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On non-abelian generalisation of Born-Infeld action in string theory

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Abstract

We show that the part of the tree-level open string effective action for the non-abelian vector field which depends on the field strength but not on its covariant derivatives, is given by the symmetrised trace of the direct non-abelian generalisation of the Born-Infeld invariant. We comment on possible applications to D-brane dynamics.

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Non-locality of string theory (i.e. the presence of a tower of massive states) implies that the low-energy effective action for massless modes is an infinite power series of all orders in α' [1]. In particular, this applies to the tree-level Lagrangian L_{eff} for the gauge vectors in the open bosonic or type I string theory. In the case of an abelian Chan-Paton gauge group all terms in the action which depend on the field strength F_{mn} but not on its derivatives sum up into the Born-Infeld (BI) Lagrangian [2,3,4,5,6,7]

$$L_{BI} = c_0 \sqrt{\det(\delta_{mn} + T^{-1} F_{mn})} \ , \qquad T^{-1} = 2\pi\alpha' \ . \tag{1}$$

Derivative corrections to this action were discussed in [8,9,10].

In the non-abelian case, the tree-level (disc) effective Lagrangian in the open string theory can be represented as an expansion in powers of the field strength and its covariant derivatives,

$$L_{eff} = \mathrm{Tr}\big(a_0 F^2 + a_1 F D^2 F + a_2 F^4 + a_3 F^2 D^2 F + ...\big) = L(F) + O(DF) \ , \tag{2}$$

where $L(F)$ is the part not containing covariant derivatives of $F_{mn} = \partial_m A_n - \partial_n A_m - i[A_m, A_n]$ (F is assumed to be a hermitian matrix with indices in the fundamental representation of the gauge algebra). Previously, only the terms up to order F^4 in (2) were completely determined [3,11] (there was also a discussion of F^5 terms in [10]). The question we shall address below is about the structure of L in (2), i.e. of a non-abelian analogue of the BI action (NBI action for short).

In contrast to the abelian case where the separation between derivative-independent and derivative-dependent terms in $L_{eff}(F, \partial F)$ is completely unambiguous, this is not true in the non-abelian case. Since $[D_m, D_n]F_{kl} = [F_{mn}, F_{kl}]$ some of the derivative terms may be traded for some of non-derivative ones, and vice versa. We shall resolve this ambiguity by assuming that all $[F, F]$ ('commutator') terms should be treated as a part of the DF-dependent terms in L_{eff} and thus should *not* be included into $L(F)$ in (2). The effective Lagrangian will then be dominated by $L(F)$ under the circumstances when the covariant derivatives of F are much smaller than the powers F.

Adopting such a definition of $L(F)$ or NBI Lagrangian, we shall prove below that, both in the bosonic and the superstring theory, it is given by the following natural generalisation of the Born-Infeld action (1)

$$L(F) = L_{NBI} = c_0 \mathrm{STr} \sqrt{\det(\delta_{mn} + T^{-1} F_{mn})} \ . \tag{3}$$

Here δ_{mn} implicitly includes a factor of the unit matrix in internal space, the determinant is computed with respect to the mn indices only, and STr is the symmetrised trace in the fundamental representation, $\mathrm{STr}(A_1...A_n) \equiv \frac{1}{n!}\mathrm{Tr}(A_1...A_n + \text{all permutations})$. This Lagrangian is thus equal to the same sum of even powers of F_{mn} as appearing in the expansion of BI Lagrangian (1), with each factor of field strength being replaced by a hermitian matrix F and all possible orderings of the matrices included with equal weight. The same invariant was previously conjectured to be a part of a non-abelian generalisation of BI Lagrangian in [12], where, however, an additional term with STr replaced by the antisymmetrised trace was also suggested to be present.[1] The latter is given by the sum of traces of odd powers of F which always contain a factor of $[F,F]$ (as follows from $F_{mn} = -F_{nm}$) and thus should not be included into NBI Lagrangian according to the definition given above.

