Ref.TH.1333-CERN

UPPER BOUNDS ON PARTICLE-ANTIPARTICLE ELASTIC CROSS-SECTION DIFFERENCES

S.M. Roy *)
CERN - Geneva

A B S T R A C T

Assuming isotopic spin invariance, we prove the following axiomatic upper bounds on the difference between $\pi^- p$ and $\pi^+ p$ integrated unpolarized elastic cross-sections, and compare them with experiment.

(1) $$\left[\sigma_{el}^{\pi^- p}(s) - \sigma_{el}^{\pi^+ p}(s)\right]^2 \leq 4\sigma^{\pi^- p \to \pi^0 n}(s) \left[\sigma_{el}^{\pi^- p}(s) + \sigma_{el}^{\pi^+ p}(s) - \sigma^{\pi^- p \to \pi^0 n}(s)\right],$$

valid at any c.m. energy $\sqrt{s}$.

(2) If $\sigma_{el}^{\pi^- p}(s) + \sigma_{el}^{\pi^+ p}(s) \underset{s\to\infty}{\leq} \text{Const}$, $\left[\sigma_{tot}^{\pi^- p}(s) + \sigma_{tot}^{\pi^+ p}(s)\right]/(\ln s) \underset{s\to\infty}{\longrightarrow} 0$,

and, $$\lim_{s\to\infty}\left[\sigma_{tot}^{\pi^- p}(s) - \sigma_{tot}^{\pi^+ p}(s)\right] = \Delta\sigma_{tot},$$

then,

$$\left[\sigma_{el}^{\pi^- p}(s) - \sigma_{el}^{\pi^+ p}(s)\right]^2 \underset{s\to\infty}{\leq} 4\left[\sigma^{\pi^- p\to\pi^0 n}(s) - \frac{2m_\pi^2}{\pi^3}(\Delta\sigma_{tot})^2\right]\left[\sigma_{el}^{\pi^- p}(s) + \sigma_{el}^{\pi^+ p}(s) - \sigma^{\pi^- p\to\pi^0 n}(s)\right]$$

where m_π = pion-mass. Similar results for KN scattering are also given.

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*) On leave of absence from the Tata Institute of Fundamental Research, Bombay.

Ref.TH.1333-CERN
23 May 1971

1. INTRODUCTION

The Serpukhov total cross-section experiments [1] have generated a great deal of theoretical interest [2)-4)] in the consequences of asymptotically unequal particle and antiparticle total cross-sections. In particular, for πN scattering, it has been shown that [3], if

$$\lim_{s\to\infty}\left[\sigma_{tot}^{\pi^- p}(s) - \sigma_{tot}^{\pi^+ p}(s)\right] = \Delta\sigma_{tot} \neq 0, \tag{1}$$

then

$$\left[\Delta\sigma_{tot}\right]^2 \leq \limsup_{s\to\infty} \frac{\pi^3}{2m_\pi^2}\,\sigma^{\pi^- p\to\pi^0 n}(s)$$

This result implies in particular that the πN charge exchange cross-section cannot vanish for $s \to \infty$ if the Pomeranchuk theorem [5] is violated. It is of obvious interest to ask a corresponding question about the difference between integrated elastic cross-sections for $\pi^- p$ and $\pi^+ p$ scattering. The answers given in this paper may be conveniently summarized by the following theorems. Theorem 1 is a consequence of isospin invariance alone, and for the remaining theorems we assume in addition unitarity, crossing and the analyticity properties and asymptotic bounds given by axiomatic field theory [6].

Theorem 1

Isospin invariance alone implies that the unpolarized integrated cross-sections for πN scattering at any c.m. energy $\sqrt{s}$ must obey the bound,

$$\left[\sigma_{el}^{\pi^- p}(s) - \sigma_{el}^{\pi^+ p}(s)\right]^2 \leq 4\,\sigma^{\pi^- p\to\pi^0 n}(s)\left[\sigma_{el}^{\pi^- p}(s) + \sigma_{el}^{\pi^+ p}(s) - \sigma^{\pi^- p\to\pi^0 n}(s)\right] \tag{2}$$

This result enables us to understand the small elastic cross-section differences at high energies in terms of the small charge exchange cross-sections. In particular, it requires that if

$$\sigma^{\pi^- p \to \pi^0 n}(s) \left[\sigma_{el}^{\pi^- p}(s) + \sigma_{el}^{\pi^+ p}(s) \right] \xrightarrow[s \to \infty]{} 0 \ ,$$

then

$$\left[\sigma_{el}^{\pi^- p}(s) - \sigma_{el}^{\pi^+ p}(s) \right] \xrightarrow[s \to \infty]{} 0 \ .$$

In addition, at low energies, comparison with experiment (see the Figure) reveals the unsuspected result that the upper bound given by Eq. (2) is actually equal to the experimental value of $\left[\sigma_{el}^{\pi^- p} - \sigma_{el}^{\pi^+ p}\right]^2$ within the experimental errors, for energies below the one-pion production threshold. At high energies, this result can be improved if the Pomeranchuk theorem is violated, to yield the following result.

