CERN-TH/97-258
PSI-PR-97-30
hep-ph/9710521

Non-factorizable photonic corrections to $e^+e^- \to WW \to 4\,$fermions

A. Denner

Paul-Scherrer-Institut, Würenlingen und Villigen
CH-5232 Villigen PSI, Switzerland

S. Dittmaier

Theory Division, CERN
CH-1211 Geneva 23, Switzerland

M. Roth

Paul-Scherrer-Institut, Würenlingen und Villigen
CH-5232 Villigen PSI, Switzerland
and
Institut für Theoretische Physik, ETH-Hönggerberg
CH-8093 Zurich, Switzerland

Abstract:

We study the non-factorizable corrections to W-pair-mediated four-fermion production in e^+e^- annihilation in double-pole approximation. We show how these corrections can be combined with the known corrections to the production and the decay of on-shell W-boson pairs, and how the full off-shell Coulomb singularity is included. Moreover, we find that the actual form of the real non-factorizable corrections depends on the parametrization of phase space, more precisely, on the definition of the invariant masses of the resonant W bosons. For the usual parametrization the full analytical results for the non-factorizable corrections are presented. Our analytical and numerical results for the non-factorizable corrections agree with a recent calculation, which was found to differ from a previous one. The detailed numerical discussion covers the invariant-mass distribution, various angular distributions, and the lepton-energy distribution for leptonic final states.

October 1997

1 Introduction

After the successful experimental investigation of the Z boson at LEP1, at present the W boson is studied via its pair production at the upgraded collider LEP2. Since W bosons decay very rapidly, the actual reaction observed is $e^+e^- \to 4\,\text{fermions}$. The virtual W bosons can be reconstructed by selecting those events in which the invariant masses of the final-state fermion pairs are close to the W-boson mass. The expected integrated luminosity of LEP2 is $500\,\text{pb}^{-1}$ [1]. Accordingly, one expects to produce about 8000 W pairs, and the typical experimental accuracy is of the order of one to a few per cent. For the measurement of the W-boson mass the expected accuracy is even below the per-mille level [2].

This experimental accuracy should be matched or better exceeded by the precision of the theoretical predictions. The calculation of the cross section for $e^+e^- \to 4\,\text{fermions}$ with an accuracy of 1% or better is a non-trivial task. In the LEP2 energy region, the lowest-order cross section is dominated by the diagrams that involve two resonant W bosons. All other lowest-order diagrams are typically suppressed by a factor Γ_W/M_W, but may be enhanced in certain regions of phase space. All these contributions are required at the 1% level (see Ref. [3] and references therein). At the level of $\mathcal{O}(\alpha)$ corrections, on the other hand, it is sufficient to take into account those contributions that are enhanced by two resonant W propagators (doubly-resonant contributions). These can be split in two types: factorizable and non-factorizable corrections [4]. The factorizable corrections are those that correspond either to the production of the W-boson pair or to the decay of one of the W bosons. The corrections to on-shell W-pair production [5] and decay [6] are available. The non-factorizable corrections are those in which the production subprocess and the decay subprocesses are not independent; typically an additional particle is exchanged between two of the subprocesses. Simple power-counting reveals that among the non-factorizable corrections only those Feynman diagrams contribute to the doubly-resonant contributions in which a soft photon is exchanged between the subprocesses.

In this paper we explicitly calculate these non-factorizable photonic corrections. It has been shown that they vanish in inclusive quantities, i.e. if the invariant masses of both W bosons are integrated out [7]. However, for non-inclusive quantities these corrections do not vanish in general. The non-factorizable photonic corrections have already been investigated by two groups. Melnikov and Yakovlev [8] have given the analytical results only in an implicit form and restrict the numerical evaluation to a special phase-space configuration. Beenakker, Berends and Chapovsky have provided both the complete formulae and an adequate numerical evaluation [9, 10], but do not find agreement with all results of Ref. [8]. For this reason, it is worth-while to present the results of a third independent calculation.

We start out by discussing the definition of the virtual and real photonic non-factorizable corrections in double-pole approximation in detail. Since only soft photons are relevant in double-pole approximation, the virtual non-factorizable correction is just a factor to the lowest-order cross section. For the corresponding real correction, the situation is similar, but in addition an integration over the photon momentum has to be performed. This requires a specification of the phase-space parametrization, which includes, in particular, the invariant masses of the W bosons. Usually these are defined via the invariant masses of the respective final-state fermion pairs and are chosen as independent variables [8, 9, 10]. Experimentally, however, the invariant mass of a W boson is identified with the invariant mass of the associated jet pair that necessarily includes soft and collinear photons. Therefore, the influence of the choice for the invariant masses of the W bosons on the non-factorizable corrections should be investigated in order to provide sound predictions for physical situations.

Besides the non-factorizable doubly-resonant corrections, the most important effect of the instability of the W bosons is the modification of the Coulomb singularity. Since the off-shell Coulomb singularity results from a scalar integral that also contributes to the doubly-resonant non-factorizable corrections, it seems to be natural to approximate this integral in such a way that both effects are simultaneously included. This requires going beyond the strict double-pole approximation.

The paper is organized as follows: in Sect. 2 we discuss the basis of the double-pole approximation, with particular emphasis on the real corrections, and formulate a general gauge-independent definition of the non-factorizable doubly-resonant corrections. In Sect. 3 we sketch the calculation of the scalar integrals and describe the evaluation of the 5-point functions in detail. Section 4 summarizes our analytic results for the correction factor; in particular, the cancellations between virtual and real photonic corrections are inspected, and the off-shell Coulomb singularity is embedded in the approach. Section 5 is devoted to the discussion of numerical results. The appendices provide more information on the evaluation of the real bremsstrahlung integrals and the explicit results for the required scalar integrals.

2 Definition of the approximation

2.1 Conventions and notations

In this paper we discuss corrections to the process

$$\mathrm{e}^+(p_+) + \mathrm{e}^-(p_-) \;\to\; \mathrm{W}^+(k_+) + \mathrm{W}^-(k_-) \;\to\; f_1(k_1) + \bar{f}_2(k_2) + f_3(k_3) + \bar{f}_4(k_4). \tag{2.1}$$

The relative charges of the fermions f_i are represented by Q_i with $i = 1, \dots, 4$. The masses of the external fermions, $m_i^2 = k_i^2$ and $m_\mathrm{e}^2 = p_\pm^2$, are neglected, except where this would lead to mass singularities. The momenta of the intermediate W bosons are defined by

$$k_+ = k_1 + k_2, \qquad k_- = k_3 + k_4, \tag{2.2}$$

their complex mass squared and their respective invariant masses are denoted by

$$M^2 = M_\mathrm{W}^2 - \mathrm{i} M_\mathrm{W} \Gamma_\mathrm{W}, \qquad M_\pm = \sqrt{k_\pm^2}, \tag{2.3}$$

respectively, and we introduce the variables

$$K_+ = k_+^2 - M^2, \qquad K_- = k_-^2 - M^2. \tag{2.4}$$

Furthermore, we define the following kinematical invariants

$$\begin{aligned} s &= (p_+ + p_-)^2 = (k_+ + k_-)^2, \\ s_{ij} &= (k_i + k_j)^2, \\ s_{ijk} &= (k_i + k_j + k_k)^2, \qquad i,j,k = 1,2,3,4, \end{aligned} \tag{2.5}$$

which obey the relations

$$\begin{aligned} s &= k_+^2 + k_-^2 + s_{13} + s_{14} + s_{23} + s_{24}, \qquad s_{12} = k_+^2, \qquad s_{34} = k_-^2, \\ s_{ijk} &= s_{ij} + s_{ik} + s_{jk}, \qquad i,j,k = 1,2,3,4. \end{aligned} \tag{2.6}$$

2.2 Doubly-resonant virtual corrections

The aim of the present paper is to evaluate the non-factorizable corrections to the process (2.1) in double-pole approximation (DPA). The DPA takes into account only the leading terms in an expansion around the poles originating from the two resonant W propagators.

In DPA, the lowest-order matrix element for the process (2.1) factorizes into the matrix element for the production of the two on-shell W bosons, $\mathcal{M}_{\mathrm{Born}}^{\mathrm{e^+e^-\to W^+W^-}}(p_+, p_-, k_+, k_-)$, the (transverse parts of the) propagators of these bosons, and the matrix elements for the decays of these on-shell bosons, $\mathcal{M}_{\mathrm{Born}}^{\mathrm{W^+}\to f_1\bar{f}_2}(k_+, k_1, k_2)$ and $\mathcal{M}_{\mathrm{Born}}^{\mathrm{W^-}\to f_3\bar{f}_4}(k_-, k_3, k_4)$:

$$\mathcal{M}_{\mathrm{Born}} = \sum_{\lambda_+,\lambda_-} \frac{\mathcal{M}_{\mathrm{Born}}^{\mathrm{e^+e^-\to W^+W^-}} \mathcal{M}_{\mathrm{Born}}^{\mathrm{W^+}\to f_1\bar{f}_2} \mathcal{M}_{\mathrm{Born}}^{\mathrm{W^-}\to f_3\bar{f}_4}}{K_+ K_-}. \tag{2.7}$$

The sum runs over the physical polarizations $\lambda_\pm$ of the $\mathrm{W}^\pm$ bosons.

The higher-order corrections to (2.1) can be separated into factorizable and non-factorizable contributions [4]. In the factorizable contributions the production of two W bosons and their subsequent decays are independent. The corresponding Feynman diagrams can be split into three parts by cutting only the two W-boson lines. The corresponding matrix element factorizes in the same way as the lowest-order matrix element (2.7).

The non-factorizable corrections comprise all those contributions in which W-pair production and/or the subsequent W decays are not independent. Obviously, this includes all Feynman diagrams in which a particle is exchanged between the production subprocess and one of the decay subprocesses or between the decay subprocesses. Examples for such manifestly non-factorizable corrections are the diagrams (a), (b), and (c) in Fig. 1. If the additional exchanged particle is massive, the corresponding correction has no double pole for on-shell W bosons. However, if a photon is exchanged between the different subprocesses, this leads to a doubly-resonant contribution originating from the soft-photon region. This can be directly seen from the usual soft-photon approximation (SPA), which yields contributions proportional to the (doubly-resonant) lowest-order contribution.

The doubly-resonant contributions can be extracted on the basis of a simple power-counting argument. For instance, the loop integral corresponding to diagram (c) in Fig. 1 is of the following form:

$$I = \int \mathrm{d}^4 q \frac{N(q,k_i)}{(q^2-\lambda^2)[(q-k_3)^2-m_3^2][(q-k_-)^2-M_\mathrm{W}^2][(q+k_+)^2-M_\mathrm{W}^2][(q+k_2)^2-m_2^2]}$$

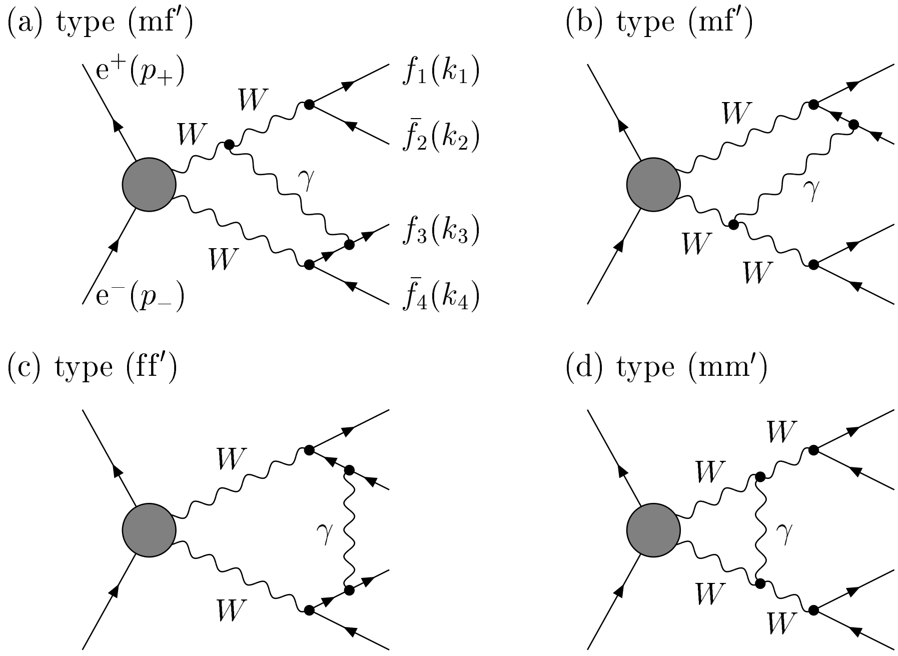

Figure 1: Examples of non-factorizable photonic corrections in $\mathcal{O}(\alpha)$. The shaded blobs stand for all tree-level graphs contributing to $\mathrm{e^+e^-} \to \mathrm{W^+W^-}$. Whenever Feynman diagrams with intermediate would-be Goldstone bosons $\phi^\pm$ instead of $W^\pm$ bosons are relevant, the inclusion of such graphs is implicitly understood.

$$= \int \mathrm{d}^4 q \frac{N(q,k_i)}{(q^2-\lambda^2)(q^2-2qk_3)(q^2-2qk_- + k_-^2 - M_\mathrm{W}^2)(q^2+2qk_+ + k_+^2 - M_\mathrm{W}^2)(q^2+2qk_2)}, \tag{2.8}$$

where we have introduced an infinitesimal photon mass λ to regularize the infrared (IR) singularity. The function $N(q,k_i)$ involves the numerator of the Feynman integral, i.e. a polynomial in the momenta q and k_i, and possible further denominator factors originating from propagators (hidden in the blob of the diagrams) that are regular for $q=0$ and $k_\pm^2 = M_\mathrm{W}^2$. For on-shell W bosons ($k_\pm^2 = M_\mathrm{W}^2$), the integral has a quadratic IR singularity. For off-shell W bosons, part of the IR singularity is regularized by the off-shellness, $k_\pm^2 - M_\mathrm{W}^2 \neq 0$, such that the usual logarithmic IR singularity remains. Vice versa, the off-shell result develops a pole if either W boson becomes on shell, and is thus doubly-resonant. Thus, the quadratic IR singularity in the on-shell limit is characteristic of the doubly-resonant non-factorizable contributions. All terms that involve a factor q in the numerator are less IR-singular and therefore do not lead to doubly-resonant contributions and can be omitted. Similarly, q can be neglected in all denominator factors included in $N(q,k_i)$. In summary, q can be put to zero in $N(q,k_i)$ in DPA. We have checked this for various examples explicitly. As a consequence, we are left with only scalar integrals, and the non-factorizable virtual corrections are proportional to the lowest-order matrix element. We call the resulting approximation *extended soft-photon approximation* (ESPA). It differs from the usual SPA only by the fact that q is not neglected in the resonant W propagators. In ESPA, diagram (c) in Fig. 1 gives the following contribution to the matrix element:

$$\mathcal{M}^{\bar{f}_2 f_3} = \mathrm{i}e^2 Q_2 Q_3 \mathcal{M}_{\mathrm{Born}} \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{4k_2k_3}{(q^2-\lambda^2)(q^2-2qk_3)(q^2+2qk_2)} \times \frac{(k_-^2-M_\mathrm{W}^2)(k_+^2-M_\mathrm{W}^2)}{[(q-k_-)^2-M_\mathrm{W}^2][(q+k_+)^2-M_\mathrm{W}^2]}. \tag{2.9}$$

The q^2 terms in the last four denominators are not relevant in the soft-photon limit and were omitted in Refs. [8, 9, 10]. In fact, using the above power-counting argument it can easily be seen that the differences of doubly-resonant contributions with and without these q^2 terms are non-doubly-resonant. We have chosen to keep the q^2 terms, because we want to use the standard techniques for the evaluation of virtual scalar integrals [11]. In DPA, i.e. if we perform the limit $k_\pm^2 \to M_\mathrm{W}^2$ after evaluating the integral, we should obtain the same result.

In order to arrive at physical results, we have to incorporate the finite width of the W bosons. In DPA this can be done in at least two different ways:

As a first possibility, we perform the integrals for zero width and afterwards put $k_\pm^2 = M_\mathrm{W}^2$ where this does not give rise to singularities. In all other places, i.e. in the resonant propagators and in logarithms of the form $\ln(k_\pm^2 - M_\mathrm{W}^2 + \mathrm{i}\epsilon)$, we replace $k_\pm^2 - M_\mathrm{W}^2 + \mathrm{i}\epsilon$ by $K_\pm = k_\pm^2 - M_\mathrm{W}^2 + \mathrm{i}M_\mathrm{W}\Gamma_\mathrm{W}$. Since the width is only relevant in the on-shell limit, it is clear that the (physical) on-shell width has to be used.

Alternatively, we introduce the width in the W propagators before integration. This has to be done with caution. If we introduce the finite width by resumming W-self-energy insertions, the width depends on the invariant mass of the W boson and thus on the integration momentum. Fortunately, the contribution we are interested in results only from the soft-photonic region where the virtual W bosons are almost on shell. Therefore, we can insert the on-shell width inside the loop integral. After performing the integral, we put $k_\pm^2 = M_\mathrm{W}^2$ and $\Gamma_\mathrm{W} = 0$ where this does not lead to singularities. In DPA this gives the same results as the above treatment.

In the following we write $M^2 = M_\mathrm{W}^2 - \mathrm{i}M_\mathrm{W}\Gamma_\mathrm{W}$ instead of M_W^2 in the loop integrals. It is always understood that M^2 and $k_\pm^2$ are replaced by M_W^2 where possible after evaluation of the integrals.

If we implement the width into the integrand, it is clear that only the part of the integration region with $|q_0| \lesssim \Gamma_\mathrm{W}$ contributes in DPA. If $|q_0| \gg \Gamma_\mathrm{W}$, one of the W propagators must be non-resonant and the contribution becomes negligible.

Once the width is introduced, it becomes evident that the relative error of the DPA is of the order of Γ_W/scale. Let E_CM be the centre-of-mass (CM) energy and $\Delta E = E_\mathrm{CM} - 2M_\mathrm{W}$ be the available kinetic energy of the W bosons. Then, for $\Delta E \gtrsim M_\mathrm{W}$ the scale is given by M_W, for $\Gamma_\mathrm{W} \lesssim \Delta E \lesssim M_\mathrm{W}$ it is given by ΔE and for $\Delta E \lesssim \Gamma_\mathrm{W}$ it is given by Γ_W. This shows that the DPA is only sensible several Γ_W's above threshold. This is simply due to the fact that, close to threshold, the phase space where both W propagators can become doubly-resonant is very small, and the singly-resonant diagrams become important.

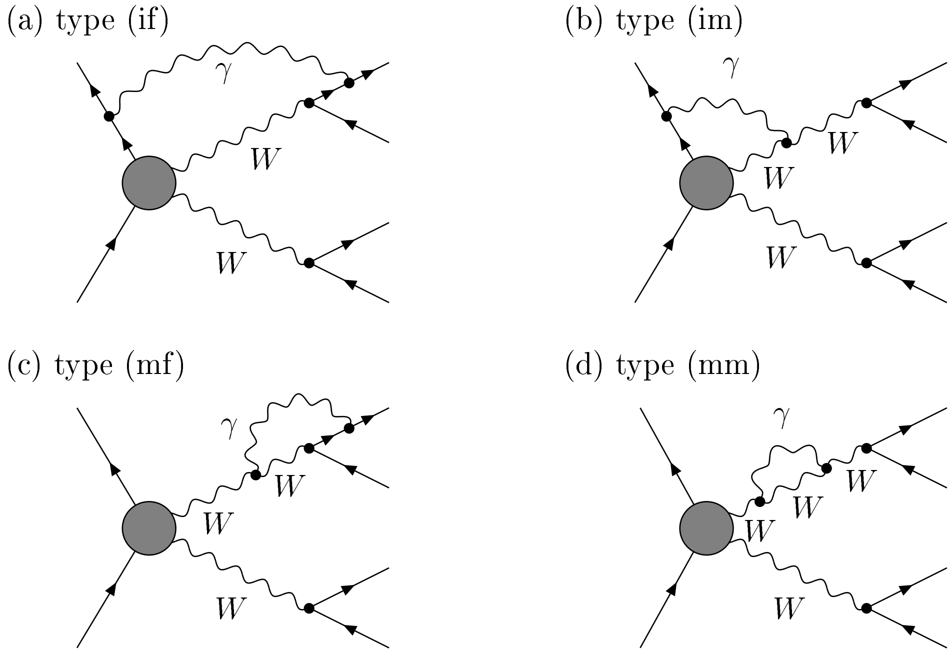

Figure 2: Further examples of non-factorizable photonic corrections in $\mathcal{O}(\alpha)$.

