

ESI 780 (1999)  
ULB-TH-99/24  
FTUV/99-66  
CERN-TH/99-320  
MPI/PhT-97-48  
SPIN-1999/24  
hep-th/9910201

## Cohomological analysis of gauged-fixed gauge theories

Glenn Barnich<sup>a,b</sup>, Marc Henneaux<sup>b,c</sup>, Tobias Hurth<sup>d,e</sup>  
and Kostas Skenderis<sup>f</sup>

<sup>a</sup> *Departament de Física Teòrica, Universitat de València,  
E-46100 Burjassot (València), Spain.*

<sup>b</sup> *Physique Théorique et Mathématique, Université Libre de Bruxelles,  
Campus Plaine C.P. 231, B-1050 Brussels, Belgium*

<sup>c</sup> *Centro de Estudios Científicos de Santiago,  
Casilla 16443, Santiago 9, Chile*

<sup>d</sup> *CERN, Theory Division,  
CH-1211 Geneva 23, Switzerland*

<sup>e</sup> *Max-Planck-Institute for Physics, Werner-Heisenberg-Institute,  
Föhringer Ring 6, D-80805 Munich, Germany*

<sup>f</sup> *Spinoza Insitute, University of Utrecht, Leuvenlaan 4,  
3584 CE Utrecht, The Netherlands*

### Abstract

The relation between the gauge-invariant local BRST cohomology involving the antifields and the gauge-fixed BRST cohomology is clarified. It is shown in particular that the cocycle conditions become equivalent once it is imposed, on the gauge-fixed side, that the BRST cocycles should yield deformations that preserve the nilpotency of the (gauge-fixed) BRST differential. This shows that the restrictions imposed on local counterterms by the Quantum Noether condition in the Epstein–Glaser construction of gauge theories are equivalent to the restrictions imposed by BRST invariance on local counterterms in the standard Lagrangian approach.

\* gbarnich@ulb.ac.be, henneaux@ulb.ac.be, tobias.hurth@cern.ch, K.Skenderis@phys.uu.nl

# 1 Introduction

In the BRST approach to perturbative gauge theories [1, 2], the possible counterterms are restricted by Ward–Slavnov–Taylor identities [3, 4, 5], which have a cohomological interpretation. If one follows the path-integral approach and takes into account the renormalization of the BRST symmetry à la Zinn-Justin [6] by introducing sources coupled to the BRST variation of the fields and the ghosts, it may be shown [6, 1, 7, 8, 9] that the counterterms must fulfil the BRST invariance condition

$$sA = 0, \tag{1.1}$$

where  $s$  is the BRST differential acting in the space of fields, ghosts and associated sources (“antifields” [10]). The counterterms are local, so  $A$  in (1.1) is given by the integral of a local  $n$ -form  $a$ , in terms of which the BRST invariance condition becomes

$$sa + db = 0, \tag{1.2}$$

for some  $(n - 1)$ -form  $b$ . Following an initial investigation by Joglekar and Lee [11], the general solution of (1.2) for Yang–Mills gauge models has been determined in [12], where it was shown that up to trivial terms of the form  $sc + de$ , the counterterm  $a$  in (1.2) is equal to a strictly gauge-invariant operator, plus Chern–Simons terms in odd space-time dimensions (in the absence of  $U(1)$  factors for which there are further solutions [13], also dealt with in [12]). This guarantees renormalizability of the theory in the “modern sense” [14] in any number of spacetime dimensions, and in the standard power-counting sense in 4 dimensions.

If one follows instead the operator formalism and the Quantum Noether method based on the gauge-fixed BRST formulation [15, 16], one finds that the counterterms are constrained by the condition

$$\gamma^g a + db \approx 0, \tag{1.3}$$

where  $\gamma^g$  is the “gauge-fixed” BRST differential acting on the fields and where both  $a$  and  $b$  involve only the fields (no antifield). The symbol  $\approx$  means “equal when the (gauge-fixed) equations of motion hold”.

The question then arises as to whether (1.2) and (1.3) are equivalent. It may be shown that the antifield and gauge-fixed local cohomologies are equivalent [17], so that any solution  $a$  of  $sa = 0$  defines a solution  $a'$  of  $\gamma^g a' \approx 0$  and vice versa. This is *not* true, however, for the cohomologies modulo  $d$  [18]. In particular, there are solutions of (1.3) that have *no* analogue in the antifield cohomology and which, therefore, do *not* correspond to an integrated, gauge-invariant operator. An example is given by the Curci–Ferrari mass term [19]

$$-\frac{1}{2}A_\mu^a A_a^\mu + \bar{C}_a C^a, \tag{1.4}$$

which is a solution of (1.3) in the gauge where the equation of motion for the auxiliary  $b$ -field is  $b_a + \partial_\mu A_a^\mu - \frac{1}{2}\bar{C}_b f_{ac}^b C^c = 0$ , but which does not define an integrated gauge-invariant

operator. The properties of (1.4) have been studied in [20, 21, 22, 23, 24, 25]. Thus, (1.2) and (1.3) are in general not equivalent<sup>1</sup>.

If, however, the cocycle condition (1.3) is supplemented by the requirement of nilpotency of the deformed BRST differential (which is required if we want the theory to be unitary) the Curci–Ferrari mass term is excluded. It is the purpose of this letter to show that, quite generally, the gauged-fixed cocycle condition (1.3), supplemented by the requirement that the deformation generated by the permissible counterterms should preserve (on-shell) nilpotency of the BRST symmetry, is equivalent to the antifield cocycle condition (1.2), which controls the counterterms in the Zinn-Justin approach.

This letter is organized as follows. In the next section, we recall some salient properties of the gauge-fixed action. The equivalence of the two cocycle conditions is shown in section 3, while a discussion of trivial solutions is presented in section 4. In section 5, we review the analysis of counterterms in the Quantum Noether method and show how nilpotency of the deformed BRST differential arises in that context. Finally, in an appendix we present an analysis of the relation between the antifield and the weak gauged-fixed cohomology using methods of homological algebra.