<u>Theorem 2</u>

If

$$\lim_{s \to \infty} \left[\sigma_{tot}^{\pi^- p}(s) + \sigma_{tot}^{\pi^+ p}(s) \right] / \ln s = 0 , \tag{3}$$

$$\left[\sigma_{el}^{\pi^- p}(s) + \sigma_{el}^{\pi^+ p}(s) \right] \underset{s \to \infty}{\leq} \text{Const} \ , \tag{4}$$

and

$$\lim_{s \to \infty} \left[\sigma_{tot}^{\pi^- p}(s) - \sigma_{tot}^{\pi^+ p}(s) \right] = \Delta\sigma_{tot} \neq 0 , \tag{5}$$

then there must exist a sequence of values of $s \to +\infty$, such that,

$$\left[\sigma_{el}^{\pi^- p}(s) - \sigma_{el}^{\pi^+ p}(s) \right]^2 \underset{s \to \infty}{\leq} 4 \left[\sigma^{\pi^- p \to \pi^0 n}(s) - 2 \frac{m_\pi^2}{\pi^3} (\Delta\sigma_{tot})^2 \right] \left[\sigma_{el}^{\pi^- p}(s) + \sigma_{el}^{\pi^+ p}(s) - \sigma^{\pi^- p \to \pi^0 n}(s) \right] . \tag{6}$$

Comparison with the inequalities (1) and (2) shows that the result (6) improves both the unitarity-analyticity bound (1) and the isotopic spin result (2). The bound (6) shows also the interesting feature that for given values of $\sigma^{\pi^- p \to \pi^0 n}$ and $(\sigma_{e\ell}^{\pi^- p} + \sigma_{e\ell}^{\pi^+ p})$, the larger the Pomeranchuk theorem violation, the smaller is the allowed difference $|\sigma_{e\ell}^{\pi^- p} - \sigma_{e\ell}^{\pi^+ p}|$. In particular, <u>if the violation of the Pomeranchuk theorem is the largest permitted by the unitarity-analyticity bound (1), then the elastic cross-section difference $[\sigma_{e\ell}^{\pi^- p}(s) - \sigma_{e\ell}^{\pi^+ p}(s)]$ must vanish for $s \to \infty$.</u> This result is particularly interesting when we note that, in this case, the individual elastic cross-sections have to be non-zero because of the unitarity-analyticity bound [3),7)],

$$|\Delta\sigma_{tot}| \leq \frac{\pi^{3/2}}{m_\pi} \min\left[\limsup_{s\to\infty}\sqrt{\sigma_{e\ell}^{\pi^- p}(s)}, \limsup_{s\to\infty}\sqrt{\sigma_{e\ell}^{\pi^+ p}(s)}\right]. \tag{7}$$

Hence, the elastic $\pi^- p$ and $\pi^+ p$ cross-sections are in this case non-zero and equal asymptotically. These results should be compared with the recent results of Cornille and Martin [8)] who show that the particle-antiparticle integrated elastic cross-section differences must vanish asymptotically if certain conditions on the elastic differential cross-sections are fulfilled.

The isotopic spin bound (2) can also be improved at high energies if $\sigma_{tot}^{\pi^\pm p}(s)/(\ell n s) \geq$ const. for $s \to \infty$. This is expressed by the following theorem.

<u>Theorem 3</u>

If
$$[\sigma_{tot}^{\pi^- p}(s) + \sigma_{tot}^{\pi^+ p}(s)]/\ell n s \underset{s\to\infty}{\geq} \text{Const.} \tag{8}$$

and
$$[\sigma_{e\ell}^{\pi^- p}(s) + \sigma_{e\ell}^{\pi^+ p}(s)] \underset{s\to\infty}{\leq} \text{Const}, \tag{9}$$

then,

$$\left(\sigma_{el}^{\pi^-p}(s) - \sigma_{el}^{\pi^+p}(s)\right)^2 \underset{s\to\infty}{\leq} 4\sigma^{\pi^-p\to\pi^0 n}(s)\left[\sigma_{el}^{\pi^-p}(s) + \sigma_{el}^{\pi^+p}(s) - \sigma^{\pi^-p\to\pi^0 n}(s)\right] -$$

$$- \frac{4m_\pi^2}{\pi(\ln s)^2}\left[\sigma_{tot}^{\pi^-p}(s)\sigma_{tot}^{\pi^+p}(s)\left\{2\sigma^{\pi^-p\to\pi^0 n}(s) - \sigma_{el}^{\pi^-p}(s) - \sigma_{el}^{\pi^+p}(s) + 2\sqrt{\sigma_{el}^{\pi^-p}(s)\sigma_{el}^{\pi^+p}(s)}\right\} + \right.$$

$$\left. + \left\{\sigma_{tot}^{\pi^-p}(s)\sqrt{\sigma_{el}^{\pi^+p}(s)} - \sigma_{tot}^{\pi^+p}(s)\sqrt{\sigma_{el}^{\pi^-p}(s)}\right\}^2\right] . \qquad (10)$$

It may be noted that the expression in the square bracket in (10) is positive definite because

$$\left\{2\sigma^{\pi^-p\to\pi^0 n}(s) - \sigma_{el}^{\pi^-p}(s) - \sigma_{el}^{\pi^+p}(s) + 2\sqrt{\sigma_{el}^{\pi^-p}(s)\,\sigma_{el}^{\pi^+p}(s)}\right\} \geq 0, \qquad (11)$$

which follows from (2). Hence the result (10) improves the isospin bound (2).

Encouraged by the saturation of the isotopic spin bound (2) at low energies, we now state some improvements due to unitarity of this bound which are valid at all energies, but require the use of pion-nucleon phase shifts as additional input. The usefulness of these results is suggested to us by the recent work of Höhler and Jakob [9] on comparison of π N unitarity bounds with data.

