U(1)$ commutative connection A_μ to the $U(1)$ *NC* one $\widehat{A}_\mu$ is

$$\widehat{A}_\mu = A_\mu + \frac{1}{2}\theta^{\rho\sigma} A_\sigma (2\partial_\rho A_\mu - \partial_\mu A_\rho) + \ldots . \tag{4.1}$$

In the following discussions we keep terms only up to the first order of θ and higher power terms of θ indicated by ... are omitted.

Under a commutative $U(1)$ transformation of $\delta A_\mu = \partial_\mu \lambda$, $\widehat{A}_\mu$ transforms as

$$\delta \widehat{A}_\mu \;=\; \partial_\mu\{\lambda + \frac{1}{2}\theta^{\rho\sigma} A_\sigma \partial_\rho \lambda\} + \theta^{\rho\sigma}\partial_\sigma\lambda\partial_\rho A_\mu \;=\; \widehat{D}_\mu \widehat{\lambda}. \tag{4.2}$$

Although the field $\widehat{A}_\mu$ defined above transforms as $U(1)$ *NC* gauge potentials the gauge transformation parameter function $\widehat{\lambda}$ is now gauge field dependent

$$\widehat{\lambda}(\lambda, A) = \lambda + \frac{1}{2}\theta^{\rho\sigma} A_\sigma \partial_\rho \lambda. \tag{4.3}$$

The field strength $\widehat{F}_{\mu\nu}$ defined as in (3.2) is, in terms of the commutative fields A_μ and $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$, as

$$\widehat{F}_{\mu\nu} = F_{\mu\nu} + \theta^{\rho\sigma} F_{\rho\mu} F_{\sigma\nu} - \theta^{\rho\sigma} A_\rho \partial_\sigma F_{\mu\nu} \tag{4.4}$$

and transforms under $\delta A_\mu = \partial_\mu \lambda$ covariantly as

$$\delta \widehat{F}_{\mu\nu} = -\theta^{\rho\sigma} \partial_\rho \lambda \partial_\sigma F_{\mu\nu} = -i[F_{\mu\nu}, \lambda] = -i[\widehat{F}_{\mu\nu}, \widehat{\lambda}]. \tag{4.5}$$

In the d+1 dimensional Hamiltonian formalism we can regard the mapping (4.1) as a canonical transformation. Denoting the d+1 dimensional potentials $\widehat{\mathcal{A}}_\mu(t,x)$ and $\mathcal{A}_\mu(t,x)$ corresponding to d dimensional ones $\widehat{A}_\mu(t,\mathbf{x})$ and $A_\mu(t,\mathbf{x})$ respectively, the generating function is

$$W(\mathcal{A}, \widehat{\Pi}) = \int d^d x \; \widehat{\Pi}^\mu \left(\mathcal{A}_\mu + \frac{1}{2} \theta^{\rho\sigma} \mathcal{A}_\sigma (2\partial_\rho \mathcal{A}_\mu - \partial_\mu \mathcal{A}_\rho) \right) \; + \; W^0(\mathcal{A}), \tag{4.6}$$

where $W^0(\mathcal{A})$ is a function of $\mathcal{A}_\mu$ of order θ. It reproduces the transformation of $\mathcal{A}_\mu$ as in (4.1)

$$\widehat{\mathcal{A}}_\mu \;\; = \;\; \mathcal{A}_\mu + \frac{1}{2} \theta^{\rho\sigma} \mathcal{A}_\sigma (2\partial_\rho \mathcal{A}_\mu - \partial_\mu \mathcal{A}_\rho) \tag{4.7}$$

and determine the relation between Π^μ and $\widehat{\Pi}^\mu$, conjugate momenta of $\mathcal{A}_\mu$ and $\widehat{\mathcal{A}}_\mu$ respectively, as

$$\Pi^\mu = \widehat{\Pi}^\mu + \frac{1}{2} \widehat{\Pi}^\sigma \theta^{\rho\mu} (2\partial_\rho \mathcal{A}_\sigma - \partial_\sigma \mathcal{A}_\rho) - \partial_\rho (\theta^{\rho\sigma} \mathcal{A}_\sigma \widehat{\Pi}^\mu) + \frac{1}{2} \partial_\rho (\widehat{\Pi}^\rho \theta^{\mu\sigma} \mathcal{A}_\sigma) + \frac{\delta W^0(\mathcal{A})}{\delta \mathcal{A}_\mu}. \tag{4.8}$$

It can be inverted, to first order in θ, as

$$\widehat{\Pi}^\mu = \Pi^\mu + \theta^{\mu\rho} \Pi^\sigma \mathcal{F}_{\rho\sigma} + \Pi^\mu \frac{1}{2} \theta^{\rho\sigma} \mathcal{F}_{\rho\sigma} + \theta^{\rho\sigma} \mathcal{A}_\sigma \partial_\rho \Pi^\mu - \frac{1}{2} (\partial_\rho \Pi^\rho) \theta^{\mu\sigma} \mathcal{A}_\sigma - \frac{\delta W^0(\mathcal{A})}{\delta \mathcal{A}_\mu}. \tag{4.9}$$

Note that the canonical transformation, (4.7) and (4.9), is independent of the concrete theories we are considering.

