PROOF OF DISPERSION RELATIONS AND UNITARITY OF S MATRIX

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A B S T R A C T

The proof of dispersion relation is extended to higher momentum transfer by making use of the unitarity of S matrix for the energy region below the threshold for three-particle states. For the pion-nucleon scattering, dispersion relation can be proved up to the momentum transfer $\Delta^2 < 4.7\,\mu^2$. While Mandelstam was able to show an extended analyticity of the scattering amplitude for equal mass particles, we cannot apply his method to unequal mass cases. Therefore, we follow essentially the procedure of Streater with a view to apply the Jost-Lehmann-Dyson integral representation explicitly to the final expression of the absorptive part.

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I. Introduction

During the past few years, dispersion relations have been rigorously proved for a certain class of the strongly interacting processes, however, only up to certain limited values of the momentum transfer [1),2)]. As to the failure to establish them for a higher momentum transfer of these processes, it has been attributed to our inability so far to incorporate all the information contained in the underlying basic axioms into the analysis. For instance, the content of the unitarity of the S matrix has not been sufficiently applied in the hitherto treatment of the problem, due to the difficulty arising from the non-linear character of the unitarity.

In fact, Mandelstam succeeded recently to show that the region of analyticity of the scattering amplitude for equal mass particles could be extended by making use of the unitarity [3)]. In this paper, we shall investigate whether one can improve the restriction on the momentum transfer for the validity of dispersion relations for unequal mass-particle scatterings by making use of the information contained in the unitarity of the S matrix. To be specific, we shall consider the pion-nucleon scattering.

First of all, we start with the observation that the absorptive part of the scattering amplitude $A(W, \Delta^2, \zeta)$ is analytic in the region $Z = \{ \zeta : -R < \mathrm{Re}\, \zeta \leq \mu^2 , |\mathrm{Im}\, \zeta| < \delta \}$, provided that $W > m+2\mu$ and $\Delta^2 < 4.7\,\mu^2$, as shown by Lehmann [1)]. In order to prove dispersion relation up to $\Delta^2 < 4.7\,\mu^2$, one has now to consider only the elastic contribution to the dispersion integral. In the elastic region, the unitarity of S matrix takes the simplest form $A(W, \Delta^2, \zeta) \sim \int d\Omega\, T(W, \Delta_1^2, \zeta, \mu^2) T^*(W, \Delta_2^2, \zeta, \mu^2)$. Then, following Streater [4)], we derive dispersion relations for the intermediate transitions, before applying the Jost-Lehmann-Dyson representation to $T(W, \Delta_i^2, \zeta, \mu^2)$ as it was the case in Refs. [1)] and [2)]. Now in turn, we apply the Jost-Lehmann-Dyson representation to $A(W_i, \Delta_i, \zeta, \mu^2)$ which are absorptive parts of the intermediate transition amplitudes to carry out the angular integration $\int d\Omega$ over the intermediate states. With it we obtain an explicit dependence of $A(W, \Delta^2, \varsigma)$ on W as well as on ς and Δ^2. Unfortunately, this expression for $A(W, \Delta^2, \varsigma)$ cannot be shown to be analytic in the required strip Z in the ς plane, however, for the proof of dispersion relation it suffices to prove the analyticity of the dispersion integral. By making use of the explicit W dependence of $A(W, \Delta^2, \varsigma)$, the method of Polkinghorne et al. shows that the dispersion integral is in fact analytic in Z, though $A(W, \Delta^2, \varsigma)$ itself might not be so.

II. Preliminaries

We consider the pion-nucleon scattering. Let the four-momenta of the incoming and outgoing pions be k_1 and k_2, respectively, while those of the nucleons be p_1 and p_2. As usual, we neglect the spin and isospin of these particles, since its inclusion does not affect the analyticity considerations. With the definition of an additional variable $\varsigma = k_1^2 = k_2^2$, for a fictitious pion mass $\varsigma < -\Delta^2$, we can derive the following dispersion relation

$$T(W, \Delta, \varsigma) = \frac{1}{\pi}\int_{(m+\mu)^2}^{\infty} dW'^2 \left(\frac{1}{W'^2 - W^2 - i\epsilon} + \frac{1}{W'^2 + W^2 - 2m^2 - 2\varsigma - 4\Delta^2}\right) A(W', \Delta, \varsigma) \tag{1}$$

$$+ \text{ single nucleon terms,}$$

with

$$W^2 = (p_i + k_i)^2, \quad 4\Delta^2 = -(p_1 - p_2)^2, \quad \cos\theta = 1 - \frac{2\Delta^2}{K^2(W, \varsigma)},$$

$$K^2(W, \varsigma) = \frac{(W^2 + m^2 - \varsigma)^2 - 4W^2 m^2}{4W^2}, \quad K^2(W) \equiv K^2(W, \mu^2). \tag{2}$$

As it is well known [1], the absorptive part of the scattering amplitude $A(W, \Delta, \varsigma)$ is analytic in the strip $Z(W)$ in the ς plane defined by

$$Z(W) = \{ \varsigma : -R < \mathrm{Re}\, \varsigma \leq \mu^2 \qquad |\mathrm{Im}\, \varsigma| < \delta(W) \} \quad {}^{5)}$$

provided that

$$\Delta^2 < K^2(W) + \frac{8\mu^3(2m+\mu)}{W^2 - (m-2\mu)^2} \tag{3}$$

We now separate the dispersion integral in (1) into the following two parts

$$T(W, \Delta, \varsigma) = T_1(W, \Delta, \varsigma) + T_2(W, \Delta, \varsigma) + \tag{4}$$

+ single nucleon terms,

with

$$T_1(W, \Delta, \varsigma) = \frac{1}{\pi} \int_{(m+\mu)^2}^{(m+2\mu)^2} dW'^2 \left(\frac{1}{W'^2 - W^2 - i\epsilon} + \frac{1}{W'^2 + W^2 - 2m^2 - 2\varsigma - 4\Delta^2} \right) A(W', \Delta, \varsigma)$$

$$T_2(W, \Delta, \varsigma) = \frac{1}{\pi} \int_{(m+2\mu)^2}^{\infty} dW'^2 \left(\frac{1}{W'^2 - W^2 - i\epsilon} + \frac{1}{W'^2 + W^2 - 2m^2 - 2\varsigma - 4\Delta^2} \right) A(W', \Delta, \varsigma)$$

Then according to (3), $T_2(W, \Delta, \varsigma)$ is shown to be analytic in $Z(W)$, if the momentum transfer is restricted by

$$\Delta^2 < \left[K^2 \right]_{W=m+2\mu} + \frac{8\mu^3\,(2m+\mu)}{(m+2\mu)^2-(m-2\mu)^2} = 4.7\,\mu^2 . \qquad (5)$$

Now, what we attempt to show in this paper is that $T_1(W,\Delta,\zeta)$ is also analytic in Z under the condition (5) as well as $T_2(W,\Delta,\zeta)$.

To this end we start with the following expression for the absorptive part [1)]

$$A(W',\Delta,\zeta) = \sum_{\gamma} \langle p_2 | j(0) | \gamma \rangle \langle \gamma | j(0) | p_1 \rangle , \qquad (6)$$

where $j(x)$ is the interaction current of the pion. The summation should be extended over all the physical states $|\gamma\rangle$ with the total four-momentum p_i+k_i $(i = 1,2)$ and with the same quantum number as the states $|p_i,k_i\rangle$. Since we are now only interested in $T_1(W,\Delta,\zeta)$, namely the contribution from the elastic region $m+\mu < W' < m+2\mu$, the reduction formula of Lehmann et al.[6)] allows us to rewrite (6) in the following form

$$A(W',\Delta,\zeta) = \sum_{\gamma} \int d^4x_1 \int d^4x_2 \; e^{\frac{i}{2}\{(k_2(\zeta)+k_3)x_2 - (k_1(\zeta)+k_3)x_1\}} . \qquad (7)$$

$$\cdot\, \theta(x_{20})\,\theta(-x_{10}) \langle p_2 | [j(\tfrac{1}{2}x_2), j(-\tfrac{1}{2}x_2)] | p_3,\gamma \rangle \langle p_3,\gamma | [j(\tfrac{1}{2}x_1), j(-\tfrac{1}{2}x_1)] | p_1 \rangle$$

