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THE VENEZIANO MODEL AND UNITARITY SUM RULES *)

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A B S T R A C T

An attempt is made to place an additional constraint on the Veneziano model for $\pi\pi$ scattering by using a non-linear sum rule which incorporates unitarity. This sum rule, which was first derived by Arbab and Slansky, we call a unitarity sum rule (USR). We discuss various uses of USR's, using the Veneziano model as an illustration.

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1. INTRODUCTION

Recent studies of finite-energy sum rule (FESR) bootstraps together with the assumption of straight line trajectories are leading to new ways of analyzing the S matrix theory of strong interactions [1]. Of particular interest has been a proposal by Veneziano [2] for the construction of a simple Regge-behaved, crossing symmetric amplitude, which is a solution of the FESR bootstrap. Recent evidence seems to show that this model provides a reasonable parametrization for many strong interaction processes [3], despite having some unsatisfactory features, especially in connection with unitarity. However, the model can only give ratios of coupling constants and hence widths (not over-all magnitudes), due to its homogenous nature.

In this paper we discuss the possibility of imposing a further constraint on the Veneziano amplitude by using a unitarity sum rule (USR), first derived by Arbab and Slansky [4]. We consider $\pi\pi$ scattering, for which we take the following Veneziano amplitudes [5] with no satellite terms:

$$A^{I=0}(t,s,u) = \frac{3}{2}\left[V(t,s)+V(t,u)\right] - \frac{1}{2}V(s,u)$$

$$A^{I=1}(t,s,u) = V(t,s) - V(t,u)$$

$$A^{I=2}(t,s,u) = V(s,u) \tag{1.1}$$

where

$$V(t,s) = \frac{-\lambda\,\Gamma(1-\alpha(t))\,\Gamma(1-\alpha(s))}{\Gamma(1-\alpha(s)-\alpha(t))} \tag{1.2}$$

and I is the isospin in the t channel. s, t and u are the usual Mandelstam variables and we take for the Regge trajectory

$$\mathrm{Re}\ \alpha(t) = \frac{1}{2} + a(t-\mu^2) \quad \text{with} \quad a = \frac{1}{2(m_\rho^2-\mu^2)} \tag{1.3}$$

in order to satisfy the Adler self-consistency condition [5] and $\mathrm{Re}\,\alpha(m_\rho^2) = 1$.

Arbab and Slansky [4], assuming analyticity, elastic unitarity, Regge asymptotic behaviour for $s \geq N$, and a resonance approximation for $s \leq N$, derived a sum rule which can be written in the following form:

$$\sin\left[2\,\mathrm{Im}\,\alpha(t)\,\ln.(2z_N)\right] = \sum_{r=1}^{p(m_p<N^{1/2})} a_r(t)\,\Gamma_r \tag{1.4}$$

where $z_N = 1+(2N)/(t-4\mu^2)$, and the sum is over all resonances with mass $m_r < N^{\frac{1}{2}}$. The $a_r(t)$ are known kinematical factors and the Γ_r are the elastic widths of these resonances.

They then suggested applying Eq. (1.4) at $t = m_i^2$, where m_i is the mass of an elastic resonance with width Γ_i, in order to obtain a relation between high and low energy parameters. Then $\mathrm{Im}\,\alpha(t=m_i^2)$ is given by $\mathrm{Im}\,\alpha(m_i^2) = am_i\Gamma_i$ and so Eq. (1.4) becomes a non-linear relation between resonance widths.

We investigate a possibility of using the amplitudes given by Eqs. (1.1) and (1.2) as a model. Then the ratio of each resonance width to λ is given and Eq. (1.4) becomes a non-linear equation for λ of the form

$$\sin(c\lambda) = \Upsilon\lambda \tag{1.5}$$

thus determining λ and the over-all scale of the interactions. We note that the amplitudes given by Eqs. (1.1) and (1.2) have Regge asymptotic behaviour and the conditions under which Eq. (1.4) were derived hold with some small modifications.

In Section 2 we derive a modified version of Eq. (1.4) and discuss the approximations involved. We find that there are some uncertainties which make it difficult to determine λ. This is gone into in detail in Section 3 for the $I = 1$ amplitude. In Section 4 we discuss the difficulties involved in applying Eq. (1.4) to the $I = 0$ amplitude. These difficulties illustrate the care which must be exercised in applying USRs.