In the last section the generator of $U(1)$ *NC* theory was obtained in (3.18) as

$$G[\widehat{\Lambda}] \;\; = \;\; \int dx \left[\widehat{\Pi}^\mu \widehat{\mathcal{D}}_\mu \widehat{\Lambda} \; + \; \frac{i}{4} \epsilon(x^0) \widehat{\mathcal{F}}_{\mu\nu} \; [\widehat{\mathcal{F}}^{\mu\nu}, \widehat{\Lambda}] \right]. \tag{4.10}$$

The last term appeared since the original Lagrangian L^{non} changes as a surface term as in (3.8) under the gauge transformation. Now we want to see how this generator transforms under the SW map. It is straightforward to show that, for $W^0(\mathcal{A}) = 0$,

$$\int dx [\; \widehat{\Pi}^\mu \widehat{\mathcal{D}}_\mu \widehat{\Lambda}(\Lambda, \mathcal{A}) \;] \;\; = \;\; \int dx \; [\; \Pi^\mu \partial_\mu \Lambda \;], \tag{4.11}$$

where

$$\widehat{\Lambda}(\Lambda,\mathcal{A}) = \Lambda + \frac{1}{2}\theta^{\rho\sigma}\mathcal{A}_\sigma\partial_\rho\Lambda, \qquad \dot{\Lambda} = \partial_{x^0}\Lambda. \tag{4.12}$$

These results are independent of the specific form of Lagrangian for $U(1)$ NC and commutative gauge theories. On the other hand the term $\delta(\sigma)\mathcal{K}(t,\sigma)$ appearing in (2.13) depends on the specific theory we are considering. For the $U(1)$ NC theory, (3.1), it is the Lagrangian dependent term in (4.10) and is up to the first order of θ

$$\frac{i}{4}\int dx\ \epsilon(x^0)\widehat{\mathcal{F}}_{\mu\nu}[\widehat{\mathcal{F}}^{\mu\nu},\widehat{\Lambda}] \ = \ \frac{1}{4}\int dx\ \delta(x^0)\theta^{0i}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}\partial_i\Lambda. \tag{4.13}$$

In this case the generator of $U(1)$ NC transformations can be mapped to that of commutative one

$$\begin{aligned} G[\widehat{\Lambda}(\Lambda,\mathcal{A})] \ &= \ \int dx\ \{\Pi^0\partial_0\Lambda \ + (\Pi^i + \frac{1}{4}\delta(x^0)\theta^{0i}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu})\partial_i\Lambda\} - \int dx\frac{\delta W^0(\mathcal{A})}{\delta\mathcal{A}_\mu}\partial_\mu\Lambda \\ &= \ \int dx\ [\ \Pi^\mu\partial_\mu\Lambda\ ] \end{aligned} \tag{4.14}$$

for a choice of the canonical transformation with

$$W^0(\mathcal{A}) \ = \ \frac{1}{4}\int dx\ \delta(x^0)\ \theta^{0\mu}\mathcal{A}_\mu\mathcal{F}_{\rho\sigma}\mathcal{F}^{\rho\sigma}. \tag{4.15}$$

The right hand side of (4.14) is the generator for any $U(1)$ commutative theory which is invariant under the $U(1)$ gauge transformations. See the appendix for case of the $U(1)$ commutative Maxwell theory.

Now we would like to see what is the form of the $U(1)$ Hamiltonian obtained from (3.9) under the SW map, (4.7) and (4.9). The $U(1)$ commutative Hamiltonian results to be

$$H^{(c)} = \int dx\ [\Pi^\nu(t,x)\mathcal{A}'_\nu(t,x) \ - \ \delta(x^0)\mathcal{L}^{(c)}(t,x)] \tag{4.16}$$

where

$$\mathcal{L}^{(c)}(t,x) = -\frac{1}{4}\mathcal{F}^{\nu\mu}\mathcal{F}_{\nu\mu} - \frac{1}{2}\mathcal{F}^{\mu\nu}\theta^{\rho\sigma}\mathcal{F}_{\rho\mu}\mathcal{F}_{\sigma\nu} + \frac{1}{8}\theta^{\nu\mu}\mathcal{F}_{\nu\mu}\mathcal{F}_{\rho\sigma}\mathcal{F}^{\rho\sigma}. \tag{4.17}$$

It is the d+1 dimensional Hamiltonian for an abelian $U(1)$ gauge theory with the Lagrangian

$$L^{(c)}(t,\mathbf{x}) \ = \ -\frac{1}{4}F^{\nu\mu}F_{\nu\mu} - \frac{1}{2}F^{\mu\nu}\theta^{\rho\sigma}F_{\rho\mu}F_{\sigma\nu} + \frac{1}{8}\theta^{\nu\mu}F_{\nu\mu}F_{\rho\sigma}F^{\rho\sigma} \tag{4.18}$$

in the d dimension. One can check that this Lagrangian is, up to a total derivative, the expansion of BI action up to order F^3 terms when written in terms of the open string parameters [5] [5].

$$L^{(c)} \ \sim \ 1 - \sqrt{-\det(\eta_{\mu\nu} - \theta_{\mu\nu} + F_{\mu\nu})} \ \sim \ 1 - \sqrt{-\det(\eta_{\mu\nu} + \widehat{F}_{\mu\nu})}. \tag{4.19}$$

[5] We acknowledge discussions with Joan Simón on this point.

5 BRST symmetry

In this section we study the BRST symmetry [22][23] at classical and quantum levels for the $U(1)$ NC gauge theory. We construct the BRST charge and the BRST invariant Hamiltonian working with the d+1 dimensional formulation. We check the nilpotency of the BRST generator. The SW map will be generalized in the superphase space. The BRST charges and Hamiltonians of the $U(1)$ NC and commutative $U(1)$ gauge theories are mapped each other.

We will also study the BRST symmetry at Lagrangian level using the field-antifield formalism [24][25], for a review see [26][27][28]. We will construct the solution of the classical master equation in the classical and gauge fixed basis. We will also realize the SW map as an antibracket canonical transformation.

5.1 Hamiltonian BRST charge

The BRST symmetry at classical level encodes the classical gauge structure through the nilpotency of the BRST transformations of the classical fields and ghosts [29][30][31]. The BRST symmetry of the classical fields is constructed from the gauge transformation by changing the gauge parameters by ghost fields.

The BRST transformations are

$$\delta_B \widehat{A}_\mu \;=\; \widehat{D_\mu}\widehat{C}, \qquad \delta_B \widehat{C} \;=\; -\,i\widehat{C} * \widehat{C}, \tag{5.1}$$

$$\delta_B \widehat{\overline{C}} \;=\; \widehat{B}, \qquad \delta_B \widehat{B} \;=\; 0, \tag{5.2}$$

where $\widehat{C}, \widehat{\overline{C}}, \widehat{B}$ are the ghost, antighost and auxiliary field respectively.