Let us first compare the α'-expansion of (3) ($c_1 = \pi^2\alpha'^2 c_0$)

$$L'_{NBI} = c_0 \mathrm{STr}[\sqrt{\det(\delta_{mn} + T^{-1}F_{mn})} - I]$$

$$= c_1 \mathrm{STr}\big[F^2_{mn} - \tfrac{1}{2}(2\pi\alpha')^2\big(F^4 - \tfrac{1}{4}(F^2)^2\big) + O(\alpha'^3)\big]$$

$$= c_1 \mathrm{Tr}\big[F^2_{mn} - \tfrac{1}{3}(2\pi\alpha')^2\big(F_{mn}F_{rn}F_{ml}F_{rl} + \tfrac{1}{2}F_{mn}F_{rn}F_{rl}F_{ml} - \tfrac{1}{4}F_{mn}F_{mn}F_{rl}F_{rl} - \tfrac{1}{8}F_{mn}F_{rl}F_{mn}F_{rl}\big) + O(\alpha'^4)\big] \ , \tag{4}$$

with the known perturbative results. The two leading orders in α' in (4) indeed give the full form of the non-abelian open superstring effective action to order $O(\alpha'^3)$ (all α'^2-terms with covariant derivatives have field redefinition dependent coefficients [3]). The F^4 terms were originally found in the STr-form in [11] and in the equivalent Tr-form in [3].

As for the bosonic theory, there (4) does not represent the full effective Lagrangian to α'^3-order: the bosonic L_{eff} contains $\alpha' F^3$ term [1] and the coefficients of the F^4 invariants are somewhat different from the ones in (4) [3]. However, it is easy to see that both F^3 and the excess of F^4 terms are the 'commutator' terms, i.e. they can be represented as $\mathrm{Tr}(\frac{4}{3}i\alpha' F_{mn}[F_{ml}, F_{nl}] + 2\alpha'^2 \mathrm{Tr}(F^{mn}F^{rl}[F_{mn}, F_{rl}])$ and thus, according to our definition, belong to the covariant derivative part of L_{eff} and not to the NBI part. Similar remark applies to the F^5 terms [10][2] and, in general, to all terms of odd power in F.

Let us now give the general argument demonstrating that the covariant derivative independent part of the open string effective action is indeed given by the NBI action (3). The starting point is the expression for the generating functional for the vector amplitudes on the disc. In the bosonic case [2,3]

$$Z(A) =< \mathrm{Tr} P \exp[i \int d\varphi \ \dot{x}^m A_m(x)] > \tag{5}$$

[1] Some other ad hoc generalisations of BI action to non-abelian case where considered in [13] but because of their different trace structure they cannot appear in the tree-level open string effective action.

[2] The F^5-terms have the coefficients proportional to $\zeta(3)$ [10] and should rather not appear in any simple NBI action.

$$= \int d^D x_0 < \mathrm{Tr} P \exp[i \int d\varphi \ \dot\xi^m A_m(x_0+\xi)] > ,$$

where $x = x_0 + \xi(\varphi)$, $0 < \varphi \leq 2\pi$ and the averaging is done with the free string propagator restricted to the boundary of the disc ($\epsilon \to +0$ is a world-sheet UV regularisation)

$$< ... >= \int [d\xi] \ e^{-\frac{1}{2}T \int \xi G^{-1} \xi} ... , \qquad G(\varphi, \varphi') = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{e^{-n\epsilon}}{n} \cos n(\varphi - \varphi') . \qquad (6)$$

As explained in [5,8], the low-energy effective action is given by the renormalised value of (5), computed by expanding in powers of α', $S_{eff}(A) = Z(A(\epsilon), \epsilon)$.[3]

Using the radial gauge $\xi^m A_m(x_0+\xi) = 0$, $A_m(x_0) = 0$ (see, e.g., [15]) we get the following expansion in terms of symmetrised products of covariant derivatives of F at x_0,

$$\int d\varphi \ \dot\xi^m A_m(x_0+\xi) = \int d\varphi \ \dot\xi^m \big[\tfrac{1}{2}\xi^n F_{nm} + \tfrac{1}{3}\xi^n \xi^l D_l F_{nm} + \tfrac{1}{8}\xi^n \xi^l \xi^s D_{(s} D_{l)} F_{nm} + ... \big] . \quad (7)$$