<u>Theorem 4</u>

$$\left[\sigma_{el}^{\pi^-p}(s) - \sigma_{el}^{\pi^+p}(s)\right]^2 \leq 4\sigma^{\pi^-p\to\pi^0 n}(s)\left[\sigma_{el}^{\pi^-p}(s) + \sigma_{el}^{\pi^+p}(s) - \sigma^{\pi^-p\to\pi^0 n}(s)\right] -$$

$$- 2\left[\sqrt{\sigma_{el}^{\pi^-p}(s)\sigma_{el}^{\pi^+p}(s)} - \sqrt{\sigma_{el,Re}^{\pi^-p}(s)\,\sigma_{el,Re}^{\pi^+p}(s)}\right] \times$$

$$\times\left[2\sigma^{\pi^-p\to\pi^0 n}(s) - \sigma_{el}^{\pi^-p}(s) - \sigma_{el}^{\pi^+p}(s) + 2\sqrt{\sigma_{el}^{\pi^-p}(s)\sigma_{el}^{\pi^+p}(s)}\right] , \qquad (12)$$

where,

$$\sigma_{el,Re}^{\pi^\pm p}(s) \equiv \frac{4\pi}{k^2}\sum_{\ell=0}^{\infty}(\ell+1)\left[\left(\mathrm{Re}\, f_{\ell+}^{\pi^\pm p}(s)\right)^2+\left(\mathrm{Re}\, f_{(\ell+1)-}^{\pi^\pm p}(s)\right)^2\right], \qquad (13)$$

k is the c.m. momentum, and the $f_{\ell\pm}(s)$ are the pion-nucleon partial wave amplitudes related to the inelasticity parameters $\eta_{\ell\pm}$ and the phase-shifts $\delta_{\ell\pm}$ by the relation,

$$f_{\ell\pm}(s) = \frac{\eta_{\ell\pm}(s)\; e^{2i\delta_{\ell\pm}(s)} - 1}{2i}. \qquad (14)$$

<u>Theorem 5</u>

Let the phase shifts for all ℓ in some set S be known. Then,

$$\left[\sigma_{el}^{\pi^-p}(s)-\sigma_{el}^{\pi^+p}(s)\right]^2 \leq 4\sigma^{\pi^-p\to\pi^0 n}(s)\left[\sigma_{el}^{\pi^-p}(s)+\sigma_{el}^{\pi^+p}(s)-\sigma^{\pi^-p\to\pi^0 n}(s)\right]-$$
$$-\left[4\left(a\,\sigma_{el}^{\pi^-p}(s)+b\,\sigma_{el}^{\pi^+p}(s)\right)+c^2-4ab+2c\left(2\sigma^{\pi^-p\to\pi^0 n}(s)-\sigma_{el}^{\pi^-p}(s)-\sigma_{el}^{\pi^+p}(s)\right)\right], \qquad (15)$$

where

$$a \equiv \frac{4\pi}{k^2}\sum_{\ell\in S}(\ell+1)\left[\left|f_{\ell+}^{\pi^+p}(s)\right|^2+\left|f_{(\ell+1)-}^{\pi^+p}\right|^2\right], \qquad (16)$$

$$b \equiv \frac{4\pi}{k^2}\sum_{\ell\in S}(\ell+1)\left[\left|f_{\ell+}^{\pi^-p}(s)\right|^2+\left|f_{(\ell+1)-}^{\pi^-p}(s)\right|^2\right], \qquad (17)$$

and

$$c \equiv \frac{8\pi}{k^2}\sum_{\ell\in S}(\ell+1)\,\mathrm{Re}\left[\left(f_{\ell+}^{\pi^+p}(s)\right)^* f_{\ell+}^{\pi^-p}(s)+\left(f_{(\ell+1)-}^{\pi^+p}(s)\right)^* f_{(\ell+1)-}^{\pi^-p}(s)\right]. \qquad (18)$$

It may be remarked that, as expected, the larger the number of phase shifts that are known, the better the bound (15).

It would be interesting to compare the bounds given by theorems 4 and 5 with experiment.

Before passing to the proof of these results, we wish to note that Theorems 1, 4 and 5 do not use crossing symmetry and hence are independent of the fact that the π^- and π^+ are antiparticles of each other. Hence they can be generalized to obtain, for example, bounds on $\left[\sigma_{e\ell}^{K^+p} - \sigma_{e\ell}^{K^0p}\right]$ in terms of $\left[\sigma_{e\ell}^{K^+p} + \sigma_{e\ell}^{K^0p}\right]$ and $\sigma^{K^+n\to K^0p}$. Secondly, all the Theorems 1 to 5 generalize to the case of KN scattering when the following replacements are made :

$$\sigma^{\pi^-p\to\pi^0n}_{(s)} \longrightarrow 2\,\sigma^{K_Lp\to K_sp}_{(s)} \tag{19}$$

$$\pi^+p \longrightarrow K^+n \tag{20}$$

$$\pi^-p \longrightarrow K^-n\,. \tag{21}$$

2. ISOTOPIC SPIN BOUND ON ELASTIC CROSS-SECTION DIFFERENCE

The integrated unpolarized elastic cross-sections have the partial wave expansions :

$$\sigma_{e\ell}^{\pi^\pm p}(s) = \frac{4\pi}{k^2}\sum_{\ell=0}^{\infty}(\ell+1)\left[|f_{\ell+}^{\pi^\pm p}(s)|^2 + |f_{(\ell+1)-}^{\pi^\pm p}(s)|^2\right]. \tag{22}$$