The Lagrangian associated with (3.1) changes under the BRST transformations as

$$\delta_B L \;=\; \frac{i}{2}\,[\widehat{F}_{\mu\nu}, \widehat{C}]\;\widehat{F}^{\mu\nu}. \tag{5.3}$$

We construct the gauge fixing Lagrangian $\widehat{L}_{gf+FP}$ by introducing the gauge fixing fermion

$$\widehat{\Psi} = \widehat{\overline{C}}\;(\partial^\mu \widehat{A}_\mu + \alpha\widehat{B}) \tag{5.4}$$

as $\delta_B\widehat{\Psi}$ up to total derivative. The $\widehat{L}_{gf+FP}$ is given by

$$\widehat{L}_{gf+FP} \;=\; -\,\partial^\mu\widehat{\overline{C}}\;\widehat{D_\mu}\widehat{C} \;+\; \widehat{B}\;(\partial^\mu \widehat{A}_\mu + \alpha\widehat{B}). \tag{5.5}$$

We have

$$\delta_B \widehat{L}_{gf+FP} \;=\; \partial^\mu(\widehat{B}\widehat{D_\mu}\widehat{C}). \tag{5.6}$$

In order to construct the generator of the BRST transformations and the BRST invariant Hamiltonian we should use the d+1 dimensional formulation. We denote the

d+1 dimensional fields corresponding to the d dimensional ones $\widehat{C}, \widehat{\overline{C}}, \widehat{B}$, using with the calligraphic letters, as $\widehat{\mathcal{C}}, \widehat{\overline{\mathcal{C}}}, \widehat{\mathcal{B}}$ respectively. The BRST invariant Hamiltonian is giving by

$$
\begin{aligned}
H(t) &= H^{(0)} + H^{(1)} && (5.7)\\
H^{(0)} &= \int dx\, [\widehat{\Pi}^{\nu}(t,x)\widehat{\mathcal{A}}'_{\nu}(t,x) + \widehat{\mathcal{P}_c}(t,x)\widehat{\mathcal{C}}'(t,x) - \delta(x^0)\widehat{\mathcal{L}}^0(t,x)], && (5.8)\\
H^{(1)} &= \int dx\, [\widehat{\mathcal{P}}_{\mathcal{B}}\widehat{\mathcal{B}}'(t,x) + \widehat{\mathcal{P}_{\overline{\mathcal{C}}}}(t,x)\widehat{\overline{\mathcal{C}}}'(t,x) - \delta(x^0)\widehat{\mathcal{L}}_{gf+FP}(t,x)]. && (5.9)
\end{aligned}
$$

The BRST charge is

$$
\begin{aligned}
Q_B &= Q_B^{(0)} + Q_B^{(1)} && (5.10)\\
Q_B^{(0)} &= \int dx \left[\widehat{\Pi}^{\mu}\widehat{\mathcal{D}}_{\mu}\widehat{\mathcal{C}} - i\,\widehat{\mathcal{P}_{\mathcal{C}}} * \widehat{\mathcal{C}} * \widehat{\mathcal{C}} + \frac{1}{2}\epsilon(x^0)\delta_B\widehat{\mathcal{L}}^0(t,x)\right]. && (5.11)\\
Q_B^{(1)} &= \int dx \left[\widehat{\mathcal{P}_{\overline{\mathcal{C}}}}\,\widehat{\mathcal{B}} + \frac{1}{2}\epsilon(x^0)\delta_B\widehat{\mathcal{L}}_{gf+FP}(t,x)\right], && (5.12)
\end{aligned}
$$

It is an analogue of the BFV charge [32][33] for $U(1)$ NC theory. $H^{(0)}$, $Q_B^{(0)}$ are the "gauge unfixed" and the H, Q_B are "gauge fixed" Hamiltonians and BRST charges.

Using the graded symplectic structure of the superphase space [34]

$$
\begin{aligned}
\{\widehat{\mathcal{A}}_{\mu}(t,x), \widehat{\Pi}^{\nu}(t,x')\} &= \delta_{\mu}{}^{\nu}\, \delta^{(d)}(x-x'), \quad \{\widehat{\mathcal{C}}(t,x), \widehat{\mathcal{P}}_{\widehat{\mathcal{C}}}(t,x')\} = \delta^{(d)}(x-x'),\\
\{\widehat{\overline{\mathcal{C}}}(t,x), \widehat{\mathcal{P}}_{\widehat{\overline{\mathcal{C}}}}(t,x')\} &= \delta^{(d)}(x-x'), \qquad \{\widehat{\mathcal{B}}(t,x), \widehat{\mathcal{P}}_{\widehat{\mathcal{B}}}(t,x')\} = \delta^{(d)}(x-x') \quad (5.13)
\end{aligned}
$$

we have

$$
\{H^{(0)}, Q_B^{(0)}\} = \{Q_B^{(0)}, Q_B^{(0)}\} = 0, \tag{5.14}
$$

and

$$
\{H, Q_B\} = \{Q_B, Q_B\} = 0. \tag{5.15}
$$

Thus the BRST charges are nilpotent and the Hamiltonians are BRST invariant both in the gauge unfixed and the gauge fixed levels.

5.2 Seiberg-Witten map in superphase space

Now we would like to see how the BRST charges and the BRST invariant Hamiltonians of the NC and commutative gauge theories are related. In order to do that we will extend the SW map to a canonical transformation in the superphase space $(\mathcal{A}, \mathcal{C}, \overline{\mathcal{C}}, \mathcal{B}, \Pi, \mathcal{P}_{\mathcal{C}}, \mathcal{P}_{\overline{\mathcal{C}}}, \mathcal{P}_{\mathcal{B}})$. We introduce the generating function