2.3 Classification and gauge-independent definition of the non-factorizable doubly-resonant virtual corrections

Manifestly non-factorizable corrections arise from photon exchange between the final states of the two W bosons (ff′), between initial and final state (if), and between one of the intermediate resonant W bosons and the final state of the other W boson (mf′). Examples for these types of corrections are shown in Fig. 1 (c), Fig. 2 (a), and Fig. 1 (a,b), respectively. In addition, there are diagrams where the photon does not couple to uniquely distinguishable subprocesses. These contributions can be classified into photon-exchange contributions between one of the intermediate resonant W bosons and the final state of the same W boson (mf), between the intermediate and the initial state (im), between the two intermediate W bosons (mm′), and within a single W-boson line, i.e. the photonic part of the W-self-energy corrections (mm). Diagrams contributing to these types of corrections are given in Fig. 2 (c), Fig. 2 (b), Fig. 1 (d), and Fig. 2 (d), respectively. Because the photon coupling to the W boson can be attributed to the decay or the production subprocesses, these diagrams involve both factorizable and non-factorizable corrections.

In order to define the non-factorizable corrections, we have to specify how the factorizable contributions are split off. This should be done in such a way that the non-factorizable corrections become gauge-independent. In Refs. [9, 10] this was reached by exploiting the fact that in ESPA the matrix element can be viewed as a product of the lowest-order matrix element with two conserved currents. Taking all interferences between the positively and the negatively charged currents arising from the outgoing W bosons and fermions gives a gauge-independent result.

We have chosen a different definition of the non-factorizable corrections, which, however, turns out to be equivalent to the one of Refs. [9, 10] in DPA. Our approach has the advantage of providing a clear procedure how to combine factorizable and non-factorizable contributions to the full $\mathcal{O}(\alpha)$ correction in DPA. Because the complete matrix element is gauge-independent order by order, the sum of all doubly-resonant $\mathcal{O}(\alpha)$ corrections must be gauge-independent. On the other hand, the factorizable doubly-resonant corrections can be defined by the product of gauge-independent on-shell matrix elements for W-pair production and W decays and the (transverse parts of the) W propagators,

$$\mathcal{M}_{\mathrm{f}} = \sum_{\lambda_+,\lambda_-} \frac{1}{K_+K_-}\Big(\delta\mathcal{M}^{\mathrm{e^+e^-\to W^+W^-}}\mathcal{M}_{\mathrm{Born}}^{\mathrm{W^+}\to f_1\bar{f}_2}\mathcal{M}_{\mathrm{Born}}^{\mathrm{W^-}\to f_3\bar{f}_4} \\ + \mathcal{M}_{\mathrm{Born}}^{\mathrm{e^+e^-\to W^+W^-}}\delta\mathcal{M}^{\mathrm{W^+}\to f_1\bar{f}_2}\mathcal{M}_{\mathrm{Born}}^{\mathrm{W^-}\to f_3\bar{f}_4} + \mathcal{M}_{\mathrm{Born}}^{\mathrm{e^+e^-\to W^+W^-}}\mathcal{M}_{\mathrm{Born}}^{\mathrm{W^+}\to f_1\bar{f}_2}\delta\mathcal{M}^{\mathrm{W^-}\to f_3\bar{f}_4}\Big), \tag{2.10}$$

where $\delta\mathcal{M}^{\mathrm{e^+e^-\to W^+W^-}}$, $\delta\mathcal{M}^{\mathrm{W^+}\to f_1\bar{f}_2}$, and $\delta\mathcal{M}^{\mathrm{W^-}\to f_3\bar{f}_4}$ denote the one-loop amplitudes of the respective subprocesses. We can define the non-factorizable doubly-resonant corrections by subtracting the factorizable doubly-resonant corrections from the complete doubly-resonant corrections. This definition allows us to calculate the complete doubly-resonant corrections by simply adding the factorizable corrections, defined via the on-shell matrix elements, to our results. Our definition can be applied diagram by diagram. In this way, all diagrams that are neither manifestly factorizable nor manifestly non-factorizable can be split. Such diagrams receive doubly-resonant contributions from the complete range of the photon momentum q, and not only from the soft-photon region. This is obviously due to the presence of two explicit resonant propagators. However, after subtracting the factorizable contributions, all doubly-resonant terms that are not IR-singular in the on-shell limit cancel exactly, i.e. only the soft-photon region contributes. Consequently, also in this case q can be neglected everywhere except for the denominators that become IR-singular in the on-shell limit. As an example, we give the non-factorizable correction originating from diagram (d) of Fig. 1:[1]

$$\mathcal{M}_{\mathrm{nf}}^{\mathrm{W^+W^-}} \sim \mathrm{i}e^2\mathcal{M}_{\mathrm{Born}}\bigg\{\int\frac{\mathrm{d}^4q}{(2\pi)^4}\frac{4k_+k_-}{q^2[(q+k_+)^2-M^2][(q-k_-)^2-M^2]} \\ - \bigg[\int\frac{\mathrm{d}^4q}{(2\pi)^4}\frac{4k_+k_-}{(q^2-\lambda^2)(q^2+2qk_+)(q^2-2qk_-)}\bigg]_{k_\pm^2=M_{\mathrm{W}}^2}\bigg\}. \tag{2.11}$$

This example shows that the on-shell subtraction introduces additional IR singularities. If the IR singularities in the non-factorizable real corrections are regularized in the same way, they cancel in the sum. In (2.11) an infinitesimal photon mass λ is used as IR regulator, but we have repeated the same calculation also by using a finite W-decay width as IR regulator instead of λ, leading to the same results in the sum of virtual and real photonic corrections.

We illustrate our definition of the non-factorizable corrections also for the photonic contribution to the W-self-energy correction [diagram (d) of Fig. 2]. The non-factorizable part of the $\mathrm{W^+}$ self-energy reads

$$\mathcal{M}_{\mathrm{nf}}^{\mathrm{W^+W^+}} \sim -\mathrm{i}e^2\mathcal{M}_{\mathrm{Born}}\bigg\{\int\frac{\mathrm{d}^4q}{(2\pi)^4}\frac{4k_+^2}{q^2[(q+k_+)^2-M^2](k_+^2-M^2)}$$

[1]We use the sign $\sim$ to indicate an equality within DPA, i.e. up to non-doubly-resonant terms.

$$
- \Big[\int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{4k_+^2}{q^2(q^2+2qk_+)}\Big]_{k_+^2=M^2} \frac{1}{k_+^2-M^2}
$$

$$
+ \Big[\int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{4k_+^2}{(q^2-\lambda^2)(q^2+2qk_+)^2}\Big]_{k_+^2=M_\mathrm{W}^2} \Big\}. \tag{2.12}
$$

The first integral results from the off-shell self-energy diagram, the second from the corresponding mass-renormalization term, and the third integral is the negative of the on-shell limit of the first two integrals. The integrals in (2.12) are UV-divergent and can be easily evaluated in dimensional regularization.

The gauge independence of the non-factorizable corrections has been ensured by construction. The consistent evaluation of gauge theories requires, besides gauge independence of the physical matrix elements, the validity of Ward identities. It was found in Ref. [12] that the violation of Ward identities can lead to completely wrong predictions. The procedure described above for extracting the non-factorizable corrections from the full matrix element does not lead to problems with the Ward identities that rule the gauge cancellations inside matrix elements. This is due to the fact that the non-factorizable corrections are proportional to the Born matrix element. Therefore, if Ward identities and gauge cancellations are under control in lowest order, the same is true for the non-factorizable corrections.

Finally, we show how our definition of the non-factorizable corrections can be rephrased in terms of products of appropriately defined currents. By using

$$
\frac{1}{(q\pm k_\pm)^2-M^2} = \frac{1}{q^2\pm 2qk_\pm}\Big[1-\frac{k_\pm^2-M^2}{(q\pm k_\pm)^2-M^2}\Big], \tag{2.13}
$$

and the fact that in DPA $k_\pm^2$ can be put to M_W^2 before integration in integrals that do not depend on M^2, the contribution (2.11) can be expressed as

$$
\mathcal{M}_\mathrm{nf}^{\mathrm{W^+W^-}} \sim \mathrm{i}e^2\mathcal{M}_\mathrm{Born}\int\frac{\mathrm{d}^4 q}{(2\pi)^4}\frac{4k_+k_-}{(q^2-\lambda^2)(q^2+2qk_+)(q^2-2qk_-)}\Big[-\frac{k_+^2-M^2}{(q+k_+)^2-M^2}
- \frac{k_-^2-M^2}{(q-k_-)^2-M^2} + \frac{k_+^2-M^2}{(q+k_+)^2-M^2}\frac{k_-^2-M^2}{(q-k_-)^2-M^2}\Big]. \tag{2.14}
$$

The other non-factorizable corrections that involve photons coupled to W bosons can be rewritten in a similar way. Finally, all non-factorizable virtual corrections can be cast into the following form:

$$
\mathcal{M}_\mathrm{nf}^\mathrm{virt} \sim \mathrm{i}\mathcal{M}_\mathrm{Born}\int\frac{\mathrm{d}^4 q}{(2\pi)^4}\frac{1}{q^2-\lambda^2}\Big[j_\mathrm{virt}^{\mathrm{e^+e^-\to W^+W^-},\mu} j_{\mathrm{virt},\mu}^{\mathrm{W^+}\to f_1\bar f_2} + j_\mathrm{virt}^{\mathrm{e^+e^-\to W^+W^-},\mu} j_{\mathrm{virt},\mu}^{\mathrm{W^-}\to f_3\bar f_4}
+ j_\mathrm{virt}^{\mathrm{W^+}\to f_1\bar f_2,\mu} j_{\mathrm{virt},\mu}^{\mathrm{W^-}\to f_3\bar f_4}\Big] \tag{2.15}
$$

with

$$
j_{\mathrm{virt},\mu}^{\mathrm{e^+e^-\to W^+W^-}} = e\Big(\frac{2k_{+\mu}}{q^2+2qk_+}+\frac{2k_{-\mu}}{q^2-2qk_-}-\frac{2p_{-\mu}}{q^2-2qp_-}-\frac{2p_{+\mu}}{q^2+2qp_+}\Big),
$$

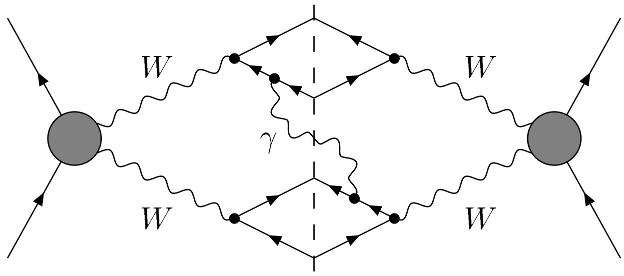

Figure 3: Example of a non-factorizable real correction.

$$\begin{aligned} j^{\mathrm{W}^+\to f_1\bar{f}_2}_{\mathrm{virt},\mu} &= e\biggl(Q_1\frac{2k_{1\mu}}{q^2+2qk_1} - Q_2\frac{2k_{2\mu}}{q^2+2qk_2} - \frac{2k_{+\mu}}{q^2+2qk_+}\biggr)\frac{k_+^2-M^2}{(k_++q)^2-M^2}, \\ j^{\mathrm{W}^-\to f_3\bar{f}_4}_{\mathrm{virt},\mu} &= -e\biggl(Q_3\frac{2k_{3\mu}}{q^2-2qk_3} - Q_4\frac{2k_{4\mu}}{q^2-2qk_4} + \frac{2k_{-\mu}}{q^2-2qk_-}\biggr)\frac{k_-^2-M^2}{(k_--q)^2-M^2}. \end{aligned} \tag{2.16}$$

The last term in (2.15) originates from the Feynman graphs shown in Fig. 1 and those where the final-state fermions are appropriately interchanged [interference terms (ff′), (mf′), and (mm′)]. The contributions involving the current $j^{\mathrm{e}^+\mathrm{e}^-\to\mathrm{W}^+\mathrm{W}^-}_{\mathrm{virt},\mu}$ contain the interference terms (if), (mf), (im), (mm), and the remaining contributions of (mf′) and (mm′). The contribution of the W^+ self-energy is given, for instance, by the product of the two terms involving $k_{+\mu}$ in $j^{\mathrm{e}^+\mathrm{e}^-\to\mathrm{W}^+\mathrm{W}^-}_{\mathrm{virt},\mu}$ and $j^{\mathrm{W}^+\to f_1\bar{f}_2}_{\mathrm{virt},\mu}$.

In DPA, the q^2 terms in the denominators of (2.16) can be neglected, and the currents are conserved. The currents $j^{\mathrm{W}^+\to f_1\bar{f}_2}_{\mathrm{virt},\mu}$ and $j^{\mathrm{W}^-\to f_3\bar{f}_4}_{\mathrm{virt},\mu}$ are the ones mentioned in Refs. [9, 10]. For the virtual corrections, this shows that our definition of non-factorizable doubly-resonant corrections coincides with the one of Refs. [9, 10] in DPA.

2.4 Doubly-resonant real corrections

The photonic virtual corrections discussed above are IR-singular and have to be combined with the corresponding real corrections in order to arrive at a sensible physical result. The real corrections originate from the process

$$\begin{aligned} \mathrm{e}^+(p_+) + \mathrm{e}^-(p_-) &\to \mathrm{W}^+(k'_+) + \mathrm{W}^-(k'_-)\,[\,+\,\gamma(q)] \\ &\to f_1(k'_1) + \bar{f}_2(k'_2) + f_3(k'_3) + \bar{f}_4(k'_4) + \gamma(q). \end{aligned} \tag{2.17}$$

Note that we have marked the fermion momenta k'_i by primes in order to distinguish them from the respective momenta without real photon emission. The momenta of the W bosons are $k'_+ = k'_1 + k'_2$ and $k'_- = k'_3 + k'_4$ if the photon is emitted in the initial state or in the final state of the other W boson, and $\bar{k}_+ = k'_1 + k'_2 + q$ or $\bar{k}_- = k'_3 + k'_4 + q$ if the photon is emitted in the final state of the respective W boson.

The non-factorizable corrections induced by the process (2.17) arise from interferences between diagrams where the photon is emitted from different subprocesses. A typical non-factorizable contribution is shown in Fig. 3. Including the integration over the photon phase space, this contribution has the following form:

$$\begin{aligned}\mathcal{I} = &\int \frac{\mathrm{d}^3\mathbf{q}}{2q_0} \frac{N(q,k_i')\delta(p_+ + p_- - k_1' - k_2' - k_3' - k_4' - q)}{[(q+k_2')^2 - m_2^2][(q+k_+')^2 - M^2](k_-'^2 - M^2)} \\ &\quad \times \left\{ \frac{1}{[(q+k_3')^2 - m_3^2](k_+'^2 - M^2)[(q+k_-')^2 - M^2]} \right\}^* \Bigg|_{q_0=\sqrt{\mathbf{q}^2+\lambda^2}} \\ = &\int \frac{\mathrm{d}^3\mathbf{q}}{2q_0} \frac{N(q,k_i')\delta(p_+ + p_- - k_1' - k_2' - k_3' - k_4' - q)}{(2qk_2')(2qk_+' + k_+'^2 - M^2)(k_-'^2 - M^2)} \\ &\quad \times \frac{1}{(2qk_3')[k_+'^2 - (M^*)^2][2qk_-' + k_-'^2 - (M^*)^2]} \Bigg|_{q_0=\sqrt{\mathbf{q}^2+\lambda^2}},\end{aligned} \tag{2.18}$$

where we again use a photon mass λ to regularize the IR singularities, and $N(q,k_i')$ has the same meaning as above. Again, the doubly-resonant contributions are characterized by a quadratic IR singularity for $k_\pm'^2 = M_\mathrm{W}^2$, $\Gamma_\mathrm{W} \to 0$, and only soft-photon emission is relevant in DPA. For this reason, the W bosons are nearly on shell, and the on-shell width is appropriate. As for the virtual corrections, the introduction of the width before or after phase-space integration leads to the same results in DPA. As already indicated in (2.18), in the following real integrals we use M^2 with the understanding that it has to be replaced by M_W^2 after integration where possible.

The aim is to integrate over the photon momentum analytically and to relate the fermion momenta k_i' to the ones of the process without photon emission, k_i. Primed and unprimed momenta differ by terms of the order of the photon momentum: $k_i' = k_i + \mathcal{O}(q_0)$. In DPA we can neglect q in $N(q,k_i')$, leading to the replacement $N(q,k_i') \to N(0,k_i)$. Moreover, we can extend the integration region for q_0 to infinity, because large photon momenta yield negligible contributions in DPA. After extension of the integration region the integral becomes Lorentz-invariant.

While the correction factor to the lowest-order cross section is universal in SPA for all observables, the correction factor is non-universal in ESPA. In order to define this correction factor in a unique way, one has to specify the parametrization of phase space, i.e. the variables that are kept fixed when the photon momentum is integrated over. This fact has not been addressed in the literature so far.

Let us consider this problem in more detail. It can be traced back to the appearance of the photon momentum q in the δ-function for momentum conservation. In the usual SPA q is neglected in this δ-function, which is sensible if the exact matrix element is a slowly varying function of q in the vicinity of $q=0$. However, in the presence of resonant propagators, in which q cannot be neglected, the simple omission of q in the momentum-conservation δ-function leads to ambiguous results: putting $q=0$ in the δ-function and identifying k_i' with k_i in (2.18) yields

$$\begin{aligned}\mathcal{I} \to &\int \frac{\mathrm{d}^3\mathbf{q}}{2q_0} \frac{N(0,k_i)\delta(p_+ + p_- - k_1 - k_2 - k_3 - k_4)}{(2qk_2)(2qk_+ + k_+^2 - M^2)(k_-^2 - M^2)} \\ &\quad \times \frac{1}{(2qk_3)[k_+^2 - (M^*)^2][2qk_- + k_-^2 - (M^*)^2]} \Bigg|_{q_0=\sqrt{\mathbf{q}^2+\lambda^2}}.\end{aligned} \tag{2.19}$$

On the other hand, eliminating k'_+ in the denominator of (2.18) with the help of the δ-function, putting $q = 0$ in the δ-function, restoring k'_+ with the modified δ-function, and setting $k'_i \to k_i$ results in

$$\mathcal{I} \to \int \frac{\mathrm{d}^3\mathbf{q}}{2q_0} \frac{N(0,k_i)\delta(p_+ + p_- - k_1 - k_2 - k_3 - k_4)}{(2qk_2)(k_+^2 - M^2)(k_-^2 - M^2)} \times \frac{1}{(2qk_3)[-2qk_+ + k_+^2 - (M^*)^2][2qk_- + k_-^2 - (M^*)^2]}\bigg|_{q_0=\sqrt{\mathbf{q}^2+\lambda^2}}. \tag{2.20}$$

Both expressions differ by a doubly-resonant contribution. The difference is in general confined to the W propagators and originates from the fact that not only soft photons but also photons with energies of the order of $|k_\pm^2 - M_\mathrm{W}^2|/M_\mathrm{W}$ or, after the inclusion of the finite width, of order Γ_W contribute in DPA. Since photons with finite energies contribute, it is evident that the integral over the photon momentum depends on the choice of the phase-space variables that are kept fixed.

As a consequence, one has to choose a definite parametrization of phase space and to exploit the δ-function carefully, in order to define the non-factorizable corrections uniquely. For instance, if the vector $k'_+ = k'_1 + k'_2$ is kept fixed, the alternative (2.20) is excluded. However, because of momentum conservation, not all external momenta can be kept fixed independently.