2. DERIVATION OF THE UNITARITY SUM RULE

We start from the generalized unitarity relation in the complex ℓ plane:

$$A^I(\ell,t) - [A^I(\ell^*,t)]^* = 2ip(t)\, A^I(\ell,t)[A^I(\ell^*,t)]^* \tag{2.1}$$

where

$$p(t) = \left(\frac{\nu_t}{\nu_t + \mu^2}\right)^{1/2} \tag{2.2}$$

in the elastic region $4\mu^2 \leq t \leq t_0$ [6],

$$\nu_t = t/4 - \mu^2 \tag{2.3}$$

μ is the pion mass and $A^I(\ell,t)$ is the signatured partial wave amplitude with isospin I in the t channel, defined for all ℓ by the Froissart-Gribov continuation

$$A^I(\ell,t) = \frac{1}{\pi \nu_t} \int_{4\mu^2}^{\infty} ds\, \tilde{A}_s^I(s,t)\, Q_\ell(1 + s/2\nu_t) \tag{2.4}$$

Here $\tilde{A}_s^I(s,t)$ is the s channel absorptive part with isospin I in the t channel and Q_ℓ is a Legendre function of the second kind.

Arbab and Slansky [4] noticed that Eq. (2.1) implies that

$$-\frac{1}{2ip(t)} = \lim_{\ell \to \alpha^*(t)} A^I(\ell,t) \tag{2.5}$$

where $\alpha(t)$ is the Regge trajectory function, and proposed that the real part of this relation would be a useful tool, independent of FESR, for relating high and low energy parameters. They thus obtained the sum rule

$$\mathrm{Re} \lim_{\ell \to \alpha^*(t)} \int_{4\mu^2}^{\infty} ds\, \tilde{A}_s^I(s,t)\, Q_\ell(1 + s/2\nu_t) = 0 \tag{2.6}$$

In the evaluation *) of this integral we divide the integration region into two parts at $s = N$ above which we assume Regge asymptotic behaviour and below which we saturate with resonances. It turns out that Eq. (2.6) depends on $\mathrm{Im}\,\alpha(t)$ so we apply it at $t = m_i^2$, where $i = \rho, f_0$, which we assume to be elastic resonances. Then $\mathrm{Im}\,\alpha(t)$ is given by the equation

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*) Since the integral in Eq. (2.6) is divergent for $\mathrm{Re}\,\alpha > \ell$, it should be evaluated for $\mathrm{Re}\,\alpha < \ell$ and continued to $\ell = \alpha^*$.

$$\text{Im}\,\alpha(m_i^2) = a\Gamma_i m_i \tag{2.7}$$

Since we know the elastic widths of all the resonances in terms of λ [see Eq. (2.14)] we know $\text{Im}\,\alpha(m_i^2)$ in terms of λ. We can now evaluate the two parts of the integral in Eq. (2.6).

<u>Evaluation of high energy part (s ≫ N)</u>

For $s \gg N$ we assume Regge asymptotic behaviour for $\tilde{A}_s^I(s,t)$ i.e.,

$$\tilde{A}_s^I(s,t) \simeq \pi(2\alpha(t)+1)\beta^I(t)\,P_{\alpha(t)}(z) \tag{2.8}$$

where

$$z = 1 + s/2\nu_t \tag{2.9}$$

We can then do the integral and obtain

$$\text{Re}\lim_{\ell\to\alpha^*(m_i^2)} \frac{1}{\pi\nu_{t=m_i^2}} \int_N^\infty ds\, \tilde{A}_s^I(s,m_i^2)\, Q_\ell(1+s/2\nu_{t=m_i^2})$$

$$= -\frac{\text{Re}\,\beta^I(m_i^2)}{2\,\text{Im}\,\alpha(m_i^2)} \sin[y^I(m_i^2)]\Big[1 + \cot[y^I(m_i^2)]\Big\{\frac{\text{Im}\,\beta^I(m_i^2)}{\text{Re}\,\beta^I(m_i^2)}$$

$$+ 2d^I\,\text{Im}\,\alpha(m_i^2) + \frac{2\,\text{Im}\,\alpha(m_i^2)}{Z_N^I}\Big\} + O\{(\text{Im}\,\alpha(m_i^2))^2\}\Big] \tag{2.10}$$

where

$$y^I(m_i^2) = 2\,\mathrm{Im}\,\alpha(m_i^2)\,\ln(2 z_N^I) \tag{2.11}$$

$$z_N^I = 1 + N/2\nu_{t=m_i^2}$$

$$d^I = \frac{2}{2\,\mathrm{Re}\,\alpha(m_i^2)+1} + \psi(\mathrm{Re}\,\alpha(m_i^2)+1/2) - \psi(\mathrm{Re}\,\alpha(m_i^2)+1) \tag{2.12}$$

with $\psi(z) = \Gamma'(z)/\Gamma(z)$.