+ terms involving equal time commutator,

where p_3 and k_3 denote the four-momenta of nucleon and pion in the intermediate states $|\gamma\rangle$, respectively. From the transitional invariance we get

$$p_1 + k_1 = p_2 + k_2 = p_3 + k_3$$

It should be noted here that $k_3^2 = \mu^2$, and $p_3^2 = m^2$, while $k_{1,2}^2 = \varsigma$. We now fix the Lorentz frame in the c.m.s. as follows

$$\begin{aligned} \underset{\sim}{p}_1(\varsigma) &= (0,\ 0,\ K(W';\varsigma)) = -\underset{\sim}{k}_1(\varsigma) \\ \underset{\sim}{p}_2(\varsigma) &= (K(W';\varsigma)\sin\theta,\ 0,\ K(W';\varsigma)\cos\theta) = -\underset{\sim}{k}_2(\varsigma) \\ \underset{\sim}{p}_3 &= (K(W')\sin\theta'\cos\varphi',\ K(W')\sin\theta'\sin\varphi',\ K(W')\cos\theta'), \end{aligned} \tag{8}$$

then the summation over the intermediate states turns out to be

$$\sum_{\gamma} \longrightarrow \frac{K(W')}{W'} \int_0^{2\pi} d\varphi' \int_{-1}^{1} d\cos\theta' \tag{9}$$

With the following definition of the transition amplitude

$$T(W';t,\varsigma,\mu^2) = \int d^4x\, e^{i\frac{1}{2}(k_1(\varsigma)+k_3)x}\,\theta(x_0) < p_3\,|[j(\tfrac{1}{2}x),\, j(-\tfrac{x}{2})]|\,p_1 > \tag{10}$$

where

$$k_1^2 = \varsigma,\quad k_3^2 = \mu^2 \qquad \text{and} \qquad 4t = -(k_1(\varsigma) - k_3)^2,$$

Eq. (7) which is nothing but the unitarity of S matrix takes the following simplest form

$$A(W;\Delta,\varsigma) = \frac{K(W')}{W'} \int d\varphi' \int d\cos\theta'\; T(W';\Delta_1,\varsigma,\mu^2)\; T^*(W';\Delta_2,\varsigma,\mu^2) \tag{7a}$$

with

$$4\Delta_i^2 = 2\{f(W', \varsigma, \mu^2) - m^2 - K(W', \varsigma)K(W')r_i\} \quad (i = 1,2) \tag{11}$$

where

$$f(W', \varsigma, \mu^2) = (4W'^2)^{-1}(W'^2 + m^2 - \varsigma)(W'^2 + m^2 - \mu^2),$$
$$r_1 = \cos\theta', \quad r_2 = \sin\theta \sin\theta' \cos\varphi' + \cos\theta \cos\theta'. \tag{12}$$

In particular,

$$f(W', \varsigma, \varsigma) = K^2(W', \varsigma) + m^2$$

III. Dispersion relations for intermediate transitions

In order to apply again the unitarity of S matrix now to the intermediate transition amplitudes $T(W, \Delta_i, \varsigma, \mu^2)$, (we shall hereafter drop the prime over W), we shall have to be able to express them in terms of their imaginary parts, and evidently it will be possible only through the dispersion relations for these amplitudes. For the purpose of obtaining dispersion relations we shall proceed along Oehme and Taylor's proof[2] of dispersion relations for inelastic scattering.

Since for the time being we keep ς at some sufficiently negative value such that $|\cos\theta| \leq 1$, we have $|r_2| \leq 1$ as well as $|r_1| \leq 1$, $T(W, \Delta_1, \varsigma, \mu^2)$ and $T(W, \Delta_2, \varsigma, \mu^2)$ can be treated in the same way. Now we fix the momentum transfer Δ_i at the value determined by $k_i^2 = \varsigma$ and $k_3^2 = \mu^2$, then clearly

$$\Delta_i^2(\varsigma, \mu^2) > 0. \tag{13}$$

Now we move both k_i^2 $(i = 1,2)$ and k_3^2 to the off-mass shell and introduce

$$k_i^2 = \varsigma_1,$$
$$k_3^2 = \varsigma_2,$$

($\varsigma_i < \varsigma$, Δ_i^2 is fixed for the time being), then we can obtain a dispersion relation

$$T(W, \Delta_i(\varsigma, \mu^2), \varsigma_1, \varsigma_2) = \frac{1}{\pi}\int_{(m+\mu)^2}^{\infty} dW_i^2 \left(\frac{1}{W_i^2 - W^2 - i\epsilon} + \frac{1}{W_i^2 + W^2 - 2m^2 - \varsigma_1 - \varsigma_2 - 4\Delta_i^2}\right) A(W_i, \Delta_i, \varsigma_1, \varsigma_2) \tag{14}$$

$+$ single nucleon terms

provided that

$$\frac{\varsigma_1 + \varsigma_2}{2} < \Delta_i^2 + \frac{(\varsigma_1 - \varsigma_2)^2}{\Delta_i^2} \equiv \alpha_0 \tag{15}$$

Fixing $\varsigma_1 - \varsigma_2$ at the value $\varsigma - \mu^2$, this inequality is satisfied by sufficiently negative values of $\varsigma_1 + \varsigma_2/2$ ($\equiv \alpha$). Then our problem turns out to prove that both sides of Eq. (14) can be analytically continued in α up to the value $\alpha = \varsigma + \mu^2/2$. As it will be shown in the next section by means of the integral representation, $A(W_i, \Delta_i, \varsigma_1, \varsigma_2)$ will be proved to be analytic in the relevant domain of the α plane, we finally arrive at the dispersion relation

$$T(W, \Delta_i (\varsigma, \mu^2), \varsigma, \mu^2) = \frac{1}{\pi} \int_{(m+\mu)^2}^{\infty} dW_i^2 \left(\frac{1}{W_i^2 - W^2 - i\epsilon} + \frac{1}{W_i^2 + W^2 - 2m^2 - \varsigma - \mu^2 - 4\Delta_i^2} \right) \times$$

$$\times A(W_i, \Delta_i, \varsigma, \mu^2) + \text{single nucleon terms} \tag{16}$$

Substituting (16) into (7a) we obtain

$$A(W, \Delta, \varsigma) = \frac{K(W)}{W} \frac{1}{\pi^2} \int_{(m+\mu)^2}^{\infty} dW_1^2 \int_{(m+\mu)^2}^{\infty} dW_2^2 \int_0^{2\pi} d\varphi' \int_{-1}^{1} d\cos\theta' \times$$

$$\times \prod_{i=1}^{2} \left\{ (D_i^{-1} + \overline{D}_i^{-1}) A(W_i, \Delta_i, \varsigma, \mu^2) \right\} + \tag{17}$$

+ terms involving single nucleon terms

with

$$D_i = W_i^2 - W^2 + (-)^i i\epsilon ,$$

$$\overline{D}_i = W_i^2 + W^2 - 2m^2 - \varsigma - \mu^2 - 4\Delta_i^2 . \tag{18}$$

IV. Integral representation of $A(W_i, \Delta_i, \varsigma_1, \varsigma_2)$

For the purpose of studying the analytic properties, we start with the Jost-Lehmann-Dyson integral representation [7] for the absorptive part

$$A(W_i, \Delta_i(\varsigma, \mu^2), \varsigma_1, \varsigma_2) = \int_{(S)} dU \int_0^{2\pi} d\alpha_i \frac{\Phi(U, \cos\alpha_i, W_i)}{Y_i - K_{i1} K_{i2} \cos(\theta_i - \alpha_i)} \tag{19}$$

$$(i = 1,2)$$

where

$$dU = du_{01}\, du_{02}\, d|\underline{u}_1|\, d|\underline{u}_2|\, d\sigma_1^2\, d\sigma_2^2\, d\beta_1\, d\beta_2 ,$$

$$K_{ij} = K(W_i, \varsigma_j), \tag{20}$$

$$K_{i1} K_{i2} \cos\theta_i = f(W_i, \varsigma, \mu^2) - f(W, \varsigma, \mu^2) + K(W, \varsigma) K(W, \mu^2)\, r_i ,$$

and

$$y_{ij} = (2 \sin\beta_j |\underline{u}_j|)^{-1} \left[|\underline{u}_j|^2 + \sigma_j^2 + K_{ij}^2 - \left\{ \frac{(m^2 - \varsigma_j)}{2W_i} - u_{j0} \right\}^2 \right] ,$$

$$Y_i = y_{i1} y_{i2} + \sqrt{y_{i1}^2 - K_{i1}^2} \sqrt{y_{i2}^2 - K_{i2}^2} \tag{20a}$$

The support for the weight function is given by

$$(S): \begin{cases} 0 \le |\underline{u}_j| \le \dfrac{W_i}{2} , \quad |u_{j0}| \le \dfrac{W_i}{2} - |\underline{u}_j| \\ \\ \sigma_j > \max\left\{ 0 \;:\; m_1 - \sqrt{\left(\frac{W_i}{2} + u_{j0}\right)^2 - \underline{u}_j^2} \;:\; m_2 - \sqrt{\left(\frac{W_i}{2} - u_{j0}\right)^2 - \underline{u}_j^2} \right\}. \end{cases} \tag{21}$$