It is, however, possible to keep, for instance, the invariant masses of the final-state fermion pairs $k'^2_+ = (k'_1 + k'_2)^2$ and $k'^2_- = (k'_3 + k'_4)^2$ fixed when integrating over the photon momentum. If we require $(k'_1+k'_2)^2 = k'^2_+ = k^2_+ = (k_1+k_2)^2$, we obtain for the denominator of the W^+ boson

$$\begin{aligned}(q + k'_+)^2 - M^2 &= 2qk'_+ + k'^2_+ - M^2 = 2qk_+ + k_+^2 - M^2 + \mathcal{O}(q_0^2)\\ &= (q + k_+)^2 - M^2 + \mathcal{O}(q_0^2),\end{aligned} \tag{2.21}$$

where $k'_i = k_i + \mathcal{O}(q_0)$ was used. Based on the power-counting argument given above, the terms of order q_0^2 can be neglected in DPA, and we find

$$(q + k'_+)^2 - M^2 \sim (q + k_+)^2 - M^2. \tag{2.22}$$

If we choose to eliminate k'_+, as done in the derivation of (2.20), we find, on the other hand,

$$\begin{aligned}(q + k'_+)^2 - M^2 &= (p_+ + p_- - k'_-)^2 - M^2 = (p_+ + p_- - k_-)^2 - M^2 + \mathcal{O}(q_0)\\ &= k_+^2 - M^2 + \mathcal{O}(q_0).\end{aligned} \tag{2.23}$$

The $\mathcal{O}(q_0)$ terms are relevant in DPA [and in fact given by (2.21)]. As a consequence, (2.20) is not correct if we choose to fix $k'^2_+ = k^2_+$ when integrating over the photon momentum. For fixed $k'^2_+ = k^2_+$, (2.21) leads to the unique result (2.19) for the W^+ propagator in DPA, independently of the other phase-space parameters. If we choose, on the other hand, to fix $\bar{k}^2_+ = k^2_+$, which corresponds to a different definition of the invariant mass of the W^+ boson, we obtain

$$(q + k'_+)^2 - M^2 = \bar{k}_+^2 - M^2, \tag{2.24}$$

and thus (2.20) instead of (2.19). Consequently, the different approaches (2.19) and (2.20) correspond to different definitions of the invariant mass of the W^+ boson which decays into the fermion pair $(f_1, \bar{f}_2)$. In order to define the DPA for real radiation, one has to specify at least the definition of the invariant masses of the W bosons that are kept fixed. In the following we always fix $k_+^2 = (k_1'+k_2')^2 = (k_1+k_2)^2$ and $k_-^2 = (k_3'+k_4')^2 = (k_3+k_4)^2$, as it was also implicitly done in Refs. [8, 9, 10]. Once the invariant masses of the W bosons are fixed in this way, the resulting formulae for the non-factorizable corrections hold independently of the choice of all other phase-space variables.

We stress that the results obtained within this parametrization of phase space differ from those in other parametrizations by doubly-resonant corrections. As already indicated in the introduction, in an experimentally more realistic approach the invariant masses of W bosons are identified with invariant masses of jet pairs, which also include part of the photon radiation. Since this situation can only be described with Monte Carlo programs, our results (as well as those of Refs. [8, 9, 10]) should be regarded as an estimate of the non-factorizable corrections. Moreover, our analytic results provide benchmarks for such future Monte Carlo calculations.

2.5 Classification and gauge-independent definition of the non-factorizable doubly-resonant real corrections

The doubly-resonant real corrections can be classified in exactly the same way as the virtual corrections. For each virtual diagram there is exactly a real contribution, which we denote in the same way, e.g. ff$'$ refers to all interferences where the photon is emitted by two fermions corresponding to the two different W bosons.

As in the case of the virtual corrections, one has to define the non-factorizable real corrections in a gauge-independent way. To this end, we proceed analogously and define the non-factorizable real corrections as the difference of the complete real corrections and the factorizable real corrections in DPA. The factorizable corrections are defined by the products of the matrix elements for on-shell W-pair production and decay with additional photon emission. There are three contributions to the factorizable real corrections, one where the photon is radiated during the production process and two where it is emitted during one of the W-boson decays. The corresponding matrix elements read

$$
\begin{aligned}
\mathcal{M}_{\mathrm{real},1} &= \sum_{\lambda_+,\lambda_-} \frac{\mathcal{M}_{\mathrm{Born}}^{e^+e^-\to W^+W^-\gamma}\mathcal{M}_{\mathrm{Born}}^{W^+\to f_1\bar{f}_2}\mathcal{M}_{\mathrm{Born}}^{W^-\to f_3\bar{f}_4}}{(k_+^2-M^2)(k_-^2-M^2)}, \\
\mathcal{M}_{\mathrm{real},2} &= \sum_{\lambda_+,\lambda_-} \frac{\mathcal{M}_{\mathrm{Born}}^{e^+e^-\to W^+W^-}\mathcal{M}_{\mathrm{Born}}^{W^+\to f_1\bar{f}_2\gamma}\mathcal{M}_{\mathrm{Born}}^{W^-\to f_3\bar{f}_4}}{[(k_++q)^2-M^2](k_-^2-M^2)}, \\
\mathcal{M}_{\mathrm{real},3} &= \sum_{\lambda_+,\lambda_-} \frac{\mathcal{M}_{\mathrm{Born}}^{e^+e^-\to W^+W^-}\mathcal{M}_{\mathrm{Born}}^{W^+\to f_1\bar{f}_2}\mathcal{M}_{\mathrm{Born}}^{W^-\to f_3\bar{f}_4\gamma}}{(k_+^2-M^2)[(k_-+q)^2-M^2]},
\end{aligned}
\tag{2.25}
$$

in analogy to (2.7). Note that the matrix elements $\mathcal{M}_{\mathrm{real},2}$ and $\mathcal{M}_{\mathrm{real},3}$ involve an explicit q-dependent propagator. The factorizable corrections are given by the squares of these three matrix elements, and include by definition no interferences between them.

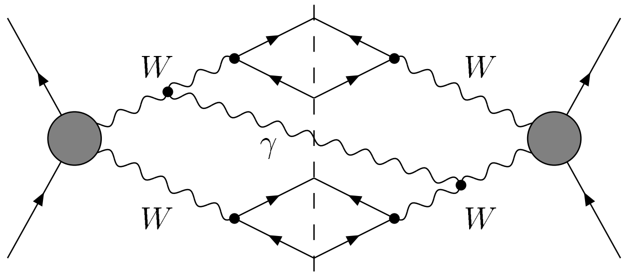

Figure 4: Real bremsstrahlung diagram containing non-factorizable and factorizable contributions.

As an example for the extraction of non-factorizable corrections from real diagrams that involve both factorizable and non-factorizable corrections, we consider the contribution of the diagram in Fig. 4. After subtraction of the factorizable contribution, which originates from $|\mathcal{M}_{\mathrm{real},1}|^2$, it gives rise to the following correction factor to the square $|\mathcal{M}_{\mathrm{Born}}|^2$ of the lowest-order matrix element (2.7):

$$\begin{aligned}\delta_{\mathrm{real}}^{\mathrm{W^+W^-}} &= e^2 \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3 2q_0}\, 2\,\mathrm{Re}\biggl\{ \frac{4k_+k_-}{[(k_++q)^2-M^2][(k_-+q)^2-(M^*)^2]} \\ &\quad - \biggl[\frac{4k_+k_-}{2qk_+ 2qk_-}\biggr]_{k_\pm^2=M_\mathrm{W}^2}\biggr\}\bigg|_{q_0=\sqrt{\mathbf{q}^2+\lambda^2}}. \end{aligned} \tag{2.26}$$

Note that the form of the correction factor is only correct for fixed $(k_1'+k_2')^2$ and $(k_3'+k_4')^2$. For other conventions the off-shell contribution changes, whereas the on-shell contribution stays the same. Using the relations (2.13) for $q^2=0$, in DPA we can rewrite (2.26) as

$$\begin{aligned}\delta_{\mathrm{real}}^{\mathrm{W^+W^-}} &\sim -e^2 \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3 2q_0}\, 2\,\mathrm{Re}\biggl\{ \frac{k_+k_-}{(qk_+)(qk_-)} \biggl[\frac{k_+^2-M^2}{(k_++q)^2-M^2} + \frac{k_-^2-(M^*)^2}{(k_-+q)^2-(M^*)^2} \\ &\quad - \frac{k_+^2-M^2}{(k_++q)^2-M^2}\,\frac{k_-^2-(M^*)^2}{(k_-+q)^2-(M^*)^2}\biggr]\biggr\}\bigg|_{q_0=\sqrt{\mathbf{q}^2+\lambda^2}}. \end{aligned} \tag{2.27}$$

In the same way, all other contributions that originate from a photon coupled to a W boson can be rewritten such that the complete real non-factorizable corrections can finally be expressed as the following correction factor to the lowest-order cross section,

$$\begin{aligned}\delta_{\mathrm{real,nf}} &\sim -\int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3 2q_0}\, 2\,\mathrm{Re}\Bigl[j_{\mathrm{real}}^{\mathrm{e^+e^-\to W^+W^-},\mu} \bigl(j_{\mathrm{real},\mu}^{\mathrm{W^+}\to f_1\bar f_2}\bigr)^* + j_{\mathrm{real}}^{\mathrm{e^+e^-\to W^+W^-},\mu} \bigl(j_{\mathrm{real},\mu}^{\mathrm{W^-}\to f_3\bar f_4}\bigr)^* \\ &\quad + j_{\mathrm{real}}^{\mathrm{W^+}\to f_1\bar f_2,\mu} \bigl(j_{\mathrm{real},\mu}^{\mathrm{W^-}\to f_3\bar f_4}\bigr)^*\Bigr]\Big|_{q_0=\sqrt{\mathbf{q}^2+\lambda^2}} \end{aligned} \tag{2.28}$$

with the currents

$$\begin{aligned} j_{\mathrm{real},\mu}^{\mathrm{e^+e^-\to W^+W^-}} &= e\biggl(\frac{k_{+\mu}}{qk_+} - \frac{k_{-\mu}}{qk_-} + \frac{p_{-\mu}}{qp_-} - \frac{p_{+\mu}}{qp_+}\biggr), \\ j_{\mathrm{real},\mu}^{\mathrm{W^+}\to f_1\bar f_2} &= e\biggl(Q_1\frac{k_{1\mu}}{qk_1} - Q_2\frac{k_{2\mu}}{qk_2} - \frac{k_{+\mu}}{qk_+}\biggr)\frac{k_+^2-M^2}{(k_++q)^2-M^2}, \\ j_{\mathrm{real},\mu}^{\mathrm{W^-}\to f_3\bar f_4} &= e\biggl(Q_3\frac{k_{3\mu}}{qk_3} - Q_4\frac{k_{4\mu}}{qk_4} + \frac{k_{-\mu}}{qk_-}\biggr)\frac{k_-^2-M^2}{(k_-+q)^2-M^2}. \end{aligned} \tag{2.29}$$

The factor (2.28) for the non-factorizable correction can be viewed as the interference contributions in the square of the matrix element (ε denotes the polarization vector of the photon)

$$\mathcal{M}_{\mathrm{real}} = \mathcal{M}_{\mathrm{Born}}\, \varepsilon^{\mu} \left(j_{\mathrm{real},\mu}^{\mathrm{e^+e^-\to W^+W^-}} + j_{\mathrm{real},\mu}^{\mathrm{W^+}\to f_1\bar{f}_2} + j_{\mathrm{real},\mu}^{\mathrm{W^-}\to f_3\bar{f}_4} \right), \tag{2.30}$$

which is just the sum of the three matrix elements $\mathcal{M}_{\mathrm{real},i}$, $i = 1,2,3$, in ESPA (including radiation from the external fermions and the internal W bosons). The respective squares of these three contributions correspond to the factorizable corrections. Note that $\mathcal{M}_{\mathrm{Born}}^{\mathrm{e^+e^-\to W^+W^-}} \varepsilon^{\mu} j_{\mathrm{real},\mu}^{\mathrm{e^+e^-\to W^+W^-}}$ is the soft-photon matrix element for on-shell W-pair production. Similarly, $\mathcal{M}_{\mathrm{Born}}^{\mathrm{W^+}\to f_1\bar{f}_2} \varepsilon^{\mu} j_{\mathrm{real},\mu}^{\mathrm{W^+}\to f_1\bar{f}_2}$ and $\mathcal{M}_{\mathrm{Born}}^{\mathrm{W^-}\to f_3\bar{f}_4} \varepsilon^{\mu} j_{\mathrm{real},\mu}^{\mathrm{W^-}\to f_3\bar{f}_4}$ correspond to the soft-photon matrix elements for the decays of the on-shell W bosons, apart from the extra factors $(k_\pm^2 - M^2)/[(k_\pm + q)^2 - M^2]$. These factors result from the definition of the lowest-order matrix element in terms of $k_\pm$ and from the fact that we impose $k_\pm'^2 = k_\pm^2$ and not $\bar{k}_\pm^2 = k_\pm^2$.

Obviously, the currents (2.29) are conserved, and the corresponding Ward identity for the $\mathrm{U(1)_{em}}$ symmetry of the emitted photon is fulfilled.

3 Calculation of scalar integrals

In this section we set our conventions for the scalar integrals. We describe the reduction of the virtual and real 5-point functions to 4-point functions and indicate how the scalar integrals were evaluated. More details and the explicit results for the scalar integrals can be found in the appendix.

3.1 Reduction of 5-point functions

3.1.1 Reduction of virtual 5-point functions

The virtual 3-, 4-, and 5-point functions are defined as

$$\begin{aligned} C_0(p_1,p_2,m_0,m_1,m_2) &= \frac{1}{\mathrm{i}\pi^2}\int \mathrm{d}^4 q\, \frac{1}{N_0 N_1 N_2}, \\ D_{\{0,\mu\}}(p_1,p_2,p_3,m_0,m_1,m_2,m_3) &= \frac{1}{\mathrm{i}\pi^2}\int \mathrm{d}^4 q\, \frac{\{1,q_\mu\}}{N_0 N_1 N_2 N_3}, \\ E_{\{0,\mu\}}(p_1,p_2,p_3,p_4,m_0,m_1,m_2,m_3,m_4) &= \frac{1}{\mathrm{i}\pi^2}\int \mathrm{d}^4 q\, \frac{\{1,q_\mu\}}{N_0 N_1 N_2 N_3 N_4}, \end{aligned} \tag{3.1}$$

with the denominator factors

$$N_0 = q^2 - m_0^2 + \mathrm{i}\epsilon, \qquad N_i = (q+p_i)^2 - m_i^2 + \mathrm{i}\epsilon, \qquad i = 1,\dots,4, \tag{3.2}$$

where $\mathrm{i}\epsilon$ ($\epsilon > 0$) denotes an infinitesimal imaginary part.

The reduction of the virtual 5-point function to 4-point functions is based on the fact that in four dimensions the integration momentum depends linearly on the four external momenta p_i [13]. This gives rise to the identity

$$0 = \begin{vmatrix} 2q^2 & 2qp_1 & \dots & 2qp_4 \\ 2p_1q & 2p_1^2 & \dots & 2p_1p_4 \\ \vdots & \vdots & \ddots & \vdots \\ 2p_4q & 2p_4p_1 & \dots & 2p_4^2 \end{vmatrix} = \begin{vmatrix} 2N_0 + Y_{00} & 2qp_1 & \dots & 2qp_4 \\ N_1 - N_0 + Y_{10} - Y_{00} & 2p_1^2 & \dots & 2p_1p_4 \\ \vdots & \vdots & \ddots & \vdots \\ N_4 - N_0 + Y_{40} - Y_{00} & 2p_4p_1 & \dots & 2p_4^2 \end{vmatrix} \tag{3.3}$$

with

$$Y_{00} = 2m_0^2, \quad Y_{i0} = Y_{0i} = m_0^2 + m_i^2 - p_i^2, \quad Y_{ij} = m_i^2 + m_j^2 - (p_i - p_j)^2, \quad i,j = 1,2,3,4. \tag{3.4}$$

Dividing this equation by $N_0 N_1 \cdots N_4$ and integrating over $\mathrm{d}^4 q$ yields

$$0 = \frac{1}{\mathrm{i}\pi^2} \int \mathrm{d}^4 q \, \frac{1}{N_0 N_1 \cdots N_4} \begin{vmatrix} 2N_0 + Y_{00} & 2qp_1 & \dots & 2qp_4 \\ N_1 - N_0 + Y_{10} - Y_{00} & 2p_1^2 & \dots & 2p_1p_4 \\ \vdots & \vdots & \ddots & \vdots \\ N_4 - N_0 + Y_{40} - Y_{00} & 2p_4p_1 & \dots & 2p_4^2 \end{vmatrix}. \tag{3.5}$$

Expanding the determinant along the first column, we obtain

$$\begin{aligned} 0 = {} & \Big[2D_0(0) + Y_{00}E_0\Big] \begin{vmatrix} 2p_1p_1 & \dots & 2p_1p_4 \\ \vdots & \ddots & \vdots \\ 2p_4p_1 & \dots & 2p_4p_4 \end{vmatrix} \\ & + \sum_{k=1}^{4} (-1)^k \Big\{ D_\mu(k) - \Big[D_\mu(0) + p_{4\mu} D_0(0)\Big] + p_{4\mu} D_0(0) + (Y_{k0} - Y_{00}) E_\mu \Big\} \\ & \times \begin{vmatrix} 2p_1^\mu & \dots & 2p_4^\mu \\ 2p_1p_1 & \dots & 2p_1p_4 \\ \vdots & \ddots & \vdots \\ 2p_{k-1}p_1 & \dots & 2p_{k-1}p_4 \\ 2p_{k+1}p_1 & \dots & 2p_{k+1}p_4 \\ \vdots & \ddots & \vdots \\ 2p_4p_1 & \dots & 2p_4p_4 \end{vmatrix}, \end{aligned} \tag{3.6}$$

where $D_0(k)$ denotes the 4-point function that is obtained from the 5-point function E_0 by omitting the kth propagator N_k^{-1}. The terms involving $p_{4\mu} D_0(0)$ have been added for later convenience.

All integrals in (3.6) are UV-finite and Lorentz-covariant. Therefore, the vector integrals possess the following decompositions

$$
\begin{aligned}
E_\mu &= \sum_{i=1}^{4} E_i p_{i\mu}, \\
D_\mu(k) &= \sum_{\substack{i=1\\ i\neq k}}^{4} D_i(k) p_{i\mu}, \qquad k=1,2,3,4, \\
D_\mu(0) + p_{4\mu} D_0(0) &= \sum_{i=1}^{3} D_i(0)(p_i - p_4)_\mu.
\end{aligned} \tag{3.7}
$$

The last decomposition becomes obvious after performing a shift $q \to q - p_4$ in the integral. From (3.7) it follows immediately that the terms in (3.6) that involve $D_\mu(k)$ drop out when multiplied with the determinants, because the resulting determinants vanish. Similarly, $D_\mu(0) + p_{4\mu} D_0(0)$ vanishes after summation over k. Finally, the term $p_{4\mu} D_0(0)$ contributes only for $k = 4$, where it can be combined with the first term in (3.6). Rewriting the resulting equation as a determinant and reinserting the explicit form of the tensor integrals leads to

$$
0 = \frac{1}{\mathrm{i}\pi^2} \int \mathrm{d}^4 q \, \frac{1}{N_0 N_1 \cdots N_4}
\begin{vmatrix}
N_0 + Y_{00} & 2qp_1 & \ldots & 2qp_4 \\
Y_{10} - Y_{00} & 2p_1p_1 & \ldots & 2p_1p_4 \\
\vdots & \vdots & \ddots & \vdots \\
Y_{40} - Y_{00} & 2p_4p_1 & \ldots & 2p_4p_4
\end{vmatrix}. \tag{3.8}
$$

Using

$$
\begin{aligned}
2p_ip_j &= Y_{ij} - Y_{i0} - Y_{0j} + Y_{00}, \\
2qp_j &= N_j - N_0 + Y_{0j} - Y_{00},
\end{aligned} \tag{3.9}
$$

adding the first column to each of the other columns, and then enlarging the determinant by one column and one row, this can be written as

$$
0 = \begin{vmatrix}
1 & Y_{00} & \ldots & Y_{04} \\
0 & D_0(0) + Y_{00} E_0 & \ldots & D_0(4) + Y_{04} E_0 \\
0 & Y_{10} - Y_{00} & \ldots & Y_{14} - Y_{04} \\
\vdots & \vdots & \ddots & \vdots \\
0 & Y_{40} - Y_{00} & \ldots & Y_{44} - Y_{04}
\end{vmatrix}. \tag{3.10}
$$

Equation (3.10) is equivalent to

$$0 = \begin{vmatrix} -E_0 & D_0(0) & D_0(1) & D_0(2) & D_0(3) & D_0(4) \\ 1 & Y_{00} & Y_{01} & Y_{02} & Y_{03} & Y_{04} \\ 1 & Y_{10} & Y_{11} & Y_{12} & Y_{13} & Y_{14} \\ 1 & Y_{20} & Y_{21} & Y_{22} & Y_{23} & Y_{24} \\ 1 & Y_{30} & Y_{31} & Y_{32} & Y_{33} & Y_{34} \\ 1 & Y_{40} & Y_{41} & Y_{42} & Y_{43} & Y_{44} \end{vmatrix}, \tag{3.11}$$

which expresses the scalar 5-point function E_0 in terms of five scalar 4-point functions

$$E_0 = \frac{1}{\det(Y)} \sum_{i=0}^{4} \det(Y_i)\, D_0(i), \tag{3.12}$$

where $Y = (Y_{ij})$, and Y_i is obtained from Y by replacing all entries in the ith column with 1.