<u>Evaluation of low energy part $(s \leq N)$</u>

Here we saturate with resonances and can safely substitute the zero-width approximation for $\tilde{A}_s^I(s,t)$ into Eq. (2.6) and retain sufficient accuracy, as $\mathrm{Im}\,\alpha(m_i^2)$ is small. Thus $\tilde{A}_s^I(s,t)$ is given by

$$\tilde{A}_s^I(s,t) = \sum_{I'=0}^{1} X_{II'} \sum_{\ell'=0}^{\ell_R(m_R<N^{1/2})} \pi(2\ell'+1)\Gamma_R m_R\,\delta(s-m_R^2)\left(\frac{\nu_R+\mu^2}{\nu_R}\right)^{1/2} P_{\ell'}(1+t/2\nu_R) \tag{2.13}$$

where

$$X_{II'} = \begin{pmatrix} 1/3 & 1 \\ 1/3 & 1/2 \end{pmatrix}$$

is a 2x2 sub-matrix of the isospin crossing matrix. (There are no isospin two resonances in our model, of course).

One can explicitly calculate elastic widths Γ_R of resonances R from Eqs. (1.1) and (1.2), e.g.,

$$\Gamma_\rho = \frac{\lambda}{3}\,\frac{(m_\rho^2-4\mu^2)^{3/2}}{m_\rho^2}\,, \quad \Gamma_\varepsilon = \frac{3\lambda}{2}(m_\rho^2-4\mu^2)^{1/2}\,,$$

$$\Gamma_{f_0} = \frac{\lambda\,(m_{f_0}^2-4\mu^2)^{5/2}}{20\,m_{f_0}^2(m_\rho^2-\mu^2)} \quad \ldots . \tag{2.14}$$

and so on [7].

Thus, if we substitute Eqs. (2.10)-(2.14) into Eq. (2.6), drop terms of [8] $0\left\{(\mathrm{Im}\,\alpha(m_i^2))^2\right\}$ and put [9]

$$\rho(m_i^2)\,\mathrm{Re}\,\beta^I(m_i^2) = \mathrm{Im}\,\alpha(m_i^2) \tag{2.15}$$

we obtain

$$\sin(c^I\lambda) + K^I\lambda\cos(c^I\lambda) = \lambda\sum_{j=1}^{J_N}\gamma_j^I\cos(\lambda\delta_j^I) \tag{2.16}$$

where $y^I = c^I\lambda$, γ_j^I and δ_j^I [10] are calculable,

$$\lambda K^I = \frac{\mathrm{Im}\,\beta^I(m_i^2)}{\mathrm{Re}\,\beta^I(m_i^2)} + 2d^I\,\mathrm{Im}\,\alpha(m_i^2) + \frac{2\,\mathrm{Im}\,\alpha(m_i^2)}{z_N^I}$$

$$= O(\mathrm{Im}\,\alpha(m_i^2)) \tag{2.17}$$

and J_N is the integer immediately below $\alpha(N)$. Eq. (2.16) is the modified version of Eq. (1.5). The difficulty in using Eq. (2.16) to fix the value of λ is that we do not know K^I, because we do not know $(\mathrm{Im}\,\beta^I(m_i^2))/(\mathrm{Re}\,\beta^I(m_i^2))$. However, we do know (from unitarity) that the unknown term is $O(\mathrm{Im}\,\alpha(m_i^2))$ and this enables us to get some information from the sum rule.

We must comment here on why we dropped some terms, while we have retained terms of the same (or even lower) order in $\mathrm{Im}\,\alpha(m_i^2)$. Since we have not taken the limit $\mathrm{Im}\,\alpha(m_i^2) \to 0$ we must study the coefficient of $\mathrm{Im}\,\alpha(m_i^2)$ or $(\mathrm{Im}\,\alpha(m_i^2))^2$ before we can neglect terms. This is why we must retain higher order terms in $\mathrm{Im}\,\alpha(m_i^2)$ which arise from expanding $\sin\left[2\,\mathrm{Im}\,\alpha(m_i^2)\,\ell n(2z_N)\right]$ since when $\mathrm{Im}\,\alpha(m_i^2)$ is multiplied by $2\,\ell n(2z_N)$ the product is of order one.