Separating in this way the dependence of Z on covariant derivatives we get

$$Z(A) = \int d^D x_0 \big[\mathcal{L}(F) + O(D_{(k} ... D_{l)} F) \big] , \qquad (8)$$

$$\mathcal{L}(F) =< \mathrm{Tr} P \exp \big[\tfrac{1}{2} i F_{nm} \int d\varphi \ \dot\xi^m \xi^n \big] > . \qquad (9)$$

The path integral in (9) is effectively non-gaussian[4] because of the normal ordering of the $F_{nm}(x_0)(\dot\xi^m \xi^n)(\varphi)$ factors which is non-trivial if the matrices F_{mn} do not commute. It may still be possible to compute it explicitly. In the abelian case the path ordering is trivial and one finds

$$L(F) = c_0 \big[\det(\delta_{mn} + T^{-1} F_{mn}) \big]^{\nu} , \qquad (10)$$

$$\nu = -\pi \int_0^{2\pi} \dot G^2 = - \big(\sum_{n=1}^{\infty} e^{-2\epsilon n} \big)_{\epsilon \to 0} = -\tfrac{1}{2\epsilon} + \tfrac{1}{2} , \qquad (11)$$

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[3] The logarithmic renormalisation of the 'coupling' A_m corresponds to a subtraction of the massless poles in the amplitudes [14,3,5] (the field redefinition ambiguity in the effective action corresponds to the renormalisation scheme ambiguity in this framework [3]). In addition, one is to subtract (or absorb into the renormalisation of the tachyon coupling) the leading linear divergence. This is equivalent to a subtraction of the $SL(2,R)$ Möbius group volume factor. Power divergences are absent in the superstring case where the super-Möbius volume is finite [8].

[4] It may be re-written as a standard 1-dimensional path integral by introducing the auxiliary fields to represent the path-ordered exponent as, e.g., in [16,8].

so that $\nu = \frac{1}{2}$ after the subraction of the Möbius volume divergence [5] (which is done effectively when using the ζ-function prescription [2]). As a result, one finds the BI expression (1).

Since we defined the DF-independent part $L(F)$ of the effective Lagrangian as not containing terms with commutators of F, to determine it we may treat the matrices F_{mn} in (9) as commuting, or, equivalently, symmetrise over all of their orderings in each monomial F^n. Then the path ordering becomes trivial just as in the case of the abelian gauge group, so that instead of (8) we get

$$Z(A) = \int d^D x_0 \big[L(F) + O(D_k ... D_l F)\big] \ , \tag{12}$$

and

$$\mathcal{L}(F) \ \to \ L(F) =< \mathrm{STr} P \exp[\tfrac{1}{2} i F_{nm} \int d\varphi \ \dot{\xi}^m \xi^n] > \tag{13}$$

$$= \mathrm{STr} < \exp[\tfrac{1}{2} i F_{nm} \int_0^{2\pi} d\varphi \ \dot{\xi}^m \xi^n] >= c_0 \mathrm{STr} \big[\det(\delta_{mn} + T^{-1} F_{mn})\big]^\nu \ .$$

Since $\nu_{ren} = \frac{1}{2}$, we finish with the NBI Lagrangian (3).

This discussion is readily generalised to the superstring case, where the gauge-invariant expression for the generating functional is given by the following manifestly 1-d supersymmetric expression [8]

$$Z(A) =< \mathrm{Tr} \hat{P} \exp[i \int d\hat{\varphi} \ \mathcal{D}\hat{x}^m A_m(\hat{x})] > \ . \tag{14}$$