$$
\begin{aligned}
W(\mathcal{A}, \mathcal{C}, \overline{\mathcal{C}}, \mathcal{B}, \widehat{\Pi}, \widehat{\mathcal{P}}_{\mathcal{C}}, \widehat{\mathcal{P}}_{\overline{\mathcal{C}}}, \widehat{\mathcal{P}}_{\mathcal{B}}) &= \int dx \left[\widehat{\Pi}^{\mu}\left(\mathcal{A}_{\mu} + \frac{1}{2}\theta^{\rho\sigma}\mathcal{A}_{\sigma}(2\partial_{\rho}\mathcal{A}_{\mu} - \partial_{\mu}\mathcal{A}_{\rho})\right)\right.\\
&\quad \left. + \widehat{\mathcal{P}}_{\mathcal{C}}\left(\mathcal{C} + \frac{1}{2}\theta^{\rho\sigma}\mathcal{A}_{\sigma}\partial_{\rho}\mathcal{C}\right) + \widehat{\mathcal{P}}_{\overline{\mathcal{C}}}\overline{\mathcal{C}} + \widehat{\mathcal{P}}_{\mathcal{B}}\mathcal{B}\right]\\
&\quad + W^0(\mathcal{A}, \mathcal{C}) + W^1(\mathcal{A}, \mathcal{C}, \overline{\mathcal{C}}, \mathcal{B}),
\end{aligned} \tag{5.16}
$$

where $W^0(\mathcal{A},\mathcal{C})$ depends on the specific form the $U(1)$ NC Lagrangian and $W^1(\mathcal{A},\mathcal{C},\overline{\mathcal{C}},B)$ also on the form of the gauge fixing. For the $U(1)$ NC theory and for the gauge fixing (5.4), we have

$$W^0(\mathcal{A},\mathcal{C}) \;=\; \frac{1}{4}\int dx\ \delta(x^0)\ \theta^{0\mu}\mathcal{A}_\mu\mathcal{F}_{\rho\sigma}\mathcal{F}^{\rho\sigma} \tag{5.17}$$

as in (4.15) and

$$\begin{aligned} W^1 \;&=\; \int dx\ \frac{1}{2}\epsilon(x^0)\Big[\partial^\mu\{\frac{1}{2}\theta^{\rho\sigma}\mathcal{A}_\sigma(2\partial_\rho\mathcal{A}_\mu-\partial_\mu\mathcal{A}_\rho)\}\mathcal{B} \\ &+\ \{\frac{1}{2}\theta^{\rho\sigma}\mathcal{A}_\sigma(2\partial_\rho\mathcal{A}_\mu-\partial_\mu\mathcal{A}_\rho)\partial_\sigma\mathcal{C}+\frac{1}{2}\theta^{\rho\sigma}\mathcal{A}_\sigma\partial_\mu\partial_\rho\mathcal{C}\ \}\ \partial^\mu\overline{\mathcal{C}}\ \Big]. \end{aligned} \tag{5.18}$$

The transformations are obtained by

$$\widehat{\Phi}^A=\frac{\partial_\ell W}{\partial\widehat{P}_A},\qquad\qquad P_A=\frac{\partial_r W}{\partial\Phi^A}, \tag{5.19}$$

where Φ^A represent any fields and P_A their conjugate momenta and ∂_r and ∂_ℓ are right and left derivatives respectively.

Explicitly we have

$$\widehat{\mathcal{A}}_\mu \;=\; \mathcal{A}_\mu+\frac{1}{2}\theta^{\rho\sigma}\mathcal{A}_\sigma(2\partial_\rho\mathcal{A}_\mu-\partial_\mu\mathcal{A}_\rho), \tag{5.20}$$

$$\widehat{\mathcal{C}} \;=\; \mathcal{C}+\frac{1}{2}\theta^{\rho\sigma}\mathcal{A}_\sigma\partial_\rho\mathcal{C}, \tag{5.21}$$

$$\widehat{\overline{\mathcal{C}}} \;=\; \overline{\mathcal{C}}, \tag{5.22}$$

$$\widehat{\mathcal{B}} \;=\; \mathcal{B}, \tag{5.23}$$

and

$$\begin{aligned} \widehat{\Pi}^\mu \;&=\; \Pi^\mu+\theta^{\mu\rho}\Pi^\sigma\mathcal{F}_{\rho\sigma}+\Pi^\mu\frac{1}{2}\theta^{\rho\sigma}\mathcal{F}_{\rho\sigma}+\theta^{\rho\sigma}\mathcal{A}_\sigma\partial_\rho\Pi^\mu-\frac{1}{2}(\partial_\rho\Pi^\rho)\theta^{\mu\sigma}\mathcal{A}_\sigma \\ &+\frac{1}{2}\mathcal{P}_\mathcal{C}\theta^{\mu\sigma}\partial_\sigma\mathcal{C}\ -\ \frac{\delta(W^0+W^1)}{\delta\mathcal{A}_\mu}, \end{aligned} \tag{5.24}$$

$$\widehat{\mathcal{P}}_\mathcal{C} \;=\; \mathcal{P}_\mathcal{C}+\frac{1}{2}\theta^{\rho\sigma}\partial_\rho(\mathcal{P}_\mathcal{C}\mathcal{A}_\sigma)\ -\ \frac{\delta_r(W^0+W^1)}{\delta\mathcal{C}}, \tag{5.25}$$

$$\widehat{\mathcal{P}}_{\overline{\mathcal{C}}} \;=\; \mathcal{P}_{\overline{\mathcal{C}}}\ -\ \frac{\delta_r W^1}{\delta\overline{\mathcal{C}}}, \tag{5.26}$$

$$\widehat{\mathcal{P}}_\mathcal{B} \;=\; \mathcal{P}_\mathcal{B}\ -\ \frac{\delta_r W^1}{\delta\mathcal{B}}. \tag{5.27}$$