We also point out here that the derivation of the sum rule does not depend on the detailed behaviour of $\mathrm{Im}\,\alpha(s)$. We only assumed the following:

i) $\mathrm{Im}\,\alpha(s) \to \infty$ as $s \to \infty$. This is a necessary assumption because otherwise Eq. (1.1) with Eq. (1.2) would not have Regge asymptotic behaviour;

ii) we can make the identification Eq. (2.7) which enables us to write $\mathrm{Im}\,\alpha(m_i^2)$ in terms of λ.

As is well known, modifications of the Veneziano formula with $\mathrm{Im}\,\alpha(m_i^2) \neq 0$ must have unwanted ancestors and degenerate total widths. However, as we are merely imposing unitarity at one point as a constraint, we feel that this does not affect us.

3. APPLICATIONS TO $I=1$ $\pi\pi$ SCATTERING

3.1 The value of λ for $K^{I=1} = 0$

The interesting question could be raised whether the USR, Eq. (2.16), applied for $I=1$ could fix the scale of the interaction (λ) through its non-linearity.

As shown in Eq. (2.17), the quantity $K^1\lambda$ is of the order $\mathrm{Im}\,\alpha(m_\rho^2)$. Therefore, if $\cot(c^1\lambda)$ is less than one, the second term on the left-hand side of Eq. (2.16) is less than the order $\mathrm{Im}\,\alpha(m_\rho^2)$ compared with the first term.

Thus, one method to obtain information on λ is to solve Eq. (2.16) neglecting the $K^1\lambda$ term and in previous applications [11] this has been done. An important point to note here is that after Eq. (2.16) is solved by neglecting the $K^1\lambda$ term we must check our answer for λ and ascertain that it does not give too large a value for $\cot(c^1\lambda)$, otherwise this would definitely invalidate our approximation.

The solutions of Eq. (2.16) for λ with $K^1 = 0$ are given in the Table for various values of the separation points [12]. As can be seen the solutions appear reasonable for the highest values of N shown (to give a ρ width of 112 MeV we require $\lambda = 0.55$). This decrease of λ with increasing N continues and for $N = N_{30}$ we obtain $\lambda = 0.44$ However, these results for large N should not be taken so seriously as eventually our narrow resonance approximation breaks down. As is also evident from the Table, values for $\cot(c^1 \lambda)$ are less than one for calculated λ but increase as N increases.

Since the magnitude of K^1 (though it is known to be of the order $\mathrm{Im}\,\alpha(m_\rho^2)$) is not known precisely we have no way of knowing the reliability of the value of λ obtained above. The 10% error of the left-hand side of the equation could make the non-linearity of Eq. (2.16) meaningless since the above error can grow to a 100% error for the value of λ.

So at least, in the present stage, we cannot claim that this method will fix the scale of the interaction, unless we have a model to calculate higher order terms in $\mathrm{Im}\,\alpha(m_i^2)$ like $\mathrm{Im}\,\beta^I(m_i^2)/\mathrm{Re}\,\beta^I(m_i^2)$. The reasonable values we obtained for λ must be due to a cancellation which makes K^1 small.

3.2 USR's as a relation between $(\mathrm{Im}\,\beta^I)/(\mathrm{Re}\,\beta^I)$ and λ

Another way to view the USR, Eq. (2.16), is as a relation between λ and $\mathrm{Im}\,\beta^{I=1}(m_\rho^2)/\mathrm{Re}\,\beta^{I=1}(m_\rho^2)$. Therefore, we can say that the USR gives a good determination of $\mathrm{Im}\,\beta^{I=1}(m_\rho^2)/$ $/\mathrm{Re}\,\beta^{I=1}(m_\rho^2)$ if the interaction scale λ is given [13].

In Fig. 2 we give the graphs of λ against K^1 for $N = N_5$, N_{10} and N_{15}. We can conclude (from the $N = N_5$, N_{10} graphs) that $K^1 < 0$. We use the $N = N_5$ plot to determine $\mathrm{Im}\,\beta^{I=1}(m_\rho^2)/$ $\mathrm{Re}\,\beta^{I=1}(m_\rho^2)$. We obtain

$$K^1 \approx -0.064 \tag{3.1}$$

Thus

$$\frac{\mathrm{Im}\,\beta^1(m_\rho^2)}{\mathrm{Re}\,\beta^1(m_\rho^2)} \approx -0.12\lambda \approx -0.07 \tag{3.2}$$

Therefore, we find

$$\frac{\mathrm{Im}\,\beta^1(m_\rho^2)}{\mathrm{Re}\,\beta^1(m_\rho^2)} = O(\mathrm{Im}\,\alpha(m_\rho^2)) \tag{3.3}$$

is satisfied. If this were not so, then it would be because the leading Veneziano term did not satisfy the USR.