It is well known that the minimum value of y in the region (Σ) is

$$\tilde{y}_{ij}^2 \equiv K_{ij}^2 + z_{ij}^2 = K_{ij}^2 + \frac{2(9\mu^2 - \varsigma_j)(2m+\mu)}{W_i^2 - (m-2\mu)^2} ,$$

$$\tilde{Y}_i = \sqrt{K_{i1}^2 + z_{i1}^2}\,\sqrt{K_{i2}^2 + z_{i2}^2} + z_{i1} z_{i2} \tag{22}$$

To prove the dispersion relation (16), it suffices to show that the denominator in the integral representation (19) does not vanish in the relevant regions of all the variables involved, namely

$$f(W_i, \varsigma_1, \varsigma_2) - f(W, \varsigma, \mu^2) + K(W, \varsigma) K(W, \mu^2) r_i \neq$$

$$\neq Y_i \cos\alpha_i + i\sqrt{Y_i^2 - K_{i1}^2 K_{i2}^2}\ \sin\alpha_i . \tag{23}$$

As we assumed r_i are real and $|r_i| \leq 1$, all terms in (23) are real except for the second term of the r.h.s., and we have to show only

$$f(W_i, \varsigma_1, \varsigma_2) - f(W, \varsigma, \mu^2) + K(W, \varsigma) K(W, \mu^2) r_i \neq \pm Y_i \tag{24}$$

Since $|r_i| \leq 1$, this inequality will follow immediately, if we can prove the following inequalities

$$\tilde{Y}(W_i, \varsigma_1, \varsigma_2) + f(W_i, \varsigma_1, \varsigma_2) > K(W, \varsigma) K(W, \mu^2) + f(W, \varsigma, \mu^2) , \tag{25}$$

and

$$\tilde{Y}(W_i, \varsigma_1, \varsigma_2) + f(W, \varsigma, \mu^2) > K(W, \varsigma) K(W, \mu^2) + f(W_i, \varsigma_1, \varsigma_2) , \tag{25a}$$

for the relevant regions of the variables involved, namely $m+\mu < W < m+2\mu$, $W_i \geq m+\mu$, $\alpha_0 < \alpha \leq \rho + \mu^2/2$, $\rho_1 - \rho_2/2 = \rho - \mu^2/2$. In the following we shall show that these inequalities are valid in general for any $\rho_1 \leq \rho_2$, $\rho_1 < \rho$, $\rho_2 \leq \mu^2$, which obviously include the region given above. The validity of these inequalities for $\rho_1 = \rho$, $\rho_2 = \mu^2$ will turn out to be essential in our later discussions.

i) Proof of Eq. (25)

It is clear that the l.h.s. of (25) takes its minimum value at $W_i = m+\mu$, while the r.h.s. becomes maximum at $W = m+2\mu$. Substituting these values which give rise to the worst case, it is easy to show that

$$B(\rho_1, \rho_2) \equiv z_{i_1} z_{i_2} - f(W, \rho, \mu^2) + f(W_i, \rho_1, \rho_2) > 0$$

since $B(\rho_1, \rho_2)$ can be shown to be monotone decreasing both in ρ_1 and in ρ_2, respectively, i.e.,

$$\frac{\partial}{\partial \rho_i} B(\rho_1, \rho_2) = -\frac{(2m+\mu)}{3(2m-\mu)}\sqrt{\frac{9\mu^2-\rho_j}{9\mu^2-\rho_i}} - \frac{(W_i^2+m^2-\rho_j)}{4W_i^2} < 0$$

$(i, j = 1, 2 \text{ and } i \neq j)$

and

$$B(\mu^2, \mu^2) = \left[\tilde{y}^2(\mu^2)\right]_{W_i = m+\mu} - \left[K^2(\mu^2)\right]_{W = m+2\mu} = 3\mu^2 - 2.6\mu^2 > 0 .$$

Furthermore,

$$\tilde{y}\,(W_i,\varrho_1,\varrho_2) > \tilde{y}\,(W_i,\varrho,\mu^2) > K(W,\mu^2$$

because

$$\frac{\partial}{\partial\varrho}\left\{\tilde{y}^2(W_i,\varrho) - K^2(W,\varrho)\right\} < 0,$$

$$\tilde{y}^2(W_i,\mu^2) - K^2(W,\mu^2) > 0.$$

But $\tilde{Y}$ is given by (22), so we proved the inequality (25).

ii) Proof of (25a)

(a) $m+\mu < W_i < m+2\mu$

In this case, clearly

$$\tilde{Y}(W_i,\varrho_1,\varrho_2) + f(W,\varrho,\mu^2) > \tilde{Y}(m+\mu,\varrho_1,\varrho_2) + f(m+\mu,\varrho,\mu^2),$$

and

$$K(W,\varrho)K(W,\mu^2) + f(W_i,\varrho_1,\varrho_2) < K(m+2\mu,\varrho)K(m+2\mu,\mu^2) + f(m+2\mu,\varrho_1,\varrho_2)$$

thus we can immediately apply the above argument for i).

(b) $W_i > m+2\mu$

For our purpose, it suffices to show

$$C(\varrho_1,\varrho_2) = K(W_i,\varrho_1)K(W_i,\varrho_2) - K(W,\varrho)K(W,\mu^2) - f(W_i,\varrho_1,\varrho_2) + f(W,\varrho,\mu^2) > 0,$$

since

$$\tilde{Y}(W_i, \rho_1, \rho_2) > K(W_i, \rho_1) K(W_i, \rho_2)$$

Further, one can show that $C(\rho_1, \rho_2)$ $C(\rho, \mu^2)$.
Now, from (20) one has

$$\cos\theta_i = 1 - \frac{C(\rho, \mu^2)}{K(W_i, \rho) K(W_i, \mu^2)}$$

for $r_i = 1$.

If $W_i = W$, then $C(\rho, \mu^2) = 0$, i.e., $\cos\theta_i = 1$. Therefore the point $W_i = W$ lies evidently in the physical region of the dispersion relation for $T(W_i \Delta_i, \rho, \mu^2)$.

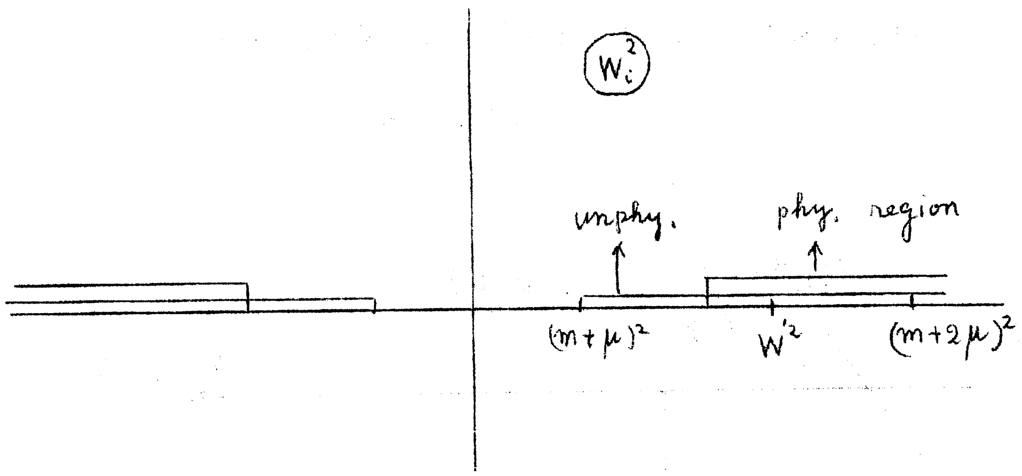

Fig. 1

As it was discussed in connection with the dispersion relation for the associated production [8], there appears only a finite unphysical region between two physical regions extending to infinities in W_i^2 plane (Fig. 2), unless the momentum transfer becomes negative. From the definition (11), we have $\Delta_i^2 > 0$. Hence, any W_i greater than W lies clearly in the physical region. Consequently, $|\cos\theta_i| \leq 1$ for $W_i > m+2\mu$, since $W \leq m+2\mu$. Thus we proved that $C(\varsigma_1, \varsigma_2) > 0$.