Using this transformation we can rewrite the BRST charge (5.10) as

$$\begin{aligned} Q_B \;&=\; Q_B^{(0)}\ +\ Q_B^{(1)}\ =\ \int dx[\Pi^\mu\partial_\mu\mathcal{C}\ +\ \mathcal{P}_{\overline{\mathcal{C}}}\mathcal{B}\ -\ \delta(x^0)\mathcal{B}\partial^0\mathcal{C}\ ] \\ &=\ \int dx[\Pi^\mu\partial_\mu\mathcal{C}\ +\ \mathcal{P}_{\overline{\mathcal{C}}}\mathcal{B}\ +\ \frac{1}{2}\epsilon(x^0)\delta_B\mathcal{L}_{gf+FP}(t,x)\ ], \end{aligned} \tag{5.28}$$

where $\mathcal{L}_{gf+FP}(t,x)$ is the abelian gauge fixing Lagrangian and is given by

$$\mathcal{L}_{gf+FP} \;=\; -\partial^\mu\overline{\mathcal{C}}\;\partial_\mu\mathcal{C} \;+\; \mathcal{B}\;(\partial^\mu\mathcal{A}_\mu + \alpha\mathcal{B}). \tag{5.29}$$

The total $U(1)$ Hamiltonian (5.7) becomes

$$H \;=\; \int dx\;[\varPi^\nu\mathcal{A}'_\nu + \mathcal{P}_\mathcal{C}\mathcal{C}' + \mathcal{P}_{\overline{\mathcal{C}}}\overline{\mathcal{C}}' + P_\mathcal{B}\mathcal{B}' - \delta(x^0)(\mathcal{L}^{(c)} + \mathcal{L}_{gf+FP})]. \tag{5.30}$$

Remember $\mathcal{L}^{(c)}$ is the $U(1)$ commutative Lagrangian given in (4.17).

5.3 Field-antifield formalism for $U(1)$ non-commutative theory

The field-antifield formalism allows us to study the BRST symmetry of a general gauge theory by introducing a canonical structure at a Lagrangian level [24][25][26][27]. The classical master equation in the classical basis encodes the gauge structure of the generic gauge theory [30][31]. The solution of the classical master equation in the gauge fixed basis gives the "quantum action" to be used in the path integral quantization. Any two solutions of the classical master equations are related by a canonical transformation in the antibracket sense [35].

Here we will apply these ideas to the $U(1)$ *NC* theory. Since we work at a Lagrangian level we will work in d dimensions. In the classical basis the set of fields and antifields are

$$\Phi^A = \{\widehat{A}_\mu, \widehat{C}\}, \qquad \Phi^*_A = \{\widehat{A}^*_\mu, \widehat{C}^*\}. \tag{5.31}$$

The solution of the classical master equation

$$(S,S) = 0, \tag{5.32}$$

is given by[6]

$$S[\Phi,\Phi^*] = I[\widehat{A}] + \widehat{A}^*_\mu\widehat{D}^\mu\widehat{C} - i\widehat{C}^*(\widehat{C} * \widehat{C}), \tag{5.33}$$

where $I[\widehat{A}]$ is the classical action and the antibracket $(\;\;,\;\;)$ is defined by

$$(X,Y) = \frac{\partial_r X}{\partial\Phi^A}\frac{\partial_l Y}{\partial\Phi^*_A} - \frac{\partial_r X}{\partial\Phi^*_A}\frac{\partial_l Y}{\partial\Phi^A}. \tag{5.34}$$

The gauge fixed basis can be analyzed by introducing the antighost and auxiliary fields and the corresponding antifields. It can be obtained from the classical basis by considering a canonical transformation, in the antibracket sense,

$$\begin{aligned} \Phi^A &\longrightarrow \Phi^A \\ \Phi^*_A &\longrightarrow \Phi^*_A + \frac{\partial_r\Psi}{\partial\Phi^A} \end{aligned} \tag{5.35}$$

generated by

$$\widehat{\Psi} = \widehat{\overline{C}}\;(\partial^\mu\widehat{A}_\mu + \alpha\widehat{B}), \tag{5.36}$$

[6]As in usual convention in the antifield formalism, d dimensional integration is understood in summations.

where $\widehat{\overline{C}}$ is the antighost and $\widehat{B}$ is the auxiliary field. We have

$$S[\Phi,\Phi^*] = \widehat{I}_\Psi + \widehat{A}^{*\mu}\widehat{D}_\mu\widehat{C} - i\widehat{C}^*(\widehat{C}*\widehat{C}) + \widehat{\overline{C}}^*\widehat{B}, \tag{5.37}$$

where $\widehat{I}_\Psi$ is the "quantum action" and is given by

$$\widehat{I}_\Psi = I[\widehat{A}] + (-\partial_\mu\widehat{\overline{C}}\ \widehat{D}^\mu\widehat{C} + \widehat{B}\ \partial_\mu\widehat{A}^\mu + \alpha\widehat{B}^2). \tag{5.38}$$

The action $\widehat{I}_\Psi$ has well defined propagators and is the starting point of the Feynman perturbative calculations.

Now we would like to study what is the SW map in the space of fields and antifields. We first consider it in the classical basis. In order to do that we construct a canonical transformation in the antibracket sense

$$\widehat{\Phi}^A = \frac{\partial_l F_{cl}[\Phi,\widehat{\Phi}^*]}{\partial\widehat{\Phi}^*_A}, \quad \Phi^*_A = \frac{\partial_r F_{cl}[\Phi,\widehat{\Phi}^*]}{\partial\Phi^A}, \tag{5.39}$$

where

$$F_{cl} = \widehat{A}^{*\mu}\left(A_\mu + \frac{1}{2}\theta^{\rho\sigma}A_\sigma(2\partial_\rho A_\mu - \partial_\mu A_\rho)\right) + \widehat{C}^*(C + \frac{1}{2}\theta^{\rho\sigma}A_\sigma\partial_\rho C). \tag{5.40}$$

The gauge structures of NC and commutative are mapped to each other

$$\widehat{A}^*_\mu\widehat{D}^\mu\widehat{C} - i\widehat{C}^*(\widehat{C}*\widehat{C}) = A^*_\mu\partial^\mu C. \tag{5.41}$$

We can generalize the previous results to the gauge fixed basis. In this case the transformations of the antighost and the auxiliary field sectors should be taken into account. The generator of the canonical transformation is modified from (5.40) to

$$F_{gf} \;=\; F_{cl} \;+\; \left(\widehat{\overline{C}}^* + \frac{1}{2}\theta^{\rho\sigma}\partial^\mu\left(A_\sigma(2\partial_\rho A_\mu - \partial_\mu A_\rho)\right)\right)\overline{C} + \widehat{B}^*B. \tag{5.42}$$