In the special case of an infrared singular 5-point function we have

$$Y_{00} = 2\lambda^2 \to 0, \qquad Y_{01} = m_1^2 - p_1^2 = 0, \qquad Y_{04} = m_4^2 - p_4^2 = 0, \tag{3.13}$$

and the determinants fulfil the relations

$$\begin{aligned} 0 &= \det(Y) - Y_{02}\det(Y_2) - Y_{03}\det(Y_3), \qquad (3.14)\\ 0 &= (Y_{24} - Y_{02})\det(Y) + Y_{02}Y_{14}\det(Y_1) + (Y_{02}Y_{34} - Y_{03}Y_{24})\det(Y_3) + Y_{02}Y_{44}\det(Y_4),\\ 0 &= (Y_{31} - Y_{03})\det(Y) + Y_{03}Y_{11}\det(Y_1) + (Y_{03}Y_{21} - Y_{02}Y_{31})\det(Y_2) + Y_{03}Y_{41}\det(Y_4), \end{aligned}$$

which allow the simplification of (3.12).

3.1.2 Reduction of real 5-point functions

The real 3-, 4-, and 5-point functions are defined as

$$\begin{aligned} \mathcal{C}_0(p_1,p_2,\lambda,m_1,m_2) &= \frac{2}{\pi}\int_{q_0<\Delta E} \frac{\mathrm{d}^3\mathbf{q}}{2q_0}\, \frac{1}{N_1'N_2'}\bigg|_{q_0=\sqrt{\mathbf{q}^2+\lambda^2}}, \\ \mathcal{D}_0(p_1,p_2,p_3,\lambda,m_1,m_2,m_3) &= \frac{2}{\pi}\int \frac{\mathrm{d}^3\mathbf{q}}{2q_0}\, \frac{1}{N_1'N_2'N_3'}\bigg|_{q_0=\sqrt{\mathbf{q}^2+\lambda^2}}, \\ \mathcal{E}_0(p_1,p_2,p_3,p_4,\lambda,m_1,m_2,m_3,m_4) &= \frac{2}{\pi}\int \frac{\mathrm{d}^3\mathbf{q}}{2q_0}\, \frac{1}{N_1'N_2'N_3'N_4'}\bigg|_{q_0=\sqrt{\mathbf{q}^2+\lambda^2}}, \qquad (3.15) \end{aligned}$$

with

$$N_i' = 2qp_i + p_i^2 - m_i^2, \qquad i = 1,\dots,4. \tag{3.16}$$

The shift of the integration boundary of q_0 to infinity leads to an artificial UV divergence in the 3-point function $\mathcal{C}_0$, which is regularized by an energy cutoff $\Delta E \to \infty$. In the

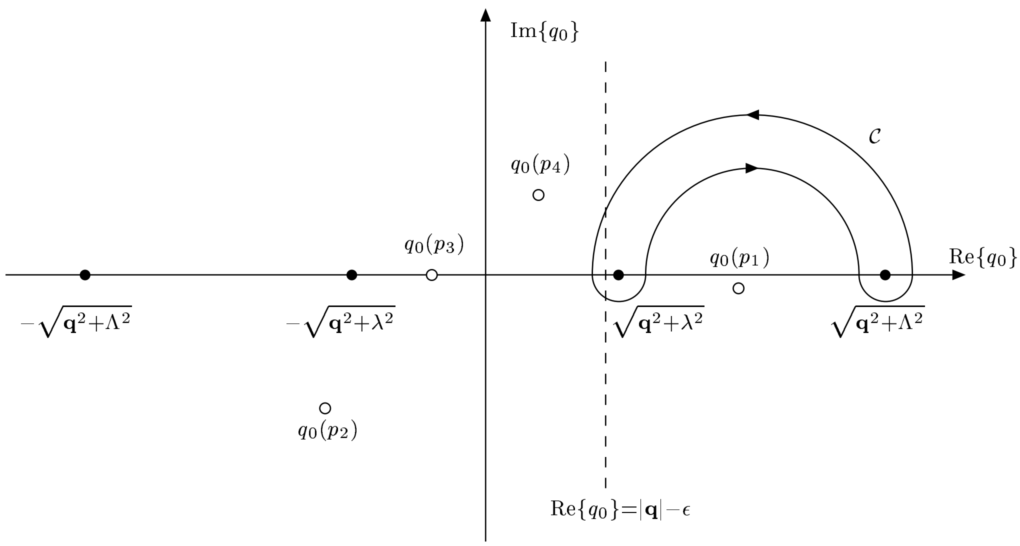

Figure 5: Illustration of the contour $\mathcal{C}$ of (3.17) in the complex q_0 plane. The open circles indicate the "particle poles" located at $q_0(p_i) = (2\mathbf{q}\mathbf{p}_i - p_i^2 + m_i^2)/(2p_{i0})$.

following we only need differences of 3-point functions that are independent of ΔE and Lorentz-invariant.

Because of the appearance of UV-singular integrals in intermediate steps, the reasoning of the previous section cannot directly be applied to $\mathcal{E}_0$. Therefore, we rewrite the real 5-point function as an integral over a closed anticlockwise contour $\mathcal{C}$ in the q_0 plane and introduce a Lorentz-invariant UV regulator Λ:

$$\mathcal{E}_0(p_1,p_2,p_3,p_4,\lambda,m_1,m_2,m_3,m_4) = \lim_{\Lambda\to\infty} \frac{1}{\mathrm{i}\pi^2}\int_{\mathcal{C}} \mathrm{d}^4q\, \frac{1}{N_0'N_1'\cdots N_4'}\frac{-\Lambda^2}{q^2-\Lambda^2} \tag{3.17}$$

with

$$N_0' = q^2 - \lambda^2. \tag{3.18}$$

The contour $\mathcal{C}$ is chosen such that it includes the poles at $q_0 = \sqrt{\mathbf{q}^2+\lambda^2}$ and $q_0 = \sqrt{\mathbf{q}^2+\Lambda^2}$, but none else (see Fig. 5)[2].

The integral (3.17) can be reduced similarly to the virtual 5-point function. Owing to the different propagators N_i', (3.3) leads to

$$0 = \lim_{\Lambda\to\infty} \frac{1}{\mathrm{i}\pi^2}\int_{\mathcal{C}} \mathrm{d}^4q\, \frac{1}{N_0'N_1'\cdots N_4'}\frac{-\Lambda^2}{q^2-\Lambda^2} \begin{vmatrix} 2N_0' & 2qp_1 & \ldots & 2qp_4 \\ N_1'+Y_{10} & 2p_1^2 & \ldots & 2p_1p_4 \\ \vdots & \vdots & \ddots & \vdots \\ N_4'+Y_{40} & 2p_4p_1 & \ldots & 2p_4^2 \end{vmatrix} \tag{3.19}$$

[2] It is straightforward to check that the naive power counting for the UV behaviour in (3.17) is valid for time-like momenta p_i; light-like p_i can be treated as a limiting case. The basic idea for the proof is to deform the contour $\mathcal{C}$ to the vertical line $\mathrm{Re}\{q_0\} = |\mathbf{q}| - \epsilon$ with a small $\epsilon > 0$, which is allowed for sufficiently large $|\mathbf{q}|$, more precisely when all particle poles appear left from the line $\mathrm{Re}\{q_0\} = |\mathbf{q}| - \epsilon$.

instead of (3.5), with Y_{ij} from (3.4), and λ^2 can be set to zero in all Y_{ij}, in particular we have $Y_{00} = 2\lambda^2 \to 0$. After expanding the determinant along the first column and using the Lorentz decompositions of the integrals where N'_k is cancelled, we see that these terms vanish, and we are left with

$$0 = \lim_{\Lambda\to\infty} \frac{1}{\mathrm{i}\pi^2} \int_{\mathcal{C}} \mathrm{d}^4 q \, \frac{1}{N'_0 N'_1 \cdots N'_4} \frac{-\Lambda^2}{q^2 - \Lambda^2} \begin{vmatrix} 2N'_0 & 2qp_1 & \dots & 2qp_4 \\ Y_{10} & 2p_1^2 & \dots & 2p_1 p_4 \\ \vdots & \vdots & \ddots & \vdots \\ Y_{40} & 2p_4 p_1 & \dots & 2p_4^2 \end{vmatrix} . \tag{3.20}$$

Using

$$\begin{aligned} 2p_i p_j &= Y_{ij} - Y_{i0} - Y_{0j}, \\ 2qp_j &= N'_j + Y_{0j}, \end{aligned} \tag{3.21}$$

adding the first column to the other columns and extending the determinant leads to

$$0 = \lim_{\Lambda\to\infty} \frac{1}{\mathrm{i}\pi^2} \int_{\mathcal{C}} \mathrm{d}^4 q \, \frac{1}{N'_0 N'_1 \cdots N'_4} \frac{-\Lambda^2}{q^2 - \Lambda^2} \begin{vmatrix} 1 & 0 & Y_{01} & \dots & Y_{04} \\ 0 & 2N'_0 & N'_1 + 2N'_0 + Y_{01} & \dots & N'_4 + 2N'_0 + Y_{04} \\ 0 & Y_{10} & Y_{11} - Y_{01} & \dots & Y_{14} - Y_{04} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & Y_{40} & Y_{41} - Y_{01} & \dots & Y_{44} - Y_{04} \end{vmatrix} . \tag{3.22}$$

Subtracting the first row from the second, adding the first row to the other rows, and exchanging the first two rows, we arrive at

$$0 = \lim_{\Lambda\to\infty} \frac{1}{\mathrm{i}\pi^2} \int_{\mathcal{C}} \mathrm{d}^4 q \, \frac{1}{N'_0 N'_1 \cdots N'_4} \frac{-\Lambda^2}{q^2 - \Lambda^2} \begin{vmatrix} -1 & 2N'_0 & N'_1 + 2N'_0 & \dots & N'_4 + 2N'_0 \\ 1 & 0 & Y_{01} & \dots & Y_{04} \\ 1 & Y_{10} & Y_{11} & \dots & Y_{14} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & Y_{40} & Y_{41} & \dots & Y_{44} \end{vmatrix} . \tag{3.23}$$

Now we perform the contour integral using power counting for $\Lambda \to \infty$ (see footnote 2). In the contribution of the pole at $q_0 = \sqrt{\mathbf{q}^2 + \lambda^2}$ we have $N'_0 = 0$ in the numerator; the limit $\Lambda \to \infty$ can be trivially taken and reproduces usual bremsstrahlung integrals, as defined in (3.15). In the contribution of the pole at $q_0 = \sqrt{\mathbf{q}^2 + \Lambda^2}$ the term containing N'_0 in the numerator survives and will be calculated below, but all other terms vanish after taking the limit $\Lambda \to \infty$.

Thus, we find

$$0 = \begin{vmatrix} -\mathcal{E}_0 & \tilde{\mathcal{D}}_0(0) & \mathcal{D}_0(1)+\tilde{\mathcal{D}}_0(0) & \mathcal{D}_0(2)+\tilde{\mathcal{D}}_0(0) & \mathcal{D}_0(3)+\tilde{\mathcal{D}}_0(0) & \mathcal{D}_0(4)+\tilde{\mathcal{D}}_0(0) \\ 1 & Y_{00} & Y_{01} & Y_{02} & Y_{03} & Y_{04} \\ 1 & Y_{10} & Y_{11} & Y_{12} & Y_{13} & Y_{14} \\ 1 & Y_{20} & Y_{21} & Y_{22} & Y_{23} & Y_{24} \\ 1 & Y_{30} & Y_{31} & Y_{32} & Y_{33} & Y_{34} \\ 1 & Y_{40} & Y_{41} & Y_{42} & Y_{43} & Y_{44} \end{vmatrix}, \tag{3.24}$$

or

$$\mathcal{E}_0 = \frac{1}{\det(Y)} \left\{ \det(Y_0)\,\tilde{\mathcal{D}}_0(0) + \sum_{i=1}^{4} \det(Y_i)\,\left[\mathcal{D}_0(i) + \tilde{\mathcal{D}}_0(0)\right] \right\}. \tag{3.25}$$

Here

$$\tilde{\mathcal{D}}_0(0) = \lim_{\Lambda\to\infty} \frac{2}{\mathrm{i}\pi^2} \int_{\mathcal{C}} \mathrm{d}^4 q \, \frac{1}{N_1' \cdots N_4'} \frac{-\Lambda^2}{q^2 - \Lambda^2}, \tag{3.26}$$

and the 4-point bremsstrahlung integrals $\mathcal{D}_0(i)$, $i = 1,2,3,4$, result from $\mathcal{E}_0$ by omitting the ith denominator N_i'. The result (3.24) differs from (3.10) only by the extra $\tilde{\mathcal{D}}_0(0)$'s added to the $\mathcal{D}_0(i)$'s.

The integral $\tilde{\mathcal{D}}_0(0)$ stems from the terms involving N_0' in the numerator in (3.23) and can be expressed as follows:

$$\tilde{\mathcal{D}}_0(0) = \frac{2}{\mathrm{i}\pi^2} \int_{\mathcal{C}'} \mathrm{d}^4 q' \, \frac{1}{2q'p_1 \cdots 2q'p_4} \frac{-1}{q'^2 - 1}, \tag{3.27}$$

where the contour $\mathcal{C}'$ surrounds $q_0' = \sqrt{\mathbf{q}'^2 + 1}$. Performing the contour integral over $\mathrm{d}q_0'$ yields

$$\tilde{\mathcal{D}}_0(0) = \frac{4}{\pi} \int \frac{\mathrm{d}^3\mathbf{q}'}{2q_0'} \, \frac{1}{2q'p_1 \cdots 2q'p_4} \bigg|_{q_0' = \sqrt{\mathbf{q}'^2+1}}. \tag{3.28}$$

Now the vector q' is time-like. Since also the vectors p_i are time-like (or at least light-like), the scalar products $q'p_i$ cannot become zero. After redefining the momenta,

$$p_i = \sigma_i \tilde{p}_i, \qquad \sigma_i = \pm 1, \qquad \tilde{p}_{i0} > 0, \tag{3.29}$$

and extracting the signs σ_i, this integral can be evaluated by a Feynman-parameter representation and momentum integration in polar coordinates resulting in:

$$\tilde{\mathcal{D}}_0(0) = -\sigma_1\sigma_2\sigma_3\sigma_4 \int_0^\infty \mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\,\mathrm{d}x_4\,\delta\Big(1 - \sum_{i=1}^4 x_i\Big) \left[\Big(\sum_{i=1}^4 x_i \tilde{p}_i\Big)^2\right]^{-2}. \tag{3.30}$$

This is just the Feynman-parameter representation of a virtual 4-point function such that we finally obtain

$$\tilde{\mathcal{D}}_0(0) = -\sigma_1\sigma_2\sigma_3\sigma_4 D_0\left(\tilde{p}_2 - \tilde{p}_1, \tilde{p}_3 - \tilde{p}_1, \tilde{p}_4 - \tilde{p}_1, \sqrt{p_1^2}, \sqrt{p_2^2}, \sqrt{p_3^2}, \sqrt{p_4^2}\right). \tag{3.31}$$

3.1.3 Explicit reduction of the virtual 5-point function for the photon exchange between $\bar{f}_2$ and f_3

For the photon exchange between $\bar{f}_2$ and f_3 the following scalar integrals are relevant:

$$\begin{aligned}
E_0 &= E_0(-k_3, -k_-, k_+, k_2, \lambda, m_3, M, M, m_2),\\
D_0(0) &= D_0(-k_4, k_+ + k_3, k_2 + k_3, 0, M, M, 0),\\
D_0(1) &= D_0(-k_-, k_+, k_2, 0, M, M, m_2),\\
D_0(2) &= D_0(-k_3, k_+, k_2, \lambda, m_3, M, m_2),\\
D_0(3) &= D_0(-k_3, -k_-, k_2, \lambda, m_3, M, m_2),\\
D_0(4) &= D_0(-k_3, -k_-, k_+, 0, m_3, M, M).
\end{aligned} \tag{3.32}$$

Since we neglect the external fermion masses, the last two relations (3.14) simplify to

$$\begin{aligned}
0 &= (s_{23} + s_{24}) \det(Y) + K_- s_{23} \det(Y_1) - \left[K_+(s_{23} + s_{24}) + K_- M_+^2\right] \det(Y_3),\\
0 &= (s_{13} + s_{23}) \det(Y) + K_+ s_{23} \det(Y_4) - \left[K_-(s_{13} + s_{23}) + K_+ M_-^2\right] \det(Y_2).
\end{aligned} \tag{3.33}$$

These relations allow us to eliminate $\det(Y_1)$ and $\det(Y_4)$ from (3.12), resulting in:

$$\begin{aligned}
E_0(-k_3, -k_-, k_+, k_2, \lambda, m_3, M, M, m_2) &= \frac{\det(Y_0)}{\det(Y)} D_0(0)\\
&+ \frac{\det(Y_3)}{\det(Y) K_- s_{23}} \Big\{[K_+(s_{23} + s_{24}) + K_- M_+^2] D_0(1) + K_- s_{23} D_0(3)\Big\}\\
&+ \frac{\det(Y_2)}{\det(Y) K_+ s_{23}} \Big\{[K_-(s_{13} + s_{23}) + K_+ M_-^2] D_0(4) + K_+ s_{23} D_0(2)\Big\}\\
&- \frac{s_{13} + s_{23}}{K_+ s_{23}} D_0(4) - \frac{s_{23} + s_{24}}{K_- s_{23}} D_0(1).
\end{aligned} \tag{3.34}$$

The matrix Y reads

$$Y = \begin{pmatrix}
0 & 0 & -K_- & -K_+ & 0\\
* & 0 & M^2 & (-K_+ - s_{13} - s_{23}) & -s_{23}\\
* & * & 2M^2 & 2M^2 - s & (-K_- - s_{23} - s_{24})\\
* & * & * & 2M^2 & M^2\\
* & * & * & * & 0
\end{pmatrix}. \tag{3.35}$$

Neglecting terms that do not contribute to the correction factor in DPA, the corresponding determinants are given by

$$\begin{aligned}
\det(Y) &\sim 2 s_{23} \left[K_+ K_- s_{14} s_{23} - (K_+ M_\mathrm{W}^2 + K_- s_{13})(K_- M_\mathrm{W}^2 + K_+ s_{24})\right],\\
\det(Y_0) &\sim -\kappa_\mathrm{W}^2,\\
\det(Y_1) &\sim K_+ \left[M_\mathrm{W}^4 (s_{23} - s_{24}) + (s_{23} + s_{24})(s_{13} s_{24} - s_{14} s_{23})\right]\\
&\quad + K_- M_\mathrm{W}^2 (-M_\mathrm{W}^4 + 2 s_{13} s_{23} + s_{13} s_{24} + s_{14} s_{23}),
\end{aligned}$$

$$
\begin{aligned}
\det(Y_2) &\sim -s_{23}\left[K_+(M_W^4+s_{13}s_{24}-s_{14}s_{23})+2K_-M_W^2 s_{13}\right],\\
\det(Y_3) &= \det(Y_2)|_{K_+\leftrightarrow K_-,s_{13}\leftrightarrow s_{24}},\\
\det(Y_4) &= \det(Y_1)|_{K_+\leftrightarrow K_-,s_{13}\leftrightarrow s_{24}},
\end{aligned}
\tag{3.36}
$$

where the shorthand κ_W is defined in Appendix A.