Since our discussions made above is valid for any $\varsigma_1 < \varsigma_2 < \mu^2$, we can assert that

$A(W_1, \Delta_1, \varsigma_1, \varsigma_2)$ is analytic in Z' and Z,

$A(W_2, \Delta_2, \varsigma_1, \varsigma_2)$ is analytic in Z',

where

$$Z' = \{\varsigma_1, \varsigma_2 : 2\alpha_0 < Re(\varsigma_1 + \varsigma_2) \leq \varsigma + \mu^2, \quad Re(\varsigma_1 - \varsigma_2) = \varsigma - \mu^2$$

$$\text{and} \quad |Im\, \varsigma_i| < \delta \}$$

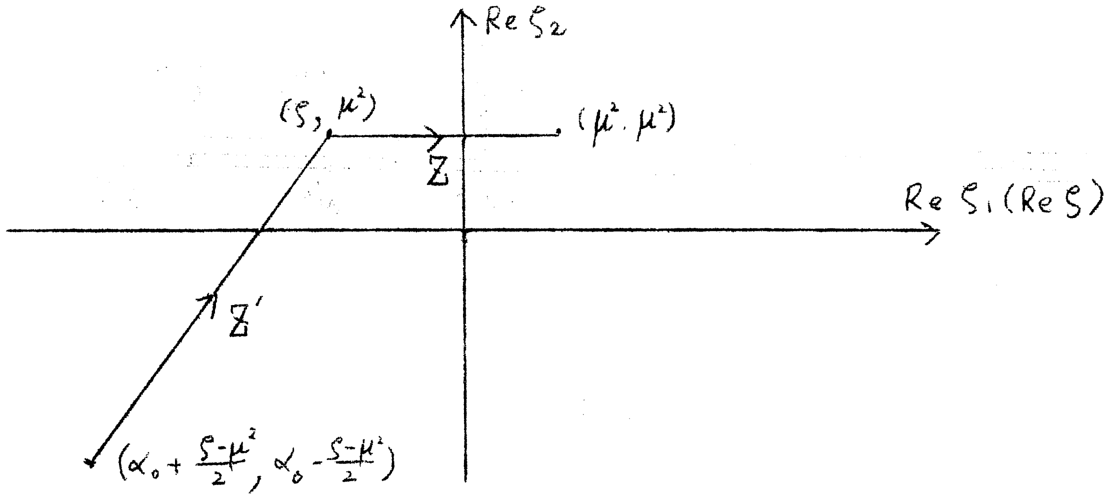

Fig. 2

Along the strip Z', the dispersion denominators $\bar{D}_i$ may vanish, however, it is immaterial, since we have a <u>uniform finite strip</u> of analyticity which allows us to shift the path of the analytic continuation through $\varsigma \to \varsigma - i\epsilon$ [5].

It should be noted here that the inequalities (25) and (25a) are valid also in the <u>uniform finite strip</u> Z although it might be small.

V. Dispersion denominators along Z

While the zeros of $\bar{D}_i$ along Z' did not cause any trouble in the derivation of the dispersion relation (16), those along Z, if any, would give rise to a singularity in the analytic continuation of $A(W,\Delta,\varsigma)$ given by (17) in ς, since there appears a product of the denominators, i.e., $\bar{D}_1\bar{D}_2$ which contains ς. In this section we shall show that there is no such zeros of $\bar{D}_i$ along Z, under a certain condition on the momentum transfer.

From (11) and (18)

$$\bar{D}_i(\varsigma,\mu^2) = W_i^2 + W^2 - \varsigma - \mu^2 - 2f(W,\varsigma,\mu^2) + 2K(W,\varsigma)K(W,\mu^2)\eta_i \tag{26}$$

i) <u>For $i = 1$</u>

The worst case arises when $\cos\theta' = -1$. Put

$$\bar{D}_1^{\circ}(\varsigma,\mu^2) = W_1^2 + W^2 - \varsigma - \mu^2 - 2f(W,\varsigma,\mu^2) - 2K(W,\varsigma)K(W,\mu^2)$$

then $\bar{D}_1^0(\varsigma, \mu^2)$ is evidently monotone increasing in ς, i.e.,

$$\frac{\partial}{\partial \varsigma} \bar{D}_1^0(\varsigma, \mu^2) < 0$$

thus it takes a minimum value at the minimum possible value of ς. By choosing this value appropriately (of course it must be $\varsigma < -\Delta^2$ such that at the starting point of the analytic continuation in ς $\cos\theta$ is physical) we can show that $\bar{D}_1^0(\varsigma, \mu^2) > 0$. Then it is obvious that $\bar{D}_1(\varsigma, \mu^2)$ does not vanish along Z up to $\varsigma = \mu^2$.

ii) For i = 2

In this case $r_2 = \sin\theta \sin\theta' \cos\varphi' + \cos\theta \cos\theta'$. For some negative values of ς, $\cos\theta$ is still in the physical region. Then we can prove $\bar{D}_2(\varsigma, \mu^2) > 0$ by the same argument as given above. But for ς greater than a certain critical value, $\cos\theta$ will take an unphysical value, i.e., $\cos\theta < -1$, thus $\sin\theta$ will become pure imaginary. Hence, in order $\bar{D}_2(\varsigma, \mu^2)$ to vanish, the following two conditions must be satisfied simultaneously, namely

$$\sin\theta' \cos\varphi' = 0,$$

and

$$Re\, \bar{D}_2 = W_2^2 + W^2 - \varsigma - \mu^2 - 2 f(W, \varsigma, \mu^2) + \\ + 2\, K(W, \varsigma)\, K(W, \mu^2) \cos\theta \cos\theta' = 0. \tag{27}$$

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But from (2) one obtains

$$\mathrm{Re}\,\overline{D}_2(\varsigma,\mu^2) > W_2^2 + W^2 - \varsigma - \mu^2 - 2f(W,\varsigma,\mu^2) + 2K(W,\varsigma)K(W,\mu^2) - 4\Delta^2$$

for $\cos\theta' = 1$ which would give the worst case possible, since $\cos\theta < -1$. One can now show that the r.h.s. is monotonously decreasing in ς, i.e.,

$$\frac{\partial}{\partial\varsigma}(\text{r.h.s.}) = -1 + \frac{W^2+m^2-\mu^2}{2W^2} - \frac{K(W,\mu^2)}{K(W,\varsigma)}\frac{(W^2+m^2-\varsigma)}{2W^2} < 0$$

thus we get

$$\mathrm{Min}\ \mathrm{Re}\,\overline{D}_2(\varsigma,\mu^2) = W_2^2 + W^2 - 2m^2 - 2\mu^2 - 4\Delta^2 . \tag{28}$$

With it we proved that

$$\overline{D}_2(\varsigma,\mu^2) \neq 0 \quad \text{(either complex or real and positive) for } \varsigma \in Z , \tag{27a}$$

provided that

$$4\Delta^2 < W^2 - (m-\mu)^2 \tag{29}$$

For $W = m+\mu$, it turns out to be

$$\Delta^2 < m\mu \approx 6.5\,\mu^2 \tag{29a}$$

which is evidently weaker than the condition $\Delta^2 < 4.7\,\mu^2$ given by (5).

VI. Integral representation of $A(W,\Delta,\zeta)$

As it has been mentioned in Section IV, $A(W_2,\Delta_2,\zeta,\mu^2)$ fails to be analytic in the strip Z, while $A(W_1,\Delta_1,\zeta,\mu^2)$ is analytic in Z as well as in Z', due to the fact that the former involves $\cos\Theta$ which becomes unphysical in a certain domain of Z. Therefore, we now proceed to investigate the analyticity of $A(W,\Delta,\zeta)$ in Z by explicitly making use of Eq. (17), instead of treating $A(W_i,\Delta_i,\zeta,\mu^2)$ separately. To this end we shall have to carry out the angular integration $\int d\varphi' \int d\cos\Theta'$ explicitly.