Note that the additional term gives rise to new terms in $A^{*\mu}$ and $\overline{C}^*$ while the others remain the same as in the classical basis. In particular

$$\widehat{\overline{C}} \;=\; \overline{C}, \qquad \widehat{B} \;=\; B. \tag{5.43}$$

Using the transformation we can express (5.37) and (5.38) as

$$S[\Phi,\Phi^*] = I_\Psi + A^{*\mu}\partial_\mu C + \overline{C}^*B \tag{5.44}$$

where

$$I_\Psi = I[\widehat{A}(A)] + (-\partial_\mu\overline{C}\ \partial^\mu C + B\ \partial_\mu A^\mu + \alpha B^2) \tag{5.45}$$

and $I[\widehat{A}(A)]$ is the classical action in terms of A_μ. This is indeed a quantum action for the commutative $U(1)$ BRST invariant action in the gauge fixed basis. In this way the canonical transformation (5.42) maps the $U(1)$ NC structure of the $S[\Phi,\Phi^*]$ into the commutative one in the gauge fixed basis.

6 Discussions

In this paper the Hamiltonian formalism of the non-local theories is discussed by using $d+1$ dimensional formulation [1][2]. For a given non-local Lagrangian in d dimensions the Hamiltonian is introduced by (2.3) on the phase space of the $d+1$ dimensional fields. The equivalence with the original non-local theory is assured by imposing two constraints (2.9) and (2.10) consistent with the time evolution. The degrees of freedom of the extra dimension (denoted by coordinate σ) has its origin in the infinite degrees of freedom associated with the non-locality. It is also applicable to *local* and higher derivative theories. In these cases the set of constraints are used to reduce the redundant degrees of freedom of the infinite dimensional phase space, reproducing the standard d dimensional formulations [12].

We have analyzed the symmetry generators of non-local theories in the Hamiltonian formalism. As an example we have considered the space-time $U(1)$ *NC* gauge theory. The gauge transformations in d dimensions are described as a rigid symmetry in $d+1$ dimensions. The generators of *rigid* transformations in $d+1$ dimensions turn out to be the generators of *gauge* transformations when the reduction to d dimensions can be performed as is shown for the $U(1)$ commutative gauge theory in the appendix.

We have extended the Seiberg-Witten map to a canonical transformation. This allows us to map the Hamiltonians and the gauge generators of non-commutative and commutative theories. We have also seen explicitly the map of the $U(1)$ *NC* and the BI actions up to F^3. The reason why we were able to discuss the SW map as a canonical transformation is that we have considered the phase space of the commutative theory also in the $d+1$ dimensions.

The BRST symmetry has been analyzed at Hamiltonian and Lagrangian levels. The relation between the commutative and $U(1)$ *NC* parameter functions is understood as a canonical transformation of the ghosts in the super phase space of the SW map. Using the field-antifield formalism we have seen how the solution of the classical master equation for non-commutative and commutative theories are related by a canonical transformation in the antibracket sense. This results shows that the antibracket cohomology classes of both theories coincide in the space of non-local functionals. The explicit forms of the antibracket canonical transformations could be useful to study the observables, anomalies, etc. in the $U(1)$ *NC* theory.

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A $U(1)$ commutative Maxwell theory in d+1 dimensions

We will show how the $U(1)$ commutative Maxwell theory is formulated using the d+1 dimensional canonical formalism developed for non-local theories in section 2 and see how it is reduced to the standard canonical formalism in d dimensions.

The canonical system is defined by the Hamiltonian (2.3) and two constraints, (2.9) and (2.10). The Hamiltonian is

$$H = \int d^dx \, [\mathit{\Pi}^\nu(t,x)\partial_{x^0}\mathcal{A}_\nu(t,x) - \delta(x^0)\mathcal{L}(t,x) \,], \tag{A.1}$$

where

$$\mathcal{L}(t,x) = -\frac{1}{4}\,\mathcal{F}_{\mu\nu}(t,x)\mathcal{F}^{\mu\nu}(t,x), \tag{A.2}$$

$$\mathcal{F}_{\mu\nu}(t,x) = \partial_\mu\mathcal{A}_\nu(t,x) - \partial_\nu\mathcal{A}_\mu(t,x). \tag{A.3}$$

The momentum constraint (2.9) is

$$\begin{aligned}\varphi^\nu(t,x) &= \mathit{\Pi}^\nu(t,x) + \int dy\; \chi(x^0,-y^0)\;\mathcal{F}^{\mu\nu}(t,y)\;\partial^y_\mu\delta(x-y) \\ &= \mathit{\Pi}^\nu(t,x) + \delta(x^0)\mathcal{F}^{0\nu}(t,x) \approx 0\end{aligned} \tag{A.4}$$

and the constraint (2.10) is

$$\tilde\varphi^\nu(t,x) = \partial_\mu\mathcal{F}^{\mu\nu}(t,x) \approx 0. \tag{A.5}$$

The generator of the $U(1)$ transformation is given by using (2.13) as

$$G[\Lambda] = \int dx[\;\mathit{\Pi}^\mu\partial_\mu\Lambda\;]. \tag{A.6}$$

It is expressed as a linear combination of the constraints,

$$G[\Lambda] = \int dx\;\Lambda\left[-(\partial_\mu\varphi^\mu) - \delta(x^0)\tilde\varphi^0\right]. \tag{A.7}$$

The Hamiltonian is expressed using the constraints and the $U(1)$ generator as

$$H = G[\mathcal{A}_0] \;+\; \int dx\;\varphi^i\;\mathcal{F}_{0i} \;+\; \int dx\;\delta(x^0)\;\{\frac{1}{2}{\mathcal{F}_{0i}}^2 + \frac{1}{4}{\mathcal{F}_{ij}}^2\}. \tag{A.8}$$

The Hamiltonian (A.8) as well as the constraints (A.4) and (A.5) contain no time (t) derivative and are functions of the canonical pairs ($\mathcal{A}_\mu(t,x), \mathit{\Pi}^\mu(t,x)$). They are conserved since the Maxwell Lagrangian in d dimensions has time translation invariance. The $U(1)$ generator is also conserved, without using constraints, for $\Lambda(t,x)$ satisfying (2.15),

$$\frac{d}{dt}G[\Lambda] = \{G[\Lambda],H\} + \frac{\partial}{\partial t}G[\Lambda] = 0, \qquad \dot\Lambda = \partial_{x^0}\Lambda. \tag{A.9}$$

in agreement with (2.18). Since the parameter Λ is subject to the last relation in (A.9) the $U(1)$ transformations in the d+1 dimensional canonical formulation are not gauge but rigid ones. We will see how the gauge transformations appear when it is written in a d dimensional form.