Here $\hat{x}^m = x^m(\varphi) + \theta\psi^m(\varphi), \ \ d\hat{\varphi} = d\varphi d\theta, \ \mathcal{D} = \frac{\partial}{\partial\theta} - \theta\frac{\partial}{\partial\varphi}$ and the supersymmetric path ordering $\hat{P}$ is defined by replacing the usual Θ-functions by the supersymmetric ones, $\hat{\Theta}(\hat{\varphi}_i, \hat{\varphi}_j) = \Theta(\hat{\varphi}_{ij}) = \Theta(\hat{\varphi}_i - \hat{\varphi}_j) + \theta_i\theta_j\delta(\varphi_i - \varphi_j), \ \hat{\varphi}_{ij} \equiv \varphi_i - \varphi_j + \theta_i\theta_j$, so that $\mathcal{D}\hat{\Theta}$ is equal to the supersymmetric δ-function $\delta(\hat{\varphi}_{ij}) = (\theta_j - \theta_i)\delta(\varphi_i - \varphi_j)$. The generating functional (14) automatically includes the contact terms necessary [17] for maintaining gauge invariance. Re-written in terms of the standard path ordering, it takes the form [8]

$$Z(A) =< \mathrm{Tr} P \exp\big(i \int d\varphi \ \big[\dot{x}^m A_m(x) - \tfrac{1}{2}\psi^m\psi^n F_{mn}(x)\big]\big) > \ , \tag{15}$$

with the $[A_m, A_n]$ term in F_{mn} appearing due to the presence of the contact $\theta_i\theta_j\delta(\varphi_i - \varphi_j)$ terms in the supersymmetric theta-functions in (14). The definition of $< ... >$ is analogous to (6) with $\xi G^{-1}\xi \to \xi G^{-1}\xi + \psi K^{-1}\psi$, where K is the restriction of the fermionic Green's function to the boundary of the disc,

$$K(\varphi, \varphi') = \frac{1}{\pi} \sum_{r=\frac{1}{2}}^{\infty} e^{-r\epsilon} \sin r(\varphi - \varphi') \ .$$

As a result, the superstring generalisation of (8),(9) has $\mathcal{L}(F)$ given by

$$\mathcal{L}(F) = < \mathrm{Tr} P \exp \left[\tfrac{1}{2} i F_{nm} \int d\varphi \ (\dot{\xi}^m \xi^n + \psi^m \psi^n) \right] > \ . \tag{16}$$

Dropping the 'commutator' terms to define $L(F)$, i.e. symmetrising the trace, we get, as in (13),(10),

$$\mathcal{L}(F) \ \to \ L(F) = \mathrm{STr} < \exp \left[\tfrac{1}{2} i F_{nm} \int_0^{2\pi} d\varphi \ (\dot{\xi}^m \xi^n + \psi^m \psi^n) \right] > \tag{17}$$

$$= c_0 \mathrm{STr} \big[\det(\delta_{mn} + T^{-1} F_{mn}) \big]^{\nu} \ ,$$

where now

$$\nu = -\pi \int_0^{2\pi} (\dot{G}^2 - K^2) = (- \sum_{n=1}^{\infty} e^{-2\epsilon n} + \sum_{r=\frac{1}{2}}^{\infty} e^{-2\epsilon r})_{\epsilon \to 0} = \tfrac{1}{2} \ . \tag{18}$$

Thus we again obtain the NBI Lagrangian (3), here in completely unambiguous way as the linear divergence in ν present in bosonic case cancels out [7] (which is a manifestation of the finiteness of the volume of the super-Möbius group [8]).

To summarise, the NBI action (3) is thus a good approximation to the effective action when all products of covariant derivatives of F are small. Since $[D, D]F = [F, F]$ that also means that the 'commutator' terms are assumed to be small, i.e. the field strength is approximately abelian.[5] There may be physically interesting cases in which such an approximation is a useful one.

There is a possible alternative expansion of L_{eff} in which one assumes that all *symmetrised* covariant derivatives are small. This does not imply smallness of commutators of F. In this case, as follows from the discussion above (see (8)), the effective Lagrangian is approximated by $\mathcal{L}(F)$ in (16) (for which, unfortunately, we do not know a closed expression).