First of all, for this purpose we rewrite the Jost-Lehmann-Dyson integral representation (19) as follows

$$A(W_i,\Delta_i,\zeta,\mu^2) = \int_{(S)} dU \int_0^{2\pi} d\alpha_i$$

$$\times \frac{(Y_i - K_{i1}K_{i2}\cos\theta_i\cos\alpha_i)\,\Phi(U,\cos\alpha_i : W_i)}{\{K_{i1}K_{i2}\cos\theta_i - Y_i\cos\alpha_i - i\sqrt{Y_i^2-K_{i1}^2K_{i2}^2}\sin\alpha_i\}\{K_{i1}K_{i2}\cos\theta_i - Y_i\cos\alpha_i + i\sqrt{Y_i^2-K_{i1}^2K_{i2}^2}\sin\alpha_i\}} \tag{30}$$

after having made use of the symmetry property

$$\int_0^{2\pi} d\alpha \, \sin\alpha \, f(\cos\alpha) = 0, \tag{31}$$

where f is an arbitrary function. Now substituting the definitions of $\cos\Theta_i$ into (30), we get

$$A(W_1, \Delta_1, \varsigma, \mu^2) = \frac{1}{K(W,\varsigma)K(W)} \int dU \int_0^{2\pi} d\alpha_1 \, \frac{\cos\alpha_1 \; \Phi(U, \cos\alpha_1 ; W_1)}{z_1 - x}$$

$$+ \frac{i\,Y_1}{K(W,\varsigma)K(W)\sqrt{Y_1^2 - K_{11}^2 K_{12}^2}} \int dU \int_0^{2\pi} d\alpha_1 \, \frac{\sin\alpha_1 \; \Phi(U, \cos\alpha_1 ; W_1)}{z_1 - x}, \tag{32}$$

and

$$A(W_2, \Delta_2, \varsigma, \mu^2) = \frac{1}{K(W,\varsigma)K(W)} \int dU \int_0^{2\pi} d\alpha_2 \, \frac{\cos\alpha_2 \; \Phi(U, \cos\alpha_2 ; W_2)}{z_2 - x\cos\theta - \sqrt{1-x^2}\,\sin\theta\cos\varphi'}$$

$$+ \frac{i\,Y_2}{K(W,\varsigma)K(W)\sqrt{Y_2^2 - K_{21}^2 K_{22}^2}} \int dU \int_0^{2\pi} d\alpha_2 \, \frac{\sin\alpha_2 \; \Phi(U, \cos\alpha_2 ; W_2)}{z_2 - x\cos\theta - \sqrt{1-x^2}\,\sin\theta\cos\varphi'} \tag{32a}$$

with

$$z_i = - \frac{f(W_i, \varsigma, \mu^2) - f(W, \varsigma, \mu^2) - Y_i \cos\alpha_i - i\sqrt{Y_i^2 - K_{i1}^2 K_{i2}^2}\,\sin\alpha_i}{K(W,\varsigma)\, K(W, \mu^2)} \tag{33}$$

and $x = \cos\theta'$. From the inequalities (25) and (25a), we have that

$$\left.\begin{array}{lll} \text{either} & z_i \text{ is real and} & |z_i| > 1 \\ \text{or} & z_i \text{ is complex} & \\ \text{for} & \varsigma \text{ in } Z. & \end{array}\right\} \tag{33a}$$

Further we introduce

$$\overline{D}_1(\rho,\mu^2) \equiv -2K(W,\rho)K(W,\mu^2)(z_{d1}-x) \tag{34}$$

$$\overline{D}_2(\rho,\mu^2) \equiv -2K(W,\rho)K(W,\mu^2)(z_{2d}-\cos\theta\cdot x-\sin\theta\sqrt{1-x^2}\cos\varphi'), \tag{34a}$$

through

$$z_{di} \equiv -\frac{W_i^2+W^2-\rho-\mu^2-2f(W,\rho,\mu^2)}{2K(W,\rho)K(W,\mu^2)} \tag{35}$$

We now substitute (32) and (34) into (17). After having carried out decomposition into partial fraction, we can easily perform the angular integrations to get a standard form. With the definition

$$g(z,z_1,z_2) \equiv \frac{1}{\sqrt{\Lambda}}\log\frac{z-z_1z_2+\sqrt{\Lambda}}{z-z_1z_2-\sqrt{\Lambda}}, \quad \Lambda = z^2+z_1^2+z_2^2-2z_1z_2z-1, \tag{36}$$

we finally get

$$\begin{aligned} A(W,\Delta^2,\rho) = {} & \frac{K(W)}{\pi^2 W}\int dW_1^2\int dW_2^2\int dU_1\int dU_2\int d\alpha_1\int d\alpha_2\;\Phi(U_i,\cos\alpha_i;W_i) \\ & \times\left\{\frac{1}{W_1^2-W^2-i\epsilon}\,\frac{1}{W_2^2-W^2+i\epsilon}+\frac{1}{W_1^2-W^2-i\epsilon}\,\frac{1}{K(W,\rho)K(W)(z_{d2}-z_2)}+\right. \\ & \left.+(1\leftrightarrow 2)+\frac{1}{(z_{d1}-z_1)(z_{d2}-z_2)}\,\frac{1}{K^2(W,\rho)K^2(W)}\right\}\frac{g(z,z_1,z_2)}{K^2(W,\rho)K^2(W,\mu^2)} \\ & + \text{terms involving } g(z,z_{d1},z_2),\; g(z,z_1,z_{d2}) \text{ and } g(z,z_{d1},z_{d2}) \end{aligned} \tag{37}$$

Here we have made use of the well-known integral

$$\int_0^{2\pi} d\varphi' \int_{-1}^{1} dx \, \frac{1}{z_1 - x} \, \frac{1}{z_2 - xz - \sqrt{1-x^2}\sqrt{1-z^2}\cos\varphi'} = \frac{1}{\sqrt{\Lambda}} \log \frac{z - z_1 z_2 + \sqrt{\Lambda}}{z - z_1 z_2 - \sqrt{\Lambda}} \; . \qquad (38)$$

We now proceed to show that the terms arising from the dispersion denominators (for crossing amplitude) are analytic in Z. Let us consider the following integral

$$\int_0^{2\pi} d\varphi' \int_{-1}^{1} dx \, \frac{1}{(z_1 - x)(z_{2d} - xz - \sqrt{1-x^2}\sqrt{1-z^2}\cos\varphi')} \quad (\text{for } \Delta^2 < m\mu). \qquad (39)$$

It is clear that both factors of the denominator do not vanish in Z. Likewise, the following integral is also analytic in Z,

$$\int_0^{2\pi} d\varphi' \int_{-1}^{1} dx \, \frac{1}{(z_{1d} - x)(z_2 - xz - \sqrt{1-x^2}\sqrt{1-z^2}\cos\varphi')}$$

since this integral is symmetric with respect to the interchange of z_2 and z_{d1}, as shown by (38).

As to the factor $(z_{di} - z_i)$ which appears in the denominators, it is easy to show that it does not give rise to any additional singularities. This factor arises from the partial fraction decomposition of the crossing amplitude and may in fact vanish, however, for $z_{d2} = z_2$ $g(z, z_1, z_2) = g(z, z_1, z_{d2})$ is analytic in Z as discussed above, provided that $\Delta^2 < m\mu$. Furthermore, this factor appears always in a pair of terms, i.e.,

$$\frac{1}{z_2 - z_{d2}} g(z, z_1, z_2) - \frac{1}{z_2 - z_{d2}} g(z, z_1, z_{2d})$$

and it is nothing but

$$\left[\frac{\partial}{\partial z_2} g(z, z_1, z_2) \right]_{z_2 = z_{2d}}$$

which appears to be also analytic in Z as well as $g(z, z_1, z_{2d})$ itself.

Now we have to investigate the analytic properties of the remaining terms of Eq. (36), which is essentially

$$\frac{1}{\sqrt{\Lambda(z)}} \log \frac{z - z_1 z_2 + \sqrt{\Lambda(z)}}{z - z_1 z_2 - \sqrt{\Lambda(z)}} \quad . \tag{40}$$

For this purpose, first of all we shall show that singularities might occur only if both z_1 and z_2 are real, so long as ζ is real. Assuming one of the of z_i to be complex, say z_2 is complex, then we see from the integral representation after φ' integration

$$\int_{-1}^{1} dx \; \frac{1}{z_1 - x} \; \frac{1}{\sqrt{z_2^2 + z^2 + x^2 - 2 z_2 z x - 1}}$$

that both factors of the denominator do not vanish. Hence, from now on we shall have to consider only the case with $\alpha_i = 0$ or π essentially, since we are to perform an analytic continuation along a small strip along the real ζ axis.

From the definition (37), one obtains the following integral representation

$$\int_{z_\pm}^{\pm\infty} \frac{1}{z' - z} \; \frac{1}{\sqrt{\Lambda(z')}} \tag{37a}$$

which shows explicitly the position of singularity. Namely, a branch cut appears and it starts from the point

$$\begin{aligned} z_+ &= z_1 z_2 + \sqrt{z_1^2 - 1}\,\sqrt{z_2^2 - 1} \;, && \text{for} \quad z_1 z_2 > 0 \\ z_- &= z_1 z_2 - \sqrt{z_1^2 - 1}\,\sqrt{z_2^2 - 1} \;, && \text{for} \quad z_1 z_2 < 0 \,. \end{aligned} \tag{41}$$

At this point we note that it suffices to study the singularity which occurs for $z < -1$, since we are to carry out the analytic continuation in ς up to $\varsigma = \mu^2$, where $z = \cos\theta$ takes unphysical value $z < -1$. This means that the right-hand cut in the z plane which starts from z_+ irrelevant to our discussion, while the left-hand cut beginning from z_- might prevent the analytic continuation desired. Therefore, we may now restrict ourselves to the case $z_1 z_2 < -1$. From the definition of z_i, this can be the case, if and only if $\sin\alpha_1 \sin\alpha_2 = -1$. To be definite, in the following we suppose $\alpha_1 = \pi$ and $\alpha_2 = 0$.