It is worth pointing out here that if we take the Regge residues from the Veneziano model [Eq. (1.1)] i.e.,

$$\beta^1(t) = \frac{\pi^{1/2}(a\nu_t)^{\alpha(t)}\lambda\alpha(t)}{\Gamma(\alpha(t)+3/2)} \tag{3.4}$$

and take this seriously for $\mathrm{Im}\,\alpha(t) \neq 0$ we get an expression for $(\mathrm{Im}\,\beta^1(t))/(\mathrm{Re}\,\beta^1(t))$ in terms of $\mathrm{Im}\,\alpha(t)$, i.e.,

$$\frac{\mathrm{Im}\,\beta^1(t)}{\mathrm{Re}\,\beta^1(t)\,\mathrm{Im}\,\alpha(t)} \approx \ln(a\nu_t) - \psi(\mathrm{Re}\,\alpha(t)+3/2) + \frac{1}{\mathrm{Re}\,\alpha(t)} \tag{3.5}$$

for small $\mathrm{Im}\,\alpha(t)$.

At $t = m_\rho^2$, we obtain

$$\frac{\mathrm{Im}\,\beta^1(m_\rho^2)}{\mathrm{Re}\,\beta^1(m_\rho^2)} \approx -1.87\,\mathrm{Im}\,\alpha(m_\rho^2) \tag{3.6}$$

$$\approx -0.14 \tag{3.7}$$

Here $\Gamma_\rho = 112$ MeV is used. This should be compared with Eq. (3.2). In fact, if we use the expression (3.6) we find that the only solution of Eq. (2.16) is $\lambda = 0$. Presumably the moral of this is that we cannot take the one term Veneziano model so seriously for $\mathrm{Im}\,\alpha(t) \neq 0$.

4. APPLICATIONS TO $I = 0$ $\pi\pi$ SCATTERING

Equation (2.16) can also be applied to the $I = 0$ amplitude by taking $t = m_{f_0}^2$. In this case if we assume the f^0 meson to be a pure elastic resonance and try to solve the equation for after assuming $K^0 = 0$ we find that the only solution [14] is $\lambda = 0$. This is because

$$c^0 < \sum_{j=1}^{I_N} \gamma_j^0 \tag{4.1}$$

We could now follow the argument of Section 3.2 and conclude that $K^0 > 0$ and hence $\mathrm{Im}\,\beta^0(m_f^2)/\mathrm{Re}\,\beta^0(m_f^2) > 0$. We could obtain a similar graph to Fig. 2, relating λ to K^0. However, we maintain that the result would have very little meaning because of uncertainties in the inelasticity at $t = m_{f_0}^2$. This affects the sum rule as follows: the relation between $\mathrm{Im}\,\alpha(m_{f_0}^2)$ and λ is changed from

$$\mathrm{Im}\,\alpha(m_{f_0}^2) = \frac{\lambda a^2}{10\, m_{f_0}} (m_{f_0}^2 - 4\mu^2)^{5/2} \tag{4.2}$$

to

$$\mathrm{Im}\,\alpha(m_{f_0}^2) = \frac{\lambda a^2}{10\, m_{f_0}} (m_{f_0}^2 - 4\mu^2)^{5/2}\, x(m_{f_0}^2) \tag{4.3}$$

where $x\,(m_{f_0}^2)$ is the ratio of total to elastic partial wave cross-section.

These considerations did not affect us for the $I = 1$ sum rule because the evidence suggests that up to 1 GeV $\pi\pi$ scattering is elastic [6], despite other channels being open.

Returning to the $I = 0$ sum rule we conclude that although the sum rule is not violated no information can be obtained from it. However, a better knowledge of inelastic effects near the f_0 might yield useful information via the sum rule, relating λ to $\mathrm{Im}\,\beta^0(m_{f_0}^2)/\mathrm{Re}\,\beta^0(m_{f_0}^2)$.