Using isotopic spin invariance for the scattering amplitudes T,

$$T^{\pi^-p\to\pi^0n} = \frac{T^{\pi^+p\to\pi^+p} - T^{\pi^-p\to\pi^-p}}{\sqrt{2}}, \tag{23}$$

we then obtain,

$$[\sigma_{el}^{\pi^+p}(s) + \sigma_{el}^{\pi^-p}(s) - 2\sigma^{\pi^-p\to\pi^0 n}(s)]$$
$$= \frac{4\pi}{k^2}\sum_{l=0}^{\infty}(2l+2)\,\mathrm{Re}\left[\left(f_{l+}^{\pi^+p}(s)\right)^* f_{l+}^{\pi^-p}(s) + \left(f_{(l+1)-}^{\pi^+p}(s)\right)^* f_{(l+1)-}^{\pi^-p}(s)\right]. \tag{24}$$

Schwarz inequality then yields,

$$\left|\sigma_{el}^{\pi^+p}(s) + \sigma_{el}^{\pi^-p}(s) - 2\sigma^{\pi^-p\to\pi^0 n}(s)\right| \leq 2\sqrt{\sigma_{el}^{\pi^+p}(s)\,\sigma_{el}^{\pi^-p}(s)}\,, \tag{25}$$

which can be re-expressed in the form of theorem 1, as an upper bound on $[\sigma_{e\ell}^{\pi^-p}(s) - \sigma_{e\ell}^{\pi^+p}(s)]^2$.

The Figure and the Table show a comparison of this upper bound with the experimental data below the one-pion production threshold. It is remarkable that in this region the upper bound coincides with the experimental elastic cross-section difference within the experimental errors. It is clear, of course, that the inequality (25) becomes an equality <u>if either</u> the $I = 1/2$ or the $I = 3/2$ amplitude vanishes <u>or if</u> the $I = 1/2$ and $I = 3/2$ phase shifts are all equal to one another. But pure $I = 3/2$ scattering is not a good approximation in the entire energy range. For example, pure $I = 3/2$ scattering requires $\sigma_{e\ell}^{\pi^+p} = 9/2\ \sigma^{\pi^-p\to\pi^0 n}$. At a lab. kinetic energy of 31.4 MeV, $\sigma_{e\ell}^{\pi^+p} = 6.15 \pm 0.12$ mb, and $9/2\ \sigma^{\pi^-p\to\pi^0 n} =$ $= 31.6 \pm 1$ mb. Even at the energy closest to the $3-3$ resonance that we have used, at 163.8 MeV kinetic energy, $\sigma_{e\ell}^{\pi^+p} = 200.7 \pm$ ± 1.9 mb, $9/2\ \sigma^{\pi^-p\to\pi^0 n} = 215.5 \pm 1.4$ mb, $(\sigma_{e\ell}^{\pi^+p} - \sigma_{e\ell}^{\pi^-p})_{expt} =$ $= 179.4 \pm 2.2$ mb, and $(\sigma_{e\ell}^{\pi^+p} - \sigma_{e\ell}^{\pi^-p})_{upper\ bound} = 182.6 \pm 1.9$ mb. Thus the saturation of the upper bound is true to a much better accuracy than explicable from the dominance of $I = 3/2$ scattering. The inequality (25) would also be an equality if the $I = 1/2$ and $I = 3/2$ phase shifts were equal. This again is not a good approximation, because, for example, the $I = 3/2$ state contains the $3 - 3$ resonance and the $I = 1/2$ state does not. It is remarkable thet the departure from pure $I = 3/2$ scattering and from the equality of $I = 1/2$ and $I = 3/2$ phase shifts gives only small corrections, comparable with the experimental errors, in this energy range.

The saturation of the bound does not hold at higher energies. For example, at 307 MeV kinetic energy, $(\sigma_{e\ell}^{\pi^+p} - \sigma_{e\ell}^{\pi^-p}) = 53.7 \pm 2.5$ mb, and the upper bound on it 64.7 ± 3.7 mb.

Comparison with triangular inequalities.

Comparison of the present isospin inequality with the well-known triangular inequalities on the differential cross-sections is facilitated when we note that Theorem 1 has the following analogue in terms of the differential cross-sections (the proof of which we omit),

$$\left[\left(\frac{d\sigma}{dt}\right)_{el}^{\pi^-p} - \left(\frac{d\sigma}{dt}\right)_{el}^{\pi^+p}\right]^2 \leq 4\left(\frac{d\sigma}{dt}\right)^{\pi^-p\to\pi^0n}\left[\left(\frac{d\sigma}{dt}\right)_{el}^{\pi^-p} + \left(\frac{d\sigma}{dt}\right)_{el}^{\pi^+p} - \left(\frac{d\sigma}{dt}\right)^{\pi^-p\to\pi^0n}\right],$$

valid at any energy and momentum transfer.

The validity of these relations at each s and t is in fact necessary and sufficient for isospin invariance. The validity of the three triangular inequalities at each s and t is also necessary and sufficient for isospin invariance. Hence the three triangular inequalities together are equivalent to the above relation between the differential cross-sections. The point of discussing the comparison with experiment of Theorem 1 is that at low energies extremely accurate data are becoming available (see for example the recent experiments of Bugg et al., and Carter et al. quoted in the Table) on the integrated elastic cross-sections, and a posteriori the fact that the inequality (2) is quite stringent, as seen from the Table and the Figure. We should like to mention that it would be of great help if accurate data on $\sigma_{e\ell}^{\pi^-p}$, $\sigma_{e\ell}^{\pi^+p}$ and $\sigma^{\pi^-p\to\pi^0n}$ are made available at the same energy, as interpolation between different data points increases the errors in the comparison of the bound with experiment.