3.1.4 Explicit reduction of the real 5-point function for the photon exchange between $\bar{f}_2$ and f_3

In this case the integrals appearing in (3.25) read

$$
\begin{aligned}
\mathcal{E}_0 &= \mathcal{E}_0(k_3,k_-,k_+,k_2,\lambda,m_3,M^*,M,m_2),\\
\tilde{\mathcal{D}}_0(0) &= -D_0(k_4,k_+-k_3,k_2-k_3,0,M_-,M_+,0),\\
\mathcal{D}_0(1) &= \mathcal{D}_0(k_-,k_+,k_2,0,M^*,M,m_2),\\
\mathcal{D}_0(2) &= \mathcal{D}_0(k_3,k_+,k_2,\lambda,m_3,M,m_2),\\
\mathcal{D}_0(3) &= \mathcal{D}_0(k_3,k_-,k_2,\lambda,m_3,M^*,m_2),\\
\mathcal{D}_0(4) &= \mathcal{D}_0(k_3,k_-,k_+,0,m_3,M^*,M).
\end{aligned}
\tag{3.37}
$$

In analogy to the virtual 5-point function, we can express the real 5-point function with the help of the relations (we denote the matrix Y for the real 5-point function with a prime, in order to distinguish it from the one for the virtual 5-point function):

$$
\begin{aligned}
0 &= (s_{23}+s_{24})\det(Y')+K_-^*s_{23}\det(Y_1')-\left[K_+(s_{23}+s_{24})-K_-^*M_+^2\right]\det(Y_3'),\\
0 &= (s_{13}+s_{23})\det(Y')+K_+s_{23}\det(Y_4')-\left[K_-^*(s_{13}+s_{23})-K_+M_-^2\right]\det(Y_2'),
\end{aligned}
\tag{3.38}
$$

by

$$
\begin{aligned}
&\mathcal{E}_0(k_3,k_-,k_+,k_2,\lambda,m_3,M^*,M,m_2) = \frac{\det(Y_0')}{\det(Y')}\tilde{\mathcal{D}}_0(0)\\
&+\frac{\det(Y_3')}{\det(Y')K_-^*s_{23}}\Big\{\left[K_+(s_{23}+s_{24})-K_-^*M_+^2\right]\left[\mathcal{D}_0(1)+\tilde{\mathcal{D}}_0(0)\right]+K_-^*s_{23}\left[\mathcal{D}_0(3)+\tilde{\mathcal{D}}_0(0)\right]\Big\}\\
&+\frac{\det(Y_2')}{\det(Y')K_+s_{23}}\Big\{\left[K_-^*(s_{13}+s_{23})-K_+M_-^2\right]\left[\mathcal{D}_0(4)+\tilde{\mathcal{D}}_0(0)\right]+K_+s_{23}\left[\mathcal{D}_0(2)+\tilde{\mathcal{D}}_0(0)\right]\Big\}\\
&-\frac{s_{13}+s_{23}}{K_+s_{23}}\left[\mathcal{D}_0(4)+\tilde{\mathcal{D}}_0(0)\right]-\frac{s_{23}+s_{24}}{K_-^*s_{23}}\left[\mathcal{D}_0(1)+\tilde{\mathcal{D}}_0(0)\right].
\end{aligned}
\tag{3.39}
$$

For the matrix Y' we find

$$
Y' = \begin{pmatrix}
0 & 0 & -K_-^* & -K_+ & 0\\
* & 0 & (M^*)^2 & (-K_++s_{13}+s_{23}) & s_{23}\\
* & * & 2(M^*)^2 & (-2K_+-2K_-^*+s-(M^*)^2-M^2) & (-K_-^*+s_{23}+s_{24})\\
* & * & * & 2M^2 & M^2\\
* & * & * & * & 0
\end{pmatrix}.
\tag{3.40}
$$

Replacing $-K_-^*$ by K_- and $(M^*)^2$ by M^2 and multiplying the second and third columns and rows by -1, this becomes equal to (3.35) in DPA.

In DPA, $\tilde{\mathcal{D}}_0(0)$ can be neglected in the terms $\mathcal{D}_0(i) + \tilde{\mathcal{D}}_0(0)$ in (3.25) and (3.39), and the reduction of the real 5-point function becomes algebraically identical to the reduction of the virtual 5-point function, apart from the differences in signs of some momenta. As a consequence, the results for the virtual corrections can be translated to the real case if we substitute $K_- \to -K_-^*$ in all algebraic factors such as the determinants and $E_0 \to \mathcal{E}_0$, $D_0(0) \to \tilde{\mathcal{D}}_0(0)$, $D_0(1) \to -\mathcal{D}_0(1)$, $D_0(2) \to -\mathcal{D}_0(2)$, $D_0(3) \to \mathcal{D}_0(3)$ and $D_0(4) \to \mathcal{D}_0(4)$. In particular, the determinants are related by

$$\begin{aligned}
\det(Y') &\sim +\det(Y)\Big|_{K_-\to -K_-^*}, &\qquad \det(Y_0') &\sim +\det(Y_0) \sim -\kappa_{\mathrm{W}}^2, \\
\det(Y_1') &\sim -\det(Y_1)\Big|_{K_-\to -K_-^*}, &\qquad \det(Y_2') &\sim -\det(Y_2)\Big|_{K_-\to -K_-^*}, \\
\det(Y_3') &\sim +\det(Y_3)\Big|_{K_-\to -K_-^*}, &\qquad \det(Y_4') &\sim +\det(Y_4)\Big|_{K_-\to -K_-^*}.
\end{aligned} \tag{3.41}$$

3.2 Calculation of 4- and 3-point functions

The scalar loop integrals have been evaluated following the methods of Ref. [11]. Our explicit results are listed in Appendix C.1. For vanishing W-boson width they agree with the general results of Ref. [11]. For finite W-boson width the virtual 4-point functions are in agreement with those of Refs. [9, 10] in DPA. This shows explicitly that the q^2 terms in the W-boson and fermion propagators are irrelevant in DPA.

An evaluation of the bremsstrahlung integrals, which follows closely the techniques for calculating loop integrals, is sketched in Appendix B. The final results in DPA are listed in Appendix C.2, and the 4-point functions agree with those of Refs. [9, 10]. We have analytically reproduced all exact results for the occurring bremsstrahlung 4- and 3-point integrals by independent methods. In addition, we have evaluated the IR-finite integrals $\mathcal{D}_0(1)$, $\mathcal{D}_0(4)$, and $\mathcal{E}_0 - \mathcal{D}_0(3)/K_+$ by a direct Monte Carlo integration over the photon momentum, yielding perfect agreement with our exact analytical results for these integrals. Note that this, in particular, checks the reduction of the bremsstrahlung 5-point function described in the previous section.

Our results for the 3-point functions cannot directly be compared with those of Refs. [9, 10], because different approaches have been used. While our results are IR-singular owing to the subtracted on-shell integrals, the results of Refs. [9, 10] are artificially UV-singular owing to the neglect of q^2 in the W propagators. However, when adding the real and virtual 3-point functions the two results agree. This confirms that our definition of the non-factorizable corrections is equivalent to the one of Refs. [9, 10] in DPA.

Thus, it turns out that in DPA the subtraction of the on-shell contribution is effectively equivalent to the neglect of the q^2 terms in all but the photon propagators.

4 Analytic results for the non-factorizable corrections

4.1 General properties of non-factorizable corrections

In Ref. [8] it was shown from the integral representation that the non-factorizable corrections associated with photon exchange between initial and final state vanish in DPA. This was confirmed in Refs. [9, 10]. Via explicit evaluation of all integrals we have checked that the cancellation between virtual and real integrals takes place diagram by diagram once the factorizable contributions are subtracted. In this way all interference terms (if), (mf), (im), and (mm) drop out. Examples for the virtual Feynman diagrams contributing to these types of corrections are shown in Fig. 2.

The only non-vanishing non-factorizable corrections are due to the contributions (ff′), (mf′), and (mm′). The corresponding virtual diagrams are shown in Fig. 1, apart from permutations of the final-state fermions. Two of the corresponding real diagrams are pictured in Figs. 3 and 4. Since these corrections depend only on s-channel invariants, the non-factorizable corrections are independent of the production angle of the W bosons, as was also pointed out in Refs. [9, 10].

4.2 Generic form of the correction factor

The non-factorizable corrections $d\sigma_{\rm nf}$ to the fully differential lowest-order cross-section $d\sigma_{\rm Born}$ resulting from the matrix element (2.7) take the form of a correction factor to the lowest-order cross-section:

$$d\sigma_{\rm nf} = \delta_{\rm nf}\, d\sigma_{\rm Born}. \tag{4.1}$$

Upon splitting the contributions that result from photons coupled to the W bosons according to $1 = Q_{\rm W^+} = Q_1 - Q_2$ and $1 = -Q_{\rm W^-} = Q_4 - Q_3$ into contributions associated with definite final-state fermions, the complete correction factor to the lowest-order cross-section can be written as

$$\delta_{\rm nf} = \sum_{a=1,2}\sum_{b=3,4} (-1)^{a+b+1}\, Q_a Q_b\, \frac{\alpha}{\pi}\, {\rm Re}\{\Delta(k_+,k_a;k_-,k_b)\}. \tag{4.2}$$

In the following, only $\Delta = \Delta(k_+,k_2;k_-,k_3)$ is given; the other terms follow by obvious substitutions. As discussed above, Δ gets contributions from *intermediate–intermediate* ($\Delta_{\rm mm'}$), *intermediate–final* ($\Delta_{\rm mf'}$), and *final–final* ($\Delta_{\rm ff'}$) interactions:

$$\Delta = \Delta_{\rm mm'} + \Delta_{\rm mf'} + \Delta_{\rm ff'}. \tag{4.3}$$

The quantity ($\Delta_{\rm mm'}$ is independent of the final-state fermions. The individual contributions read

$$\begin{aligned}\Delta_{\rm mm'} \sim (2M_{\rm W}^2 - s)\Big\{ & C_0(k_+,-k_-,0,M,M) - \Big[C_0(k_+,-k_-,\lambda,M_{\rm W},M_{\rm W})\Big]_{k_\pm^2=M_{\rm W}^2}\\ & - \mathcal{C}_0(k_+,k_-,0,M,M^*) + \Big[\mathcal{C}_0(k_+,k_-,\lambda,M_{\rm W},M_{\rm W})\Big]_{k_\pm^2=M_{\rm W}^2}\Big\},\end{aligned} \tag{4.4}$$

$$\Delta_{\rm mf'} \sim -(s_{23}+s_{24})\Big[K_+D_0(1) - K_+\mathcal{D}_0(1)\Big] - (s_{13}+s_{23})\Big[K_-D_0(4) - K_-^*\mathcal{D}_0(4)\Big], \tag{4.5}$$

$$\Delta_{\rm ff'} \sim -K_+ s_{23}\Big[K_-E_0 - K_-^*\mathcal{E}_0\Big], \tag{4.6}$$

with the arguments of the 5- and 4-point functions as defined in (3.32) and (3.37).

The sum $\Delta_{\rm mf'} + \Delta_{\rm ff'}$ can be simplified by inserting the decompositions of the 5-point functions (3.34) and (3.39). In DPA this leads to

$$\begin{aligned}\Delta_{\rm mf'} + \Delta_{\rm ff'} \sim & -\frac{K_+K_-s_{23}\det(Y_0)}{\det(Y)}D_0(0) + \frac{K_+K_-^*s_{23}\det(Y_0')}{\det(Y')}\tilde{\mathcal{D}}_0(0)\\ & - \frac{K_+\det(Y_3)}{\det(Y)}\Big\{[K_+(s_{23}+s_{24}) + K_-M_{\rm W}^2]D_0(1) + K_-s_{23}D_0(3)\Big\}\\ & - \frac{K_-\det(Y_2)}{\det(Y)}\Big\{[K_-(s_{13}+s_{23}) + K_+M_{\rm W}^2]D_0(4) + K_+s_{23}D_0(2)\Big\}\\ & + \frac{K_+\det(Y_3')}{\det(Y')}\Big\{[K_+(s_{23}+s_{24}) - K_-^*M_{\rm W}^2]\mathcal{D}_0(1) + K_-^*s_{23}\mathcal{D}_0(3)\Big\}\\ & + \frac{K_-^*\det(Y_2')}{\det(Y')}\Big\{[K_-^*(s_{13}+s_{23}) - K_+M_{\rm W}^2]\mathcal{D}_0(4) + K_+s_{23}\mathcal{D}_0(2)\Big\}.\end{aligned} \tag{4.7}$$

Note that $\Delta_{\rm mf'}$ is exactly cancelled by the contributions of the last two terms in (3.34) and (3.39).

Inserting the expressions for the scalar integrals from Appendix C and using the first of the relations (3.14), various terms, notably all IR divergences and mass-singular logarithms, cancel between the real and virtual corrections, and in DPA we are left with

$$\begin{aligned}\Delta_{\mathrm{mm'}} \sim \frac{2M_\mathrm{W}^2 - s}{s\beta_\mathrm{W}}\Big[&-\mathcal{L}i_2\Big(\frac{K_-}{K_+}, x_\mathrm{W}\Big) + \mathcal{L}i_2\Big(\frac{K_-}{K_+}, x_\mathrm{W}^{-1}\Big) + \mathcal{L}i_2\Big(-\frac{K_-^*}{K_+}, x_\mathrm{W}\Big)\\ &- \mathcal{L}i_2\Big(-\frac{K_-^*}{K_+}, x_\mathrm{W}^{-1}\Big) + 2\pi\mathrm{i}\ln\Big(\frac{K_+ + K_-^* x_\mathrm{W}}{\mathrm{i}M_\mathrm{W}^2}\Big)\Big] \;+\; \text{imaginary parts}, \qquad (4.8)\end{aligned}$$

$$\begin{aligned}\Delta_{\mathrm{mf'}} + \Delta_{\mathrm{ff'}} \sim &-\frac{K_+K_-s_{23}\det(Y_0)}{\det(Y)}D_0(0) - \frac{K_+\det(Y_3)}{\det(Y)}F_3 - \frac{K_-\det(Y_2)}{\det(Y)}F_2\\ &+ \frac{K_+K_-^*s_{23}\det(Y_0')}{\det(Y')}\tilde{\mathcal{D}}_0(0) + \frac{K_+\det(Y_3')}{\det(Y')}\mathcal{F}_3 + \frac{K_-^*\det(Y_2')}{\det(Y')}\mathcal{F}_2\\ &+ \text{imaginary parts}, \qquad (4.9)\end{aligned}$$

with $D_0(0)$ and $\tilde{\mathcal{D}}_0(0)$ given in (C.3) and (C.10), respectively, and

$$\begin{aligned}F_3 =& -2\,\mathcal{L}i_2\Big(\frac{K_+}{K_-}, -\frac{s_{23}+s_{24}}{M_W^2} - \mathrm{i}\epsilon\Big) + \sum_{\tau=\pm1}\Big[\mathcal{L}i_2\Big(\frac{K_+}{K_-}, x_\mathrm{W}^\tau\Big) - \mathcal{L}i_2\Big(-\frac{M_W^2}{s_{23}+s_{24}} + \mathrm{i}\epsilon, x_\mathrm{W}^\tau\Big)\Big]\\ &- \mathrm{Li}_2\Big(-\frac{s_{24}}{s_{23}}\Big) + 2\ln\Big(-\frac{s_{23}}{M_W^2} - \mathrm{i}\epsilon\Big)\ln\Big(-\frac{K_-}{M_W^2}\Big) - \ln^2\Big(-\frac{s_{23}+s_{24}}{M_W^2} - \mathrm{i}\epsilon\Big),\\ \mathcal{F}_3 =& F_3\Big|_{K_-\to -K_-^*} + 2\pi\mathrm{i}\Big[2\ln\Big(1 - \frac{K_+}{K_-^*}\frac{s_{23}+s_{24}}{M_\mathrm{W}^2}\Big) - \ln\Big(1 + \frac{K_+}{K_-^*x_\mathrm{W}}\Big) - \ln\Big(1 + \frac{x_\mathrm{W}M_\mathrm{W}^2}{s_{23}+s_{24}}\Big)\\ &+ \ln\Big(\frac{\mathrm{i}K_-^*}{s_{23}+s_{24}}\Big)\Big],\\ F_2 =& F_3\Big|_{K_+\leftrightarrow K_-, s_{24}\leftrightarrow s_{13}},\\ \mathcal{F}_2 =& F_2\Big|_{K_-\to -K_-^*} + 2\pi\mathrm{i}\Big\{\ln\Big(1 + \frac{K_-^*x_\mathrm{W}}{K_+}\Big) - \ln\Big(1 + \frac{x_\mathrm{W}M_\mathrm{W}^2}{s_{13}+s_{23}}\Big) + \ln\Big[\frac{K_+s_{23}}{\mathrm{i}M_\mathrm{W}^2(s_{13}+s_{23})}\Big]\Big\}.\end{aligned} \qquad (4.10)$$

The variables β_W, x_W and the dilogarithms Li_2, $\mathcal{L}i_2$ are defined in Appendix A.

The above results contain logarithms and dilogarithms the arguments of which depend on $K_\pm$. It is interesting to see whether enhanced logarithms of the form $\ln(K_\pm/M_{\mathrm{W}}^2)$ appear in the limits $K_\pm \to 0$. It turns out that such logarithms are absent from the non-factorizable corrections, irrespective of the ratio in which the two limits $K_\pm \to 0$ are realized.

4.3 Inclusion of the exact off-shell Coulomb singularity

For non-relativistic W bosons, i.e. for a small on-shell velocity β_{W}, the long range of the Coulomb interaction leads to a large radiative correction, known as the Coulomb singularity. For on-shell W bosons, this correction behaves like $1/\beta_{\mathrm{W}}$ near threshold, but including the instability of the W bosons the long-range interaction is effectively truncated, and the $1/\beta_{\mathrm{W}}$ singularity is regularized. Therefore, for realistic predictions in the threshold region, the on-shell Coulomb singularity should be replaced by the corresponding off-shell correction. The precise form of this off-shell Coulomb singularity [14] reveals that corrections of some per cent occur even a few W-decay widths above threshold. As explained above, the DPA becomes valid only several widths above threshold. Nevertheless, there exists an overlapping region in which the inclusion of the Coulomb singularity within the double-pole approach is reasonable.

The Feynman graph relevant for the Coulomb singularity is diagram (d) of Fig. 1. The non-factorizable corrections contain just the difference between the off-shell and the on-shell contributions of this diagram. Therefore, the difference between off-shell and on-shell Coulomb singularity is in principle included in $\Delta_{\mathrm{mm'}}$, as defined in (4.4). The genuine form of $\Delta_{\mathrm{mm'}}$ in DPA, which is given by (4.8), does not contain the full effect of the Coulomb singularity, because in both C_0 functions of (4.4) the on-shell limit $K_\pm \to 0$ was taken under the assumption of a finite β_{W}. In order to include the correct difference between off-shell and on-shell Coulomb singularity in $\Delta_{\mathrm{mm'}}$, the on-shell limit of the C_0 functions of (4.4) has to be taken for arbitrary β_{W}. The full off-shell Coulomb singularity can be included by adding

$$\frac{2M_{\mathrm{W}}^2 - s}{s}\left[\frac{2\pi\mathrm{i}}{\bar\beta}\ln\left(\frac{\beta+\Delta_M-\bar\beta}{\beta+\Delta_M+\bar\beta}\right) - \frac{2\pi\mathrm{i}}{\beta_{\mathrm{W}}}\ln\left(\frac{K_+ + K_- + s\beta_{\mathrm{W}}\Delta_M}{2\beta_{\mathrm{W}}^2 s}\right)\right] \tag{4.11}$$

to the genuine DPA result (4.8) for $\Delta_{\mathrm{mm'}}$. The quantities β, $\bar\beta$, and Δ_M are defined in (A.1). After combination with the factorizable doubly-resonant corrections, all doubly-resonant corrections and the correct off-shell Coulomb singularity are included. The on-shell Coulomb singularity contained in the factorizable corrections is compensated by a corresponding contribution in the non-factorizable ones. Note, however, that this subtracted on-shell Coulomb singularity appears as an artefact if the non-factorizable corrections are discussed separately.

4.4 Non-local cancellations

In Ref. [7] it was pointed out that the non-factorizable photonic corrections completely cancel in DPA if the phase-space integration over both invariant masses of the W bosons is performed. This cancellation is due to a symmetry of the non-factorizable corrections.