## 2 Gauge-fixed action

The starting point is the solution  $S[\phi, \phi^*]$  of the master equation

$$(S, S) \equiv 2 \frac{\delta^R S}{\delta \phi^A} \frac{\delta^L S}{\delta \phi_A^*} \equiv -2 \frac{\delta^R S}{\delta \phi_A^*} \frac{\delta^L S}{\delta \phi^A} = 0. \quad (2.1)$$

We use DeWitt’s condensed notations. The solution  $S$  is a local functional, as are all functionals without free indices occurring below. The “fields”  $\phi^A$  include the original fields, the ghosts, as well as the auxiliary fields and the antighosts of the non-minimal sector. We assume that the canonical transformation necessary for gauge-fixing has already been performed, so that the gauge-fixed action is simply obtained by setting the antifields equal to zero,  $S^g[\phi] = S[\phi, \phi^* = 0]$ . The gauge-fixed BRST differential  $\gamma^g$  is defined by

$$\gamma^g \phi^A = - \frac{\delta^R S}{\delta \phi_A^*} \Big|_{\phi^* = 0}, \quad (2.2)$$

where right and left derivatives are defined by  $\delta F = (\delta^R F / \delta z^\alpha) \delta z^\alpha = \delta z^\alpha (\delta^L F / \delta z^\alpha)$ . We use the conventions of [17], but the derivations are taken to act from the left (so  $sF = (S, F)$  etc.). The transformation generated by  $\gamma^g$  leaves the gauge-fixed action invariant because of the master equation. As a result, the functional derivatives of  $S^g$  transform into themselves:

$$\gamma^g \frac{\delta^R S^g}{\delta \phi^A} = - \frac{\delta^R S^g}{\delta \phi^B} \frac{\delta^R}{\delta \phi^A} \left( \frac{\delta^L S}{\delta \phi_B^*} \right); \quad (2.3)$$

---

<sup>1</sup>We keep here the auxiliary  $b$ -field, but similar considerations can be made if one eliminates the auxiliary fields, since the gauge-fixed BRST cohomological groups are invariant under such an elimination.

here and below, it is understood that  $\phi^*$  is set equal to zero after the second derivatives have been computed. The gauge-fixed BRST differential is weakly nilpotent,

$$(\gamma^g)^2 \phi^A = \frac{\delta^R S^g}{\delta \phi^B} \frac{\delta^L}{\delta \phi_B^*} \left( \frac{\delta^R S}{\delta \phi_A^*} \right). \quad (2.4)$$

Both (2.3) and (2.4) are direct consequences of the definition (2.2) and the master equation.

The BRST differential in the space of the fields and the antifields is defined by

$$sF = (S, F) \quad (2.5)$$

for any  $F(\phi, \phi^*)$ . It is related to  $\gamma^g$  as  $s\phi^A = \gamma^g \phi^A +$  antifield-dependent terms or, which is the same,  $\gamma^g \phi^A = s\phi^A |_{\phi^*=0}$ . It is strictly nilpotent,  $s^2 = 0$ . It will be useful in the sequel to give a special name to the terms linear in  $\phi^*$  in the expansion of  $s\phi^A$ ,

$$s\phi^A = \gamma^g \phi^A + \lambda^g \phi^A + O((\phi^*)^2), \quad (2.6)$$

with

$$\begin{aligned} \lambda^g \phi^A &= (S, \phi^A) |_{\text{linear in } \phi^*} \\ &= -\phi_B^* \frac{\delta^L}{\delta \phi_B^*} \left( \frac{\delta^R S}{\delta \phi_A^*} \right). \end{aligned} \quad (2.7)$$

The action of  $s$  on the antifields can also be expanded in powers of the antifields. One has  $s\phi_A^* = \delta^g \phi_A^* + \gamma^g \phi_A^* + O((\phi^*)^2)$  where  $\delta^g$  is the Koszul differential associated with the gauge-fixed stationary surface [17],

$$\delta^g \phi_A^* = (S, \phi_A^*) |_{\phi^*=0} = \frac{\delta^R S^g}{\delta \phi^A} \quad (2.8)$$

and where

$$\gamma^g \phi_A^* = (S, \phi_A^*) |_{\text{linear in } \phi^*} = \phi_B^* \frac{\delta^R}{\delta \phi^A} \left( \frac{\delta^L S}{\delta \phi_B^*} \right). \quad (2.9)$$

One easily verifies the relations  $(\delta^g)^2 = 0$ ,  $\delta^g \gamma^g + \gamma^g \delta^g = 0$ ,  $(\delta^g \lambda^g + \lambda^g \delta^g) \phi^A + (\gamma^g)^2 \phi^A = 0$  from the definitions of the derivations  $\delta^g$ ,  $\gamma^g$  and  $\lambda^g$ . These relations are actually the first ones to arise in the expansion of  $s^2 = 0$  in powers of the antifields.

The canonical transformation appropriate to gauge-fixing does not modify the cohomology of  $s$  neither in the space of local functions nor in the space of local functionals, because it is just a change of variables. So, in the case of Yang–Mills theory, the cohomology group  $H^0(s, \mathcal{F})$  of the BRST differential in the space  $\mathcal{F}$  of local functionals is still given by the analysis of [12]. In  $H^0(s, \mathcal{F})$ , the superscript 0 is the total ghost number. Note, however, that the expansion  $s = \delta^g + \gamma^g + \lambda^g + \dots$  is not the standard expansion arising prior to gauge-fixing, since the degree involved here is the total antifield number that gives equal weight to each antifield, irrespective of its “antighost” number. This is why it is the Koszul resolution associated with the gauge-fixed stationary surface that

arises in the present analysis, and not the Koszul–Tate resolution associated with the gauge-invariant equations of motion.

Since the equations of motion following from the gauge-fixed action have no gauge invariance (by assumption), one may invoke the general results of [26, 27, 28] to assert that

$$H_k(\delta^g, \mathcal{F}) = 0, \quad k \geq 2 \quad (2.10)$$

where  $k$  is the total antifield number used in the above expansions. In words: any local functional  $A \in \mathcal{F}$  ( $A = \int a$ ) that solves  $\delta^g A = 0$  ( $\delta^g a + db = 0$ ) and is at least quadratic in the antifields has the form  $A = \delta^g C$  ( $a = \delta^g c + dm$ ).