3. POMERANCHUK THEOREM VIOLATION AND PARTICLE-ANTIPARTICLE ELASTIC CROSS-SECTION DIFFERENCES

Let the total cross-sections obey the conditions (3) and (5) and the helicity non-flip amplitudes be normalized such that

$$\mathrm{Im}\, T^{\pi^\pm p}_{0\frac{1}{2};0\frac{1}{2}}(s,t=0) = \frac{k\sqrt{s}}{4\pi}\,\sigma^{\pi^\pm p}_{tot}(s)\,, \tag{26}$$

where t denotes the c.m. momentum transfer squared.

The forward dispersion relations then imply [10] that there must exist a sequence of values of $s \to +\infty$ such that

$$\lim_{s\to\infty} \mathrm{Re}\left[\frac{T^{\pi^- p}_{0\frac{1}{2};0\frac{1}{2}}(s,t=0)}{s\,\ln s}\right] = -\lim_{s\to\infty} \mathrm{Re}\left[\frac{T^{\pi^+ p}_{0\frac{1}{2};0\frac{1}{2}}(s,t=0)}{s\,\ln s}\right] = -\frac{\Delta\sigma_{tot}}{8\pi^2}\,. \tag{27}$$

We will show that this restriction can be used to improve the bound on the elastic cross-section difference given by isospin invariance. For this, we will need the additional result that in the partial wave expansion for the forward scattering amplitude it is sufficient to keep orbital angular momenta up to $\ell_0 = \frac{\sqrt{s}}{4m_\pi}\ln s$, for $s \to \infty$. To be precise, we will need the fact that,

$$\lim_{s\to\infty}\left[\frac{\mathrm{Re}\, T^{\pi^\pm p}_{0\frac{1}{2};0\frac{1}{2}}(s,t=0) - \mathrm{Re}\,\hat{T}^{\pi^\pm p}_{0\frac{1}{2};0\frac{1}{2}}(s,t=0)}{s\,\ln s}\right] = 0, \tag{28}$$

where $\hat{T}_{0,1/2;0,1/2}$ denotes the truncated partial wave series,

$$\hat{T}^{\pi^\pm p}_{0\frac{1}{2};0\frac{1}{2}}(s,t=0) \equiv \frac{\sqrt{s}}{k}\sum_{\ell=0}^{\ell_0}(\ell+1)\left[f^{\pi^\pm p}_{\ell+}(s) + f^{\pi^\pm p}_{(\ell+1)-}(s)\right]. \tag{29}$$

We defer the proof of Eq. (28) to the Appendix. We proceed to find upper and lower bounds on $[\sigma_{e\ell}^{\pi^+p}(s) + \sigma_{e\ell}^{\pi^-p}(s) - 2\sigma^{\pi^-p\to\pi^0n}(s)]$ which has the partial wave expansion (24), given Re $\hat{T}^{\pi^\pm p}_{0,1/2;0,1/2}(s,t=0)$, and $\sigma_{e\ell}^{\pi^\pm p}(s)$. This is most easily done by pretending that the π^-p partial wave amplitudes are all given and varying the π^+p partial waves according to Lagrange's method of undetermined multipliers [11] subject to the two constraints that $\sigma_{e\ell}^{\pi^+p}(s)$ and Re $\hat{T}^{\pi^+p}_{0,1/2;0,1/2}(s,t=0)$ are given. The stationary values of

$$\left[\sigma_{e\ell}^{\pi^+p}(s) + \sigma_{e\ell}^{\pi^-p}(s) - 2\sigma^{\pi^-p\to\pi^0n}(s)\right]$$

are reached when,

$$f_{\ell\pm}^{\pi^+p}(s) = \hat{f}_{\ell\pm}^{\pi^+p}(s)$$

where,

$$\begin{aligned}
\mathrm{Re}\,\hat{f}_{\ell\pm}^{\pi^+p}(s) &\equiv \alpha(s)\,\mathrm{Re}\,f_{\ell\pm}^{\pi^-p}(s) + \beta, \quad 0 \le \ell \le \ell_0, \\
\mathrm{Re}\,\hat{f}_{\ell\pm}^{\pi^+p}(s) &\equiv \alpha(s)\,\mathrm{Re}\,f_{\ell\pm}^{\pi^-p}(s), \quad \ell > \ell_0, \\
\mathrm{Im}\,\hat{f}_{\ell\pm}^{\pi^+p}(s) &\equiv \alpha(s)\,\mathrm{Im}\,f_{\ell\pm}^{\pi^-p}(s), \quad \underline{\text{for all } \ell},
\end{aligned} \tag{30}$$

and α and β are to be determined to fit the given values of $\sigma_{e\ell}^{\pi^+p}(s)$ and Re $\hat{T}^{\pi^+p}_{0,1/2;0,1/2}(s)$. We will now show that, in fact, the desired bounds are obtained with the choice (30) of the partial waves, the upper and lower bounds being obtained respectively in the cases $\alpha(s) > 0$ and $\alpha(s) < 0$. Denoting the cross-sections with the partial waves (30) by $\hat{\sigma}_{e\ell}^{\pi^+p}$ and $\hat{\sigma}^{\pi^-p\to\pi^0n}$, we obtain, after using (30) and the two constraints,

$$\begin{aligned}
&\left[\hat{\sigma}_{e\ell}^{\pi^+p}(s) + \sigma_{e\ell}^{\pi^-p}(s) - 2\hat{\sigma}^{\pi^-p\to\pi^0n}(s)\right] - \left[\sigma_{e\ell}^{\pi^+p}(s) + \sigma_{e\ell}^{\pi^-p}(s) - 2\sigma^{\pi^-p\to\pi^0n}(s)\right] \\
&= \frac{1}{\alpha}\frac{4\pi}{k^2}\sum_{\ell=0}^{\infty}(\ell+1)\left[\left|\hat{f}_{\ell+}^{\pi^+p}(s) - f_{\ell+}^{\pi^+p}(s)\right|^2 + \left|\hat{f}_{(\ell+1)-}^{\pi^+p}(s) - f_{(\ell+1)-}^{\pi^+p}(s)\right|^2\right].
\end{aligned} \tag{31}$$