In case Lagrangians are local or higher derivative ones it is often convenient to make expansion of the canonical variables using the Taylor basis[36] in reducing them to d dimensional forms. We expand the canonical variables as

$$\mathcal{A}_\mu(t,x) \equiv \sum_{m=0}^{\infty} e_m(x^0)\ A_\mu^{(m)}(t,\mathbf{x}), \qquad \mathit{\Pi}^\mu(t,x) \equiv \sum_{m=0}^{\infty} e^m(x^0)\ \mathit{\Pi}^\mu_{(m)}(t,\mathbf{x}), \tag{A.10}$$

where $e^\ell(x^0)$ and $e_\ell(x^0)$ are orthonormal basis

$$e^\ell(x^0) = (-\partial_{x^0})^\ell \delta(x^0), \qquad e_\ell(x^0) = \frac{(x^0)^\ell}{\ell!}, \tag{A.11}$$

$$\int dx^0\ e^\ell(x^0)\ e_m(x^0) = \delta^\ell_m, \qquad \sum_{\ell=0}^{\infty} e^\ell(x^0)\ e_\ell(x^{0\prime}) = \delta(x^0 - x^{0\prime}). \tag{A.12}$$

The $(A_\mu^{(m)}(t,\mathbf{x}), \mathit{\Pi}^\mu_{(m)}(t,\mathbf{x}))$ are d dimensional fields and are the new symplectic coordinates

$$\Omega(t) = \int dx\ \delta\mathit{\Pi}^\mu(t,x) \wedge \delta\mathcal{A}_\mu(t,x) = \sum_{m=0}^{\infty} \int d\mathbf{x}\ \delta\mathit{\Pi}^\mu_{(m)}(t,\mathbf{x}) \wedge \delta A_\mu^{(m)}(t,\mathbf{x}). \tag{A.13}$$

In terms of them the constraint (A.4) is expressed as

$$\varphi^\mu(t,x) = \sum_{m=0}^{\infty} e^m(x^0)\varphi^\mu_{(m)}(t,\mathbf{x}), \tag{A.14}$$

$$\varphi^0_{(m)}(t,\mathbf{x}) = \mathit{\Pi}^0_{(m)}(t,\mathbf{x}) = 0, \qquad (m \geq 0), \tag{A.15}$$

$$\varphi^i_{(0)}(t,\mathbf{x}) = \mathit{\Pi}^i_{(0)}(t,\mathbf{x}) - (\mathcal{A}_i^{(1)}(t,\mathbf{x}) - \partial_i\mathcal{A}_0^{(0)}(t,\mathbf{x})) = 0, \tag{A.16}$$

$$\varphi^i_{(m)}(t,\mathbf{x}) = \mathit{\Pi}^i_{(m)}(t,\mathbf{x}) = 0, \qquad (m \geq 1). \tag{A.17}$$

The constraint (A.5) is

$$\tilde{\varphi}^\mu(t,x) = \sum_{m=0}^{\infty} e_m(x^0)\tilde{\varphi}^{\mu(m)}(t,\mathbf{x}), \tag{A.18}$$

$$\tilde{\varphi}^{i(m)}(t,\mathbf{x}) = \partial_j(\partial_j\mathcal{A}_i^{(m)}(t,\mathbf{x}) - \partial_i\mathcal{A}_j^{(m)}(t,\mathbf{x})) - (\mathcal{A}_i^{(m+2)}(t,\mathbf{x}) - \partial_i\mathcal{A}_0^{(m+1)}(t,\mathbf{x})) = 0, \qquad (m \geq 0), \tag{A.19}$$

$$\tilde{\varphi}^{0(m)}(t,\mathbf{x}) = \partial_i(\mathcal{A}_i^{(m+1)}(t,\mathbf{x}) - \partial_i\mathcal{A}_0^{(m)}(t,\mathbf{x})) = 0, \qquad (m \geq 0). \tag{A.20}$$

It must be noted the identities

$$\tilde{\varphi}^{0(m+1)}(t,\mathbf{x}) = \partial_i\tilde{\varphi}^{i(m)}(t,\mathbf{x}), \qquad (m \geq 0). \tag{A.21}$$

Thus the only independent constraint of (A.20) is $m = 0$ case. It is expressed, using (A.16), as the gauss law constraint,

$$\tilde{\varphi}^{0(0)}(t, \mathbf{x}) \;=\; \partial_i \mathit{\Pi}^i_{(0)}(t, \mathbf{x}) \;=\; 0. \tag{A.22}$$