In order that the left-hand cut does not prevent our desired analytic continuation in Z, we must require

$$Z > Z_- = Z_1 Z_2 - \sqrt{Z_1^2 - 1}\,\sqrt{Z_2^2 - 1} \tag{42}$$

throughout the relevant regions of variables involved.

i) $\underline{m + \mu < W_1 < m + 2\mu, \text{ and } m + \mu < W_2 < m + 2\mu}$

In this case, through an elementary calculation which is essentially the same as that given for the proof of (25), one can easily find that (42) is in fact valid, provided that

$$\Delta^2 < \mathrm{Min}\ \frac{K^2(W,\varsigma)}{2}\left\{1 - Z_1 Z_2 + \sqrt{Z_1^2 - 1}\,\sqrt{Z_2^2 - 1}\right\}$$

Putting the numerical values of masses, we find it is greater than the desired maximum value of $\Delta^2 = 4.7\ \mu^2$.

ii) $W_1 > m+2\mu$, and $W_2 > m+2\mu$

It is clear from the definition (33) that z_1 takes its minimum value at $W_1 = m+2\mu$, while z_2 takes its minimum value $z_2 = 1$ for $W_2 \to \infty$ at $\varsigma = \mu^2$. Then it turns out that the inequality (42) will be satisfied, if

$$\Delta^2 < \left[\tilde{y}^2(W_1) \right]_{W_1 = m+2\mu} = 4.7\,\mu^2$$

which is the desired value of Δ^2_{max}.

So far we have in fact considered only the analyticity of the integrand in (37) with fixed value of W_1 and W_2, and in general it is not sufficient to guarantee the analyticity of the integral itself. However, the argument by Minguzzi and Streater [5)] suffices to establish the analyticity of the integral.

iii) $m+\mu < W_1 < m+2\mu$, and $W_2 > m+2\mu$ (or W_1, W_2 interchanged)

In this case, our argument is exactly the same as that for ii), namely the worst case occurs when $W_1 = m+\mu$, $W_2 \to \infty$ and $\varsigma = \mu^2$, where z_1 and z_2 take their minimum values, respectively. However, unfortunately putting these values into the inequality (42) which turns out to be true only if

$$\Delta^2 < \left[\tilde{y}^2(W_1) \right]_{W_1 = m+\mu} \simeq 3\,\mu^2$$

which is nothing but the value obtained by Lehmann. So long as we are concerned with the proof of dispersion relations only, our result up to this point is not different from that by Lehmann, however, Lehmann's is much more fruitful than ours in the following respect.

In fact, we cannot show the analyticity of $A(W, \Delta, \varsigma)$ in the Lehmann ellipse, since W_1 and W_2 extend to infinity. This fact is not at all unreasonable, since we have lost some information by introducing the dispersion relation (17) for the intermediate transitions. This is due to the fact that the Lehmann ellipse in the $\cos\Theta$ plane shrinks down around the physical region at high energies, for instance in our case $z_2 \to 1$ for $W_2 \to \infty$. As it was emphasized by Mandelstam, so long as one is employing the ordinary dispersion relation in W variable in connection with unitarity, this situation appears always there.

However, in our representation of $A(W, \Delta^2, \varsigma)$ in the elastic region, we have a new information which was absent in Lehmann's. While in Lehmann's representation W dependence of $A(W, \Delta^2, \varsigma)$ was hidden in the weight function, in ours we have a W dependence explicitly, although at the cost of the new variables W_1 and W_2 which are clearly hidden in the weight functions. In addition to it, for the proof of dispersion relation (1) for $\varsigma = \mu^2$, it is necessary and sufficient to show that the <u>integral itself</u>

$$T_1(W, \Delta, \varsigma) = \frac{1}{\pi}\int_{(m+\mu)^2}^{(m+2\mu)^2} dW'^2 \left(\frac{1}{W'^2 - W^2 - i\epsilon} + \frac{1}{W'^2 + W^2 - 2m^2 - 2\varsigma - 4\Delta^2} \right) A(W', \Delta, \varsigma)$$

is analytic in Z, though $A(W', \Delta, \varsigma)$ might not necessarily be analytic for some values of W'. As it will be shown in the next section, by exploring these two facts, the difficulty described above can be avoided at least for the proof of ordinary dispersion relations, though not for the analyticity in the $\cos\Theta$ plane.

VII. Analyticity of dispersion integral in Z

We now turn back to the very starting point of the whole discussion. Confining ourselves to the region

$$\Delta^2 < 4.7\,\mu^2 - \varepsilon \tag{5a}$$

(ε : arbitrary but fixed small positive number)

we consider the dispersion integral $T_1(W,\Delta,\zeta)$ given above. In the usual proof of dispersion relation, we start with the definition

$$T(W,\Delta^2,\zeta) = \int d^4x \, \exp\{i\tfrac{1}{2}(k_1+k_2)x\}\,\theta(x_0)\langle p_2|[j(\tfrac{x}{2}),j(-\tfrac{x}{2})]|p_1\rangle\,, \tag{43}$$

which is analytic in D given by

$$D \equiv \{\omega,\zeta \;:\; |\,\mathrm{Im}\,\omega\,| > |\,\mathrm{Im}\sqrt{\omega^2-\zeta-\Delta^2}\,|\} \tag{44}$$

(which is R_1 in Ref. [1]), where

$$\omega = \frac{W^2 - m^2 - \zeta - 2\Delta^2}{2\sqrt{m^2+\Delta^2}} \tag{45}$$

Then we show that the dispersion integral given by the r.h.s. of (1) is analytic in a finite strip Z, and the physical mass value $\zeta = \mu^2$ is on the boundary of the intersection of D and Z. As it is well known, it is possible for Re $\omega > \Delta^2 + \mu^2$, but is not the case otherwise. In terms of the c.m.s. energy, the physical mass value can be reached through $D \cap Z$ only if

$$W^2 > W_{ph}^2 \equiv (\sqrt{m^2+\Delta^2} + \sqrt{\mu^2+\Delta^2}\,)^2 . \tag{46}$$

For the range of integration in $T_1(W, \Delta^2, \mu^2)$ is well below this onset for the physical region, so long as $3\mu^2 < \Delta^2 < 4.7\mu^2$, the denominator W'^2-W^2 does not vanish. Besides, the crossing denominator never vanishes for $\Delta^2 < m\mu$.

Although we have an explicit W' dependence of $A(W', \Delta^2, \zeta)$ given by (37), we are unable to carry out the integration in W' explicitly. Therefore, in order to obtain an analytic property of $T_1(W, \Delta^2, \zeta)$ in ζ, we shall apply the method developed by Polkinghorne et al., which has been intensively used in the perturbation theory, to study the analyticity without carrying out the integrations involved. As regards our case under consideration we shall apply the following lemma given by Polkinghorne [9].

<u>Lemma</u>: the singularities of the function

$$f_2(s,t) = \int ds' \frac{\text{disc. } f_2(s',t)}{s-s'} + \text{crossed term,} \tag{47}$$

(the subscript 2 denotes the two-particle term, and disc. stand for the discontinuity across the normal threshold branch cut) are given by:

i) a pinch between a singularity of disc. f and the pole of $1/(s-s')$.

ii) a singularity of disc. f at one of the end points of integration.

iii) a pinch between two coincident singularities of disc. f.