This equation demonstrates the assertion that the choice (30) yields the upper or the lower bound according to whether $\alpha(s)$ is positive or negative. Evaluation of the bounds and use of Eqs. (27) and (28) then yield Theorem 2 stated before.

We have already noted the qualitative consequences of this theorem, viz., that $[\sigma_{e\ell}^{\pi^-p}(s) - \sigma_{e\ell}^{\pi^+p}(s)]$ must vanish for $s \to \infty$, either if $\sigma^{\pi^-p\to\pi^0n}(s) \underset{s\to\infty}{\to} 0$ or if the Pomeranchuk theorem violation is the maximum possible. A valid numerical comparison with data is not possible at the moment because $\sigma^{\pi^-p\to\pi^0n}(s)$ is decreasing rapidly and measurements at very high energies of the elastic and charge exchange cross-sections are not available. To get a feeling for this bound, we note, however, that using the Serpukhov data [1] on total cross-sections and the 16.86 GeV lab. kinetic energy data [12] on $\sigma^{\pi^-p\to\pi^0n}$ we obtain the value (0.8 ± 0.2) mb for the upper bound on $[\sigma_{e\ell}^{\pi^-p} - \sigma_{e\ell}^{\pi^+p}]$. This number will improve if $\sigma^{\pi^-p\to\pi^0n}$ continues to decrease as the energy increases, and may give a useful bound on the poorly measured quantity $[\sigma_{e\ell}^{\pi^-p}(s) - \sigma_{e\ell}^{\pi^+p}(s)]$ at high energies.

We shall omit the detailed proof of Theorem 3 which is obtained by an analogous variational procedure with $\mathrm{Im}\, T_{0,1/2;0,1/2}^{\pi^+p}(s,t=0)$ held fixed instead of $\mathrm{Re}\, T_{0,1/2;0,1/2}^{\pi^+p}(s,t=0)$.

4. USE OF PHASE SHIFTS.

An improved bound on the elastic π^-p, π^+p cross-section difference valid at all energies can be obtained by supplementing isotopic spin invariance with the positivity property due to unitarity of the imaginary parts of the elastic partial wave amplitudes $f_{\ell\pm}$. We then obtain from (24),

$$\begin{aligned}
&[\sigma_{e\ell}^{\pi^+p}(s) + \sigma_{e\ell}^{\pi^-p}(s) - 2\sigma^{\pi^-p\to\pi^0n}(s)] \\
&\geqslant (4\pi/k^2)\sum_{\ell=0}^{\infty}(2\ell+2)\left[\mathrm{Re}\, f_{\ell+}^{\pi^+p}(s)\, \mathrm{Re}\, f_{\ell+}^{\pi^-p}(s) + \mathrm{Re}\, f_{(\ell+1)-}^{\pi^+p}(s)\, \mathrm{Re}\, f_{(\ell+1)-}^{\pi^-p}(s)\right] \\
&\geqslant -2\sqrt{\sigma_{e\ell,\mathrm{Re}}^{\pi^+p}(s)\, \sigma_{e\ell,\mathrm{Re}}^{\pi^-p}(s)}\,,
\end{aligned} \tag{32}$$

where the second step follows from the Schwarz inequality. Combining (25) and (32), we then obtain the improved bound given by Theorem 4. Theorem 5, the proof of which we shall omit, is obtained by using a variational procedure in which some of the partial wave amplitudes are held fixed. For a proper test of the bounds given by Theorem 4 and 5, the phase shifts should be calculated from the $\pi^- p$ and $\pi^+ p$ data alone (without the use of isospin invariance and the data on charge exchange scattering). The bounds can then be used as a test of isospin invariance, unitarity, and of the truncation after a finite number of partial waves which is necessary in any partial wave analysis.

ACKNOWLEDGEMENTS

I wish to thank A. Martin for communicating the results of his work with H. Cornille on particle-antiparticle elastic cross-section differences, G.V. Dass, J. Finkelstein and A. Martin for clarifying the relationship of the isotopic spin bound (2) with the usual triangular inequalities, and J.L. Petersen for referring me to the recent accurate πN data of Bugg et al. and Carter et al.

APPENDIX 1. PROOF OF EQ. (28).