We have of course  $H_k(\delta^g, \mathcal{L}) = 0$  for  $k \geq 1$  in the space  $\mathcal{L}$  of local functions [17], but we shall need the version valid for local functionals below. This is a direct consequence of Theorem 8.3 or 10.1 of [26], which states that there can be no non-trivial higher-order conservation laws for an action having no gauge symmetries. This theorem is also known as the “Vinogradov two-line theorem”. Because higher-order conservation laws and elements of  $H_k(\delta^g, \mathcal{F})$  (also denoted  $H_k(\delta^g|d)$ ) are in bijection, the property (2.10) follows. In general, however, the homological group  $H_1(\delta^g, \mathcal{F})$  does not vanish (even though  $H_1(\delta^g, \mathcal{L}) = 0$  in the space of local functions) and is related to the global symmetries of the gauge-fixed action [26].

### 3 Reconstruction Theorem

We now have all the required tools to show that a local counterterm of the gauge-fixed formalism that preserves nilpotency defines a local counterterm of the antifield Zinn-Justin approach. That is, the condition

$$\gamma^g A_0 \approx 0 \quad (3.1)$$

for the local functional  $A_0[\phi] = \int a_0$  (which implies (1.3) for the integrand  $a_0$ ), together with the fact that the associated deformed BRST symmetry  $\gamma^g + e\Delta$  *should remain weakly nilpotent* (for the new equations of motion) to  $O(e^2)$  in the deformation parameter  $e$ ,

$$(\gamma^g + e\Delta)^2 \approx' O(e^2), \quad (3.2)$$

determines a local functional cocycle  $A[\phi, \phi^*]$  of the antifield cohomology

$$sA = 0. \quad (3.3)$$

In (3.2), the symbol  $\approx'$  means “equal when the deformed equations of motion  $\delta^R(S + eA_0)/\delta\phi^A = 0$  hold”. The relationship between  $A$  and  $A_0$  is

$$A = A_0 + A_1 + A_2 + O((\phi^*)^3), \quad (3.4)$$

where  $A_1$  (respectively,  $A_2$ ) is linear (respectively, quadratic) in the antifields.

The above derivation  $\Delta$  is the deformation of the BRST-symmetry and is related to the deformation  $A_0$  of the action as follows. When one adds  $eA_0$  to the gauge-fixed action,  $S^g[\phi] \rightarrow S^g[\phi] + eA_0[\phi]$ , one modifies the gauge-fixed BRST symmetry as  $\gamma^g \rightarrow \gamma^g + e\Delta$

in such a way that  $(\gamma^g + e\Delta)(S^g + eA_0) = \mathcal{O}(e^2)$ . The existence of  $\Delta$  is guaranteed by the cocycle condition (3.1), which we can rewrite as

$$\gamma^g A_0 + \delta^g A_1 = 0 \quad (3.5)$$

for some local functional  $A_1$  linear in the antifields. We have  $\Delta\phi^A = \Delta^A$ ,  $A_1 = \phi_A^* \Delta^A (-1)^{g_A}$ , where  $g_A$  is the Grassman parity of  $\phi^A$ .

As (3.4) shows, the relationship between  $A[\phi, \phi^*]$  and  $A_0[\phi]$  is that  $A[\phi, \phi^*]$  starts like  $A_0[\phi]$  to zeroth order in the antifields. Thus, the question is whether any local functional  $A_0$  that fulfils both (3.1) and (3.2) can be completed by terms of higher orders in the antifields to yield a (local functional) solution of (3.3).

The converse statement, namely, that any local functional  $A[\phi, \phi^*]$  solution of (3.3) defines, when setting the antifields equal to zero, a cocycle of the weak cohomology (3.1) fulfilling (3.2), is rather obvious. Indeed, if  $sA = 0$ , then  $\gamma^g A_0 \approx 0$  (term independent of the antifields in  $sA = 0$ ). Furthermore, at the next order,

$$\lambda^g A_0 + \gamma^g A_1 + \delta^g A_2 = 0, \quad (3.6)$$

a relation that is seen to be equivalent to (3.2) by rephrasing the condition (3.2) in terms of  $A_0$  and  $A_1$ . On the one hand, direct calculations yield

$$\begin{aligned} \gamma^g A_1 &= \phi_A^* \Delta^B \frac{\delta^L}{\delta\phi^B} \left( \frac{\delta^R S}{\delta\phi_A^*} \right) - \phi_A^* (\gamma^g \Delta^A) \\ \lambda^g A_0 &= \phi_A^* \frac{\delta^R A_0}{\delta\phi^B} \frac{\delta^L}{\delta\phi_B^*} \left( \frac{\delta^R S}{\delta\phi_A^*} \right). \end{aligned} \quad (3.7)$$

On the other hand, if one replaces the weak equality by a strong equality in  $(\gamma^g + e\Delta)^2 \phi^A \approx' \mathcal{O}(e^2)$ , one gets, in view of (2.4),

$$(\gamma^g + e\Delta)^2 \phi^A = \left( \frac{\delta^R S^g}{\delta\phi^B} + e \frac{\delta^R A_0}{\delta\phi^B} \right) \left( \frac{\delta^L}{\delta\phi_B^*} \left( \frac{\delta^R S}{\delta\phi_A^*} \right) + e\mu^{AB} \right) + \mathcal{O}(e^2) \quad (3.8)$$

for some  $\mu^{AB}$ . Thus, (3.2) becomes to order  $e$ ,

$$\gamma^g \Delta^A - \Delta^B \frac{\delta^L}{\delta\phi^B} \left( \frac{\delta^R S}{\delta\phi_A^*} \right) \approx \frac{\delta^R A_0}{\delta\phi^B} \frac{\delta^L}{\delta\phi_B^*} \left( \frac{\delta^R S}{\delta\phi_A^*} \right), \quad (3.9)$$

which shows that (3.2) is indeed equivalent to the statement that  $\gamma^g A_1 + \lambda^g A_0$  vanishes weakly, or which is the same, (3.6).

Accordingly, to each counterterm of the antifield Zinn-Justin approach corresponds a counterterm of the BRST-Noether method.

Conversely, given a solution of (3.1) – or (3.5) – which also fulfils (3.2), the question is whether one can construct a local functional  $A$  that starts like  $A_0 + A_1$  and is BRST-invariant. That (3.1) (or (3.5)) by itself does not guarantee the existence of  $A$  is illustrated by the Curci-Ferrari mass term and has been explained in [18].