To be specific in our case, the range of integration runs from $m+\mu$ to $m+2\mu$, and disc. $f_2(s',t)$ is nothing else but the absorptive part $A(W', \Delta^2, \zeta)$. First of all, a singularity of type i) is excluded, since the dispersion denominators do not vanish as discussed above. Now we proceed to discuss the possibility of the singularity of type ii):

ii)a <u>at $W' = m+\mu$</u>

For, at this point $K(W', \mu^2) = 0$, we shall have to prove rather

$$K^2(W',\mu^2)K^2(W',\varsigma)\,z > K^2(W',\varsigma)K^2(W',\mu^2)\left(z_1 z_2 - \sqrt{z_1^2-1}\sqrt{z_2^2-1}\right) \quad (42a)$$

than (42). It is easy to see that this inequality is indeed valid for any value of W_1 and W_2, since

$$\text{l.h.s.} = K^2(W',\mu^2)\left\{K^2(W',\varsigma) - 2\Delta^2\right\} = 0$$

and

$$\text{r.h.s.} < K^2(W',\mu^2)K^2(W',\varsigma)\,z_1 z_2$$

$$= \left\{f(W_1,\varsigma,\mu^2) - f(W',\varsigma,\mu^2) + Y_1\right\} \times$$

$$\times \left\{f(W_2,\varsigma,\mu^2) - f(W',\varsigma,\mu^2) - Y_2\right\} < 0$$

At this point it should be noted that for $W_2 \to \infty$ and $\varsigma = \mu^2$ the r.h.s. might approach to zero, since at $\varsigma = \mu^2$

$$f(W_2,\varsigma,\mu^2) - m^2 - Y_2 = K^2(W_2,\mu^2) - K^2(W_2,\mu^2) - \frac{16\mu^3(2m+\mu)}{W_2^2 - (m-2\mu)^2}$$

However, it does not give rise to any trouble, because this expression tends to a non-vanishing finite value, if the associated dispersion denominator $(W_2^2-W'^2)$ is multiplied.

ii)b <u>at $W' = m+2\mu$</u>

So long as the momentum transfer is restricted by $\Delta^2 < 4.7\,\mu^2-\varepsilon$, $A(W',\Delta^2,\zeta)$ at this point is clearly analytic in Z, as it was the very starting point of our discussion (see Eqs. (3) and (5)).

Finally, we now turn to the possibility of the singularity of type iii). As it was discussed in the last section, the curve of singularities of $A(W',\Delta,\zeta)$ is given by

$$Z = Z_1 Z_2 - \sqrt{Z_1^2-1}\sqrt{Z_2^2-1}\,, \quad \text{or} \quad \Lambda(Z) = 0 \tag{48}$$

Then the possible singularity in ζ of $T_1(W,\Delta^2,\zeta)$ might appear only from the term

$$\tilde{T}_1(W,\Delta,\zeta) = \frac{1}{\pi^2}\int_{(m+\mu)^2}^{(m+2\mu)^2} dW_1^2 \int_{(m+2\mu)^2}^{\infty} dW_2^2 \int_{(m+\mu)^2}^{(m+2\mu)^2} dW'^2 \int dU_1 \int dU_2 \,\times$$
$$\times \frac{K(W')\;\Phi(U_1,U_2,W_1,W_2)}{W'(W_1^2-W'^2-i\epsilon)(W_2^2-W'^2+i\epsilon)(W'^2-W^2-i\epsilon)}\cdot\frac{1}{\sqrt{\Lambda(Z)}}\log\frac{Z-Z_1Z_2+\sqrt{\Lambda(Z)}}{Z-Z_1Z_2-\sqrt{\Lambda(Z)}}\,, \tag{49}$$

since the contributions from other ranges of W_1 and W_2 were shown to be analytic, as discussed in the last section. As it is proved in the appendix, we have

$$\frac{\partial}{\partial W'^2} \Lambda\left(z(W'), z_1(W'), z_2(W')\right) \neq 0 , \quad \text{for } \Lambda = 0 , \tag{50}$$

which implies that $g(z, z_1, z_2)$ itself does not give any pinching singularities in the W' plane. In other words, it means that the singularities in the W' plane is disconnected from the real axis, except for those singularities which move along the real axis as ς increases along the real axis toward the physical mass value. However, we have unfortunately the dispersion denominator $W_1^2 - W'^2$ which may vanish in the relevant interval and together with $\Lambda(z) = 0$ can give rise to a coincidence singularity.

To investigate this point in detail, we now turn back to the definition of $g(z(W'^2), z_1(W'^2), z_2(W'^2))$ (see Eq. (36)). It is obvious that for some negative values of ς we have $|z| < 1$. From this it follows that $g(z(W'^2), z_1(W'^2), z_2(W'^2))$ is analytic in the W'^2 plane, at least around the elastic interval. In addition to this, $(W_2 - W')(W' - W)$ does not vanish in the elastic interval. Therefore, we are now entitled to move the contour of integration in W'^2 a bit upward (small but finite) off the real axis to avoid the pole term $(W_1^2 - W'^2 - i\epsilon)$ (see Fig. 3). As to the pole term $(W_1^2 - W'^2 + i\epsilon)$ we must deform the path of integration correspondingly downward off the real axis. Thus, we may now drop $i\epsilon$ and we have finally

$$\tilde{T}_1(W, \Delta^2, \varsigma) = \frac{1}{\pi^2} \int dW_i \int dU_i\, \Phi(U_i : W_i) \tag{49a}$$

$$\times \int_{C_+} \frac{dW'^2 \; K(W') \; g(z, z_1, z_2)}{W' (W_1^2 - W'^2)(W_2^2 - W'^2)(W'^2 - W^2)}$$

Because now $W_1 - W' = 0$ is excluded, a possible coincidence singularity might appear, only if [10]

$$\Lambda(z, z_1, z_2) = 0 \text{ and } \frac{\partial}{\partial W'} \Lambda(z, z_1, z_2) = 0 \quad \text{on} \quad C_+ \tag{51}$$

however, it is excluded in virtue of (50), since $g(z)$ is analytic in W'.

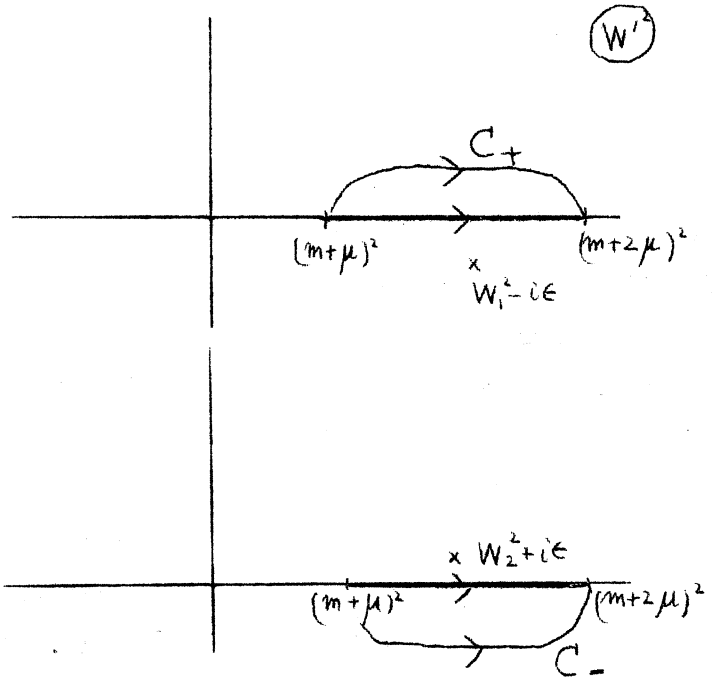

Fig. 3

Collecting all the results obtained above, we may now conclude that

$T_1(W, \zeta, \Delta)$, which was defined by (4) for $\zeta < -\Delta^2$

is in fact analytic in Z, as well as $T_2(W, \Delta, \zeta)$, though we could not show the analyticity of the absorptive part $A(W', \Delta^2, \zeta)$ itself in Z. Now carrying out the analytic continuation of both sides of (1) with respect to ζ

up to the physical mass $\zeta = \mu^2$, we can write down the following dispersion relation

$$T(W,\Delta) = T_1(W,\Delta) + \frac{1}{\pi}\int^{\infty} dW'^2 \left(\frac{1}{W'^2 - W^2 - i\epsilon} + \frac{1}{W'^2 + W^2 - 2m^2 - 2\zeta - 4\Delta^2}\right) A(W',\Delta) + \tag{1a}$$

$+$ single nucleon terms.

where $T_1(W,\Delta)$ denotes only symbolically the analytic continuation of $T_1(W,\Delta,\zeta)$ along Z to $\zeta = \mu^2$, while we could not give sense to $A(W',\Delta,\zeta)$ for $m+\mu < W' < m+2\mu$.