It should be noted again that it is $L(F) = L_{NBI}$ and not $\mathcal{L}(F)$ that reproduces the full expression for the superstring effective action at the order α'^2 (4). This suggests that L_{NBI} is the relevant object to consider as a DF-independent part of L_{eff}. Another indication of this is provided by the existence of the $D = 10$ space-time supersymmetric extension of the Lagrangian $\mathrm{Tr}F^2 + c\mathrm{STr}[F^4 - \frac{1}{4}(F^2)^2]$ with the *symmetrised* trace [19]. It should be possible to find a supersymmetric version of the full non-abelian Born-Infeld action (3)

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[5] There is also another choice for a translationally invariant non-abelian gauge field: $A_m =$ const [18]. It would be interesting to compute the value of the effective action, i.e. the partition function (14),(15) in this case.

generalising the action found in the abelian case in [20]. One indirect approach could be to repeat the above analysis using the light-cone Green-Schwarz formalism with the fermionic partner of A_m included in the world-sheet action (cf. [21,3,22]).

Let us now comment on a possible application of NBI action (3) to the description of D-branes [23,24]. The form of (bosonic, parity-even, part of) the D-brane effective action [25] is essentially determined (via T-duality) by the abelian $D = 10$ open superstring effective action (see [26,27]). In the 'small acceleration' approximation it is thus given by the BI action for the $D = 10$ vector potential $A_m = (A_s, A_a = TX_a)$ reduced to $p+1$ dimensions. In the static gauge on a flat background

$$I_p = T_p \int d^{p+1}x \sqrt{-\det(\eta_{mn} + T^{-1}F_{mn})} \tag{19}$$

$$= T_p \int d^{p+1}x \sqrt{-\det(\eta_{rs} + \partial_r X_a \partial_s X_a + T^{-1}F_{rs})} \ .$$

In the 'non-relativistic' approximation, i.e. to the leading quadratic order, this action is the same as the dimensional reduction of the $D = 10$ $U(1)$ Maxwell action for A_m [28]. As argued in [28], for a system of N parallel D-branes the fields (A_s, X_a) become $U(N)$ matrices and the Maxwell action is generalised to the $D = 10$ Yang-Mills action reduced to $p+1$ dimensions. This action should, in general, be corrected by higher-order terms which, as in the abelian case, should be determined by the dimensional reduction of the open string effective action.[6] It is natural to expect that the most important part of these corrections is represented by the NBI action (3), i.e. by the following generalisation of (19)

$$I_p = T_p' \int d^{p+1}x \ \mathrm{STr}\sqrt{-\det(\eta_{mn} + T^{-1}F_{mn})} \tag{20}$$

$$= T_p' \int d^{p+1}x \ \mathrm{STr}\big[\sqrt{-\det(\eta_{rs} + D_r X_a(\delta_{ab} + T[X_a, X_b])^{-1} D_s X_b + T^{-1}F_{rs})}$$

$$\times \sqrt{\det(\delta_{ab} + T[X_a, X_b])}\ \big] \ .$$

Here STr applies to the products of components of the field strength F_{mn}, i.e. $F_{ab} = T^2[X_a, X_b]$, $F_{ra} = TD_r X_a = T(\partial_r X_a - i[A_r, X_a])$ and $F_{rs}(A)$. Expanding in powers of $[X_a, X_b]$ (e.g., assuming that the D-branes are not too close) we find (cf. [24])

$$I_p = T_p' \int d^{p+1}x \ \mathrm{STr}\big[\sqrt{-\det(\eta_{rs} + D_r X_a D_s X_a + T^{-1}F_{rs})} + \tfrac{1}{4}T^2([X_a, X_b])^2 + ...\big] \ . \tag{21}$$

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[6] For a discussion of D-brane equations in the non-abelian case (using the conformal invariance approach as in [25]) see [29]. One may also generalise to the non-abelian case the equivalent but more straightforward partition function approach of [27].

Like the abelian BI action (19) with all higher-order F^n terms included which grasps some important features of D-brane dynamics (e.g., a relation between the existence of limiting velocity and maximal field strength [26]) the NBI action (20) may also find some useful applications, provided one understands the regions of applicability of different expansions used.

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