The lowest-order cross section in DPA is symmetric with respect to the “reflections”

$$\begin{aligned} K_+ &= (k_+^2 - M_\mathrm{W}^2) + \mathrm{i} M_\mathrm{W}\Gamma_\mathrm{W} \;\leftrightarrow\; -(k_+^2 - M_\mathrm{W}^2) + \mathrm{i} M_\mathrm{W}\Gamma_\mathrm{W} = -K_+^*, \\ K_- &= (k_-^2 - M_\mathrm{W}^2) + \mathrm{i} M_\mathrm{W}\Gamma_\mathrm{W} \;\leftrightarrow\; -(k_-^2 - M_\mathrm{W}^2) + \mathrm{i} M_\mathrm{W}\Gamma_\mathrm{W} = -K_-^*. \end{aligned} \qquad (4.12)$$

Therefore, Δ can be symmetrized in $K_+ \to -K_+^*$ or $K_- \to -K_-^*$ if the respective invariant mass is integrated out. For instance, if k_-^2 is integrated out, Δ can be replaced by

$$\begin{aligned} &\frac{1}{2}\left(\Delta + \Delta\Big|_{K_- \leftrightarrow -K_-^*}\right) \\ &\sim\; \mathrm{i}\pi \Bigg\{ \frac{s - 2M_\mathrm{W}^2}{\beta_\mathrm{W} s} \ln\left(\frac{K_- x_\mathrm{W} + K_+^*}{K_- x_\mathrm{W} - K_+}\right) \\ &\qquad + K_+ s_{23} \kappa_\mathrm{W} \left[\frac{K_-}{\det(Y)} - \frac{K_-^*}{\det(Y')}\right] \left[\ln\left(-x_\mathrm{W}\frac{s_{23}}{M_\mathrm{W}^2}\right) + \ln\left(1 + \frac{s_{13}}{s_{23}}(1-z)\right) \right. \\ &\qquad\qquad \left. + \ln\left(1 + \frac{s_{24}}{s_{23}}(1-z)\right) - \ln\left(1 + \frac{s_{13}}{M_\mathrm{W}^2} z x_\mathrm{W}\right) - \ln\left(1 + \frac{s_{24}}{M_\mathrm{W}^2} z x_\mathrm{W}\right)\right] \\ &\qquad - \left[\frac{K_- \det(Y_2)}{\det(Y)} + \frac{K_-^* \det(Y_2')}{\det(Y')}\right] \left[\ln\left(x_\mathrm{W} + \frac{s_{13}+s_{23}}{M_\mathrm{W}^2}\right) - \ln\left(-x_\mathrm{W}\frac{s_{23}}{M_\mathrm{W}^2}\right)\right] \\ &\qquad - \frac{K_+ \det(Y_3)}{\det(Y)} \left[\ln\left(x_\mathrm{W} + \frac{s_{23}+s_{24}}{M_\mathrm{W}^2}\right) - 2\ln\left(1 + \frac{K_+}{K_-}\frac{s_{23}+s_{24}}{M_\mathrm{W}^2}\right)\right] \\ &\qquad - \frac{K_+ \det(Y_3')}{\det(Y')} \left[\ln\left(x_\mathrm{W} + \frac{s_{23}+s_{24}}{M_\mathrm{W}^2}\right) - 2\ln\left(1 - \frac{K_+}{K_-^*}\frac{s_{23}+s_{24}}{M_\mathrm{W}^2}\right)\right] \\ &\qquad + \frac{2 s_{23} M_\mathrm{W}^2 (K_+^2 s_{24} - K_-^2 s_{13})}{\det(Y)} \ln\left(1 - \frac{K_+}{K_- x_\mathrm{W}}\right) \\ &\qquad + \frac{2 s_{23} M_\mathrm{W}^2 (K_+^2 s_{24} - K_-^{*2} s_{13})}{\det(Y')} \ln\left(1 + \frac{K_+}{K_-^* x_\mathrm{W}}\right) \Bigg\} \\ &\quad + \text{ imaginary parts}, \end{aligned} \qquad (4.13)$$

with z defined in (C.4). Note that this expression is considerably simpler than the full result for Δ. In particular, all dilogarithms have dropped out.

Symmetrizing (4.13) also in $K_+ \leftrightarrow -K_+^*$ leads to further simplifications if the relations (3.41) for the determinants are used. Under the assumptions that $(s_{13} + s_{23}) > -M_\mathrm{W}^2 x_\mathrm{W}$, $(s_{23} + s_{24}) > -M_\mathrm{W}^2 x_\mathrm{W}$, and that κ_W is imaginary, the real part of the result vanishes. These assumptions are fulfilled on resonance, $k_\pm^2 = M_\mathrm{W}^2$; off resonance, there are regions in phase space where the assumptions are violated. The volume of those regions of phase space is suppressed by factors $|k_\pm^2 - M_\mathrm{W}^2|/M_\mathrm{W}^2$ and thus negligible in DPA. Therefore we can use the above assumptions and find that Δ vanishes in DPA after averaging over the four points in the (k_+^2, k_-^2) plane that are related by the reflections (4.12):

$$\Delta + \Delta\Big|_{K_+ \to -K_+^*} + \Delta\Big|_{K_- \to -K_-^*} + \Delta\Big|_{K_\pm \to -K_\pm^*} \sim 0. \qquad (4.14)$$

This explicitly confirms the results of Ref. [7]. In particular, the non-factorizable corrections vanish in the limit $|k_\pm^2 - M_\mathrm{W}^2| \ll \Gamma_\mathrm{W} M_\mathrm{W}$, i.e. for on-shell W bosons.

The above considerations lead to the following simplified recipe for the non-factorizable corrections to single invariant-mass distributions, i.e. as long as at least one of the invariant masses $k_\pm^2$ is integrated out: the full factor Δ can be replaced according to

$$\Delta \;\to\; \frac{1}{2}\left(\Delta + \Delta\Big|_{K_-\leftrightarrow -K_-^*}\right) + \frac{1}{2}\left(\Delta + \Delta\Big|_{K_+\leftrightarrow -K_+^*}\right), \tag{4.15}$$

where the first term on the r.h.s. is given in (4.13), and the second follows from the first by interchanging $K_+ \leftrightarrow K_-$ and $s_{13} \leftrightarrow s_{24}$. Note that no double counting is introduced, since the first (second) term does not contribute if k_+^2 (k_-^2) is integrated out. In order to introduce the exact Coulomb singularity, one simply has to add the additional term of (4.11) to (4.13) and (4.15).

4.5 Non-factorizable corrections to related processes

Since all non-factorizable corrections involving the initial e^+e^- state cancel, the above results for the correction factor also apply to other W-pair production processes, such as $\gamma\gamma \to \mathrm{WW} \to 4\,$fermions and $q\bar{q} \to \mathrm{WW} \to 4\,$fermions.

The presented analytical results can also be carried over to Z-pair-mediated four-fermion production, $e^+e^- \to \mathrm{ZZ} \to 4\,$fermions. In this case, $\Delta_{\mathrm{ff'}}$ yields the complete non-factorizable correction, where all quantities such as M_W and Γ_W defined for the W bosons are to be replaced by the ones for the Z bosons. The fact that $Q_1 = Q_2$ and $Q_3 = Q_4$ has two important consequences. Firstly, it implies the cancellation of mass singularities contained in $\Delta_{\mathrm{ff'}}$ when all contributions are summed as in (4.2). Secondly, it leads to the antisymmetry of δ_{nf} under each of the interchanges $k_1 \leftrightarrow k_2$ and $k_3 \leftrightarrow k_4$. As was also noted in Refs. [9, 10], the latter implies that the non-factorizable corrections to these processes vanish after integration over all angles except the production angle. It is interesting to note that (4.2) with (4.3) are directly applicable, since $\Delta_{\mathrm{mf'}}$ and $\Delta_{\mathrm{mm'}}$ cancel in the sum of (4.2). Therefore, a practical way to calculate the non-factorizable corrections to $e^+e^- \to \mathrm{ZZ} \to 4\,$fermions consists in taking $\Delta_{\mathrm{ff'}} + \Delta_{\mathrm{mf'}}$ from (4.9) and (4.10), and setting $\Delta_{\mathrm{mm'}}$ to zero.

5 Numerical results

If not stated otherwise, we used the parameters

$$\alpha^{-1} = 137.0359895, \quad M_\mathrm{Z} = 91.187\,\mathrm{GeV}, \quad M_\mathrm{W} = 80.22\,\mathrm{GeV}, \quad \Gamma_\mathrm{W} = 2.08\,\mathrm{GeV}, \tag{5.1}$$

which coincide with those of Ref. [9], for the numerical evaluation.

In order to exclude errors, we have written two independent programs for the correction factor (4.2) and compared all building blocks numerically. These subroutines are implemented in the Monte Carlo generator `EXCALIBUR` [15] as a correction factor to the three (doubly-resonant) W-pair-production signal diagrams. In all numerical results below, only these signal diagrams are included, and no phase-space cuts have been applied. We restrict our numerical analysis to the correction factor for purely leptonic final states, $e^+e^- \to WW \to \nu_\ell \ell^+ \ell'^- \nu_{\ell'}$. Results for other final states can easily be obtained from our formulae and computer programs. As also pointed out in Refs. [9, 10], the relative non-factorizable corrections are the same for all final states after integration over the decay angles.

The results for the figures have been obtained from 50 million phase-space points using the histogram routine of `EXCALIBUR` with 40 bins for each figure. For each entry in the tables, 10 million phase-space points were generated.

5.1 Comparison with existing results

The non-factorizable photonic corrections have already been evaluated by two groups. As was noted in the introduction, the authors of Refs. [9, 10] have not confirmed[3] the results of Ref. [8]. Therefore, we first compare our findings with the results of these two groups.

Melnikov and Yakovlev [8] give the relative non-factorizable corrections only to the completely differential cross section for the process $e^+e^- \to WW \to \nu_e e^+ e^- \bar{\nu}_e$ as a function of the invariant mass M_+ of the $\nu_e e^+$ system for all other phase-space parameters fixed. All momenta of the final-state fermions lie in a plane, and the momenta $\mathbf{k}_2$ and $\mathbf{k}_3$ of the final-state positron and electron point into opposite directions. The angle between the W^- boson and the positron is fixed to $\theta_{W^- e^+} = 150°$, and the CM energy is chosen as $\sqrt{s} = 180\,\text{GeV}$. The invariant mass of the $e^- \bar{\nu}_e$ system takes the values $M_- = 78$ and $82\,\text{GeV}$. The other parameters are $\alpha = 1/137$, $M_W = 80\,\text{GeV}$, and $\Gamma_W = 2.0\,\text{GeV}$. In Fig. 6 we show our results for the non-factorizable photonic corrections for this phase-space configuration. The intermediate–intermediate (mm′) corrections agree reasonably with those of Ref. [8], but the other curves differ qualitatively and quantitatively[4]. We mention that Fig. 6 has been reproduced [17] by the authors of Refs. [9, 10] within the expected level of accuracy. While the individual contributions shown in Fig. 6 are at the level of 10% owing to mass singularities, the sum, which is free of mass singularities, is below 1.2%.

[3]While the result of Refs. [9, 10] (as our result) for the complete non-factorizable correction is free of mass-singular logarithms, the result of Ref. [8] contains logarithms of ratios of fermion masses. However, the authors of Ref. [8] have informed us [16] that the results of Ref. [8] and Refs. [9, 10] agree for equal fermion pairs in the final state.

[4]Although not stated in Ref. [8], mass-singular parts have been dropped there in the numerical evaluation [16] rendering a thorough comparison of the (mf′) and (ff′) parts impossible. Comparing the sum of all three contributions, i.e. the complete non-factorizable correction factor, our result differs from the sum read off from Ref. [8].

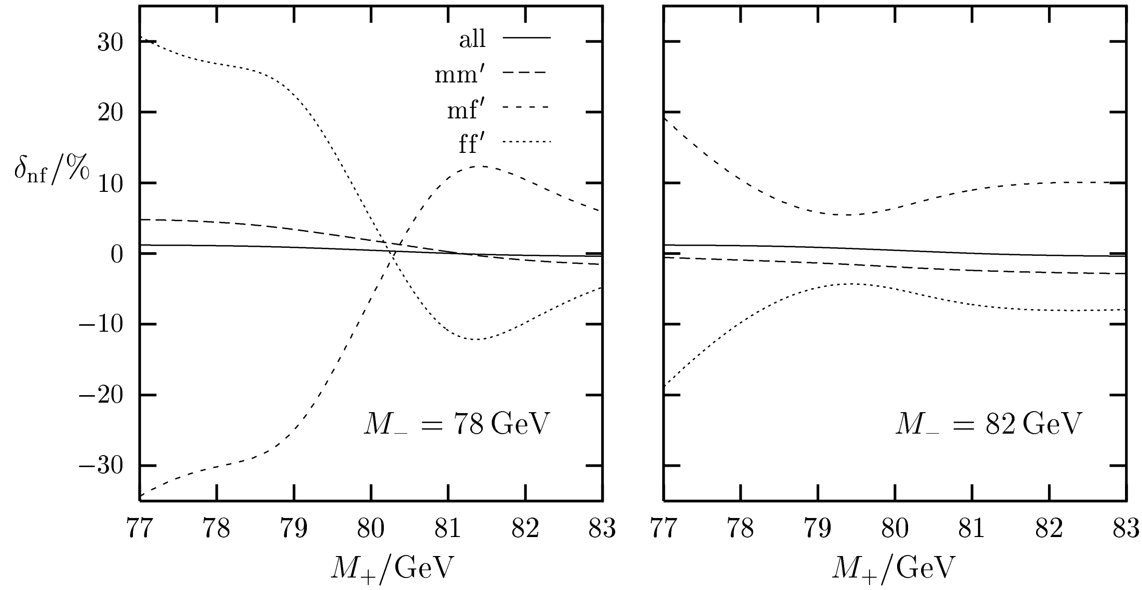

Figure 6: Relative non-factorizable correction factor to the differential cross section for the phase-space configuration given in the text. The curves labelled mm′, mf′, and ff′ correspond to the curves A, B, and C of Ref. [8], respectively.

Beenakker et al. [9] have evaluated the relative non-factorizable corrections to the distributions $\mathrm{d}\sigma/\mathrm{d}M_+\mathrm{d}M_-$, $\mathrm{d}\sigma/\mathrm{d}M_+$, and $\mathrm{d}\sigma/\mathrm{d}M_{\mathrm{av}}$, where $M_{\mathrm{av}} = (M_- + M_+)/2$. Our corresponding results for the set of parameters given in (5.1) are shown in Fig. 7 for the single invariant-mass distributions and in Table 1 for the double invariant-mass distribution. The deviation between the distributions $\mathrm{d}\sigma/\mathrm{d}M_+$ and $\mathrm{d}\sigma/\mathrm{d}M_-$ in Fig. 7, which should be identical, gives an indication on the Monte Carlo error of our calculation. The single and double invariant-mass distributions agree very well with those of Ref. [9]. The worst agreement is found for small invariant masses and amounts to 0.03%. In fact, the agreement is better than expected, in view of the fact that our results differ from those of Ref. [9] by non-doubly resonant corrections. In the numerical evaluations of Ref. [9] the phase space and the Born matrix element are taken entirely on shell [17]. Moreover, the scalar integrals are parametrized by scalar invariants different from ours, leading to differences of the order of $|k_\pm^2 - M_{\mathrm{W}}^2|/M_{\mathrm{W}}^2$.

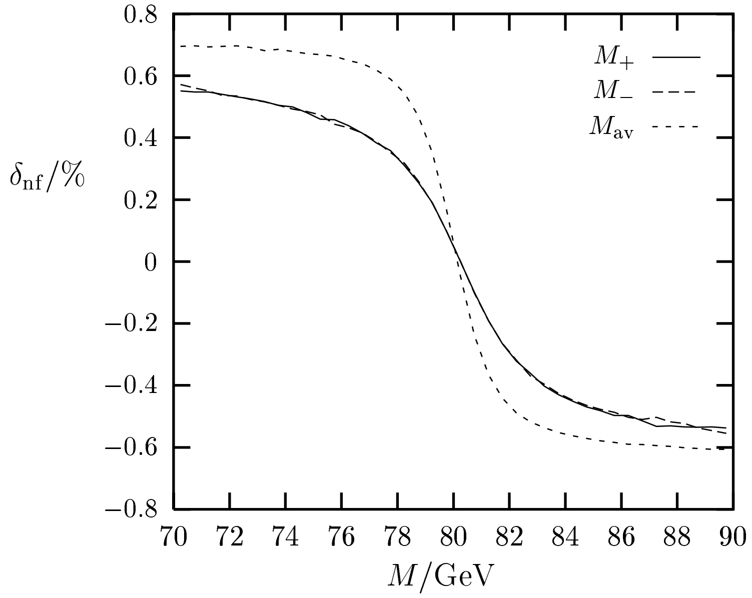

Figure 7: Relative non-factorizable corrections to the single invariant-mass distributions $\mathrm{d}\sigma/\mathrm{d}M_\pm$ and $\mathrm{d}\sigma/\mathrm{d}M_{\rm av}$ for the CM energy $\sqrt{s} = 184\,\mathrm{GeV}$.

Δ_+	Δ_-						
	-1	$-1/2$	$-1/4$	0	$1/4$	$1/2$	1
-1	0.81	0.64	0.52	0.38	0.22	0.07	−0.16
$-1/2$	0.64	0.52	0.40	0.24	0.08	−0.07	−0.25
$-1/4$	0.52	0.40	0.28	0.13	−0.02	−0.15	−0.31
0	0.38	0.24	0.13	0.00	−0.13	−0.24	−0.37
$1/2$	0.22	0.08	−0.02	−0.13	−0.24	−0.32	−0.43
$1/4$	0.07	−0.07	−0.15	−0.24	−0.32	−0.39	−0.48
1	−0.16	−0.25	−0.31	−0.37	−0.43	−0.48	−0.54

Table 1: Relative non-factorizable corrections in per cent to the double invariant-mass distribution $\mathrm{d}\sigma/\mathrm{d}M_+\mathrm{d}M_-$ for the CM energy $\sqrt{s} = 184\,\mathrm{GeV}$ and various values of $M_\pm$ specified in terms of their distance from $M_{\rm W}$ in units of $\Gamma_{\rm W}$, i.e. $\Delta_\pm = (M_\pm - M_{\rm W})/\Gamma_{\rm W}$.

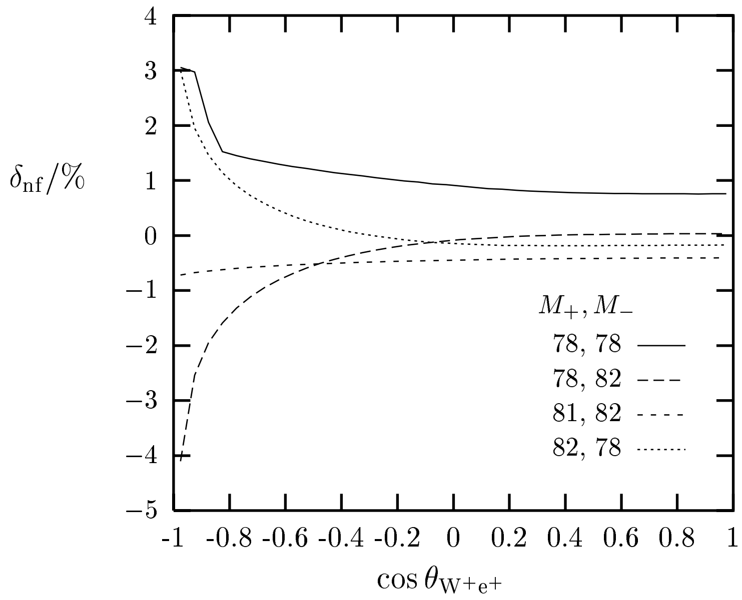

Figure 8: Relative non-factorizable corrections to the decay-angular distribution $\mathrm{d}\sigma/\mathrm{d}M_-\mathrm{d}M_+\mathrm{d}\cos\theta_{\mathrm{W^+e^+}}$ for fixed values of the invariant masses $M_\pm$ and the CM energy $\sqrt{s} = 184\,\mathrm{GeV}$.

In Ref. [9], additionally, the decay-angular distribution $\mathrm{d}\sigma/\mathrm{d}M_-\mathrm{d}M_+\mathrm{d}\cos\theta_{\mathrm{W^+e^+}}$ has been considered, where $\theta_{\mathrm{W^+e^+}}$ is the decay angle between $\mathbf{k}_+$ and $\mathbf{k}_2$ in the laboratory system. Our results for this angular distribution are shown in Fig. 8. The cross section is small for $\cos\theta_{\mathrm{W^+e^+}} \sim -1$, where the corrections are largest. Unfortunately the corresponding figure in Ref. [9] is not correct [17]. The authors of Ref. [9] have provided a corrected figure, which agrees reasonably well with Fig. 8, but does not show the kinks in the curve for $M_\pm = 78\,\mathrm{GeV}$. The kinks are due to a logarithmic Landau singularity in the 4-point functions. If one employs the on-shell parametrization of phase space, as in Ref. [9], the Landau singularities appear at the boundary of phase space. Although no kinks appear in the physical phase space in this case, the Landau singularities still give rise to large corrections for $\cos\theta_{\mathrm{W^+e^+}} \sim -1$. Since the kinks appear in a region where the cross section is small, they are not relevant for phenomenology. The issue of the kinks is further discussed in Sect. 5.3.

5.2 Further results

In Fig. 9 we show the non-factorizable corrections to the single invariant-mass distribution $\mathrm{d}\sigma/\mathrm{d}M_+$ for various CM energies. While the corrections reach up to 1.3% for $\sqrt{s} = 172\,\mathrm{GeV}$, they decrease with increasing energy and are less than 0.04% for $\sqrt{s} = 300\,\mathrm{GeV}$. Note that the shape of the corrections is exactly what is naively expected. If a photon is emitted in the final state, the invariant mass of the fermion pair is smaller than the invariant mass of the resonant W boson, which is given by the invariant mass of the fermion pair plus photon. Since we calculate the corrections as a function of the invariant masses of the fermion pairs, the cross section tends to increase for small invariant masses and decrease for large invariant masses.