5. CONCLUDING REMARKS

So far, discussions have been focused on the real part of Eq. (2.5). If we take the imaginary part of this equation the following sum rule can also be derived:

$$\frac{1}{2\rho(t)} = \mathrm{Im} \lim_{\ell \to \alpha^*(t)} \frac{1}{\pi \nu_t} \int_{4\mu^2}^{\infty} ds\, \tilde{A}_s^I(s,t)\, Q_\ell(1 + s/2\nu_t) \tag{5.1}$$

However, in evaluating the low energy integral we find that the term $\mathrm{Im}\,\tilde{A}_s^I(s,t)\,\mathrm{Re}\,Q_{\alpha^*(t)}(1+s/2\nu_t)$ cannot be neglected as it is of the same order in $\mathrm{Im}\,\alpha(t)$ as $\mathrm{Re}\,\tilde{A}_s^I(s,t)\,\mathrm{Im}\,Q_{\alpha^*}(1+s/2\nu_t)$. Since there is no reliable method of calculating the $\mathrm{Im}\,\tilde{A}_s^I(s,t)\,\mathrm{Re}\,Q_{\alpha^*(t)}(1+s/2\nu_t)$ term the above sum rule is not very useful at the present stage.

We must also remark that USRs and FESRs give independent information. Arbab and Slansky found this numerically but we can see it as follows: the model given by Eqs. (1.1) and (1.2) would satisfy FESRs for any value of λ, although λ must be positive so that the residues of the resonances are positive. However, any positive value of λ would not satisfy the USRs, even allowing for uncertainties in $\mathrm{Im}\,\beta^I(m_i^2)/\mathrm{Re}\,\beta^I(m_i^2)$. This is, of course, the use for USRs proposed by Arbab and Slansky, albeit in a weaker form because of the uncertainties. Despite this we were able to conclude that the model given by Eqs. (1.1) and (1.2) with $\lambda = 0.55$ (which we need to obtain the correct ρ width) was consistent with both the $I = 0$ and $I = 1$ USRs.

Also we have shown that the $I = 1$ USR can be used to determine $\mathrm{Im}\,\beta^1(m_\rho^2)/\mathrm{Re}\,\beta^1(m_\rho^2)$ when λ is given. This use is not restricted to the Veneziano model as we could insert experimental widths instead of the widths obtained from the model. This value for $\mathrm{Im}\,\beta^1(m_\rho^2)/\mathrm{Re}\,\beta^1(m_\rho^2)$ could be useful in the dispersion relations for $\beta^1(t)$ and $\alpha(t)$.

To illustrate the uses of USRs we have used the Veneziano model with only leading terms. It is, however, possible that for a realistic model we need satellite terms, perhaps even an infinite number of them [15].

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T A B L E

N	N_1	N_2	N_3	N_4	N_5	N_6	N_7	N_8
c	0.67	0.83	0.94	1.03	1.07	1.12	1.16	1.20
$\sum_{i=1}^{n} \gamma_i$	0.54	0.72	0.84	0.92	0.98	1.03	1.07	1.10
λ	2.23	1.38	1.06	0.90	0.81	0.74	0.71	0.69
$y(m_\rho^2)$	1.49	1.15	0.99	0.91	0.87	0.83	0.82	0.82
Γ_ρ (in MeV)	459	283	213	185	167	152	147	142
	5.5	10	14.5	19	23.4	27.8	32.3	36.7

Solutions of Eq. (2.16) for various values of N and $K^1 = 0$

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6) From a purely theoretical point of view $t_0 = 16\mu^2 \approx 0.3$ GeV^2 but experiment seems to indicate that we can take $t_0 \approx 1$ GeV^2. See
P.B. Johnson et al., Phys.Rev. 176, 1651 (1968).

7) There is a small problem here since the width of the ε' (see Fig. 1) turns out to be small and negative for $\lambda > 0$. Therefore we ignore its contribution. We consider all of the other resonances shown in Fig. 1.

8) This is a good approximation for $i = \rho, f_0$, since, inserting experimental values in Eq. (2.7) we obtain $\mathrm{Im}\,\alpha(m_\rho^2) \approx 0.09$, $\mathrm{Im}\,\alpha(m_{f_0}^2) \approx 0.15$.

9) This involves as approximation since Eq. (2.15) is only true if we neglect terms of $0\{(\mathrm{Im}\,\alpha(m_i^2))^2\}$. However, as we do not know the corrected version of Eq. (2.15) we assume it as exact. This makes very little difference as we can consider the correction as being absorbed in the unknown term of K^I, i.e., $\mathrm{Im}\beta^I(m_i^2)/\mathrm{Re}\beta^I(m_i^2)$. However, if we use Eq. (2.15) to determine $\mathrm{Im}\beta^I(m_i^2)/\mathrm{Re}\beta^I(m_i^2)$ (see Section 3.2) the result will be presumably distorted. However it seems reasonable to lump the unknowns together as "background".