The problem arises because the perturbative construction, yielding successively  $A_2$ ,  $A_3$ , etc., given the “initial data”  $A_0$  and  $A_1$  along the lines of homological perturbation

theory applied to the antifield formalism [29, 30, 17] can be obstructed in the space of local functionals. The obstructions are in the homological groups  $H_k(\delta^g, \mathcal{F})$  (also denoted by  $H_k(\delta^g|d)$ ) The point is that the equations defining the higher-order terms  $A_2, A_3$  etc. take the form

$$\delta^g A_k = B_{k-1}, \quad (3.10)$$

where the local functional  $B_{k-1}$  involves only the lower-order terms  $A_i$  ( $i < k$ ) and can be shown to be  $\delta^g$ -closed. To infer that  $B_{k-1}$  is exact, one needs either  $H_{k-1}(\delta^g, \mathcal{F}) = 0$  or, if  $H_{k-1}(\delta^g, \mathcal{F})$  does not vanish, additional information guaranteeing that  $B_{k-1}$  is in the zero class.

As we recalled above,  $H_j(\delta^g, \mathcal{F}) = 0$  for  $j > 1$ . Thus the only obstructions may arise for  $k - 1 = 1$ , i.e. for  $A_2$ . If it can be proven that  $A_2$  exists, there cannot be any further obstruction at the next orders, and  $A$  also exists. The strategy of the construction of  $A$  from  $A_0$  and  $A_1$  consists, then, in showing that one avoids the obstruction for  $A_2$ . It is here that the condition (3.2) is necessary.

The equation (3.10) for  $A_2$  is actually (3.6) with

$$B_1 = -\lambda^g A_0 - \gamma^g A_1. \quad (3.11)$$

We must show that  $B_1$  is  $\delta^g$ -exact, i.e. that it vanishes weakly. But this is guaranteed because (3.6) and (3.2) have been shown to be equivalent, so that (3.2) implies (3.6) or (3.11). Therefore, the obstruction for  $A_2$  is avoided, as announced.

One may understand the equivalence between (3.2) and (3.6) more directly, in terms of the master equation itself. As is known [31], the elements of  $H^0(s, \mathcal{F})$  can be viewed as consistent, first-order deformations of the master equation,  $S \rightarrow S' = S + eA$ ,  $(S, S) = 0 \rightarrow (S', S') = O(e^2)$ . As we have indicated, given  $A_0$ , the obstruction to the construction of  $A$  can only occur for  $A_2$ , i.e. we must verify that the term (3.6) is zero. But this term is the term linear in the antifields in the master equation. So, the absence of obstruction is equivalent to the statement that  $(S', S')|_{\text{linear in } \phi^*}$  vanishes, or  $((S', S'), \phi^A)|_{\phi^*=0} = 0$ . This is precisely the statement that the deformed BRST symmetry remains nilpotent, as the Jacobi identity for the antibrackets easily shows.

## 4 Trivial Solutions

The map between the antifield cohomology and the gauged-fixed cohomology fails to be surjective, since only classes with representatives fulfilling the extra condition (3.6) are in the image of the map. The map fails also to be injective, because there are non-trivial cocycles of the antifield cohomology that are mapped on trivial cocycles of the gauged-fixed cohomology. This is best seen on a simple example. Consider electromagnetism with a neutral scalar field  $\phi$  and impose the gauge condition  $\partial_\mu A^\mu = \mu\phi$  through the equation of motion for the auxiliary  $b$ -field, where  $\mu$  is a constant with dimension  $L^{-1}$ . With that gauge choice, the nontrivial cocycle  $\int d^4x \phi$  of the gauge-invariant cohomology becomes trivial in the weak gauge-fixed cohomology since one has  $\phi \approx s\bar{C} + \partial_\mu A^\mu$ . Similar considerations would apply to any function  $f(\phi)$  in the gauge  $\partial_\mu A^\mu = f(\phi)$ . Although

we will not provide a precise argument, we note that these mod- $d$  coboundaries of the gauge-fixed cohomology, which are present in peculiar gauges, are not expected to be physically trivial. The reason is that correlation functions of gauge-invariant operators do not change in different gauges (for a proof within the EG framework, see [36]).

Note that the gauge-fixed action has a nontrivial global symmetry acting on the unphysical variables, namely the shift  $\bar{C} \rightarrow \bar{C} + \theta$ , where  $\theta$  is a constant Grassmann odd parameter, corresponding to the cohomology class  $\int d^4x \bar{C}^*$ . This phenomenon is precisely related in the appendix to the non-injectivity of the above map.

## 5 Counterterms in the Quantum Noether method

We show in this section how the nilpotency condition arises in the Quantum Noether method. This method [15, 16] is a general method for constructing theories with global symmetries using the Epstein-Glaser (EG) approach to quantum field theory. In this approach, which was introduced by Bogoliubov and Shirkov [32] and developed by Epstein and Glaser [33, 34], the (perturbative) S-matrix is directly constructed in the Fock space of asymptotic fields by imposing causality and Poincaré invariance. The method can be regarded as an “inverse” of the cutting rules: one builds  $n$ -point functions by appropriately “gluing” together  $m$ -point functions ( $m < n$ ). Moreover, this method directly yields a finite perturbation theory; one avoids UV infinities altogether by proper treatment of  $n$ -point functions as operator-valued distributions. The coupling constants of the theory,  $e$ , are replaced by tempered test functions  $g(x)$  (i.e. smooth functions rapidly decreasing at infinity), which switch on the interactions. The iterative construction of the S-matrix starts by giving a number of free fields satisfying (gauged-fixed) fields equation (so that there are propagators) and the first term,  $T_1$ , in the perturbative expansion of the S-matrix. Ultimately, one is interested in the theory in which  $g(x)$  becomes again constant,  $g(x) \rightarrow e$ . This is the so-called adiabatic limit. We use the convention to still keep  $e$  explicit, in which case the adiabatic limit is  $g(x) \rightarrow 1$ . We work before the adiabatic limit is taken, as the latter does not always exist because of physical infrared singularities.

Causality and Poincaré invariance completely fix the S-matrix up to local terms. The remaining local ambiguity is further constrained by symmetries. It is the purpose of our analysis to determine the precise restrictions imposed on these local terms by Ward identities. At tree level the local terms are equal to the Lagrangian of the conventional approach [15], but new local terms may be introduced at each order in perturbation theory. The local terms at loop level correspond to the counterterms in the Lagrangian approach, although their role is not to subtract infinities, as the perturbative expansion is already finite. If the form of these local terms remains the same to all orders in perturbation theory then the theory is renormalizable.