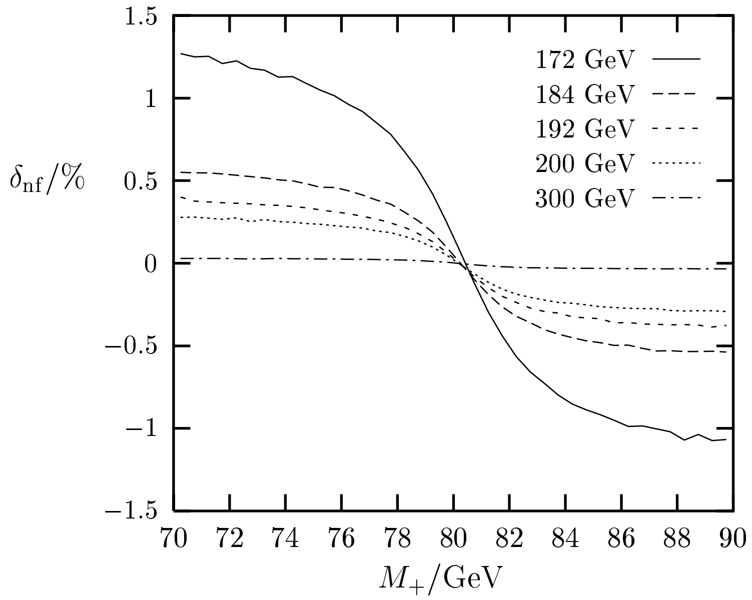

Figure 9: Relative non-factorizable corrections to the single invariant-mass distribution $\mathrm{d}\sigma/\mathrm{d}M_+$ for various CM energies.

$\sqrt{s}/\mathrm{GeV}$	172	184	192	200	300
$\Delta M_+/\mathrm{MeV}$	−2.0	−1.1	−0.8	−0.6	−0.09

Table 2: Shift of the maximum of the single invariant-mass distributions $\mathrm{d}\sigma/\mathrm{d}M_+$ induced by the non-factorizable corrections at various CM energies.

The non-factorizable corrections distort the invariant-mass distribution and thus lead to a shift in the W-boson mass determined from the direct reconstruction of the decay products with respect to the actual W-boson mass. This shift can be estimated by the displacement of the maximum of the single-invariant-mass distribution caused by the corrections shown in Fig. 9. To this end, we determine the slope of the corrections for $M_+ = M_\mathrm{W}$, multiply this linearized correction to a simple Breit-Wigner factor, and determine the shift ΔM_+ of the maximum. The smallness of the correction allows us to evaluate ΔM_+ in linear approximation, leading to the simple formula

$$\Delta M_+ = \left(\frac{\mathrm{d}\delta_{\mathrm{nf}}}{\mathrm{d}M_+}\right)\Big|_{M_+=M_\mathrm{W}} \frac{\Gamma_\mathrm{W}^2}{8}. \tag{5.2}$$

Extracting the slope from our numerical results we obtain the mass shifts shown in Table 2.

In Figs. 10 and 11 we show the effect of the non-factorizable corrections on various angular distributions. Since the non-factorizable corrections are independent of the production angle of the W bosons, it suffices to consider distributions involving the angles of the final-state fermions. We define all angles in the laboratory system, which is the CM system of the production process. The distribution over the angle ϕ between the two planes spanned by the momenta of the two fermion pairs in which the W bosons decay, i.e.

$$\cos\phi = \frac{(\mathbf{k}_1 \times \mathbf{k}_2)(\mathbf{k}_3 \times \mathbf{k}_4)}{|\mathbf{k}_1 \times \mathbf{k}_2||\mathbf{k}_3 \times \mathbf{k}_4|}, \tag{5.3}$$

is presented in Fig. 10. The corrections are of the order of 1% or less. Like the ϕ distribution, the distribution over the angle between positron and electron $\theta_{\mathrm{e^+e^-}}$ (Fig. 11) is symmetric under the interchange of M_+ and M_-. As for the $\theta_{\mathrm{e^+W^+}}$ distribution (Fig. 8), the corrections reach several per cent in the region where the cross section is small.

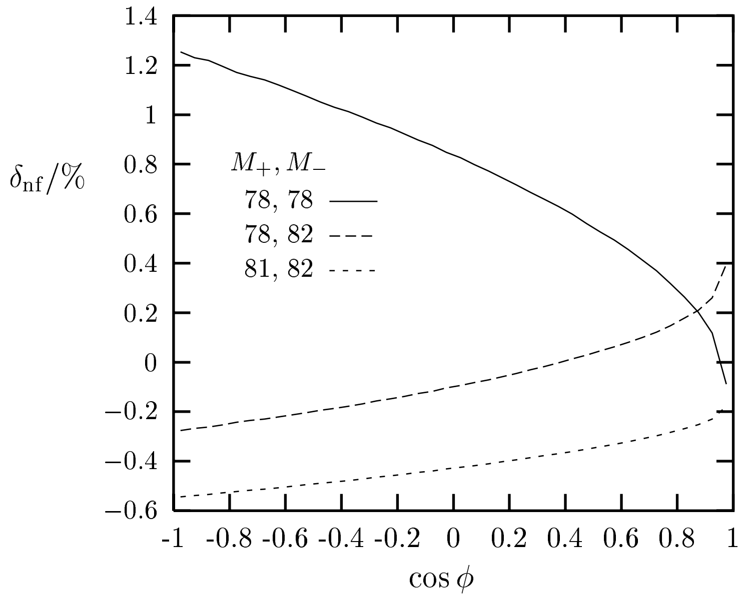

Figure 10: Relative non-factorizable corrections to the angular distribution $\mathrm{d}\sigma/\mathrm{d}M_-\mathrm{d}M_+\mathrm{d}\cos\phi$ for fixed values of the invariant masses $M_\pm$ and the CM energy $\sqrt{s} = 184\,\mathrm{GeV}$.

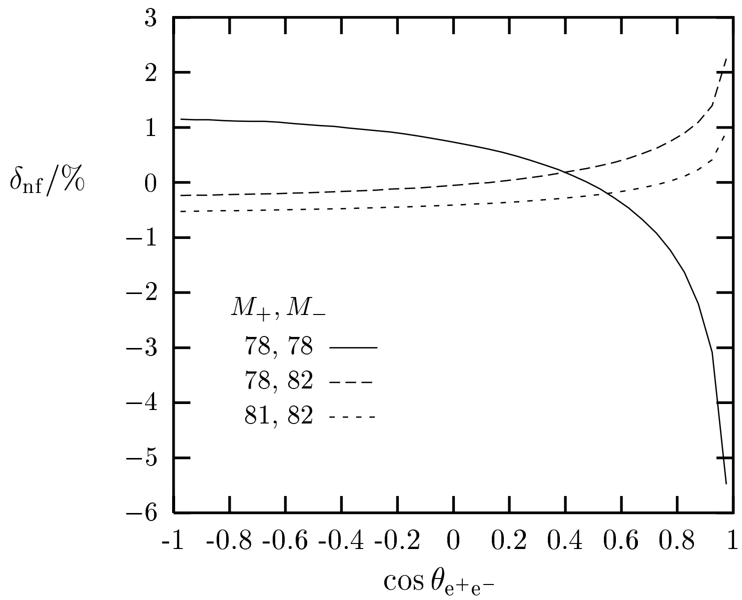

Figure 11: Relative non-factorizable corrections to the angular distribution $\mathrm{d}\sigma/\mathrm{d}M_-\mathrm{d}M_+\mathrm{d}\cos\theta_{\mathrm{e^+e^-}}$ for fixed values of the invariant masses $M_\pm$ and a CM energy $\sqrt{s} = 184\,\mathrm{GeV}$.

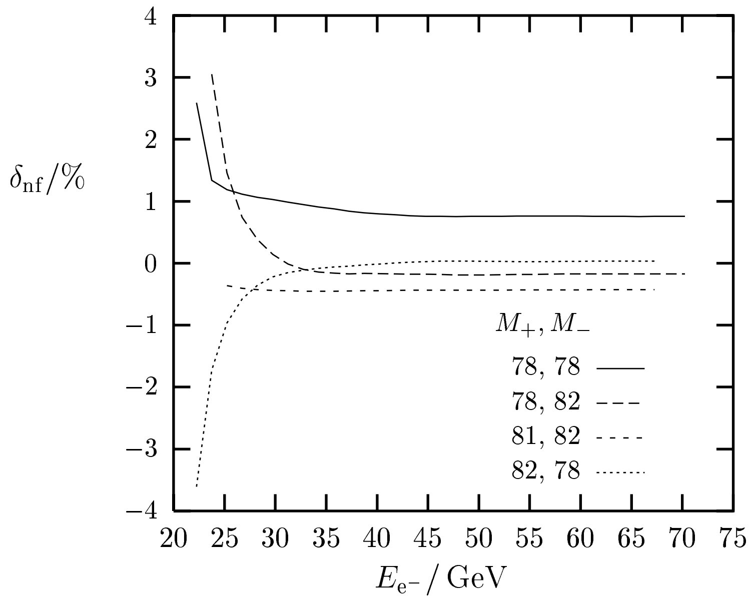

Figure 12: Relative non-factorizable corrections to the electron-energy distribution $\mathrm{d}\sigma/\mathrm{d}M_-\mathrm{d}M_+\mathrm{d}E_{\rm e^-}$ for fixed values of the invariant masses $M_\pm$ and a CM energy $\sqrt{s} =$ 184 GeV.

The distribution in the electron energy $E_{\rm e^-}$ is considered in Fig. 12. The corrections are typically of the order of 1%. Again the corrections become large where the cross section is small.

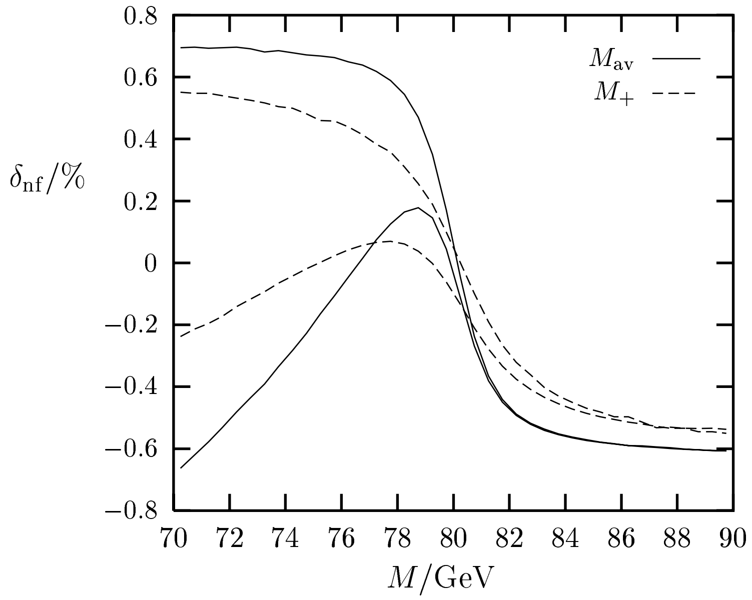

Figure 13: Relative non-factorizable corrections to the single invariant-mass distributions $\mathrm{d}\sigma/\mathrm{d}M_+$ and $\mathrm{d}\sigma/\mathrm{d}M_{\rm av}$ for a CM energy $\sqrt{s} = 184\,\mathrm{GeV}$ with (lower curves) and without (upper curves) improved Coulomb-singularity treatment.

Δ_+	Δ_-						
	−1	−1/2	−1/4	0	1/4	1/2	1
−1	0.39	0.31	0.23	0.15	0.05	−0.04	−0.20
−1/2	0.31	0.28	0.20	0.10	−0.02	−0.13	−0.27
−1/4	0.23	0.20	0.13	0.03	−0.09	−0.19	−0.33
0	0.15	0.10	0.03	−0.08	−0.18	−0.27	−0.38
1/2	0.05	−0.02	−0.09	−0.18	−0.27	−0.35	−0.44
1/4	−0.04	−0.13	−0.19	−0.27	−0.35	−0.41	−0.49
1	−0.20	−0.27	−0.33	−0.38	−0.44	−0.49	−0.54

Table 3: Same as in Table 1 but with improved Coulomb-singularity treatment.

In Sect. 4.3 we have introduced a correction term that includes the full off-shell Coulomb singularity. The results for the non-factorizable corrections with this improvement are compared with those of the pure DPA in Fig. 13. For $\sqrt{s} = 184\,\mathrm{GeV}$ the additional terms shift the non-factorizable corrections by up to 1.4% for $\mathrm{d}\sigma/\mathrm{d}M_{\mathrm{av}}$ and by up to 0.8% for $\mathrm{d}\sigma/\mathrm{d}M_+$ for small invariant masses, whereas for large invariant masses there is practically no effect. The difference originates essentially from the differences between $1/\bar{\beta}$ and $1/\beta_{\mathrm{W}}$ in (4.11). For large invariant masses, the explicit logarithms in (4.11) are small, i.e. the Coulomb singularity correction is minuscule, and this difference practically makes no effect. For small invariant masses, the logarithms are approximately $\mathrm{i}\pi$ and the difference causes the effect seen in Fig. 13. In Table 3 we show the non-factorizable corrections to the double invariant-mass distribution, as in Table 1, but now with the improved Coulomb-singularity treatment. We find a difference of up to half a per cent for small invariant masses but no effect for large ones. We mention that the difference between the entries in Tables 3 and 1 is directly given by the contribution (4.11) to $\Delta_{\mathrm{mm'}}$, without any influence of the phase-space integration.

5.3 Discussion of intrinsic ambiguities

In the results presented so far, all scalar integrals were parametrized by s, s_{23}, s_{13}, s_{24}, and $k_\pm^2$ (parametrization 1). In DPA, however, the parameters of the scalar integrals are only fixed up to terms of order $k_\pm^2 - M_{\mathrm{W}}^2$. We can for example parametrize the scalar integrals in terms of s, s_{23}, s_{123}, s_{234}, and $k_\pm^2$ (parametrization 2) instead. As a third parametrization, we fix all scalar invariants except for $k_\pm^2$ by their on-shell values, corresponding exactly to the approach of Ref. [9]. The results of these three parametrizations differ by non-doubly-resonant corrections.

The difference between parametrizations 1 and 2 is illustrated in Fig. 14 for the single invariant-mass distribution. The difference amounts to $\sim 0.1\%$. Note that for an invariant mass $M_+ = 70\,\mathrm{GeV}$ we have $\alpha|M_{\mathrm{W}}^2 - k_+^2|/M_{\mathrm{W}}^2 \sim 0.002$ and would thus expect absolute changes in the non-factorizable corrections at this level.

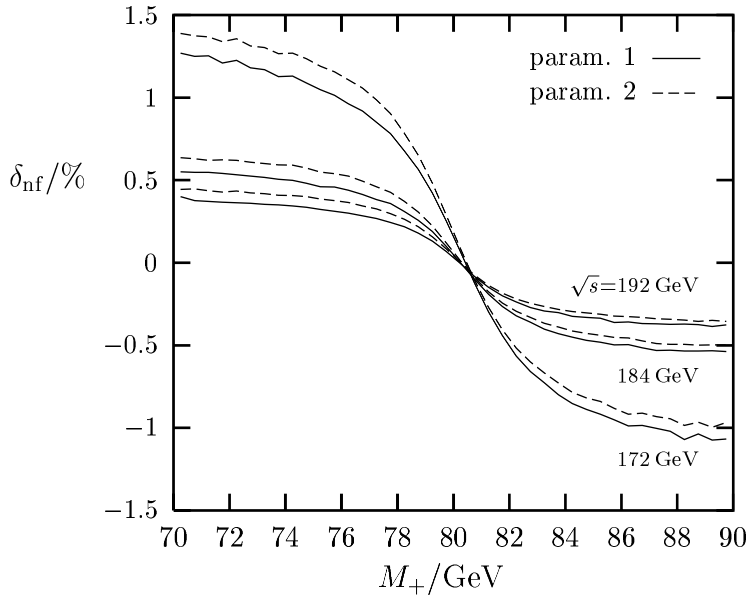

Figure 14: Relative non-factorizable corrections to the single invariant-mass distribution $\mathrm{d}\sigma/\mathrm{d}M_+$ for the CM energies 172, 184, and 192 GeV and two different parametrizations.

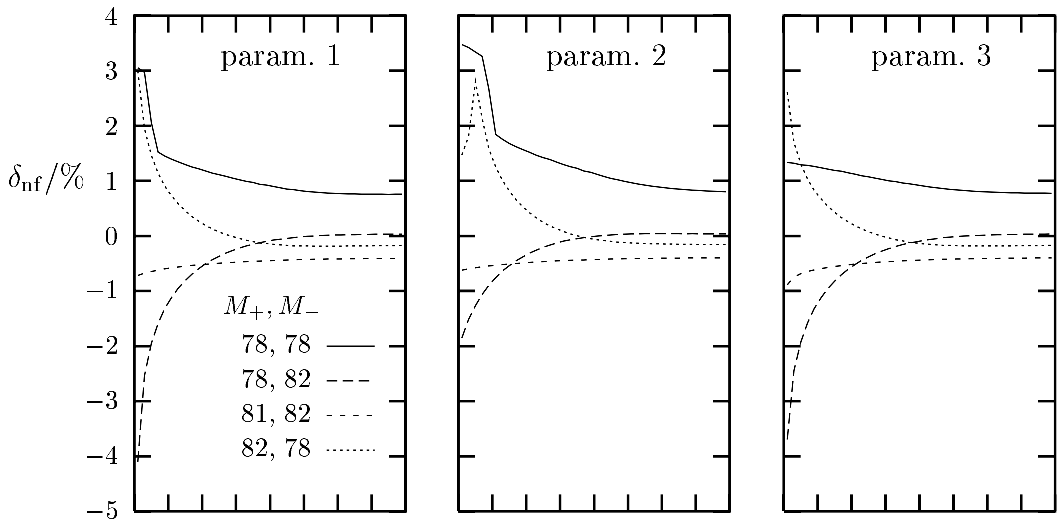

Figure 15: Relative non-factorizable corrections to the decay-angular distribution $\mathrm{d}\sigma/\mathrm{d}M_-\mathrm{d}M_+\mathrm{d}\cos\theta_{\mathrm{W^+e^+}}$ for fixed values of the invariant masses $M_\pm$ and a CM energy $\sqrt{s} = 184\,\mathrm{GeV}$ using three different parametrizations, as specified in the text.

For the non-factorizable corrections to the angular distributions, uncertainties of the same order are to be expected. The only exceptions are the distributions over the decay angles $\theta_{\mathrm{W^+e^+}}$ and $\theta_{\mathrm{W^-e^-}}$. Let us explain this for $\theta_{\mathrm{W^+e^+}}$ in more detail: the non-factorizable correction contains the term $2\pi\mathrm{i}\ln[1+x_{\mathrm{W}}M_{\mathrm{W}}^2/(s_{13}+s_{23})]$, which can be evaluated by taking $(s_{13}+s_{23})$ directly or $(s_{123}-M_{\mathrm{W}}^2)$ as input. This parametrization ambiguity can lead to larger uncertainties, because the above logarithm can become singular, and the location of this Landau singularity is shifted by the ambiguity. Since there is a one-to-one correspondence between s_{123} and $\theta_{\mathrm{W^+e^+}}$ for fixed s and k_+^2, this logarithmic singularity is washed out if the angular integration over $\theta_{\mathrm{W^+e^+}}$ is performed, but appears as a kink structure in the angular distribution over this angle. Figure 15 shows the non-factorizable corrections to this angular distribution for the three parametrizations. For $M_+ = 78\,\mathrm{GeV}$ we still have $\alpha|k_+^2 - M_{\mathrm{W}}^2|/M_{\mathrm{W}}^2 \sim 0.0004$. Apart from the regions where the Landau singularities appear, this is indeed of the order of the differences between the three parametrizations. When considering the Landau singularities, one should realize that the parametrization ambiguity of the locations of the singularities is not suppressed by a factor α, i.e. different parametrizations shift the locations at the level of $|k_+^2 - M_{\mathrm{W}}^2|/M_{\mathrm{W}}^2 \sim 0.05$. However, the impact of the corresponding kinks on observables is again suppressed with $\alpha|k_+^2-M_{\mathrm{W}}^2|/M_{\mathrm{W}}^2$ if the angles are integrated over, since the singularities can only appear near the boundary of phase space and disappear from phase space exactly on resonance. Since the cross section is small where the Landau singularity appears, the effect is phenomenologically irrelevant[5].

6 Conclusion

A reasonable approach for the evaluation of the $\mathcal{O}(\alpha)$ corrections to W-pair-mediated four-fermion production consists in an expansion around the poles of the W propagators. For most of the corrections, the leading terms in this expansion, the doubly-resonant corrections, are sufficient. Combining the results for on-shell W-pair production and decay, one obtains the factorizable doubly-resonant corrections, which account for the largest part of the corrections. The remaining doubly-resonant corrections comprise the effects of virtual soft-photon exchange and real soft-photon interference where W-pair production and W decay are not independent. Those corrections are called non-factorizable.