10) The $\cos(\lambda\delta_j^I)$ term in Eq. (2.16) comes from $\mathrm{Re}\, Q_{\alpha^*}(z) \approx \approx Q_{\mathrm{Re}\,\alpha}(z)\cos(\mathrm{Im}\,\alpha\, \ln(2z))$ for z corresponding to the resonances.

11) E.g., Ref. 4) and the original version of this paper. We are grateful to Drs. R.C. Slansky and C. Rebbi for pointing out to us that the argument given in the original version [footnote 9)] to justify dropping the $\mathrm{Im}\beta^1(m_\rho^2)/\mathrm{Re}\beta^1(m_\rho^2)$ term is incorrect.

12) The separation points $N = N_1, N_2, \ldots, N_8$ are defined by Fig. 1. Each N_i is placed half-way between two sets of resonances.

13) Apart from the limitations mentioned in Ref. 9). Of course, this does not affect Eq. (3.3).

14) This has also been found by Slansky and Rebbi (private communication).

15) See, for instance, S. Matsuda, Phys.Rev. (to be published).

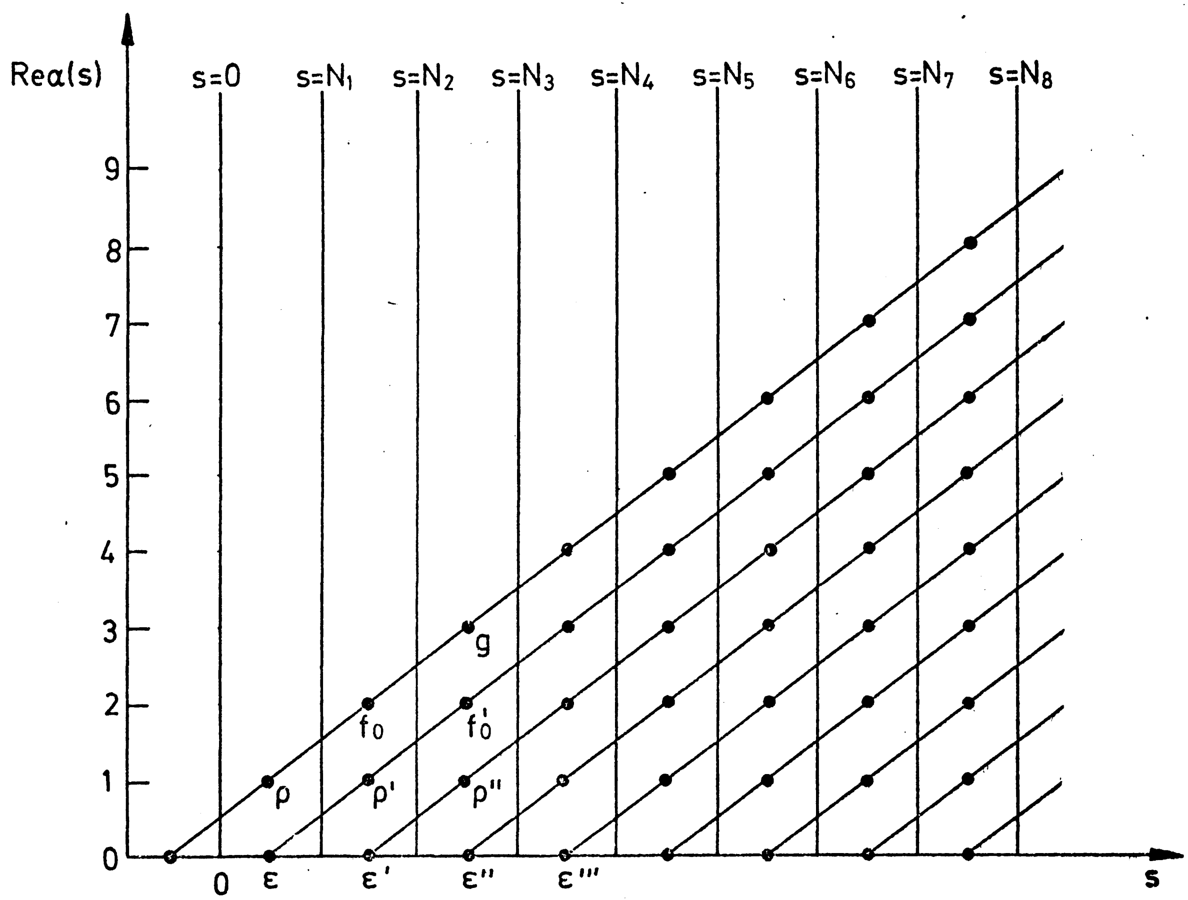

FIG. 1 Resonances used to saturate the sum rule (marked •) and definition of $N_1, N_2 \ldots\ldots N_8$