The Quantum Noether method consists of adding a coupling to the Noether current  $j_0^\mu$  that generates the asymptotic (and hence linear) symmetry in the theory and then requiring that this current be conserved inside correlation functions. There are a number of equivalent ways to present this condition [15, 16]. Here we follow [16], where the condition was formulated in terms of the interacting Noether current. The Ward identity,



formula (3.1) in [16], contains terms that vanish in the (naive) adiabatic limit,  $g(x) \rightarrow 1$ . Their explicit form, which can be found in [16], is not important for the present analysis. Here we will schematically denote them by  $\partial_\mu g \tilde{j}^\mu$ . Due to these terms the interacting BRST charge is not conserved before the adiabatic limit is taken. For a discussion of the implications of this fact (and also of other difficulties encountered when attempting to construct the interacting BRST charge) we refer to [35]. We note, however, that considerations involving only currents are sufficient in order to derive all consequences of nonlinear symmetries for time-ordered products. The Quantum Noether condition reads

$$\partial_\mu^x T[j_0^\mu(x) T_1(x_1) \cdots T_1(x_n)] = \partial_\mu g \tilde{j}^\mu. \quad (5.1)$$

Working out the consequences of this condition to all orders, one recovers the non-linear structure in a manner similar to the way the Noether method works in classical field theory [15, 16]. Further consistency requirements on the theory follow by considering multi-current correlation functions. In particular, the two-current equation is

$$\partial_\mu^x T[j_0^\mu(x) j_0^\nu(y) T_1(x_1) \cdots T_1(x_n)] = \partial_\mu g \tilde{j}^{\mu\nu}, \quad (5.2)$$

where again we have only schematically included terms that vanish in the naive adiabatic limit. The explicit form of these terms, as well as an all-order analysis of (5.2), will be presented in [36].

We are interested in gauge theories. In this case the relevant symmetry is BRST symmetry. We now present the analysis of (5.1), (5.2) for this case to first non-trivial order. This is sufficient in order to connect with the analysis of the preceding sections. Equation (5.1) at first order yields the following condition on  $\mathcal{L}_1 = (\hbar/i)T_1$  :

$$\gamma^g \mathcal{L}_1 = \partial_\mu \mathcal{L}_1^\mu + e \Delta \phi^A \mathcal{K}_{AB}^{(0)} \phi^B, \quad (5.3)$$

where  $\phi^A$  denotes collectively all the fields;  $\gamma^g = [f j^0, \cdot]$  generates the asymptotic transformation rules;  $\Delta \phi^A$  is defined by eq. (5.3). It was shown in [15, 16] that  $\Delta \phi^A$  is the next-order symmetry transformation rule;  $\mathcal{L}_1^\mu$  is some local function of  $\phi^A$  and its first derivative  $\partial_\mu \phi^A$ , and  $\mathcal{K}_{AB}^{(0)} \phi^B$  are the free-field equations.

To work out the consequences of condition (5.2), we first note that since  $j_0^\mu$  is the gauged-fixed BRST current it satisfies

$$\gamma^g j_0^\mu = \partial_\nu T_0^{\mu\nu} + J_0^{\mu A} \mathcal{K}_{AB}^{(0)} \phi^A, \quad (5.4)$$

where  $T_0^{\mu\nu}$  is antisymmetric in  $\mu, \nu$ , and  $J_0^{\mu A}$  may contain derivatives acting on the free-field equations. Equation (5.4) guarantees that (5.2) is satisfied at  $n = 0$  (i.e. no  $T_1$  involved). At  $n = 1$  one finds the following condition:

$$J_0^{\mu A} \frac{\delta \mathcal{L}_1}{\delta \phi^A} + \gamma^g j_1^\mu + \Delta j_0^\mu = \partial_\mu T_1^{\mu\nu} + J_1^{\mu A} \mathcal{K}_{AB}^{(0)} \phi^B, \quad (5.5)$$

for some  $T_1^{\mu\nu}$  and  $J_1^{\mu A}$  (also possibly containing derivatives acting on  $\mathcal{K}_{AB}^{(0)} \phi^B$ );  $\delta \mathcal{L}_1 / \delta \phi^A$  is the Euler derivative of  $\mathcal{L}_1$ , and if  $J_0^{\mu A}$  contained derivatives in (5.4) they now act on

$\delta\mathcal{L}_1/\delta\phi^A$ ;  $j_1^\mu$  arises as a local normalization term of the correlation function  $T[j_0^\mu(x_1)T_1(x_2)]$ . It was shown in [16] that it is the Noether current that generates the symmetry transformation rules  $\Delta\phi^A$ . Combining with (5.4) we obtain

$$(\gamma^g + e\Delta)(j_0^\mu + ej_1^\mu) = \partial_\mu(T_0^{\mu\nu} + eT_1^{\mu\nu}) + (J_0^{\mu A} + eJ_1^{\mu A})\mathcal{K}_{AB}^{(1)}\phi^A + O(e^2), \quad (5.6)$$

where  $\mathcal{K}_{AB}^{(1)}\phi^A$  are the field equations that follow from the Lagrangian  $\mathcal{L}_0 + e\mathcal{L}_1$ , where  $\mathcal{L}_0$  generates the free field equations.

Conditions (5.3) and (5.6) are equivalent to conditions (3.1) and (3.2) we analysed in section 3.

## 6 Conclusions

In this letter, we have shown that the restrictions imposed on counterterms by the Quantum Noether condition in the Epstein–Glaser construction of gauge theories are equivalent to those imposed in the Zinn-Justin (“antifield”) approach to the renormalization of gauge theories. The crucial requirement that guarantees the equivalence of the restrictions on the counterterms (“cocycle conditions”) is the nilpotency of the deformed BRST generator. We have also analysed how this requirement arises in the EG approach. Similar considerations apply to anomalies. This will be discussed elsewhere [36].

## Acknowledgements

The authors acknowledge the hospitality of the Erwin Schrödinger International Institute for Mathematical Physics in Vienna, where this collaboration has been started. This work has been partly supported by the “Actions de Recherche Concertées” of the “Direction de la Recherche Scientifique - Communauté Française de Belgique”, by IISN - Belgium (convention 4.4505.86), by Proyectos FONDECYT 1970151 and 7960001 (Chile). TH was supported by the DOE under grant No. DE-FG03-92-ER40701 during a visit of the theory group at CALTECH where part of this work was done. KS is supported by the Netherlands Organization for Scientific Research (NWO).