[5] Although it would be possible to remove all Landau singularities from the physical phase space in DPA by choosing an appropriate parametrization (e.g. parametrization 3), we have kept these structures, because we view this kind of ambiguity as unavoidable in DPA. Only a complete off-shell calculation in $\mathcal{O}(\alpha)$ can resolve this uncertainty. We have checked that the Landau singularities are definitely present in the scalar integrals appearing in the exact off-shell calculation and we do not see a reason why these should drop out in the complete off-shell $\mathcal{O}(\alpha)$ calculation. In any case, the kink structures will be smeared out by the finite decay width of the W boson.

We have studied the non-factorizable doubly-resonant corrections to W-pair production analytically as well as numerically. By defining these corrections in such a way that the complete doubly-resonant corrections are obtained by adding the factorizable corrections, the non-factorizable corrections become manifestly gauge-independent. We have modified the double-pole approximation in such a way that the non-factorizable corrections also include the correct off-shell Coulomb singularity, which is the most important effect of the instability of the W bosons.

It turns out that the actual form of the real non-factorizable corrections depends on the parametrization of phase space, i.e. on whether the photon momentum is included in the definition of the invariant masses of the resonant W bosons or not. This ambiguity is due to the fact that not only extremely soft photons but also photons with energies of the order of the W width contribute to the doubly-resonant corrections. For the parametrization where the invariant masses of the final-state fermion pairs are fixed when integrating over the photon momentum, we have listed all relevant analytical results. We have sketched the calculation of the real bremsstrahlung integrals and the reduction of the real 5-point functions. The final analytical results are presented as transparent as possible so that they can be easily implemented in Monte Carlo generators. Our correction factor for the non-factorizable corrections agrees analytically with the one of Refs. [9, 10] in double-pole approximation.

We have numerically discussed various distributions and, in particular, compared to the results of Refs. [8, 9] as far as possible. Our numerical results for the correction factor differ from those of Ref. [8], but agree with those of Ref. [9] at the level of the expected intrinsic uncertainty of the double-pole approach. These small deviations are due to different embeddings of the non-factorizable correction in the off-shell phase space, which are equivalent in double-pole approximation.

For the adopted phase-space parametrization the non-factorizable doubly-resonant corrections are independent of the W-production angle, because all non-factorizable virtual and real soft-photonic effects that are related to the initial state cancel. For this reason, the presented correction factor for the non-factorizable doubly-resonant corrections is directly applicable to other reactions with the same final state, such as $\gamma\gamma \to \mathrm{WW} \to 4\,\mathrm{fermions}$ and $q\bar{q} \to \mathrm{WW} \to 4\,\mathrm{fermions}$. The non-factorizable corrections to Z-pair-mediated four-fermion production, $\mathrm{e^+e^-} \to \mathrm{ZZ} \to 4\,\mathrm{fermions}$, can also be read off from our analytical results. As already mentioned in Refs. [9, 10], the corrections to pure invariant-mass distributions vanish in this case.

The non-factorizable corrections are non-vanishing only for distributions in the invariant masses of the W bosons and vanish after integration over both invariant masses. Therefore, they do not modify the usual genuine differential distributions that are used for the investigation of the triple gauge couplings. For invariant-mass distributions, the non-factorizable doubly-resonant corrections in the LEP2 energy region are small with respect to the experimental accuracy at LEP2, but should become relevant for possible future linear colliders with higher luminosity. At high energies the non-factorizable corrections are negligible.

Acknowledgements

We thank R. Pittau for help in implementing the corrections in EXCALIBUR. We are grateful to W. Beenakker, F.A. Berends, A.P. Chapovsky, K. Melnikov, R. Pittau, D. Wackeroth and O. Yakovlev for useful discussions. Finally, we thank A.P. Chapovsky for his effort to perform a detailed comparison with our numerical results for the observables discussed in Ref. [9].

Appendix

A Useful definitions

In the main text, we have already used the following short-hand expressions,

$$\begin{aligned} \beta_{\mathrm{W}} &= \sqrt{1-\frac{4M_{\mathrm{W}}^2}{s}+\mathrm{i}\epsilon}, \\ x_{\mathrm{W}} &= \frac{\beta_{\mathrm{W}}-1}{\beta_{\mathrm{W}}+1}, \\ \beta &= \sqrt{1-\frac{4M^2}{s}}, \\ \bar{\beta} &= \frac{\sqrt{\lambda(s,k_+^2,k_-^2)}}{s}, \\ \kappa_{\mathrm{W}} &= \sqrt{\lambda(M_{\mathrm{W}}^4, s_{13}s_{24}, s_{14}s_{23})-\mathrm{i}\epsilon}, \\ \Delta_M &= \frac{|k_+^2-k_-^2|}{s}, \end{aligned} \qquad \text{(A.1)}$$

where

$$\lambda(x,y,z) = x^2+y^2+z^2-2xy-2xz-2yz. \qquad \text{(A.2)}$$

The evaluation of one-loop n-point functions naturally leads to the usual dilogarithm,

$$\mathrm{Li}_2(z) = -\int_0^z \frac{\mathrm{d}t}{t}\ln(1-t), \qquad |\arg(1-z)|<\pi, \qquad \text{(A.3)}$$

and its analytically continued form

$$\begin{aligned} \mathcal{L}i_2(x,y) = {} & \mathrm{Li}_2(1-xy) + [\ln(xy)-\ln(x)-\ln(y)]\ln(1-xy), \\ & |\arg(x)|, |\arg(y)| < \pi. \end{aligned} \qquad \text{(A.4)}$$

B Calculation of the real bremsstrahlung integrals

In this appendix we describe a general method for evaluating the bremsstrahlung 3- and 4-point integrals defined in (3.15). We make use of the generalized Feynman-parameter representation [18]

$$1 \quad \Gamma(n) \int^{\infty} \quad \delta(1-\textstyle\sum_{i=1}^{n}\alpha_i x_i) \qquad \text{(B.1)}$$

We first consider the 3-point integral $\mathcal{C}_0$. We only need the difference between the general IR-finite integral and the corresponding IR-divergent on-shell integral,

$$\begin{aligned}
&\mathcal{C}_0(p_1,p_2,0,m_1,m_2) - \mathcal{C}_0\Big(p_1,p_2,\lambda,\sqrt{p_1^2},\sqrt{p_2^2}\Big) \\
&= \int \frac{\mathrm{d}^3\mathbf{q}}{\pi q_0}\left\{\frac{1}{(2p_1q+p_1^2-m_1^2)(2p_2q+p_2^2-m_2^2)} - \frac{1}{(2p_1q)(2p_2q)}\right\}\bigg|_{q_0=\sqrt{\mathbf{q}^2+\lambda^2}},
\end{aligned} \tag{B.2}$$

which is UV-finite and Lorentz-invariant. Extracting the signs σ_i of p_{i0} via definition (3.29), and assuming for the moment $\sigma_i \,\mathrm{Re}(p_i^2-m_i^2)>0$, the Feynman-parameter representation (B.1) can be applied to each term in the integrand of (B.2). The integration over $\mathrm{d}^3\mathbf{q}$ can be carried out and yields

$$\begin{aligned}
&\mathcal{C}_0(p_1,p_2,0,m_1,m_2) - \mathcal{C}_0\Big(p_1,p_2,\lambda,\sqrt{p_1^2},\sqrt{p_2^2}\Big) \\
&= \sigma_1\sigma_2\int_0^\infty \mathrm{d}x_1\mathrm{d}x_2\,\delta\Big(1-\sum_{i=1}^2\alpha_i x_i\Big)\frac{\ln(\lambda^2)+\ln\Big[\Big(\sum_{i=1}^2\tilde{p}_i x_i\Big)^2\Big]-2\ln\Big[\sum_{i=1}^2\sigma_i(p_i^2-m_i^2)x_i\Big]}{2\Big(\sum_{i=1}^2\tilde{p}_i x_i\Big)^2}.
\end{aligned} \tag{B.3}$$

Putting for instance $\alpha_1=0$ and $\alpha_2=1$, and using

$$\int_0^\infty \mathrm{d}x\left(\frac{1}{x+z_1}-\frac{1}{x+z_2}\right)\ln(1+zx) = \mathcal{L}i_2(z_1,z)-\mathcal{L}i_2(z_2,z), \tag{B.4}$$

where $|\operatorname{arc}(z_{1,2},z,z\,z_{1,2})|<\pi$, the remaining one-dimensional integration and the analytical continuations to arbitrary complex $(p_i^2-m_i^2)$ are straightforward.

Next we consider the IR-finite 4-point function

$$\begin{aligned}
&\mathcal{D}_0(p_1,p_2,p_3,0,m_1,m_2,m_3) \\
&= \int\frac{\mathrm{d}^3\mathbf{q}}{\pi q_0}\,\frac{1}{(2p_1q+p_1^2-m_1^2)(2p_2q+p_2^2-m_2^2)(2p_3q+p_3^2-m_3^2)}\bigg|_{q_0=|\mathbf{q}|}.
\end{aligned} \tag{B.5}$$

For $\sigma_i\,\mathrm{Re}(p_i^2-m_i^2)>0$ we can proceed as above and find

$$\mathcal{D}_0(p_1,p_2,p_3,0,m_1,m_2,m_3) = \sigma_1\sigma_2\sigma_3\int_0^\infty \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\,\frac{\delta\Big(1-\sum_{i=1}^3\alpha_i x_i\Big)}{\Big[\sum_{i=1}^3\sigma_i(p_i^2-m_i^2)x_i\Big]\Big(\sum_{i=1}^3\tilde{p}_i x_i\Big)^2}. \tag{B.6}$$

Again, the two-dimensional integration over the Feynman parameters and the analytical continuations in $(p_i^2-m_i^2)$ are straightforward.

Finally, we inspect the IR-divergent 4-point function

$$\mathcal{D}_0\Big(p_1,p_2,p_3,\lambda,\sqrt{p_1^2},m_2,\sqrt{p_3^2}\Big) = \int\frac{\mathrm{d}^3\mathbf{q}}{\pi q_0}\,\frac{1}{(2p_1q)(2p_2q+p_2^2-m_2^2)(2p_3q)}\bigg|_{q_0=\sqrt{\mathbf{q}^2+\lambda^2}}. \quad \text{(B.7)}$$

Instead of applying the Feynman-parameter representation directly, it is more convenient to extract the IR singularity by subtracting and adding an IR-divergent 3-point function containing the same IR structure. Since the difference between the 4- and 3-point functions is IR-finite, we can regularize the IR divergence in these two integrals in a more convenient way. Therefore, we write

$$\begin{aligned}\mathcal{D}_0\Big(p_1,p_2,p_3,\lambda,\sqrt{p_1^2},m_2,\sqrt{p_3^2}\Big) &= \lim_{m^2\to p_1^2}\Big[\mathcal{D}_0\Big(p_1,p_2,p_3,0,m,m_2,\sqrt{p_3^2}\Big)\\ &-\frac{1}{p_2^2-m_2^2}\mathcal{C}_0\Big(p_1,p_3,0,m,\sqrt{p_3^2}\Big)+\frac{1}{p_2^2-m_2^2}\mathcal{C}_0\Big(p_1,p_3,\lambda,\sqrt{p_1^2},\sqrt{p_3^2}\Big)\Big], \quad \text{(B.8)}\end{aligned}$$

i.e. we regularize the IR divergence in the 4-point function by the off-shellness $p_1^2-m^2\neq 0$. Both the 4-point function as well as the difference of 3-point functions can be treated as above, yielding straightforward Feynman-parameter integrals according to (B.3) and (B.6), respectively. The limit $m^2\to p_1^2$ can easily be taken after the integrations have been performed.

C Explicit results for scalar integrals

C.1 Loop integrals in double-pole approximation

In the (mm′) correction the following combination of 3-point functions appears in the strict DPA:

$$\begin{aligned}&C_0(k_+,-k_-,0,M,M)-\Big[C_0(k_+,-k_-,\lambda,M_\mathrm{W},M_\mathrm{W})\Big]_{k_\pm^2=M_\mathrm{W}^2}\\ &\sim\frac{1}{s\beta_\mathrm{W}}\Big\{-\mathcal{L}i_2\Big(\frac{K_-}{K_+},x_\mathrm{W}\Big)+\mathcal{L}i_2\Big(\frac{K_-}{K_+},x_\mathrm{W}^{-1}\Big)+\mathrm{Li}_2(1-x_\mathrm{W}^2)+\pi^2+\ln^2(-x_\mathrm{W})\\ &\quad+2\ln\Big(\frac{K_+}{\lambda M_\mathrm{W}}\Big)\ln(x_\mathrm{W})-2\pi\mathrm{i}\ln(1-x_\mathrm{W}^2)\Big\}. \quad \text{(C.1)}\end{aligned}$$

In order to include the full off-shell Coulomb singularity, one has to add

$$\frac{2\pi\mathrm{i}}{s\bar\beta}\ln\Big(\frac{\beta+\Delta_M-\bar\beta}{\beta+\Delta_M+\bar\beta}\Big)-\frac{2\pi\mathrm{i}}{s\beta_\mathrm{W}}\ln\Big(\frac{K_++K_-+s\beta_\mathrm{W}\Delta_M}{2\beta_\mathrm{W}^2 s}\Big) \quad \text{(C.2)}$$

to the r.h.s. of (C.1).

For the (mf′) and (ff′) corrections we need the following 4-point functions:

$$D_0(0) = D_0(-k_4, k_+ + k_3, k_2 + k_3, 0, M, M, 0)$$
$$\sim \frac{1}{\kappa_{\mathrm{W}}} \sum_{\sigma=1,2} (-1)^\sigma \biggl\{ \mathcal{L}i_2\Bigl(-\frac{s_{13}+s_{23}}{M_{\mathrm{W}}^2} - \mathrm{i}\epsilon, -x_\sigma\Bigr) + \mathcal{L}i_2\Bigl(-\frac{M_{\mathrm{W}}^2}{s_{23}+s_{24}} + \mathrm{i}\epsilon, -x_\sigma\Bigr)$$
$$- \mathcal{L}i_2\bigl(x_{\mathrm{W}}, -x_\sigma\bigr) - \mathcal{L}i_2\bigl(x_{\mathrm{W}}^{-1}, -x_\sigma\bigr) - \ln\Bigl(1 + \frac{s_{24}}{s_{23}}\Bigr)\ln(-x_\sigma)\biggr\}, \tag{C.3}$$

$$\text{with} \qquad x_1 = \frac{s_{24} z}{M_{\mathrm{W}}^2}, \quad x_2 = \frac{M_{\mathrm{W}}^2}{s_{13} z}, \quad z = \frac{M_{\mathrm{W}}^4 + s_{13}s_{24} - s_{14}s_{23} + \kappa_{\mathrm{W}}}{2 s_{13} s_{24}}, \tag{C.4}$$

$$D_0(1) = D_0(-k_-, k_+, k_2, 0, M, M, m_2)$$
$$\sim \frac{1}{K_+(s_{23}+s_{24}) + K_- M_{\mathrm{W}}^2} \biggl\{ \sum_{\tau=\pm1} \Bigl[\mathcal{L}i_2\Bigl(\frac{K_+}{K_-}, x_{\mathrm{W}}^\tau\Bigr) - \mathcal{L}i_2\Bigl(-\frac{M_{\mathrm{W}}^2}{s_{23}+s_{24}} + \mathrm{i}\epsilon, x_{\mathrm{W}}^\tau\Bigr)\Bigr] \tag{C.5}$$
$$- 2\,\mathcal{L}i_2\Bigl(\frac{K_+}{K_-}, -\frac{s_{23}+s_{24}}{M_{\mathrm{W}}^2} - \mathrm{i}\epsilon\Bigr) - \ln\Bigl(\frac{m_2^2}{M_{\mathrm{W}}^2}\Bigr)\Bigl[\ln\Bigl(\frac{K_+}{K_-}\Bigr) + \ln\Bigl(-\frac{s_{23}+s_{24}}{M_{\mathrm{W}}^2} - \mathrm{i}\epsilon\Bigr)\Bigr]\biggr\},$$

$$D_0(2) = D_0(-k_3, k_+, k_2, \lambda, m_3, M, m_2)$$
$$\sim \frac{1}{K_+ s_{23}} \biggl\{ -\mathrm{Li}_2\Bigl(-\frac{s_{13}}{s_{23}}\Bigr) - \frac{\pi^2}{3} + 2\ln\Bigl(-\frac{s_{23}}{m_2 m_3} - \mathrm{i}\epsilon\Bigr)\ln\Bigl(\frac{-K_+}{\lambda M_{\mathrm{W}}}\Bigr) - \ln^2\Bigl(\frac{m_2}{M_{\mathrm{W}}}\Bigr)$$
$$- \ln^2\Bigl(-\frac{s_{13}+s_{23}}{m_3 M_{\mathrm{W}}} - \mathrm{i}\epsilon\Bigr)\biggr\}, \tag{C.6}$$

$$D_0(3) = D_0(-k_3, -k_-, k_2, \lambda, m_3, M, m_2) = D_0(2)\Big|_{K_+ \leftrightarrow K_-, m_2 \leftrightarrow m_3, s_{13} \leftrightarrow s_{24}}, \tag{C.7}$$

$$D_0(4) = D_0(-k_3, -k_-, k_+, 0, m_3, M, M) = D_0(1)\Big|_{K_+ \leftrightarrow K_-, m_2 \leftrightarrow m_3, s_{13} \leftrightarrow s_{24}}. \tag{C.8}$$

C.2 Bremsstrahlung integrals in double-pole approximation

In the (mm′) interference corrections the following combination of 3-point functions appears:

$$\mathcal{C}_0(k_+, k_-, 0, M, M^*) - \Bigl[\mathcal{C}_0(k_+, k_-, \lambda, M_{\mathrm{W}}, M_{\mathrm{W}})\Bigr]_{k_\pm^2 = M_{\mathrm{W}}^2}$$
$$\sim \biggl\{ C_0(k_+, -k_-, 0, M, M) - \Bigl[C_0(k_+, -k_-, \lambda, M_{\mathrm{W}}, M_{\mathrm{W}})\Bigr]_{k_\pm^2 = M_{\mathrm{W}}^2}\biggr\}\bigg|_{K_- \to -K_-^*}$$
$$- \frac{2\pi\mathrm{i}}{s\beta_{\mathrm{W}}} \ln\Bigl[\frac{K_+ + K_-^* x_{\mathrm{W}}}{\mathrm{i}\lambda M_{\mathrm{W}}(1 - x_{\mathrm{W}}^2)}\Bigr]. \tag{C.9}$$

For the (mf′) and (ff′) interference corrections the following 4-point functions are required:

$$\tilde{\mathcal{D}}_0(0) = -D_0(k_4, k_+ - k_3, k_2 - k_3, 0, M_-, M_+, 0)$$
$$\sim -D_0(0) + \frac{2\pi\mathrm{i}}{\kappa_{\mathrm{W}}} \biggl\{ \ln\Bigl(-x_{\mathrm{W}} \frac{s_{23}}{M_{\mathrm{W}}^2}\Bigr) + \ln\Bigl[1 + \frac{s_{13}}{s_{23}}(1-z)\Bigr] + \ln\Bigl[1 + \frac{s_{24}}{s_{23}}(1-z)\Bigr]$$
$$- \ln\Bigl(1 + \frac{s_{13}}{M_{\mathrm{W}}^2} z x_{\mathrm{W}}\Bigr) - \ln\Bigl(1 + \frac{s_{24}}{M_{\mathrm{W}}^2} z x_{\mathrm{W}}\Bigr)\biggr\}, \tag{C.10}$$

with z from (C.4),

$$\mathcal{D}_0(1) = \mathcal{D}_0(k_-, k_+, k_2, 0, M^*, M, m_2)$$
$$\sim D_0(1)\Big|_{K_- \to -K_-^*} + \frac{2\pi\mathrm{i}}{K_+(s_{23}+s_{24}) - K_-^* M_{\mathrm{W}}^2} \Bigl[2\ln\Bigl(1 - \frac{K_+}{K_-^*}\frac{s_{23}+s_{24}}{M_{\mathrm{W}}^2}\Bigr)$$
$$- \ln\Bigl(1 + \frac{K_+}{K_-^*}\Bigr) - \ln\Bigl(1 + \frac{x_{\mathrm{W}} M_{\mathrm{W}}^2}{\cdots}\Bigr) - \ln\Bigl(\frac{m_2^2}{M_{\mathrm{W}}^2}\Bigr)\Bigr], \tag{C.11}$$