Finally, we come to the question of the physical interpretation of the contribution from the unphysical region. Evidently, the contribution from $W' > m+2\mu$ can be well established in terms of the partial wave expansion and the analytic continuation in $z = \cos\Theta$ to its relevant unphysical value, since the large Lehmann ellipse guarantees both convergence and analyticity in $\cos\Theta$ of this expansion. However, as to the contribution from the elastic region it cannot be interpreted in that way, in the case $3\mu^2 < \Delta^2 < 4.7\mu^2$. But in principle, we can give physical meaning to this in the following manner. In our discussions made above, we have been interested in the analyticity in ζ variable. Now instead of ζ, we consider the analyticity of $T_1(W,\Delta,\zeta)$ as a function of Δ^2 (or $\cos\Theta$), fixing ζ at $\zeta = \mu^2$. Our whole discussion so far for ζ variable can be straightforwardly translated to the analyticity in Δ^2. Thus, we find that $T_1(W,\Delta,\mu^2)$ is at least analytic in Δ^2 plane in the neighbourhood around the interval $3\mu^2 < \Delta^2 < 4.7\mu^2$. Therefore, we are able, in principle, to give physical meaning to $T_1(W,\Delta)$ for these values of the momentum transfer, by performing analytic continuation in Δ^2 from the point $\Delta^2 < 3\mu^2$, where $T_1(W,\Delta)$ is well defined as described above.

So far we have treated only the pion-nucleon scattering, but the present method can be straightforwardly applied to the pion-hyperon scatterings, where the most stringent restrictions on the momentum transfer appear in the elastic region [11]. As to the kaon-nucleon scattering, the very trouble which prevents us to derive the dispersion relation comes also from the two-particle intermediate states, i.e., $\Lambda + \pi$ and $\Sigma + \pi$ states, however, in this case one can show that $z_i = 1$ will not be excluded and therefore the present approach breaks down.

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A P P E N D I X

In this appendix we shall prove that

$$\frac{\partial}{\partial W'^2} \Lambda \left(z(W'^2),\ z_1(W'^2),\ z_2(W'^2) \right) \neq 0 \tag{A1}$$

under the condition

$$\Lambda \left(z(W'^2),\ z_1(W'^2),\ z_2(W'^2) \right) = 0 \tag{A2}$$

(For simplicity, we shall drop hereafter the prime over W.) In virtue of (A2), it is clear that

$$\frac{\partial \Lambda}{\partial W'^2} = 2(z - z_1 z_2) \frac{\partial z}{\partial W'^2} + \frac{\partial}{\partial W^2}(z_1 - z_2)^2 + 2(1 - z) \frac{\partial z_1 z_2}{\partial W^2} \tag{A3}$$

Now by making use of the fact

$$\begin{aligned} \frac{\partial z}{\partial W^2} &= \frac{1 - z}{4K^2(W,\rho)} \left(1 - \frac{(m^2 - \rho)^2}{W^4} \right) \\ \frac{\partial}{\partial W^2}(z_1 z_2) &= \frac{z_1 + z_2}{4K(W,\rho)K(W,\mu^2)} \left(1 - \frac{(m^2 - \rho)(m^2 - \mu^2)}{W^4} \right) - \\ &\qquad - \frac{1}{4} z_1 z_2 \left(\frac{1 - \frac{(m^2 - \mu^2)^2}{W^4}}{K^2(W,\mu^2)} + \frac{1 - \frac{(m^2 - \rho)^2}{W^4}}{K^2(W,\rho)} \right), \end{aligned} \tag{A4}$$

and (A2), we obtain

$$\frac{\partial \Lambda}{\partial W^2} = \frac{1-z}{2}\left\{ (z - z_1 z_2)\,\frac{1-\frac{(m^2-\varrho)^2}{W^4}}{K^2(W,\varrho)} + \frac{z_1+z_2}{K(W,\varrho)K(W,\mu^2)} \cdot \left(1 - \frac{(m^2-\varrho)(m^2-\mu^2)}{W^4}\right) - \frac{1+z}{2}\left(\frac{1-\frac{(m^2-\varrho)^2}{W^4}}{K^2(W,\varrho)} + \frac{1-\frac{(m^2-\mu^2)^2}{W^4}}{K^2(W,\mu^2)}\right)\right\}$$

(A5)

We now proceed to prove the expression in the curly bracket which is in fact positive definite:

i) <u>if $z_1+z_2 > 0$</u>

From the condition $z = z_1 z_2 - \sqrt{z_1^2-1}\,\sqrt{z_2^2-1} < 0$, one can easily obtain

$$z - z_1 z_2 - \frac{(1+z)}{2} > 0. \qquad \text{(A6)}$$

Besides, $-\frac{1}{2}(1+z)$ itself is positive, therefore, we have

$$\left(z - z_1 z_2 - \frac{1+z}{2}\right)\frac{1-\frac{(m^2-\varrho)^2}{W^4}}{K^2(W,\varrho)} > 0\,, \quad -\frac{(1+z)}{2}\,\frac{1-\frac{(m^2-\mu^2)^2}{W^4}}{K^2(W,\mu^2)} > 0\,,$$

and because $z_1+z_2 > 0$, the expression in the curly bracket is positive.

ii) if $z_1+z_2 < 0$

For $\varrho \le \mu^2$, clearly

$$\frac{1-\frac{(m^2-\varrho)^2}{W^4}}{K^2(W,\varrho)} < \frac{1-\frac{(m^2-\varrho)(m^2-\mu^2)}{W^4}}{K(W,\varrho)\,K(W,\mu^2)} \quad . \tag{A7}$$

Furthermore,

$$\frac{1-\frac{(m^2-\varrho)(m^2-\mu^2)}{W^4}}{K(W,\varrho)K(W,\mu^2)} < \frac{1}{2}\left(\frac{1-\frac{(m^2-\varrho)^2}{W^4}}{K^2(W,\varrho)} + \frac{1-\frac{(m^2-\mu^2)^2}{W^4}}{K^2(W,\mu^2)}\right) \tag{A8}$$

This inequality can be proved as follows: to prove (A8), it suffices to show

$$\left\{ K^2(W,\varrho)(W^4-(m^2-\mu^2)^2) + K^2(W,\mu^2)(W^4-(m^2-\varrho)^2)\right\} -$$

$$-2\, K(W,\varrho)K(W,\mu^2)(W^4-(m^2-\varrho)(m^2-\mu^2)) > 0 \,. \tag{A8a}$$

In order to find its minimum value, we differentiate with respect to ϱ, i.e.,

$$\frac{\partial}{\partial\varrho}(A\,8a) = -\frac{W^2+m^2-\varrho}{2W^2}\left\{ W^4-(m^2-\mu^2)^2 + \frac{K(W,\mu^2)}{K(W,\varrho)}\times\right.$$

$$\left.\times(W^4-(m^2-\varrho)(m^2-\mu^2)\right\} +$$

$$+\,2\,K(W,\mu^2)\left\{K(W,\mu^2)(m^2-\varrho) - K(W,\varrho)(m^2-\mu^2)\right\} .$$

Clearly, the first bracket is negative. The second bracket also turns out to be negative, since

$$\frac{\partial}{\partial\rho}\left\{ K^2(W,\mu^2)(m^2-\rho)^2 - K^2(W,\rho)(m^2-\mu^2)^2 \right\}$$

$$= -2K^2(W,\mu^2)(m^2-\rho) + \frac{W^2+m^2-\rho}{2W^2}(m^2-\mu^2)^2$$

$$= \frac{1}{2W^2}\Big[-\left\{(W^2+m^2-\mu^2)^2 - 4W^2m^2\right\}(m^2-\rho) +$$

$$+ (W^2+m^2-\rho)(m^2-\mu^2)^2 \Big]$$

$$= \frac{1}{2}\left\{ (2m^2+2\mu^2-W^2)(m^2-\rho) + (m^2-\mu^2)^2 \right\} > 0,$$

for $W \leq m+2\mu$

and

$$\mathrm{Max}\left\{ K^2(W,\mu^2)(m^2-\rho)^2 - K^2(W,\rho)(m^2-\mu^2)^2 \right\} = 0$$

at $\rho = \mu^2$.

Further, we have

$$z - z_1 z_2 < 0, \quad z_1 + z_2 < 0, \quad -(1+z) > 0, \tag{A9}$$

thus by making use of (A7) and (A8), the curly bracket in (A5) turns out to be

$\{\cdots\cdots\}$ of (A5) $>$

$$> \left\{ (z - z_1 z_2) + z_1 + z_2 - (1+z) \right\} \frac{1 - \frac{(m^2-\rho)(m^2-\mu^2)}{W^4}}{K(W,\rho)K(W,\mu^2)}$$

$$= -(1-z_1)(1-z_2) \frac{1 - \frac{(m^2-\rho)(m^2-\mu^2)}{W^4}}{K(W,\rho)K(W,\mu^2)} \tag{A5}$$

As it was proven that $z_1 < -1$ and $z_2 > 1$, the expression (A5a) is evidently positive definite. Besides, the factor

$$1 - z > 0$$

does not vanish, hence we conclude that (A1) is valid.

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