## Appendix: Antifield (“canonical”) versus weak gauge-fixed BRST cohomology

In this appendix, the general relation in the space of local functionals  $\mathcal{F}$  between the antifield BRST cohomology computed before gauge-fixing and the weak gauge-fixed version is analysed by using standard tools from homological algebra.

As mentioned in section 2, the canonical transformation used for gauge-fixing does not modify the antifield BRST cohomology and we assume that this transformation has been done. The complete BRST differential  $s$  in canonical form then differs from that

in gauge-fixed form only in the grading used for the expansion, called generically “resolution degree” below. The grading associated to the canonical form consists in assigning antighost number 1 to the antifields of the original fields, 2 to the antifields of the ghosts, 3 to the antifields of the ghosts for ghosts, etc., while in the gauge-fixed case the grading consists in assigning antifield number 1 to all the antifields.

In both cases, we have an expansion of the form  $s = \delta' + \gamma' + \sum_{k \geq 1} s'_k$ , in the bigraded space  $V = \bigoplus_{k,g} V_k^g$ , with  $g \in \mathbf{Z}$  the ghost number and  $k \in \mathbf{N}$  the resolution degree. The ghost number of  $s$  is 1, the resolution degree of  $\delta'$ ,  $\gamma'$ ,  $s'_k$  are respectively  $-1$ ,  $0$ ,  $k$ .

Let  $V_{k \geq n}$  be the space containing only terms of resolution degree larger than  $n$ :  $A \in V_{k \geq n}$  if the expansion of  $A$  according to the resolution degree is  $A = A_n + A_{n+1} + \dots$ . In particular  $V = V_{k \geq 0}$ .

For  $n \geq 0$ , consider the spaces  $H^g(s, V_{k \geq n})$  defined by the cocycle condition  $s(A_n + A_{n+1} + \dots) = 0$  and the coboundary condition  $A_n + A_{n+1} + \dots = s(B_n + B_{n+1} + \dots)$ . In particular,  $\delta' B_n = 0$ . Consider the maps  $i_n : H^g(s, V_{k \geq n+1}) \longrightarrow H^g(s, V_{k \geq n})$  defined by  $i_n[A_{n+1} + A_{n+2} + \dots] = [A_{n+1} + A_{n+2} + \dots]$ . They are well defined because they map cocycles to cocycles and coboundaries to coboundaries. Note that the difference between  $H^g(s, V_{k \geq n+1})$  and  $\text{im } i_n$  is the coboundary condition: an element  $A = A_{n+1} + A_{n+2} + \dots$ , with  $sA = 0$ , is trivial in  $\text{im } i_n \subset H^g(s, V_{k \geq n})$ , if  $A = sB$  with  $B = B_n + B_{n+1} + \dots$ .

For  $n \geq 0$ , consider the spaces  $H_n^g(\gamma', H(\delta', V))$ . The cocycle condition for an element  $[A_n] \in H_n^g(\gamma', H(\delta', V))$  is  $\delta' A_n = 0$ ,  $\gamma' A_n + \delta' A_{n+1} = 0$  for some  $A_{n+1}$ , and the coboundary condition is  $A_n = \gamma' B_n + \delta' B_{n+1}$ , with  $\delta' B_n = 0$ . Consider the maps  $\pi_n : H^g(s, V_{k \geq n}) \longrightarrow H_n^g(\gamma', H(\delta', V))$  defined by  $\pi_n[A_n + A_{n+1} + \dots] = [A_n]$ .

Consider finally the maps  $m_n : H_n^g(\gamma', H(\delta', V)) \longrightarrow H^{g+1}(s, V_{k \geq n+1})$  defined by  $m_n[A_n] = [s(A_n + A_{n+1})]$ . It is straightforward to check that the maps  $m_n$  are well defined.

We are now in a position to prove the decomposition:

$$H^g(s, V_{k \geq n}) \simeq \ker m_n \oplus \text{im } i_n. \quad (\text{A.1})$$

The proof follows from the isomorphism (as real vector spaces)  $H^g(s, V_{k \geq n}) \simeq \text{im } \pi_n \oplus \ker \pi_n$  and by showing that  $\ker \pi_n = \text{im } i_n$  and  $\text{im } \pi_n = \ker m_n$ . From (A.1), it then follows that

$$\begin{aligned} H^g(s, V) &\simeq \ker m_0 \oplus i_0[H^g(s, V_{k \geq 1})] \\ &\simeq \ker m_0 \oplus i_0[\ker m_1 \oplus i_1[H^g(s, V_{k \geq 2})]] \simeq \dots \\ &\simeq \ker m_0 \bigoplus_{n \geq 1} i_0 \circ \dots \circ i_{n-1}[\ker m_n]. \end{aligned} \quad (\text{A.2})$$

Note that the isomorphism  $H^g(s, V_{k \geq n}) \simeq \text{im } \pi_n \oplus \ker \pi_n$  used in the proof is non-canonical in the sense that it involves a choice of supplementary subspace to  $\ker \pi_n$ .

**Discussion:** If  $V$  is the space of local functions or of horizontal forms, we have  $H_n(\delta', V) = 0$  for  $n \geq 1$ , and this both in the canonical and the gauge-fixed form. It follows that  $H_n^g(\gamma', H(\delta', V)) = 0$  and thus  $\ker m_n = 0$  for  $n \geq 1$ . Since  $H^{g+1}(s, V_{k \geq 1}) = 0$  it also follows that  $m_0 = 0$  and  $\ker m_0 = H_0^g(\gamma', H(\delta', V))$ , so that  $H^g(s, V) \simeq H_0^g(\gamma', H(\delta', V))$ . This result has been deduced in [29, 17].

If  $V$  is the space of local functionals  $\mathcal{F}$ , for the canonical form of the BRST differential (with the cohomologically trivial pairs of the non-minimal sector eliminated), there are no fields with negative pure ghost numbers. This implies that the antifield number must be larger than or equal to  $\max(0, -g) = K$ . Furthermore, if  $k > K$ , the presence of the ghosts implies [37] that  $H_k^g(\delta, \mathcal{F}) = 0$ . This implies, for  $g \geq 0$ , that  $H_k^g(\delta, \mathcal{F}) = 0$  for  $k \geq 1$ , hence  $\ker m_n = 0$ , for  $n \geq 1$  and  $m_0 = 0$ . Again we get  $H^g(s, \mathcal{F}) \simeq H_0^g(\gamma, H(\delta, \mathcal{F}))$ .