Following the Dirac's standard procedure of constraints [6] we classify them and eliminate the second class constraints. The constraints (A.17) ($m \geq 2$) are paired with the constraints (A.19) ($m \geq 0$) to form second class sets. They are used to eliminate canonical pairs $(\mathcal{A}^{(m)}_i(t, \mathbf{x}), \mathit{\Pi}^i_{(m)}(t, \mathbf{x}))$, $(m \geq 2)$ as

$$\mathcal{A}^{(m)}_i(t, \mathbf{x}) \;=\; \partial_j(\partial_j \mathcal{A}^{(m-2)}_i(t, \mathbf{x}) - \partial_i \mathcal{A}^{(m-2)}_j(t, \mathbf{x})) + \partial_i \mathcal{A}^{(m-1)}_0(t, \mathbf{x}), \tag{A.23}$$

$$\mathit{\Pi}^i_{(m)}(t, \mathbf{x})) \;=\; 0, \qquad (m \geq 2). \tag{A.24}$$

The constraints (A.17) ($m = 1$) and (A.16) are paired to a second class set and are used to eliminate $(\mathcal{A}^{(1)}_i(t, \mathbf{x}), \mathit{\Pi}^i_{(1)}(t, \mathbf{x}))$ as

$$\mathcal{A}^{(1)}_i(t, \mathbf{x}) \;=\; \mathit{\Pi}^i_{(0)}(t, \mathbf{x}) + \partial_i \mathcal{A}^{(0)}_0(t, \mathbf{x}), \tag{A.25}$$

$$\mathit{\Pi}^i_{(1)}(t, \mathbf{x}) \;=\; 0. \tag{A.26}$$

After eliminating the canonical pairs $(\mathcal{A}^{(m)}_i(t, \mathbf{x}), \mathit{\Pi}^i_{(m)}(t, \mathbf{x}))$, $(m \geq 1)$ using the second class constraints the system is described in terms of the canonical pairs $(\mathcal{A}^{(0)}_i(t, \mathbf{x}), \mathit{\Pi}^i_{(0)}(t, \mathbf{x}))$ and $(\mathcal{A}^{(m)}_0(t, \mathbf{x}), \mathit{\Pi}^0_{(m)}(t, \mathbf{x}))$, $(m \geq 0)$. The Dirac brackets among them remain same as the Poisson brackets. Remember the d dimensional fields are identified by (2.8) as

$$A_\mu(t, \mathbf{x}) \;=\; \mathcal{A}_\mu(t, 0, \mathbf{x}) \;=\; \mathcal{A}^{(0)}_\mu(t, \mathbf{x}), \qquad \Pi^\mu(t, \mathbf{x}) \;=\; \mathit{\Pi}^\mu_{(0)}(t, \mathbf{x}). \tag{A.27}$$

The remaining constraints are (A.22) and (A.15),

$$\partial_i \mathit{\Pi}^i_{(0)}(t, \mathbf{x}) \;=\; 0, \qquad \mathit{\Pi}^0_{(m)}(t, \mathbf{x}) \;=\; 0, \qquad (m \geq 0) \tag{A.28}$$

are the first class constraints. The Hamiltonian (A.8) in the reduced variables is

$$\begin{aligned} H(t) \;=\; \int d\mathbf{x} \Big[\sum_{m=0}^{\infty} \; & \mathcal{A}^{(m+1)}_0(t, \mathbf{x}) \mathit{\Pi}^0_{(m)}(t, \mathbf{x}) \;-\; \mathcal{A}^{(0)}_0(t, \mathbf{x})(\partial_i \mathit{\Pi}^i_{(0)}(t, \mathbf{x})) \\ & + \frac{1}{2}(\mathit{\Pi}^i_{(0)}(t, \mathbf{x}))^2 \;+\; \frac{1}{4}(\partial_j \mathcal{A}^{(0)}_i(t, \mathbf{x}) - \partial_i \mathcal{A}^{(0)}_j(t, \mathbf{x}))^2 \Big]. \end{aligned} \tag{A.29}$$

The $U(1)$ generator (A.7) is

$$G[\Lambda] \;=\; \int d\mathbf{x} \left[\sum_{m=0}^{\infty} \Lambda^{(m+1)}(t, \mathbf{x}) \mathit{\Pi}^0_{(m)}(t, \mathbf{x}) \;-\; \Lambda^{(0)}(t, \mathbf{x})(\partial_i \mathit{\Pi}^i_{(0)}(t, \mathbf{x})) \right], \tag{A.30}$$

where

$$\Lambda(t, \lambda) \;=\; \sum_{m=0}^{\infty} \Lambda^{(m)}(t, \mathbf{x}) e_m(x^0), \qquad \text{and} \qquad \dot{\Lambda}^{(m)}(t, \mathbf{x}) = \Lambda^{(m+1)}(t, \mathbf{x}). \tag{A.31}$$

The first class constraints $\Pi^0_{(m)}(t,\mathbf{x}) = 0, (m \geq 0)$ in (A.28) mean that $\mathcal{A}_0^{(m)}(t,\mathbf{x}), (m \geq 0)$ are the gauge degrees of freedom and we can assign them any function of $\mathbf{x}$ for all values of m at given time $t = t_0$. It is equivalent to say that we can assign any function of time to $\mathcal{A}_0^{(0)}(t,\mathbf{x})$ for all value of t, due to the equation of motion $\dot{\mathcal{A}}_0^{(m)}(t,\mathbf{x}) = \mathcal{A}_0^{(m+1)}(t,\mathbf{x})$. In this way we can understand that the Hamiltonian (A.29) is equivalent with the standard form of the canonical Hamiltonian of the Maxwell theory,

$$\begin{aligned} H(t) &= \int d\mathbf{x} \Big[\dot{A}_0(t,\mathbf{x})\Pi^0(t,\mathbf{x}) - A_0(t,\mathbf{x})(\partial_i\Pi^i(t,\mathbf{x})) \\ &\quad + \frac{1}{2}(\Pi^i(t,\mathbf{x}))^2 + \frac{1}{4}(\partial_j A_i(t,\mathbf{x}) - \partial_i A_j(t,\mathbf{x}))^2 \Big] \end{aligned} \tag{A.32}$$

in which $A_0(t,\mathbf{x})$ is arbitrary function of time. In the same manner the $U(1)$ generator (A.30) is

$$G[\Lambda] = \int d\mathbf{x} \Big[\dot{\lambda}(t,\mathbf{x})\Pi^0(t,\mathbf{x}) - \lambda(t,\mathbf{x})(\partial_i\Pi^i(t,\mathbf{x})) \Big], \tag{A.33}$$

in which the gauge parameter function $\lambda(t,\mathbf{x}) \equiv \Lambda^{(0)}(t,\mathbf{x})$ is regarded as any function of time.

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