We shall prove that if

$$\sigma_{el}^{\pi^\pm p}(s) \underset{s\to\infty}{<} \text{Const.} \tag{33}$$

then,

$$X^{\pi^\pm p}(s) \equiv \frac{1}{s \ln s} \frac{\sqrt{s}}{k} \sum_{\ell=\ell_0=\frac{\sqrt{s}}{4m_\pi}\ln s}^{\infty} (\ell+1)\left[\left|f_{\ell+}^{\pi^\pm p}(s)\right| + \left|f_{(\ell+1)-}^{\pi^\pm p}(s)\right|\right] \underset{s\to\infty}{\longrightarrow} 0 \ . \tag{34}$$

This result obviously implies the validity of Eq. (28). The main information we need is the Jin-Martin result [6] that the πN scattering amplitudes satisfy twice subtracted dispersion relations for $|t| < 4m_\pi^2$. (Actually, polynomial boundedness is sufficient for the present proof.) Using also the unitarity relations,

$$\text{Im}\, f_{\ell\pm}(s) \geqslant |f_{\ell\pm}(s)|^2 \tag{35}$$

we obtain, for $0 < t < 4m_\pi^2$

$$s^2 \underset{s\to\infty}{\geqslant} \frac{\sqrt{s}}{k} \sum_{\ell=0}^{\infty} (\ell+1)\, \text{Im}\left[f_{\ell+}^{\pi^\pm p}(s) + f_{(\ell+1)-}^{\pi^\pm p}(s)\right] P_\ell\left(1+\frac{t}{2k^2}\right)$$

$$\geqslant \frac{\sqrt{s}}{k} \sum_{\ell=\ell_0}^{\infty} (\ell+1)\left[\left|f_{\ell+}^{\pi^\pm p}(s)\right|^2 + \left|f_{(\ell+1)-}^{\pi^\pm p}(s)\right|^2\right] P_\ell\left(1+\frac{t}{2k^2}\right) \equiv G(s,t). \tag{36}$$

We then find that an upper bound on $X^{\pi^\pm p}(s)$ given $\sigma^{\pi^\pm p}_{e\ell}$ and $G(s,t)$, [using the $f_{\ell\pm}(s)$ as variational parameters], is obtained when,

$$\left|f^{\pi^\pm p}_{\ell+}(s)\right| = \left|f^{\pi^\pm p}_{(\ell+1)-}(s)\right| = \frac{\alpha(s)}{P_\ell\left(1+\frac{t}{2k^2}\right)+\beta(s)}\,, \tag{37}$$

where

$$\sigma^{\pi^\pm p}_{e\ell}(s) = \frac{8\pi}{k^2}\sum_{\ell=\ell_0}^{\infty}(\ell+1)\,\frac{\alpha^2(s)}{\left[P_\ell\left(1+\frac{t}{2k^2}\right)+\beta(s)\right]^2}\,, \tag{38}$$

and

$$G(s,t) = 2\frac{\sqrt{s}}{k}\sum_{\ell=\ell_0}^{\infty}(\ell+1)\,\frac{\alpha^2(s)\,P_\ell\left(1+\frac{t}{2k^2}\right)}{\left[P_\ell\left(1+\frac{t}{2k^2}\right)+\beta(s)\right]^2}\,. \tag{39}$$

Evaluation of the upper bound and use of (33) and (36) then yield,

$$X^{\pi^\pm p}(s) \underset{s\to\infty}{<} \text{Const.}\sqrt{\frac{\ln(\ln s)}{\ln s}}\,, \tag{40}$$

which implies the needed result (34).

TABLE CAPTION

The pion-nucleon unpolarized cross-section data on $[\sigma_{e\ell}^{\pi^- p}(s) - \sigma_{e\ell}^{\pi^+ p}(s)]$ are compared with the upper bound on it given by (2). The asterisk (*) mark on some rows indicates that slight interpolations between data points were needed to obtain values of $\sigma_{e\ell}^{\pi^- p}$, $\sigma_{e\ell}^{\pi^+ p}$ and $\sigma^{\pi^- p \to \pi^0 n}$ at the same energy. The references to the experiments appear in the last column. The last row is included to indicate that the saturation of the bound does not continue to be true at higher energies.

Pion Lab. Kinetic Energy (MeV)	$\sigma_{e\ell}^{\pi^- p}$ (millibarns)	$\sigma_{e\ell}^{\pi^+ p}$ (millibarns)	$\sigma^{\pi^- p \to \pi^0 n}$ (millibarns)	$\sigma_{e\ell}^{\pi^+ p} - \sigma_{e\ell}^{\pi^- p}$ (millibarns)	Upper bound on $\lvert \sigma_{e\ell}^{\pi^+ p} - \sigma_{e\ell}^{\pi^- p} \rvert$ (millibarns)	References
31.4	1.847 ± 0.08	6.15 ± 0.12	7.02 ± 0.22	4.3 ± 0.2	5.2 ± 1	D. Knapp et al., Rochester Report, NYO-10257 (1963)
*41.5	1.83 ± 0.17	9.16 ± 0.8	6.9 ± 1.2	7.3 ± 1	10.6 ± 3.9	S.W. Barnes et al., Phys.Rev. 117, 226 (1960) W.J. Spry et al., Phys.Rev. 95, 1295 (1964) R.A. Donald et al., Proc.Phys.Soc. 87, 445 (1966)
*65	2.9 ± 0.7	20.4 ± 2.4	8.3 ± 1.0	17.5 ± 3	22.3 ± 4.7	D. Bodansky et al., Phys.Rev. 93, 1367 (1954) C.M. York et al., Phys.Rev. 119, 1096 (1960)
*100	6.35 ± 0.3	62.4 ± 8.0	14.0 ± 1.0	56.1 ± 8.3	55.4 ± 6.7	R. Gessaroli et al., Nuovo Cimento 5, 1658 (1957) D.N. Edwards et al., Proc.Phys.Soc. 73, 856 (1959) C.M. York et al., Phys.Rev. 119, 1096 (1960)
120	10.1 ± 1.6	95 ± 6	21.5 ± 2.7	84.9 ± 7.6	84.8 ± 10	H.L. Anderson et al., Phys.Rev. 91, 155 (1953) J. Ashkin, cited in A. Loria et al., Nuovo Cimento 22, 820 (1961)
150	20 ± 1	166.0 ± 4.3	34.6 ± 1.2	146 ± 5.3	144.8 ± 6	J. Ashkin et al., Phys.Rev. 101, 1149 (1956)
*163.8	21.3 ± 1.1	200.7 ± 1.9	47.9 ± 0.3	179.4 ± 2.2	182.6 ± 1.9	D.V. Bugg et al., Nuclear Phys. B26, 588 (1971) A.A. Carter et al., Nuclear Phys. B26, 445 (1971)
*307	11.4 ± 0.8	65.1 ± 1.7	17.8 ± 1	53.7 ± 2.5	64.7 ± 3.7	V. Zinov et al., JETP 38, 1099, 1399 (1960) R. Carrara et al., Aix-en-Provence Conference, p.25 (1961)