For  $g < 0$ , the only non-vanishing cohomology group is  $H_{-g}^g(\delta, \mathcal{F})$ . This implies  $H_n^g(\gamma, H(\delta, \mathcal{F})) = 0$ , for  $n \neq -g$ , so that  $\ker m_n = 0$  for  $n \neq -g$ . Furthermore,  $H^{g+1}(s, \mathcal{F}_{k \geq -g+1}) = 0$ , so that  $m_{-g} = 0$ . Hence  $H^g(s, \mathcal{F}) \simeq i_0 \circ \dots \circ i_{-g-1} [H_{-g}^g(\gamma, H(\delta, \mathcal{F}))]$ . Finally,  $H_{-g}^g(\gamma, H(\delta|d)) \simeq H_{-g}^g(\delta|d)$ , which follows from  $H_{-g}^{g+1}(\delta, \mathcal{F}) = 0$ , and  $i_{-g-1} = \dots = i_0 = 1$  at ghost number  $-g$ , since there are no terms with antifield number less than  $-g$ , so that  $H^g(s, \mathcal{F}) \simeq H_{-g}^g(\delta, \mathcal{F})$ , which is the result obtained in [12].

As already stated in section 2, in the space of local functionals for the gauge-fixed form,  $H_k^g(\delta^g, \mathcal{F}) = 0$  for  $k \geq 2$ , with  $H_1^g(\delta^g, \mathcal{F})$  characterizing the non-trivial global symmetries of the gauge-fixed action (and their associated Noether currents) for the classical fields, the ghost fields and the fields of the gauge-fixing sector. We thus have  $\ker m_2 = \ker m_3 = \dots = 0$  and  $m_1 = 0$ , implying that

$$H^g(s, \mathcal{F}) \simeq [\ker m_0 \subset H_0^g(\gamma^g, H(\delta^g, \mathcal{F}))] \oplus i_0[H_1^g(\gamma^g, H(\delta^g, \mathcal{F}))]. \quad (\text{A.3})$$

It follows that the canonical antifield BRST cohomology  $H^g(s, \mathcal{F})$  is isomorphic to the direct sum of a subset of the weak gauge-fixed BRST cohomology  $H_0^g(\gamma^g, H(\delta^g, \mathcal{F}))$  and of a subset of the nontrivial global symmetries of the gauge-fixed action. Since  $H^{g+1}(s, \mathcal{F}_{k \geq 1}) \simeq H_1^{g+1}(\gamma, H(\delta, \mathcal{F}))$ , the condition that  $[a_0] \in \ker m_0$  becomes  $s_1 A_0 + \gamma A_1 = \gamma B_1 + \delta B_2$ , with  $\delta B_1 = 0$ . This is precisely condition (3.6) and thus equivalent to (3.2).

## References

- [1] C. Becchi, A. Rouet and R. Stora, “Renormalization of the abelian Higgs-Kibble model”, *Commun. Math. Phys.* **42** (1975) 127 ; “Renormalization of gauge theories”, *Ann. Phys.* (NY) **98** (1976) 287.
- [2] I.V. Tyutin, “Gauge invariance in field theory and statistical mechanics”, Lebedev preprint FIAN, n<sup>o</sup>39 (1975).
- [3] J.C. Ward, “An identity in Quantum Electrodynamics”, *Phys. Rev.* **78** (1950) 182.
- [4] A.A. Slavnov, “Ward identities in gauge theories”, *Theor. Math. Phys.* **10** (1972) 99.
- [5] J.C. Taylor, “Ward identities and charge renormalization of the Yang-Mills field”, *Nucl. Phys.* **B33** (1971) 436.
- [6] J. Zinn-Justin, “Renormalisation of gauge theories” in *Trends in elementary particle theory*, Lecture notes in Physics n<sup>o</sup>37 (Springer, Berlin, 1975) ; *Quantum Field Theory and Critical Phenomena*, 3<sup>rd</sup> (Edition Clarendon Press, Oxford, 1996).

- [7] H. Kluberg-Stern and J.B. Zuber, “Ward identities and some clues to the renormalization of gauge invariant operators”, *Phys. Rev.* **D12** (1975) 467 ; “Renormalization of nonabelian gauge theories in a background field gauge: 1. Green functions”, *Phys. Rev.* **D12** (1975) 482 ; “Renormalization of nonabelian gauge theories in a background field gauge. 2. Gauge invariant operators”, *Phys. Rev.* **D12** (1975) 3159.
- [8] J.A. Dixon, “Calculation of BRS cohomology with spectral sequences”, *Commun. Math. Phys.* **139** (1991) 495.
- [9] C. Itzykson and J.B. Zuber, *Quantum Field Theory* (Mc Graw-Hill, New York, 1980).
- [10] I.A. Batalin and G.A. Vilkovisky, “Gauge algebra and quantization”, *Phys. Lett.* **B102** (1981) 27 ; “Quantization of gauge theories with linearly dependent generators”, *Phys. Rev.* **D28** (1983) 2567 ; [erratum: *Phys. Rev.* **D30** (1984) 508].
- [11] S.D. Joglekar and B.W. Lee, “General theory of renormalization of gauge invariant operators”, *Ann. Phys. (NY)* **97** (1976) 160.
- [12] G. Barnich and M. Henneaux, “Renormalization of gauge invariant operators and anomalies in Yang-Mills theory”, *Phys. Rev. Lett.* **72** (1994) 1588 , hep-th/9312206 ; G. Barnich, F. Brandt and M. Henneaux, “Local BRST cohomology in the antifield formalism. II. Application to Yang-Mills theory”, *Commun. Math. Phys.* **174** (1995) 93, hep-th/9405194.
- [13] G. Bandelloni, A. Blasi, C. Becchi and R. Collina, “Nonsemisimple gauge models: 1. Classical theory and the properties of ghost states”, *Ann. Inst. Henri Poincaré*, **28** (1978) 225 ; “Nonsemisimple gauge models: 2. Renormalization”, *Ann. Inst. Henri Poincaré*, **28** (1978) 255.
- [14] J. Gomis and S. Weinberg, “Are nonrenormalizable gauge theories renormalizable?”, *Nucl. Phys.* **B469** (1996) 473, hep-th/951 0087.
- [15] T. Hurth and K. Skenderis, “Quantum Noether method”, *Nucl. Phys.* **B541** (1999) 566, hep-th/9803030.
- [16] T. Hurth and K. Skenderis, “The Quantum Noether condition in terms of interacting fields”, in *New Developments in Quantum Field Theory*, eds. P. Breitenlohner, D. Maison and J. Wess (Springer, Berlin, to appear), hep-th/9811231.
- [17] M. Henneaux, “On the algebraic structure of the BRST symmetry”, in *NATO Advanced Summer Institute and Banff Summer School in Theoretical Physics on Physics, Geometry and Topology, Banff, Canada, Aug 14-25, 1989*, H.C. Lee ed. (Plenum Press, New York, 1990) ; M. Henneaux and C. Teitelboim, *Quantization of Gauge Systems* (Princeton University Press, Princeton, 1992).
- [18] M. Henneaux, “On the gauge-fixed BRST cohomology”, *Phys. Lett.* **B 367** (1996) 163, hep-th/9510116.