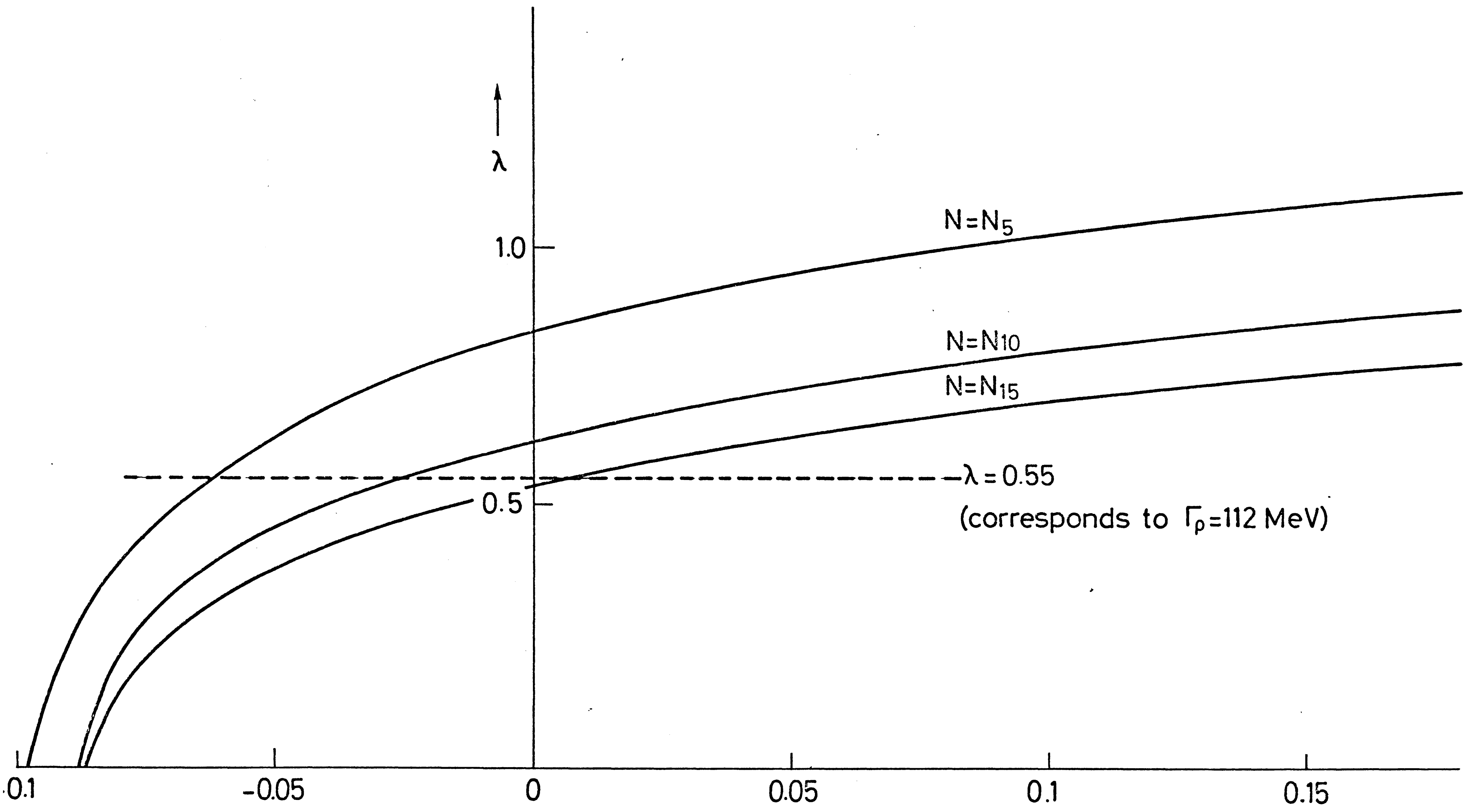

FIG.2 Plots of λ vs. K^1 for $N = N_5, N_{10}$ and N_{15}