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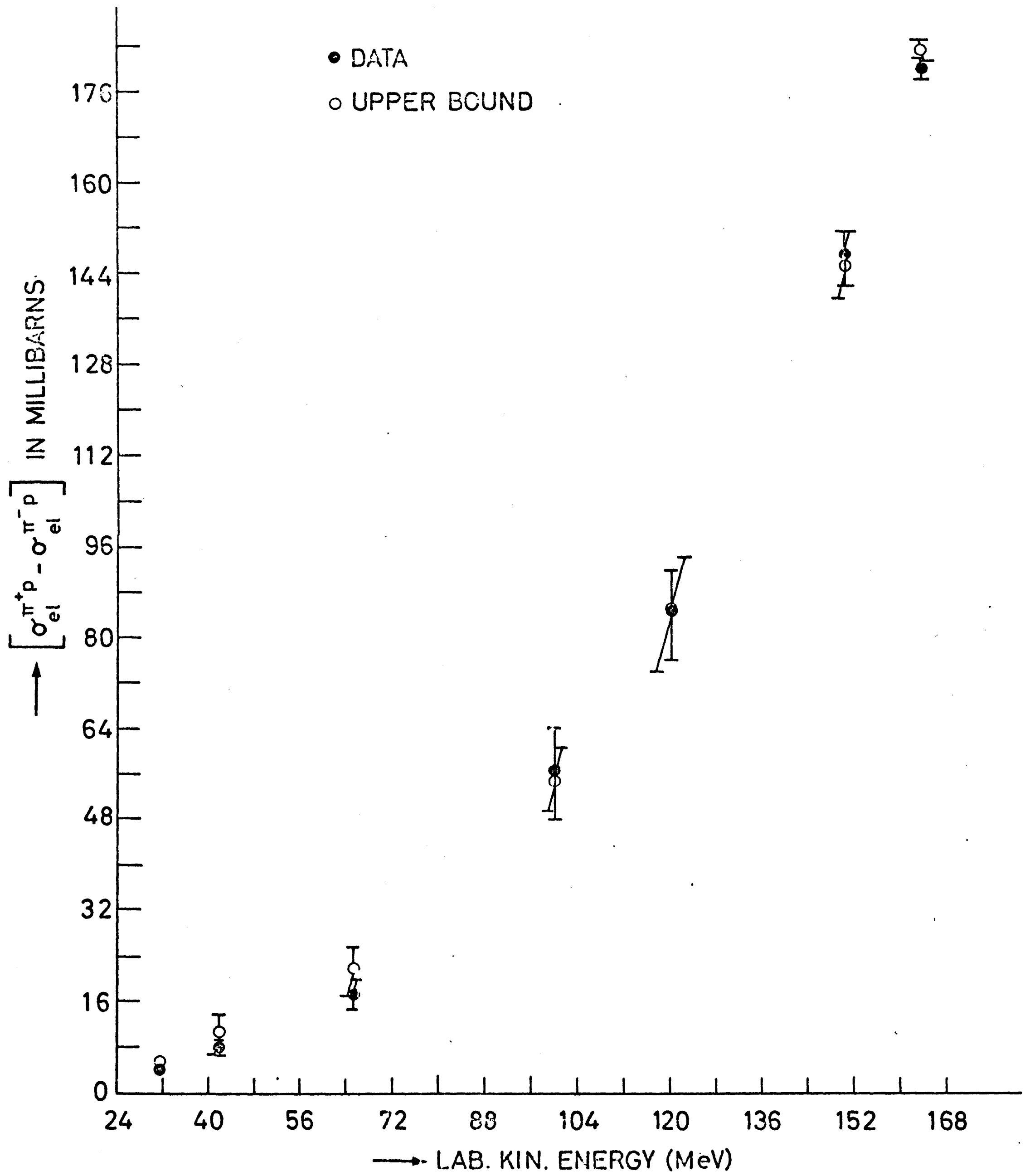
FIGURE CAPTION

The pion-nucleon unpolarized cross-section data on $[\sigma_{e\ell}^{\pi^- p}(s) - \sigma_{e\ell}^{\pi^+ p}(s)]$ at energies below the one-pion production threshold are compared with the upper bound on this difference given by (2).



DATA
UPPER BOUND
$\left[\sigma_{el}^{\pi^+ p} - \sigma_{el}^{\pi^- p}\right]$ IN MILLIBARNS
0
16
32
48
64
80
96
112
128
144
160
176
24
40
56
72
88
104
120
136
152
168
LAB. KIN. ENERGY (MeV)