- [19] G. Curci and R. Ferrari, “On a class of Lagrangian models for massive and massless Yang-Mills fields”, *Nuovo Cim.* **32 A** (1976) 151 ; “The unitarity problem and the zero - mass limit for a model of massive Yang-Mills theory”, *Nuovo Cim.* **35 A** (1976) 1.
- [20] I. Ojima, “Comments on massive and massless Yang-Mills Lagrangians with a quartic coupling of Faddeev–Popov ghosts”, *Z. Phys.* **C 13** (1982) 173.
- [21] J. de Boer, K. Skenderis, P. van Nieuwenhuizen and A. Waldron, “On the renormalizability and unitarity of the Curci-Ferrari model for massive vector bosons”, *Phys. Lett.* **B367** (1996) 175, hep-th/9510167.
- [22] T. Hurth, “Nonabelian gauge symmetry in the causal Epstein–Glaser approach”, *Int. J. Mod. Phys.* **A12** (1997) 4461, hep-th/9511139.
- [23] N. Dragon, T. Hurth and P. van Nieuwenhuizen, “Polynomial form of the Stueckelberg model”, *Nucl. Phys. Proc. Suppl.* **B56** (1997) 318, hep-th/9703017.
- [24] T. Hurth, “Higgs-free massive nonabelian gauge theories”, *Helv. Phys. Act.* **70** (1997) 406, hep-th/9511176.
- [25] F. Brandt, “Deformations of global symmetries in the extended antifield formalism”, *J. Math. Phys.* **40** (1999) 1023, hep-th/9804153.
- [26] G. Barnich, F. Brandt and M. Henneaux, “Local BRST cohomology in the anti-field formalism. I. General theorems”, *Commun. Math. Phys.* **174** (1995) 57, hep-th/9405109.
- [27] A.M. Vinogradov, “On the algebra-geometric foundations of Lagrangian field theory”, *Sov. Math. Dokl.* **18** (1977) 1200, “A spectral sequence associated with a non-linear differential equation and algebra-geometric foundations of Lagrangian field theory with constraints”, *Sov. Math. Dokl.* **19** (1978) 1028; “The theory of higher infinitesimal symmetries of non linear partial differential equations”, *Sov. Math. Dokl.* **20** (1979) 985.
- [28] R.L. Bryant and P.A. Griffiths, *Characteristic Cohomology of Differential Systems (I): General Theory* (Duke University Mathematics Preprint Series, volume 1993 n<sup>0</sup>1, January 1993).
- [29] J.M.L. Fisch and M. Henneaux, “Homological perturbation theory and the algebraic structure of the antifield - antibracket formalism for gauge theories”, *Commun. Math. Phys.* **128** (1990) 627.
- [30] M. Henneaux, “Lectures on the antifield-BRST formalism for gauge theories”, *Nucl. Phys. B (Proc. Suppl.)* **18A** (1990) 47.
- [31] G. Barnich and M. Henneaux, “Consistent couplings between fields with a gauge freedom and deformations of the master equation”, *Phys. Lett.* **B311** (1993) 123, hep-th/9304057.

- [32] N.N. Bogoliubov and D.V. Shirkov, *Introduction to the Theory of Quantized Fields* (Interscience, New York, 1959).
- [33] H. Epstein and V. Glaser, “Le rôle de la localité dans la renormalisation perturbative en théorie quantique des champs”, in *Statistical Mechanics and Quantum Field Theory*, Proceedings of the 1970 Summer School of Les Houches, eds. C. DeWitt and R. Stora (Gordon and Breach, New York, 1971); “The rôle of locality in perturbation theory”, *Ann. Inst. Poincaré* **29** (1973) 211.
- [34] H. Epstein and V. Glaser, “Adiabatic limit in perturbation theory”, in G. Velo, A.S. Wightman (eds.), *Renormalization Theory* (D. Reidel Publishing Company, Dordrecht 1976), p. 193; O. Piguet and A. Rouet, “Symmetries in Perturbative Quantum Field Theory”, *Phys. Rep.* **76** (1981) 1; H. Epstein, V. Glaser and R. Stora, “General properties of the  $n$ -point functions in local quantum field theory”, in J. Bros, D. Jagolnitzer (eds.), *Les Houches Proceedings 1975*; G. Popineau and R. Stora, “A pedagogical remark on the main theorem of perturbative renormalization theory”, unpublished; R. Stora, “Differential Algebras”, ETH-lectures (1993) unpublished; G. Scharf, “Finite Quantum Electrodynamics”, *Text and Monographs in Physics* (Springer, Berlin, 1995); T. Hurth, “NonAbelian gauge theories: the causal approach,” *Ann. Phys.* **244** (1995) 340, hep-th/9411080.
- [35] M. Dütsch and K. Fredenhagen, “A local (perturbative) construction of observables in gauge theories: The example of QED,” *Commun. Math. Phys.* **203** (1999) 71, hep-th/9807078.
- [36] T. Hurth and K. Skenderis, “Analysis of anomalies in the Quantum Noether method”, in preparation.
- [37] M. Henneaux, “Space-time locality of the BRST formalism”, *Commun. Math. Phys.* **140** (1991) 1.